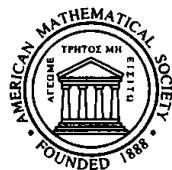


ESSAYS ON NUMBERS AND FIGURES

V. V. Prasolov



American Mathematical Society

Прасолов В. В.

Рассказы о числах, многочленах и фигурах

ФАЗИС. 1997

Translated from the Russian by A. B. Sossinski

2000 *Mathematics Subject Classification*. Primary 00A99.

ABSTRACT Each essay begins with a vivid result, accessible to high school students, which is then developed to lead up to current research in number theory, geometry, polynomial algebra, and topology. Many of the topics presented have a long and interesting history. Intended for math-oriented high school students, math teachers at the secondary school and university level, college undergraduates and all lovers of mathematics.

Library of Congress Cataloging-in-Publication Data

Prasolov, V. V. (Viktor Vasil'evich)

[*Rasskazy o chislakh, mnogochlenakh, i figurakh* English]

Essays on numbers and figures / V. V. Prasolov ; [translated from the Russian by A. B. Sossinski]

p. cm. (Mathematical world, ISSN 1055-9426 ; v. 16)

ISBN 0-8218-1944-5 (ACID-FREE)

1 Mathematics. I Title II Series

QA7.P713 2000

510 dc21

99-058690

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Assistant to the Publisher, American Mathematical Society, P. O. Box 6248, Providence, Rhode Island 02940-6248. Requests can also be made by e-mail to reprint-permission@ams.org.

© 2000 by the American Mathematical Society. All rights reserved.

The American Mathematical Society retains all rights
except those granted to the United States Government
Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.

Visit the AMS home page at URL. <http://www.ams.org/>

10 9 8 7 6 5 4 3 2 1 05 04 03 02 01 00

Contents

| | |
|--|----|
| Foreword | ix |
| Chapter 1. Conjugate Numbers | 1 |
| Chapter 2. Rational Parametrizations of the Circle | 5 |
| Chapter 3. Sums of Squares of Polynomials | 7 |
| Chapter 4. Representing Numbers as the Sum of Two Squares | 9 |
| Chapter 5. Can Any Knot Be Unraveled? | 11 |
| Chapter 6. Construction of a Regular 17-gon | 15 |
| Chapter 7. The Markov Equation | 19 |
| Chapter 8. Integer-Valued Polynomials | 21 |
| Chapter 9. Chebyshev Polynomials | 23 |
| Chapter 10. Vectors in Geometry | 25 |
| Chapter 11. The Averaging Method and Geometric Inequalities | 27 |
| Chapter 12. Intersection Points of the Diagonals of Regular Polygons | 33 |
| Chapter 13. The Chromatic Polynomial of a Graph | 39 |
| Chapter 14. Brocard Points | 43 |
| Chapter 15. Diophantine Equations for Polynomials | 51 |
| Chapter 16. The Pascal Lines | 55 |
| Chapter 17. One Butterfly and Two Butterflies Theorems | 59 |
| Chapter 18. The Van der Waerden Theorem on Arithmetical Progression | 61 |
| Chapter 19. Isogonal Conjugate Points | 65 |
| Chapter 20. Cubic Curves Related to the Triangle | 71 |

Foreword

This book contains twenty essays, each of which deals with a separate mathematical topic. It may be a brilliant mathematical statement with an interesting proof; or a simple, but effective method of problem solving; or an interesting property of polynomials; or it may refer to exceptional points of the triangle.

In its time, when I first came across each of these topics, I was seriously impressed and was led to reflect about them. I later returned, more than once, to many of them, and each time found something new to think about.

The twenty essays are, for the most part, independent of each other. The only essential exception is the last essay about cubic curves related to a triangle. This essay is based on a previous one, the one about isogonal conjugation with respect to a triangle. But the topic of cubic curves related to triangles is too difficult to be developed without preliminaries anyway.

I have lectured on all these topics to high school students in Moscow, Tel-Aviv, Haifa, and Cheliabinsk.

Conjugate Numbers

If we compute a few of the first terms of the sequence $a_n = (2 + \sqrt{3})^n$, it is easy to see that as n increases more and more, the number a_n differs from an integer less and less:

$$\begin{aligned} a_1 &\approx 3,7321 \\ a_2 &\approx 13,9282 \\ a_3 &\approx 51,9808 \\ a_4 &\approx 193,9948 \\ a_5 &\approx 723,9986 \\ a_6 &\approx 2701,9996. \end{aligned}$$

This strange behavior becomes somewhat easier to understand after we compute the first few terms of the sequence $b_n = (2 - \sqrt{3})^n$. Indeed, we have

$$\begin{aligned} b_1 &\approx 0,2679 \approx 4 - a_1 \\ b_2 &\approx 0,0718 \approx 14 - a_2 \\ b_3 &\approx 0,0192 \approx 52 - a_3 \\ b_4 &\approx 0,0052 \approx 194 - a_4 \\ b_5 &\approx 0,0014 \approx 724 - a_5 \\ b_6 &\approx 0,0004 \approx 2702 - a_6. \end{aligned}$$

The decrease of the numbers b_n is obvious, since $|2 - \sqrt{3}| < 1$. Thus, in order to explain why the numbers a_n behave as they do, it suffices to prove that all the numbers $a_n + b_n$ are integers.

To that end let us first note that no number can be represented in two different ways in the form $m + n\sqrt{3}$, where m and n are rational numbers. Indeed, let $m + n\sqrt{3} = m_1 + n_1\sqrt{3}$, where the numbers m , n , m_1 , and n_1 are rational. If $m_1 \neq m$ or $n_1 \neq n$, then

$$\sqrt{3} = \frac{m - m_1}{n_1 - n}.$$

But this equality cannot be true, because $\sqrt{3}$ is an irrational number.

Thus, for each number z of the form $m + n\sqrt{3}$, where m and n are rational numbers, we can define the conjugate number $\bar{z} = m - n\sqrt{3}$. It is easy to check that $\bar{z}_1 \bar{z}_2 = \overline{z_1 z_2}$, i.e., if

$$(a + b\sqrt{3})(c + d\sqrt{3}) = A + B\sqrt{3},$$

then

$$(a - b\sqrt{3})(c - d\sqrt{3}) = A - B\sqrt{3}.$$

Using this property, we see that

$$(a + b\sqrt{3})^n = A_n + B_n\sqrt{3}$$

implies

$$(a - b\sqrt{3})^n = A_n - B_n\sqrt{3}.$$

Therefore,

$$a_n + b_n = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n = 2A_n$$

is an integer.

The behavior of the fractional parts of the numbers $a_n = (p + \sqrt{q})^n$, where p and q are positive integers and $0 < p - \sqrt{q} < 1$ is quite similar. For the case where $-1 < p - \sqrt{q} < 0$, the situation is somewhat more complicated. This is because although the numbers $(p + \sqrt{q})^n + (p - \sqrt{q})^n$ are integers for all natural n , for even n the number $(p - \sqrt{q})^n$ is positive, whereas for odd n the number $(p - \sqrt{q})^n$ is negative.

Let a and b be rational numbers, let N be a positive integer which is not an exact square. Then the equality $(a + b\sqrt{N})^n = A + B\sqrt{N}$ implies that

$$(a - b\sqrt{N})^n = A - B\sqrt{N}.$$

We have proved this statement only for $N = 3$, but in the general case the proof is completely similar. We have already discussed one of the applications of this statement. Now let us consider a few more applications.

1. *For rational numbers x, y, z , and t the relation*

$$(x + y\sqrt{2})^2 + (z + t\sqrt{2})^2 = 5 + 4\sqrt{2}$$

is impossible.

Indeed, if the above relation holds for the rational numbers x, y, z , and t , then for the same numbers we will have

$$(x - y\sqrt{2})^2 + (z - t\sqrt{2})^2 = 5 - 4\sqrt{2}.$$

But $5 - 4\sqrt{2} < 0$, while $(x - y\sqrt{2})^2 + (z - t\sqrt{2})^2 \geq 0$.

2. *For positive integers m and n the relation $(5 + 3\sqrt{2})^m = (3 + 5\sqrt{2})^n$ cannot hold.*

Indeed, let $(5 + 3\sqrt{2})^m = (3 + 5\sqrt{2})^n$. Then $(5 - 3\sqrt{2})^m = (3 - 5\sqrt{2})^n$. A contradiction can now be obtained in different ways. The first is to notice that $|5 - 3\sqrt{2}| < 1$ and $|3 - 5\sqrt{2}| > 1$. The second is to multiply the equalities $(5 + 3\sqrt{2})^m = (3 + 5\sqrt{2})^n$ and $(5 - 3\sqrt{2})^m = (3 - 5\sqrt{2})^n$, obtaining $7^m = (-41)^n$.

3. *The equation $x^2 - 2y^2 = 1$ has infinitely many solutions.*

It is easy to find one solution, say $x_1 = 3$ and $y_1 = 2$. The equality $x_1^2 - 2y_1^2 = 1$ can be rewritten as

$$(x_1 - y_1\sqrt{2})(x_1 + y_1\sqrt{2}) = 1.$$

Clearly, we then have

$$(x_1 - y_1\sqrt{2})^n(x_1 + y_1\sqrt{2})^n = 1.$$

Further, $(x_1 \pm y_1\sqrt{2})^n = x_n \pm y_n\sqrt{2}$. Hence,

$$x_n^2 - 2y_n^2 = 1,$$

i.e., (x_n, y_n) is a solution also, and $x_{n+1} > x_n, y_{n+1} > y_n$.

Let us state without proof several properties of solutions of the equations

$$x^2 - dy^2 = c$$

in positive integers (here d is an integer which is not an exact square). The equation $x^2 - dy^2 = 1$ has solutions for all d ; if (x_1, y_1) is the smallest solution (i.e., a solution with the smallest x_1 or smallest y_1 , which of the two makes no difference), then any other positive integer solution is of the form (x_n, y_n) , where

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n.$$

For $c \neq 1$ the equation $x^2 - dy^2 = c$ may have no positive integer solution. For example, the equations $x^2 - 2y^2 = 3$ and $x^2 - 5y^2 = 2$ do not have any positive integer solutions.

If d is a prime number of the form $4k + 3$, then the equation $x^2 - dy^2 = -1$ has no positive integer solutions, while if d is a prime of the form $4k + 1$, then the equation $x^2 - dy^2 = -1$ always has positive integer solutions.

Let (x_1, y_1) be the smallest positive integer solution of the equation $x^2 - dy^2 = 1$ and (x', y') be some positive integer solution of the equation $x^2 - dy^2 = c$. Then the numbers (x'_n, y'_n) , where

$$x'_n + y'_n\sqrt{d} = (x' + y'\sqrt{d})(x_1 + y_1\sqrt{d})^{n-1},$$

will be solutions of the equation $x^2 - dy^2 = c$. Further, all solutions of the equation $x^2 - dy^2 = c$ constitute a finite family of sequences (x'_n, y'_n) , originating from the solution (x', y') . But for $c \neq 1$ there may be several such sequences. For example, the solutions of the equation $x^2 - 2y^2 = 7$ constitute two disjoint sequences, originating from the solutions $(3, 1)$ and $(5, 3)$.

Rational Parametrizations of the Circle

In order to find points on the circle $x^2 + y^2 = 1$ whose coordinates are both rational, the following construction may be used. Take an arbitrary point A with rational coordinates on the circle and draw all possible straight lines passing through it. For example, for the point A we can take $(1, 0)$ or $(3/5, 4/5)$. We shall perform the calculations for the point $(1, 0)$, but for any other point with rational coordinates on the circle the computations are similar.

A straight line passing through the point $(1, 0)$ is either given by the equation $y = t(x - 1)$, or by the equation $x = 1$ (the equation $x = 1$ may be obtained from the equation $y = t(x - 1)$ by putting $t = \infty$; hence it is convenient to agree that t can also assume the value ∞). In order to find the intersection of the line $y = t(x - 1)$ with the circle $x^2 + y^2 = 1$, we must solve the quadratic equation $x^2 + t^2(x - 1)^2 = 1$, i.e.,

$$x^2 - \frac{2t}{t^2 + 1}x + \frac{t^2 - 1}{t^2 + 1} = 0.$$

One intersection point of the line and the circle, the one corresponding to $x = 1$, is already known. Therefore the x -coordinate of the second point may be found by the Vieta theorem. As the result, we obtain

$$x = \frac{t^2 - 1}{t^2 + 1}, \quad y = t(x - 1) = -\frac{2t}{t^2 + 1}.$$

Thus the intersection point of the circle $x^2 + y^2 = 1$ and the line $y = t(x - 1)$ has the coordinates

$$\left(\frac{t^2 - 1}{t^2 + 1}, \frac{-2t}{t^2 + 1} \right).$$

This means that the point on the circle will have rational coordinates if and only if it corresponds to a rational value of the parameter t . Indeed, if the number t is rational, then so are the numbers $x = \frac{t^2 - 1}{t^2 + 1}$ and $y = \frac{-2t}{t^2 + 1}$. Conversely, if x and y are rational, then so is $t = \frac{y}{x - 1}$. It is convenient to consider the value $t = \infty$, corresponding to the point $(1, 0)$, as also being rational.

By using the rational parametrization of the circle one can describe all *Pythagorean triples*, i.e., triples of positive integers (a, b, c) , such that $a^2 + b^2 = c^2$. Namely, (a, b, c) will be Pythagorean triple if and only if there exist positive integers m and n , such that $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$ (or $a = 2mn$, $b = m^2 - n^2$, $c = m^2 + n^2$).

Let us begin the proof of this statement by getting rid of the common divisors. If two of the three numbers a, b, c are divisible by d , then the third will also be divisible by d . Hence we can assume that a/c and b/c are simple fractions. The

point $(a/c, b/c)$ lies on the circle $x^2 + y^2 = 1$ and both of its coordinates are rational. Therefore,

$$\frac{a}{c} = \pm \frac{t^2 - 1}{t^2 + 1}, \quad \frac{b}{c} = \pm \frac{2t}{t^2 + 1},$$

where t is a rational number. Let $t = m/n$ be a simple fraction. Then

$$\frac{a}{c} = \pm \frac{m^2 - n^2}{m^2 + n^2}, \quad \frac{b}{c} = \pm \frac{2mn}{m^2 + n^2}.$$

The fraction m/n is simple, so the numbers m and n cannot both be even. If one of them is even and the other odd, then the fractions $\frac{m^2 - n^2}{m^2 + n^2}$ and $\frac{2mn}{m^2 + n^2}$ are both simple. If both numbers m and n are odd, i.e., $m = 2p + 1$ and $n = 2q + 1$, then, as can readily be checked,

$$\frac{a}{c} = \pm \frac{2m_1 n_1}{m_1^2 + n_1^2}, \quad \frac{b}{c} = \pm \frac{m_1^2 - n_1^2}{m_1^2 + n_1^2},$$

where $m_1 = p + q + 1$ and $n_1 = p - q$. We have obtained simple fractions, since one of the numbers m_1 and n_1 is even, while the other is odd.

As we have already mentioned, if we take another rational point on the circle, similar calculations will yield a similar parametrization of the circle by a parameter t , and to rational values of the parameter will correspond rational points on the circle. Moreover, one may similarly parametrize any second order curve

$$(1) \quad ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

To this end one can choose an arbitrary point of the curve and draw all the straight lines passing through it.

Suppose that the following conditions are satisfied:

- the coefficients a, b, \dots, f are rational;
- the coordinates of the chosen point are rational;
- the curve (1) has infinitely many points.

Then as the result we obtain a parametrized curve (1) such that to rational values of t correspond points of the curve whose coordinates are both rational. Recall that we have used the fact that the curve (1) contains at least one point with rational coordinates. But such a point does not always exist. The simplest example is the curve $x^2 + y^2 = -1$.

A more instructive example is the curve $2x^2 + 5y^2 = 1$. Let us verify that it has no rational points. Assume that (x, y) is a rational point of this curve. We can assume that $x = p_1/q$ and $y = p_2/q$, where the numbers p_1, p_2, q have no common divisor (i.e., there is no number d that divides all three of these numbers).

The numbers p_1, p_2, q satisfy the relation $2p_1^2 + 5p_2^2 = q^2$. Hence $2p_1^2$ and q^2 have the same remainders under division by 5. But under division by 5 the number a^2 can only have the remainders ± 1 or 0. Hence $2p_1^2$ has the remainder ± 2 or 0 under division by 5, while q^2 has remainder ± 1 or 0. Therefore, both numbers $2p_1^2$ and q^2 are divisible by 5 and so are divisible by 25. But then $5p_2^2$ is divisible by 25, and so p_2 is divisible by 5. As the result we see that the numbers p_1 and p_2, q are divisible by 5, which contradicts the assumption.

Sums of Squares of Polynomials

It is easy to prove that any polynomial $p(x)$ with real coefficients that takes nonnegative values for all real x , can be represented as the sum of squares of two polynomials with real coefficients. To do this, first let us note that if $p(z) = 0$ and \bar{p} is a polynomial with real coefficients, then $p(\bar{z}) = \overline{p(z)} = 0$, i.e., \bar{z} is also a root of the polynomial p . Hence the roots of a polynomial with real coefficients are either real numbers or pairs of conjugate complex numbers. Therefore,

$$p(x) = a \prod_{j=1}^s (x - z_j)(x - \bar{z}_j) \prod_{k=1}^t (x - \alpha_k)^{m_k},$$

where the numbers α_k are real.

Now assume that we have $p(x) \geq 0$ for all real x . If x is large enough and positive, then

$$\prod_{j=1}^s (x - z_j)(x - \bar{z}_j) \prod_{k=1}^t (x - \alpha_k)^{m_k} > 0,$$

so that $a > 0$. Moreover, all the exponents m_k are even. Indeed, assume that some m_k are odd. We can suppose that only the numbers m_1, \dots, m_s are odd and $\alpha_1 < \dots < \alpha_s$. Then for $\alpha_{s-1} < x < \alpha_s$ we have the inequality

$$\prod_{k=1}^t (x - \alpha_k)^{m_k} < 0,$$

i.e., $p(x) < 0$. (If $s = 1$, then this inequality is satisfied for all $x < \alpha_s$.)

Thus the real roots also appear in pairs. Therefore,

$$p(x) = \left(\sqrt{a} \prod_{j=1}^l (x - z_j) \right) \left(\sqrt{a} \prod_{j=1}^l (x - \bar{z}_j) \right),$$

where some of the numbers z_j may be real. Let

$$\sqrt{a} \prod_{j=1}^l (x - z_j) = q(x) + ir(x),$$

where q and r are polynomials with real coefficients. Then

$$\sqrt{a} \prod_{j=1}^l (x - \bar{z}_j) = q(x) - ir(x).$$

Finally we obtain $p(x) = (q(x))^2 + (r(x))^2$.

But for polynomials in several variables a similar statement is not always true, i.e., there exist *nonnegative* polynomials (by that we mean polynomials with real

coefficients that take nonnegative values for all real values of the variables) that cannot be represented as the sum of squares of polynomials with real coefficients. This was first proved by Hilbert in 1888, but he did not give an explicit example of such a polynomial. The first simple example was presented by T. Motzkin in 1967. Namely, he showed that the polynomial

$$F(x, y) = x^2y^2(x^2 + y^2 - 3) + 1$$

is nonnegative but cannot be represented as the sum of squares of polynomials with real coefficients. The main difficulty, of course, was to find such a polynomial, while the actual proof of its properties, as we shall soon see, is not difficult.

The fact that F is nonnegative, follows from the equality

$$F(x, y) = \frac{(1 - x^2y^2)^2 + x^2(1 - y^2)^2 + x^2(1 - x^2)^2y^2}{1 + x^2}.$$

Now let us assume that $F(x, y) = \sum f_j(x, y)^2$, where the f_j are polynomials with real coefficients. Then $\sum f_j(x, 0)^2 = F(x, 0) = 1$. If at least one of the polynomials $f_j(x, 0)$ in the variable x is not a constant, then $\sum f_j(x, 0)^2$ is a polynomial whose degree is twice that of the highest degree of the polynomials $f_j(x, 0)$. Therefore, $f_j(x, 0) = c_j$ is a constant, so that $f_j(x, y) = c_j + yg_j(x, y)$. A similar argument shows that $f_j(x, y) = c'_j + xg'_j(x, y)$. Clearly $c_j = c'_j$ and $f_j(x, y) = c_j + xyh_j(x, y)$, while

$$\deg h_j = \deg f_j - 2 \leq \frac{1}{2} \deg F - 2 = 1;$$

here $\deg f$ is the degree of the polynomial f . Thus,

$$x^2y^2(x^2 + y^2 - 3) + 1 = x^2y^2 \sum h_j^2 + 2xy \sum c_j h_j + \sum c_j^2,$$

i.e.,

$$x^2y^2(x^2 + y^2 - 3) - x^2y^2 \sum h_j^2 = 2xy \sum c_j h_j + \sum c_j^2 - 1.$$

All the monomials in the right-hand side of this equation are of degree at least 3, while those in the left-hand side are of degree at most 4. Therefore, $x^2y^2(x^2 + y^2 - 3) - x^2y^2 \sum h_j^2 = 0$, and so $x^2 + y^2 - 3 = \sum h_j^2$. This is a contradiction, because $x^2 + y^2 - 3 < 0$ when $x = y = 0$.

We have discussed only the simplest known results about the sum of squares. At present a great deal is known on the subject, but many questions still remain unanswered. The German mathematician David Hilbert conjectured that any nonnegative polynomial in several variables can be represented as the sum of squares not of polynomials, but of rational functions. He included the question: "Is this true or not?" under number 17 in his famous list of 23 problems. Hilbert's Seventeenth Problem was solved in 1927 by E. Artin. He proved that any nonnegative polynomial can be represented as the sum of squares of a certain number of rational functions. In 1967 A. Pfister made Artin's theorem more precise; he showed that any nonnegative polynomial in n variables can be represented as the sum of 2^n squares of rational functions. But it is still unknown if 2^n can be replaced by a smaller number. Only for $n = 2$ is it known that the number $2^n = 4$ is in this case minimal. Actually Cassels, Ellison, and Pfister showed in 1971 that the same polynomial $F(x, y)$ that we considered above cannot be represented as the sum of squares of three rational functions. Their proof is quite difficult; it is based on deep results from the theory of elliptic curves.

Representing Numbers as the Sum of Two Squares

In one of his letters in 1658 Pierre Fermat wrote that he had obtained an “indisputable proof” of the fact that any prime number of the form $4k + 1$ is the sum of two squares and any number is the sum of no more than four squares. No written record of the proof remained after Fermat. A hundred years passed, and Leonard Euler became interested in these theorems. He proved the first one in 1747, and two years later found another elegant and relatively simple proof of the same theorem. The second theorem was proved only in 1770 by Lagrange, and then its proof was considerably simplified by Euler.

Later other interesting proofs of the fact that prime numbers of the form $p = 4k + 1$ can be represented as the sum of two squares were found; all of them, like Euler’s proof, were not quite elementary. And only recently¹, a very elegant and perfectly elementary proof of this theorem was obtained.

The main role in the proof is played by the simple but very important notion of involution. Let M be a set. A map $\sigma : M \rightarrow M$ is said to be an *involution* if $\sigma(\sigma(m)) = m$ for any element m of the set M . Any symmetry is an example of an involution. An involution combines points (elements of the set M) into pairs $\{m, \sigma(m)\}$ of “symmetric” points. For the element $\sigma(m)$ we obtain the same pair as for the element m , since $\sigma(\sigma(m)) = m$. No pair of elements arises only if $\sigma(m) = m$ (such points m are called *fixed*). Therefore if an involution of a finite set M is given, then the number of fixed points of M is even whenever the total number of points of M is even (since the other points appear in pairs, so that their number is certainly even).

The general outline of the proof is as follows: consider the set of all solutions of the equations $x^2 + 4yz = p$ in positive integers. It suffices to prove that if $p = 4k + 1$, then this equation has a solution for which $y = z$. Indeed, in that case $p = x^2 + (2y)^2$. A solution for which $y = z$ is a fixed point of the involution $\sigma(x, y, z) = (x, z, y)$. So we need only prove that the total number of solutions is odd. To this end we construct a completely different involution τ with precisely one fixed point. Here is this involution:

$$\begin{aligned} (1) \quad & (x + 2z, z, y - x - z), & x < y - z; \\ (2) \quad \tau(x, y, z) = & \begin{cases} (2y - x, y, x - y + z), & y - z < x < 2y; \\ (x - 2y, x - y + z, y), & 2y < x. \end{cases} \end{aligned}$$

First note that $x \neq 2y$ and $x \neq y - z$, since otherwise we would have

$$p = x^2 + 4yz = 4(y^2 + yz) \quad \text{or} \quad p = (y - z)^2 + 4yz = (y + z)^2.$$

¹D Zagier, *A one-sentence proof that every prime $p \equiv 1 \pmod{4}$ is a sum of two squares*, Amer Math. Monthly **97** (1990), 114.

Let us check that any solution of the equation $x^2 + 4yz = p$ is indeed taken to a solution by the involution τ . This follows from the identities

$$(x + 2z)^2 + 4z(y - x - z) = x^2 + 4yz,$$

$$(x - 2y)^2 + 4y(x - y + z) = x^2 + 4yz.$$

Let $\tau(x, y, z) = (x', y', z')$. If $x < y - z$, then $x' = x + 2z > 2z = 2y'$, i.e., a point of type (1) is taken to point of type (3). Similarly one checks that a point of type (3) is taken to a point of type (1), while a point of type (2) goes to a point of type (2). After that, it is easy to see that τ is an involution.

A point of type (1) cannot be fixed, since $x' = x + 2z > x$; for a point of type (3), we have $x' = x - 2y < x$. Hence only a point of type (2) can be fixed, and it must satisfy the relation $z = z' = x - y + z$, i.e., $x = y$. In that case $p = x^2 + 4yz = x(x + 4z)$, so that $x = y = 1$ (here we use the fact that p is prime). If $p = 4k + 1$, then $(1, 1, k)$ is a fixed point (here we use the fact that p is of the form $4k + 1$). This completes the proof.

Let us continue with a few more remarks concerning the representation of numbers as the sum of two squares. First note that a prime number of the form $4k + 3$ cannot be represented as the sum of two squares. Indeed, the square of $2m + 1$ equals $4(m^2 + m) + 1$, and so its remainder under division by 4 equals 1. Therefore, the remainder under division by 4 of the sum of two squares can equal 0, 1, or 2, but certainly not 3.

The condition that $p = 4k + 1$ be prime is also essential. For example, 21 cannot be represented as the sum of two squares.

Let us also note that if $m = a^2 + b^2$ and $n = c^2 + d^2$, then

$$mn = (ac + bd)^2 + (ad - bc)^2$$

is also the sum of two squares. Hence the product of prime numbers of the form $4k + 1$ can be represented as the sum of two squares. But if the factor $4k + 3$ raised to an odd power appears in a positive integer, then this integer cannot be represented as the sum of two squares. This follows, for example, from the fact that the divisibility of $a^2 + b^2$ by a prime number of the form $p = 4k + 3$ implies that a and b are also divisible by p . (The proof of this statement follows from Fermat's Small Theorem.)

Can Any Knot Be Unraveled?

A knot may be imagined as a string whose ends are joined together after the string has been tangled in some way. The simplest examples of knots are shown in Figure 1. The knot shown in Figure 1(b) is known as the *trefoil*.

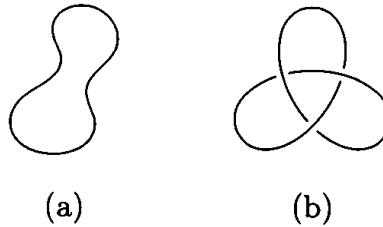


FIGURE 1

After the ends of the string have been joined, one may try to untangle it or tangle it up further, of course without tearing the string. If you try to “unravel” the trefoil (i.e., to transform it into the knot shown in Figure 1(a)), you will soon see that this cannot be done. But how does one prove that no succession of tangling and untangling manipulations will unravel the trefoil? The first proofs of this fact were based on quite complicated techniques from algebraic topology. It is only recently, in the eighties, that elementary proofs of the fact that the trefoil (and some other knots) cannot be unraveled appeared. Such a proof will now be described.

Any knot can be represented by its projection on some plane, showing which branches are higher and which are lower by interrupting the line depicting the lower branch. Such a picture is called a *knot diagram*; an example of a knot diagram for the trefoil is Figure 1(b).

Any knot diagram consists of several arcs; for instance, the diagram in Figure 1(b) consists of three arcs. Consider all possible colorings of a knot diagram in three colors, each arc being painted in one color. Let us call a coloring *correct* if at each crossing the three arcs either all have the same color, or have three different colors (i.e., two arcs of the same color and a third one of a different color cannot meet at a crossing).

THEOREM. *The number of correct colorings of any knot diagram does not change when we tangle or untangle the knot.*

PROOF. We shall use without proof the following rather obvious statement: the transformation of a knot diagram when we tangle or untangle it reduces to carrying out a succession of operations of the three types shown on Figure 2. (A rigorous proof of this statement can be obtained by replacing the knot by a closed polygonal line and following what happens to its projection in the tangling untangling process.)

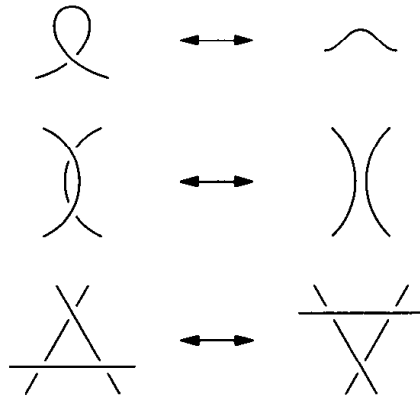


FIGURE 2

We must check that under the operations shown above each correct coloring becomes a well-defined correct coloring of the transformed diagram. Let us explain what we have in mind. Suppose a correct coloring of the diagram is given. Consider the diagram obtained from the given one by one of the three operations indicated above. This operation is performed within a circular disk D . Let us leave the coloring outside the the disk D . The arcs of the transformed diagram intersecting the boundary of the disk will be colored in some way. We must verify that this coloring can be extended to a correct coloring of all the arcs inside the disk D and that this coloring is unique.

For colorings in one color this is obvious. For all essentially distinct correct colorings in more than one color, the colorings of the corresponding transformed diagrams are shown in Figure 3. The proof of the theorem is complete. \square

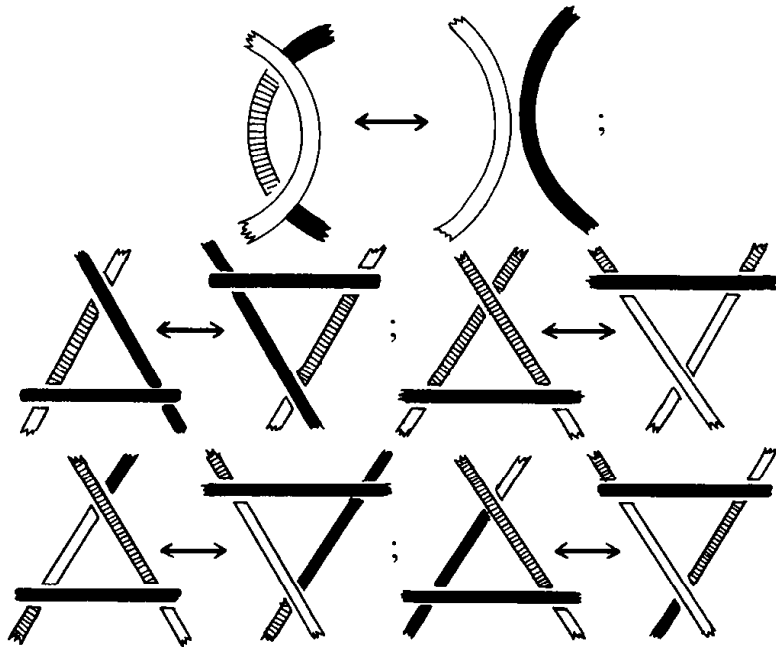


FIGURE 3

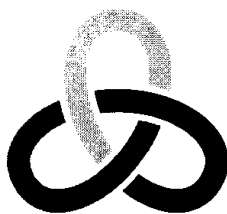


FIGURE 4

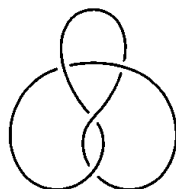


FIGURE 5



FIGURE 6

Now we can prove that the trefoil cannot be unraveled. Figure 4 shows a correct coloring of a diagram of the trefoil in which all three colors are used. But the unknotted circle shown in Figure 1(a) can only be colored in one color.

Unfortunately, by using correct colorings of knot diagrams in three colors, it is not always possible to establish that a knot cannot be unraveled. For example, the *figure eight knot*, shown in Figure 5 cannot be unraveled although all of its diagrams possess only monochromatic colorings in three colors, just as the diagrams of the unknotted circle.

In order to prove that the figure eight knot cannot be unraveled, one can use correct colorings in five colors. It is then convenient to regard the colors as remainders under division by 5. We shall say that a coloring of a knot diagram in five colors is *correct* if at each crossing we have the relation

$$b + c \equiv 2a \pmod{5}$$

(see Figure 6); this expression means that the numbers $b + c$ and $2a$ have the same remainder under division by 5. In a similar way one can define a correct coloring of a knot diagram in n colors.

By looking at Figure 7 it is easy to check that the number of correct colorings of a knot diagram in n colors does not change in the process of tangling and untangling a knot.

Figure 8 shows that the diagram of the figure eight knot has a multicolor correct coloring in five colors. Hence the figure eight knot cannot be unraveled.

It is not hard to check that all the correct colorings of the trefoil in five colors are monochromatic, so that the trefoil and the eight are different knots (they cannot be transformed into each other).

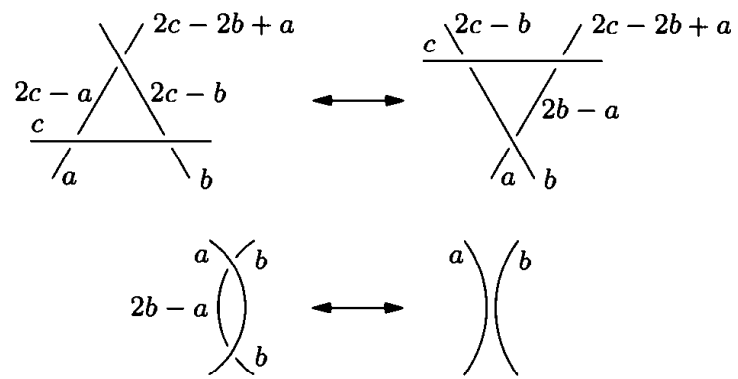


FIGURE 7

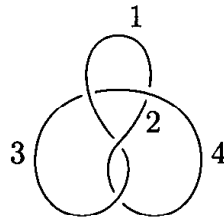


FIGURE 8

Construction of a Regular 17-gon

In Ancient Greece geometers used various instruments for constructing figures, but the main instruments used were the ruler and compass. And in the most famous ancient geometry textbook, Euclid's "Elements", no other instruments are considered. For that reason by a geometric construction one ordinarily means a construction with ruler and compass. The compass allows one to draw a circle of given radius centered at a given point, while the ruler allows one to draw the straight line passing through two given points.

Great attention is paid in Euclid's "Elements" to the construction of regular polygons. The construction of the regular triangle is described in Proposition 1 in Book I, while almost all of Book IV is devoted to the construction of other regular polygons: the square, the pentagon, the hexagon, the polygon of fifteen sides. For a very long time, nothing essentially new was added to this by geometers until 1786, when the nineteen year old Gauss showed that the regular polygon of 17 sides can be constructed by ruler and compass. Later he proved that if the number $n = 2^{2^k} + 1$ is prime, then one can construct a regular n -gon by ruler and compass. Gauss also claimed that he was able to prove that a regular n -gon can be constructed only if $n = 2^m p_1 \dots p_s$, where p_1, \dots, p_s are distinct prime numbers of the form $2^{2^k} + 1$. But none of his papers contains a proof of this claim.

The construction of the regular 17-gon involves much more algebra than geometry. The de Moivre formula

$$(\cos \alpha + i \sin \alpha)^n = \cos n\alpha + i \sin n\alpha$$

shows that the roots of the equation $x^n - 1 = 0$ are of the form

$$\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}.$$

On the complex plane, these points form the vertices of the regular n -gon. Thus the construction of the regular n -gon is closely connected with the solution of the equation $x^n - 1 = 0$. This equation has the obvious root $x = 1$. To get rid of it, let us divide the polynomial $x^n - 1$ by $x - 1$. The result will be the polynomial

$$x^{n-1} + x^{n-2} + \dots + x^2 + x + 1.$$

It is this polynomial that we shall consider below.

Let us say that the complex number $a + ib$ can be constructed if it is possible to construct line segments of lengths a and b , i.e., to construct the point $a + ib$ on the complex plane. If a unit segment is given, then by using the given segments a and b we can construct the segments $a \pm b$, ab , a/b , \sqrt{a} . It is easy to check that

$$\sqrt{a + ib} = \pm \left[\left(\frac{\sqrt{a^2 + b^2} + a}{2} \right)^{1/2} + i \left(\frac{\sqrt{a^2 + b^2} - a}{2} \right)^{1/2} \right].$$

Therefore, if a and b can be constructed, then $\sqrt{a+ib}$ can also be constructed. Thus we see that if the numbers u and v can be constructed, then so can the roots of the equation $x^2 + ux + v = 0$.

To begin with, let us describe the construction of the regular n -gons for $n = 3$ and $n = 5$ from this point of view. For $n = 3$ we obtain the equation $x^2 + x + 1 = 0$. This equation is quadratic, so its roots can be constructed.

For $n = 5$ we get the equation $x^4 + x^3 + x^2 + x + 1 = 0$. Set $u = x + x^{-1}$. It is easy to check that $u^2 + u - 1 = 0$. The roots u_1 and u_2 of this equation can be constructed. Using them, one can construct the roots of the equations $x^2 - u_1x + 1 = 0$ and $x^2 - u_2x + 1 = 0$. These roots are roots of the initial equation.

Now we can proceed to the construction of the regular 17-gon. To do this, as we have shown above, it suffices to prove that the roots of the equation

$$x^{16} + x^{15} + \cdots + x + 1 = 0$$

may be obtained by successively solving quadratic equations. It actually suffices to find one root

$$\varepsilon = \cos \frac{2\pi}{17} + i \sin \frac{2\pi}{17};$$

the others will have the form $\varepsilon^2, \varepsilon^3, \dots, \varepsilon^{16}$. Gauss' proof is based on a special numbering of these roots. He numbers them in such a way that for a fixed l , from ω_k (the root with number k) we obtain ω_{k+l} according to the same rule, namely by raising ω_k to a fixed power. Such a numbering can be obtained by setting $\omega_k = \varepsilon^{g^k}$, where the numbers $1, g, g^2, g^3, \dots, g^{15}$ all have different remainders under division by 17. Indeed, in that case

$$\omega_{k+l} = \varepsilon^{g^{k+l}} = \varepsilon^{g^k g^l} = (\omega_k)^{g^l}.$$

For g we can take, say, the number 3. Then

$$\begin{aligned} \omega_0 = \varepsilon, \omega_1 = \varepsilon^3, \omega_2 = \varepsilon^9, \omega_3 = \varepsilon^{10}, \omega_4 = \varepsilon^{13}, \omega_5 = \varepsilon^5, \omega_6 = \varepsilon^{15}, \omega_7 = \varepsilon^{11}, \\ \omega_8 = \varepsilon^{16}, \omega_9 = \varepsilon^{14}, \omega_{10} = \varepsilon^8, \omega_{11} = \varepsilon^7, \omega_{12} = \varepsilon^4, \omega_{13} = \varepsilon^{12}, \omega_{14} = \varepsilon^2, \omega_{15} = \varepsilon^6. \end{aligned}$$

First let us consider the quadratic equation with roots

$$x_1 = \sum_{k=0}^7 \omega_{2k}, \quad x_2 = \sum_{k=0}^7 \omega_{2k+1}.$$

By the Vieta theorem, $\sum \omega_i = -1$, hence $x_1 + x_2 = -1$. It is easy to check that

$$\begin{aligned} x_1 &= (\varepsilon + \varepsilon^{16}) + (\varepsilon^8 + \varepsilon^9) + (\varepsilon^2 + \varepsilon^{15}) + (\varepsilon^4 + \varepsilon^{13}) \\ &= 2(\cos \alpha + \cos 8\alpha + \cos 2\alpha + \cos 4\alpha), \end{aligned}$$

$$x_2 = 2(\cos 3\alpha + \cos 7\alpha + \cos 5\alpha + \cos 6\alpha),$$

where $\alpha = 2\pi/17$. Using the formula

$$2 \cos p\alpha \cos q\alpha = \cos(p+q)\alpha + \cos(p-q)\alpha,$$

we obtain

$$x_1 x_2 = 8(\cos \alpha + \cos 2\alpha + \cos 3\alpha + \cdots + \cos 8\alpha) = 4(x_1 + x_2) = -4.$$

Thus x_1 and x_2 satisfy the quadratic equation $x^2 + x - 4 = 0$ with integer coefficients. Therefore x_1 and x_2 can be constructed.

Now let us consider the quadratic equation with roots

$$y_1 = \sum_{k=0}^3 \omega_{4k} = 2(\cos \alpha + \cos 4\alpha), \quad y_2 = \sum_{k=0}^3 \omega_{4k+2} = 2(\cos 8\alpha + \cos 2\alpha).$$

It is easy to check that $y_1 + y_2 = x_1$ and

$$y_1 y_2 = 2(\cos \alpha + \cos 2\alpha + \cos 3\alpha + \cdots + \cos 8\alpha) = x_1 + x_2 = -1.$$

Thus y_1 and y_2 are the roots of the quadratic equation $y^2 - x_1 y + 1 = 0$. Similarly one proves that

$$y_3 = \sum_{k=0}^3 \omega_{4k+1} = 2(\cos 3\alpha + \cos 5\alpha), \quad y_4 = \sum_{k=0}^3 \omega_{4k+3} = 2(\cos 7\alpha + \cos 6\alpha)$$

are the roots of the quadratic equation $y^2 - x_2 y + 1 = 0$. Therefore, y_1, y_2, y_3 and y_4 can be constructed.

Finally, consider the quadratic equation with the roots

$$z_1 = \omega_0 + \omega_8 = 2 \cos \alpha, \quad z_2 = \omega_4 + \omega_{12} = 2 \cos 4\alpha.$$

Clearly, we have $z_1 + z_2 = y_1$ and $z_1 z_2 = 2(\cos 5\alpha + \cos 3\alpha) = y_3$. Thus z_1 and z_2 are the roots of the quadratic equation $z^2 - y_1 z + y_3 = 0$. Hence $z_1 = 2 \cos \alpha$ can be constructed, and therefore so can $\varepsilon = \cos \alpha + i \sin \alpha$. But then the regular polygon of 17 sides can be constructed.

The Markov Equation

The relation

$$(1) \quad m^2 + n^2 + p^2 = 3mnp,$$

where m , n , and p are positive integers, is known as the *Markov equation*. It is interesting that it has infinitely many solutions, and although the structure of the solutions is complicated, this structure is quite understandable.

First let us assume that among the numbers m , n , and p some are equal, say $n = p$. Then $m^2 + 2n^2 = 3mn^2$, i.e., $3m = (m/n)^2 + 2$. Therefore, $m = dn$, where d is an integer. Here $d^2 + 2 = 3nd$, i.e., $d(3n - d) = 2$. Hence $d = 1$ or 2 . In both cases $n = 1$. As the result we obtain the solutions $(1, 1, 1)$ and $(2, 1, 1)$. Let us call them *singular*.

Now take a nonsingular solution (m, n, p) for which the integers m, n, p are pairwise distinct, and consider the quadratic trinomial

$$f(x) = x^2 - 3xnp + n^2 + p^2.$$

Obviously $f(m) = 0$, i.e., one root of f equals m . Its second root m' may be found by using the Vieta theorem: $m' = 3np - m$. Clearly, (m', n, p) will then be a solution of the equation (1). Let us show that the largest of the numbers n and p lies between m and m' . To be definite, let $n > p$. Then

$$(n - m)(n - m') = f(n) = 2n^2 + p^2 - 3n^2p < 0.$$

But this means precisely that n lies between m and m' .

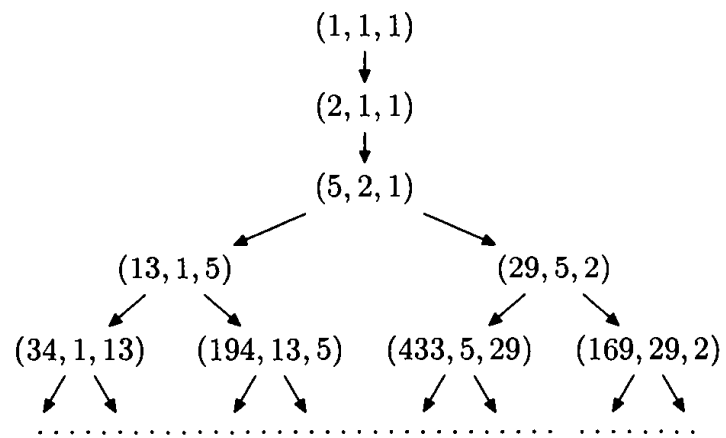
In a similar way, the solution (m, n, p) may be used to construct the solutions (m, n', p) and (m, n, p') .

Assume that m is the largest one of the numbers m, n, p . Then

$$m > \max(n, p) > m', \quad n < m = \max(m, p) < n'.$$

Thus when we pass from the solution (m, n, p) to the solution (m', n, p) , the largest of the three integers decreases, while if we pass to the solutions (m, n', p) and (m, n, p') it increases.

If we begin with the solution $(1, 1, 1)$, we obtain the following tree of solutions; this tree contains all the solutions, since an arbitrary solution will become a singular solution after several decreases of its maximal integer. By the decrease of its maximal integer we mean the passage from the solution (m, n, p) , for which $m > \max(n, p)$, to the solution (m', n, p) .



Integer-Valued Polynomials

A polynomial $p(x)$ is called *integer-valued*, if it takes integer values for all integers x .

It can be shown by induction on k that the polynomial

$$\binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!}$$

is integer-valued. Indeed, for $k = 1$ this is obvious. Assume that the polynomial $\binom{x}{k}$ is integer-valued. It is easy to verify that

$$\binom{x+1}{k+1} - \binom{x}{k+1} = \binom{x}{k}.$$

Therefore for all integers m, n the difference

$$\binom{m}{k+1} - \binom{n}{k+1}$$

is an integer. It remains to note that $\binom{0}{k+1} = 0$.

In a certain sense integer-valued polynomials all reduce to the polynomials $\binom{x}{k}$; what is more, the requirement that $p(n)$ be an integer for all integer n may be significantly weakened. Indeed, we have the following statement.

THEOREM. *Let $p_k(x)$ be a polynomial of degree k taking integer values for all $x = n, n+1, \dots, n+k$ corresponding to some integer n . Then*

$$p_k(x) = c_0 \binom{x}{k} + c_1 \binom{x}{k-1} + c_2 \binom{x}{k-2} + \dots + c_k,$$

where c_0, c_1, \dots, c_k are integers.

PROOF. By induction on k it is easy to prove that any polynomial $p_k(x)$ of degree k may be represented in the form

$$p_k(x) = c_0 \binom{x}{k} + c_1 \binom{x}{k-1} + c_2 \binom{x}{k-2} + \dots + c_k,$$

where c_0, c_1, \dots, c_k are some numbers. Indeed,

$$\binom{x}{0} = 1, \quad \binom{x}{1} = x, \quad \binom{x}{2} = \frac{x^2}{2} - \frac{x}{2}, \quad \dots, \quad \binom{x}{k} = \frac{x^k}{k!} + \dots$$

Hence if $p_k(x) = ax^k + \dots$, then the polynomial $p_k(x) - ak! \binom{x}{k}$ is of degree no higher than $k-1$, and the induction hypothesis may be applied to it. Thus we need only prove that the numbers c_0, c_1, \dots, c_k are integers. Let us prove this by induction on k . Base of induction: $k = 0$. By assumption the polynomial $p_0(x) = c_0$ assumes an integer value for $x = n$, so the coefficient c_0 must be an integer. Now suppose

that the required statement has been proved for polynomials of all degrees up to k . Let the polynomial

$$p_{k+1}(x) = c_0 \binom{x}{k+1} + \cdots + c_{k+1}$$

assume integer values for $x = n, n+1, \dots, n+k+1$. Consider the polynomial

$$\Delta p_{k+1}(x) = p_{k+1}(x+1) - p_{k+1}(x) = c_0 \binom{x}{k} + c_1 \binom{x}{k-1} + \cdots + c_k.$$

It assumes integer values for $x = n, n+1, \dots, n+k$. Hence the numbers c_0, c_1, \dots, c_k are integers, and so is

$$c_{k+1} = p_{k+1}(n) - c_0 \binom{n}{k+1} - c_1 \binom{n}{k} - \cdots - c_k \binom{n}{1}.$$

□

Chebyshev Polynomials

Chebyshev polynomials constitute one of the most remarkable families of polynomials. They often appear in various branches of mathematics, from approximation theory to number theory and three-manifold topology. We shall discuss some of the simplest, but very important, properties of Chebyshev polynomials.

The definition of Chebyshev polynomials is based on the fact that $\cos n\varphi$ can be polynomially expressed in terms of $\cos \varphi$, i.e., there exists a polynomial $T_n(x)$ such that $T_n(x) = \cos n\varphi$ for $x = \cos \varphi$. Indeed, the formula

$$\cos(n+1)\varphi + \cos(n-1)\varphi = 2 \cos \varphi \cos n\varphi$$

shows that

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

The polynomials $T_n(x)$ defined by this recurrent relation and the initial conditions $T_0(x) = 1$ and $T_1(x) = x$ possess the required property. The polynomials $T_n(x)$ are called *Chebyshev polynomials*.

It follows immediately from the relation $T_n(x) = \cos n\varphi$ (valid for $x = \cos \varphi$) that $|T_n(x)| \leq 1$ for $x \leq 1$. And now the recurrent relation implies that

$$T_n(x) = 2^{n-1}x^n + a_1x^{n-1} + \dots + a_n,$$

where a_1, \dots, a_n are integers.

The most important property of Chebyshev polynomials was discovered by Chebyshev himself. It consists of the following.

THEOREM 1. *Let $P_n(x) = x^n + \dots$ be a polynomial of degree n with coefficient 1 at the leading term, and let $|P_n(x)| \leq \frac{1}{2^{n-1}}$ for $|x| \leq 1$. Then $P_n(x) = \frac{1}{2^{n-1}}T_n(x)$. (In other words, the polynomial $\frac{1}{2^{n-1}}T_n(x)$ is the polynomial of degree n with leading coefficient 1 with the least deviation from 0 on the segment $[-1, 1]$.)*

PROOF. We shall use only one property of the polynomial $T_n(x) = 2^{n-1}x^n + \dots$, namely the fact that

$$T_n(\cos(k\pi/n)) = \cos k\pi = (-1)^k \quad \text{for } k = 0, 1, \dots, n.$$

Consider the polynomial

$$Q(x) = \frac{1}{2^{n-1}}T_n(x) - P_n(x).$$

Its degree is no greater than $n-1$ since the leading terms of the polynomials $\frac{1}{2^{n-1}}T_n(x)$ and $P_n(x)$ are equal. The inequality $|P_n(x)| \leq \frac{1}{2^{n-1}}$ (which holds for $|x| \leq 1$) implies that at the point $x_k = \cos(k\pi/n)$ the sign of $Q(x_k)$ coincides with that of $T_n(x_k)$. Thus at the extremities of each of the segments $[x_{k+1}, x_k]$ the polynomial $Q(x)$ assumes values of opposite signs, and so the polynomial $Q(x)$ has



FIGURE 1

a root in this segment. Slightly more precise arguments are required in the case for which $Q(x_k) = 0$. In this case either x_k is a root of multiplicity two, or inside one of the segments $[x_{k+1}, x_k]$ and $[x_k, x_{k-1}]$ there is one more root. This follows from the fact that at the points x_{k+1} and x_{k-1} the polynomial $Q(x)$ takes values of the same sign (Figure 1).

The number of segments $[x_{k+1}, x_k]$ equals n , therefore the polynomial $Q(x)$ has at least n roots. For a polynomial of degree no more than $n - 1$ this means that it is identically equal to zero, i.e., $P_n(x) = \frac{1}{2^{n-1}}T_n(x)$. \square

For pair of polynomials P and Q one can define their *composition* $P \circ Q(x) = P(Q(x))$. The polynomials P and Q are called *commuting* if

$$P \circ Q = Q \circ P, \quad \text{i.e.,} \quad P(Q(x)) = Q(P(x)).$$

THEOREM 2. *The Chebyshev polynomials $T_n(x)$ and $T_m(x)$ commute.*

PROOF. Let $x = \cos \varphi$. Then $T_n(x) = \cos(n\varphi) = y$ and $T_m(y) = \cos m(n\varphi)$, so that $T_m(T_n(x)) = \cos mn\varphi$. Similarly $T_n(T_m(x)) = \cos mn\varphi$. Thus the equation $T_n(T_m(x)) = T_m(T_n(x))$ holds if $|x| < 1$, and so it holds for all x . \square

In certain cases instead of the polynomial $T_n(x)$ it is convenient to consider the polynomial $P_n(x) = 2T_n(x/2)$ with leading coefficient 1. The polynomials $P_n(x)$ satisfy the recurrent relation $P_{n+1}(x) = xP_n(x) - P_{n-1}(x)$, and so $P_n(x)$ is a polynomial with integer coefficients.

If $z = \cos \varphi + i \sin \varphi = e^{i\varphi}$, then $z + z^{-1} = 2 \cos \varphi$ and $z^n + z^{-n} = 2 \cos n\varphi$. Therefore

$$P(z + z^{-1}) = 2T_n(\cos \varphi) = 2 \cos n\varphi = z^n + z^{-n},$$

i.e., the polynomial $P_n(x)$ corresponds to the polynomial expression of $z^n + z^{-n}$ in terms of $z + z^{-1}$.

Using the polynomials P_n we can prove the following statement.

THEOREM 3. *If both numbers α and $\cos(\alpha\pi)$ are rational, then the number $2\cos(\alpha\pi)$ is an integer, i.e., $\cos(\alpha\pi) = 0, \pm 1/2$ or ± 1 .*

PROOF. Let $\alpha = m/n$ be a simple fraction. Put $x_0 = 2 \cos t$, where $t = \alpha\pi$. Then

$$P_n(x_0) = 2 \cos(nt) = 2 \cos(n\alpha\pi) = 2 \cos(m\pi) = \pm 2.$$

Therefore x_0 is a root of the polynomial $P_n(x) \mp 2 = x^n + b_1x^{n-1} + \dots + b_n$ with integer coefficients. Let $x_0 = 2 \cos(\alpha\pi) = p/q$ be a simple fraction. Then

$$p^n + b_1p^{n-1}q + \dots + b_nq^n = 0,$$

and so p^n is divisible by q . But the numbers p and q are coprime, hence $q = \pm 1$, i.e., $2 \cos(\alpha\pi)$ is an integer. \square

Vectors in Geometry

Vectors are often useful for solving geometric problems. Besides, some geometric theorems can be conveniently stated in terms of vectors.

When working with vectors, one often uses the *scalar product*

$$(\overrightarrow{OA}, \overrightarrow{OB}) = OA \cdot OB \cos \angle AOB.$$

The scalar product possesses the following important properties:

- $(a, b + c) = (a, b) + (a, c)$;
- $(\lambda a, b) = \lambda(a, b)$;
- $(a, a) = |a|^2$ is the square of the length a .

By using vectors, it is easy to prove that the altitudes of any triangle ABC intersect at one point. Indeed, let O be the center of the circumscribed circle and H a point such that $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$. Then, as can easily be verified, $AH \perp BC$, $BH \perp AC$ and $CH \perp AB$, i.e., H is the intersection point of the altitudes.

This can be proved in different ways. First, we can note that

$$\begin{aligned} (\overrightarrow{AH}, \overrightarrow{BC}) &= (\overrightarrow{AO} + \overrightarrow{OH}, \overrightarrow{BO} + \overrightarrow{OC}) \\ &= (\overrightarrow{OB} + \overrightarrow{OC}, \overrightarrow{OC} - \overrightarrow{OB}) \\ &= |OC|^2 - |OB|^2 = 0. \end{aligned}$$

Second, we can note that the projections of the vectors \overrightarrow{OB} and \overrightarrow{OC} on line BC have the same length and opposite directions. Therefore, the projections of the vectors \overrightarrow{OA} and \overrightarrow{OH} on line BC coincide. This means that $AH \perp BC$.

We have not only proved that the altitudes of triangle ABC intersect at one point (H), but we have also obtained a rather convenient expression for this point: $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$, where O is the center of the circumscribed circle. Using this expression we can solve, in particular, the following problem.

PROBLEM 1. Quadrangle $ABCD$ is inscribed in a circle. Let A_1, B_1, C_1, D_1 be the intersection points of the altitudes of triangles BCD, CDA, DAB, ABC . Prove that the midpoints of the segments AA_1, BB_1, CC_1, DD_1 coincide.

HINT. If A_2 is the midpoint of the segment A_1A , then

$$\overrightarrow{OA_2} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}),$$

where O is the center of the circle. □

The vectors with origin in the center of a regular polygon and endpoints at its vertices possess the following interesting property: their sum is zero. Let us

formulate this statement more precisely. Let $A_1 \dots A_n$ be a regular n -gon, let O be its center. Then

$$(1) \quad \overrightarrow{OA_1} + \dots + \overrightarrow{OA_n} = \vec{0}.$$

To prove relation (1), consider the rotation that takes A_1 to A_2 , A_2 to A_3 , \dots , A_n to A_1 . Under this rotation the vector $\vec{x} = \overrightarrow{OA_1} + \dots + \overrightarrow{OA_n}$, on the one hand, will turn by the angle $2\pi/n$, but on the other hand, it will not change, since

$$\overrightarrow{OA_1} + \dots + \overrightarrow{OA_n} = \overrightarrow{OA_2} + \overrightarrow{OA_3} + \dots + \overrightarrow{OA_n} + \overrightarrow{OA_1}.$$

Therefore, $\vec{x} = \vec{0}$.

Using equation (1) we can prove that if the point X is located at the distance d from the center O of a regular n -gon $A_1 \dots A_n$, then

$$A_1 X^2 + \dots + A_n X^2 = n(R^2 + d^2),$$

where R is the radius of the circle circumscribed to the n -gon. Indeed,

$$\begin{aligned} \sum A_i X^2 &= \sum |\overrightarrow{A_i O} + \overrightarrow{OX}|^2 \\ &= \sum A_i O^2 + \sum OX^2 + 2(\sum \overrightarrow{A_i O}, \overrightarrow{OX}) \\ &= nR^2 + nd^2, \end{aligned}$$

since $\sum \overrightarrow{A_i O} = \vec{0}$.

Before passing to the last topic of this essay, it is useful to solve the following problem.

PROBLEM 2. Consider a convex polygon and to each of its sides assign the perpendicular unit vector pointing outwards \vec{n}_i . Let a_i be the length of the i th side of the polygon. Prove that $\sum a_i \vec{n}_i = \vec{0}$.

HINT. After a rotation by 90° , the vector $a_i \vec{n}_i$ is taken to the side of the polygon, but the sum of the side vectors of the polygon is zero. \square

A statement similar to that of Problem 2 is true for polyhedra. Indeed, let us consider a convex polyhedron and assign to its i th face the unit outward pointing normal vector \vec{n}_i . Let A_i be the area of the i th face. Then $\sum A_i \vec{n}_i = \vec{0}$.

This statement follows from the following physical considerations. Let us fill the polyhedron with gas. The pressure of the gas on the i th face is proportional to $A_i \vec{n}_i$, while the sum of all the pressure forces on the faces is zero (otherwise a *perpetuum mobile* could have been constructed).

Let us present, however, a mathematical proof. Let $\sum A_i \vec{n}_i = \vec{x}$. It suffices to show that the projection of the vector \vec{x} on any straight line l in space is equal to zero. Together with the projection of the vector \vec{x} on the line l consider the projection of the polyhedron on the plane Π , perpendicular to l . It is easy to verify that the length of the projection of the vector $A_i \vec{n}_i$ on l is equal to the area of the projection of the i th face on Π . Indeed, if the angle between the vector \vec{n}_i and the line l is α , then the angle between the i th face and the plane Π also equals α , so that both the length of the projection of the vector and the area of the projection of the face equal $A_i \cos \alpha$. It remains to note that the projection of the polyhedron on the plane Π is covered by the projections of the faces twice (by the "upper" and "lower" faces). But here opposite signs of the projected vectors correspond to the "upper" and "lower" faces.

The Averaging Method and Geometric Inequalities

In the proof of many geometric inequalities the following fact is often useful: “On the plane let there be given two systems of vectors $\vec{a}_1, \dots, \vec{a}_n$ and $\vec{b}_1, \dots, \vec{b}_m$ such that for any straight line, the sum of lengths of the projections on this line for the first system is greater than that for the second. Then the sum of lengths of the vectors of the first system is greater than that of the second.” To prove this statement, let us fix a coordinate frame on the plane and consider the line l forming the angle φ with the x -axis. If the vector \vec{a} forms the angle α with the x -axis, then the length of its projection on l equals $a|\cos(\varphi - \alpha)|$, where a is the length of \vec{a} . Thus the mean value of the length of the projection of \vec{a} equals

$$\frac{1}{\pi} \int_0^\pi a|\cos(\varphi - \alpha)| d\varphi = \frac{2a}{\pi}.$$

In particular, the mean value of the length of the projection does not depend on α , i.e., does not depend on the position of the vector \vec{a} . If the sum of lengths of the vectors $\vec{a}_1, \dots, \vec{a}_n$ on any line is greater than the sum of lengths of the projections of $\vec{b}_1, \dots, \vec{b}_m$, then the corresponding inequality is true for the mean values of the sum of lengths of the projections, and so

$$a_1 + \dots + a_n > b_1 + \dots + b_m.$$

Let us present some examples of the use of the geometric fact just proved above. We begin with a fairly simple theorem, which can also be easily proved in other ways.

1. *If a convex polygon is located inside another convex polygon, then the perimeter of the outer polygon is greater than that of the inner one.*

Indeed, let $\vec{a}_1, \dots, \vec{a}_n$ be the vectors of the sides of the outer polygon and $\vec{b}_1, \dots, \vec{b}_m$ those of the inner one. For a convex polygon the sum of lengths of the projections of its sides equals twice the length of the projection of the polygon itself. It is also obvious that the projection of the inner polygon is contained in the projection of the outer one, and so its length is less.

Now let us consider a more complicated example, for which it is hard to find a simple proof by a different method.

2. *Several convex polygons with perimeters P_1, \dots, P_n are positioned so that there is no straight line dividing them (i.e., there is no line disjoint from the polygons such that there are polygons on both sides of it). Then the given polygons can be enclosed in a convex polygon of perimeter no greater than $P_1 + \dots + P_n$.*

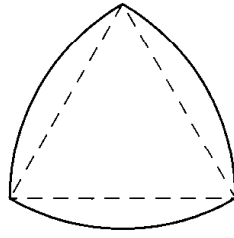


FIGURE 1

If the line l does not divide the family of polygons, then their projections on a line perpendicular to l form a single segment (rather than several). This means that the sum of lengths of the projections of the sides of the given polygons on any line is not less than that of the sides of their convex hull.¹ Therefore, the perimeter of the convex hull of the given polygons is no greater than $P_1 + \cdots + P_n$.

In some cases the fact that the mean value of the length of the projection of a vector of length a equals $2a/\pi$ can be useful.

3. *If the lengths of all the sides and diagonals of a polygon are less than 1, then its perimeter is less than π .*

Let the perimeter of the given polygon equal P . Then the mean value of the sum of lengths of the projection of its sides is $2P/\pi$. It follows from the assumption that the length of the projection of the polygon on any line is less than 1, i.e., the sum of lengths of the projections of its sides is less than 2. Hence $2P/\pi < 2$, and so $P < \pi$.

The next example involves convex *sets of constant width*. They are defined as convex sets for which the length of the projection on any line is the same. This length is then called the *width* of the set of constant width.

An obvious example of a convex set of constant width is the circular disk. But there are other sets of constant width. The simplest example is the set bounded by three arcs of circles of radius R centered at the vertices of an equilateral triangle of side R (Figure 1). It is easy to check that the length of the projection of this set on any line equals R .

4. *The perimeter of any convex set of constant width d is πd .*

The boundary of any convex set can be approximated as closely as we like by convex polygons. Then the length of the projection of the polygon on any line will lie between $d - \varepsilon$ and $d + \varepsilon$, where ε is a positive number as small as we wish. Therefore, the perimeter of such a convex polyhedron lies between $\pi(d - \varepsilon)$ and $\pi(d + \varepsilon)$. Letting ε tend to 0, we see that the perimeter of the convex set will equal πd .

5. *Let there be given vectors on the plane $\vec{a}_1, \dots, \vec{a}_n$, whose sum of lengths is L . Then among them we can choose some vectors whose sum is of length greater than or equal to L/π .*

The mean value of the sum of lengths of the projections of the given vectors is $2L/\pi$, so there exists a line for which the sum of lengths of their projections is

¹The convex hull of a family of polygons is defined as the smallest convex polygon that contains them all

no less than $2L/\pi$. Choose a direction on such a line. Then we can consider the projections with a sign, and the sum of lengths of the projections will equal $p - n$, where p is the sum of the positive projections, while n is the sum of the negative ones. Since $p - n \geq 2L/\pi$ and $p \geq 0$, $-n \geq 0$, one of the numbers p or $-n$ is not less than L/π . To be definite, let $p \geq L/\pi$. Then the sum of lengths of the vectors whose projections on the chosen line were positive is not less than L/π . Indeed, the sum of these vectors can be represented as the sum of two orthogonal vectors, one of which is of length $p \geq L/\pi$.

There is another approach to the calculation of the length of the projection of a vector. This new approach will interest us because it can be used to calculate the mean value of the projection of a vector in space as well. Suppose that to each angle φ in the plane we assign a certain number $f(\varphi)$. Then the mean value of the function f equals

$$\frac{1}{2\pi} \int_0^{2\pi} f(\varphi) d\varphi.$$

The geometric meaning of this integral is the following. Choose a point O on the plane and assign to the angle φ the end X of the unit vector \overrightarrow{OX} whose direction corresponds to the angle φ . As the result, to each angle φ a point of the unit circle centered at O will have been assigned in such a way that the difference between two angles is equal to the length of the corresponding arc. Let us divide the circle into little arcs and consider the sum

$$\sum_k f(\varphi_k) \Delta l_k,$$

where φ_k is some point on the k th arc, Δl_k is the length of this arc. If all the arcs decrease in length this sum tends to

$$\int_0^{2\pi} f(\varphi) d\varphi.$$

To obtain the mean value of f , this integral must be divided by the length of the circle, i.e., by 2π .

Now we can define the mean value of the length of the projection of a vector in space in the same way. Choose a point O in space and assign to each direction in space the end X of the unit vector \overrightarrow{OX} corresponding to this direction. As the result, we obtain the unit sphere centered at O . The function f that we consider is equal to the length of the projection of each vector \vec{a} on the ray OX . To determine its mean value, we must subdivide the sphere into sufficiently small surface elements and consider the sum $\sum_k f(X_k) \Delta S_k$, where X_k is some point of the k th element, ΔS_k is its area. Then we must calculate the limit of such sums as the subdivisions become finer and divide this limit by the area of the sphere, i.e., by 4π . Instead of subdividing the sphere into surface elements, we can consider convex polyhedra that approximate the sphere sufficiently well. Then we can assume that for a face M_k the point X_k is defined as the end of the vector \overrightarrow{OX}_k perpendicular to M_k . In that case

$$f(X_k) = a \Delta S_k |\cos \varphi_k|,$$

where ΔS_k is the area of the face M_k , φ_k is the angle between the vectors \vec{a} and \overrightarrow{OX}_k . It is easy to check that $\Delta S_k |\cos \varphi_k| = \Delta S'_k$, where $\Delta S'_k$ is the area of the projection of the face M_k on a plane orthogonal to \vec{a} . Thus the sum under consideration, $\sum_k f(X_k) \Delta S_k$, is equal to the sum of areas of the projections of

the faces of the polyhedron on the plane perpendicular to \vec{a} . In the limit this sum becomes equal to twice the area of the section of the sphere by a plane passing through the sphere's center. Thus the mean value of the length of the projection of the vector \vec{a} to lines in space equals $2aA_1/A_2$, where A_1 is the area of the equatorial section of the sphere, A_2 is its total surface area. For the unit sphere we have $A_1 = \pi$ and $A_2 = 4\pi$, so that the mean value is $a/2$.

By using the mean value of the projections of vectors in space, we can prove, by the same methods as for the plane, the following statements.

6. *If one convex polyhedron lies inside another convex polyhedron, then the surface area of the inner polyhedron is less than that of the outer one.*

In this situation we assign to each face the outward-pointing orthogonal vector of length equal to the area of the face.

7. *If the area of any projection of a convex polyhedron is not greater than 1, then its surface area is not greater than 4.*

8. *Let the vectors $\vec{a}_1, \dots, \vec{a}_n$ in space for which the sum of lengths is L be given. Then among them we can choose vectors whose sum is of length greater than or equal to $L/4$.*

In conclusion let us discuss a statement that could have been stated for the plane, but is really interesting only in space.

9. *Let a tetrahedron lie inside another tetrahedron. Then the ratio of the sum of lengths of the sides of the inner tetrahedron to a similar sum for the outer one is no greater than $4/3$, and this ratio can be as close to $4/3$ as we like. (In particular, the sum of lengths of the sides of the inner tetrahedron can be greater than that of the outer one.)*

We shall immediately prove a more general statement.

10. *Let a polyhedron with m vertices be located inside a polyhedron with n vertices. Then the ratio of the sum of all pairwise distances between vertices of the inner polyhedron to a similar sum for the outer one is not greater than $\frac{m^2}{4(n-1)}$ for even m and $\frac{m^2-1}{4(n-1)}$ for odd m .*

It suffices to prove this statement for projections of polyhedra on the straight line. In the case of the line, the required statement is an obvious consequence of the following lemma.

LEMMA. *Let k points be located on a line segment of length d , its extremities being part of the system of k points. Then the minimal sum of pairwise distances between the given points equals $(k-1)d$, while the maximal sum is $k^2d/4$ for even k and $(k^2-1)d/4$ for odd k .*

PROOF. Let A and B be the end points of the given line segment. For a point X of the segment we have $AX + BX = d$, hence the sum of pairwise distances between the given points is $(k-1)d + \Sigma$, where Σ is the sum of pairwise distances for the system of $k-2$ points obtained from the given one by eliminating A and B . The minimal value Σ is 0; it is achieved when all $k-2$ points are concentrated at one of the endpoints of AB . The maximal value of Σ can only be achieved for

the case in which the new system of $k - 2$ points includes both points A and B (we assume that $k - 2 \geq 2$). Once again let us eliminate A and B and consider the system of $k - 4$ points and so on. As the result, we see that if $k = 2s$, then the sum of pairwise distances is maximal when s points are located at one end of the segment and the s others at the other end; this sum is $s^2d = k^2d/4$. Now if $k = 2s + 1$, then there must be s points at one end and $s + 1$ at the other; in that case the sum is $s(s + 1)d = (k^2 - 1)d/4$. \square

Intersection Points of the Diagonals of Regular Polygons

Quite a few problems about triangles with integer angles are known. Here are two examples of such problems.

PROBLEM 1. In an isosceles triangle ABC with base BC the angle at A equals 80° . A point M is chosen inside the triangle so that $\angle MBC = 30^\circ$ and $\angle MCB = 10^\circ$ (Figure 1(a)). Prove that $\angle AMC = 70^\circ$.

PROBLEM 2. In an isosceles triangle ABC with base AC the angle at B equals 20° . Points D and E are chosen on the sides BC and AB respectively so that $\angle DAC = 60^\circ$ and $\angle ECA = 50^\circ$ (Figure 1(b)). Prove that $\angle ADE = 30^\circ$.

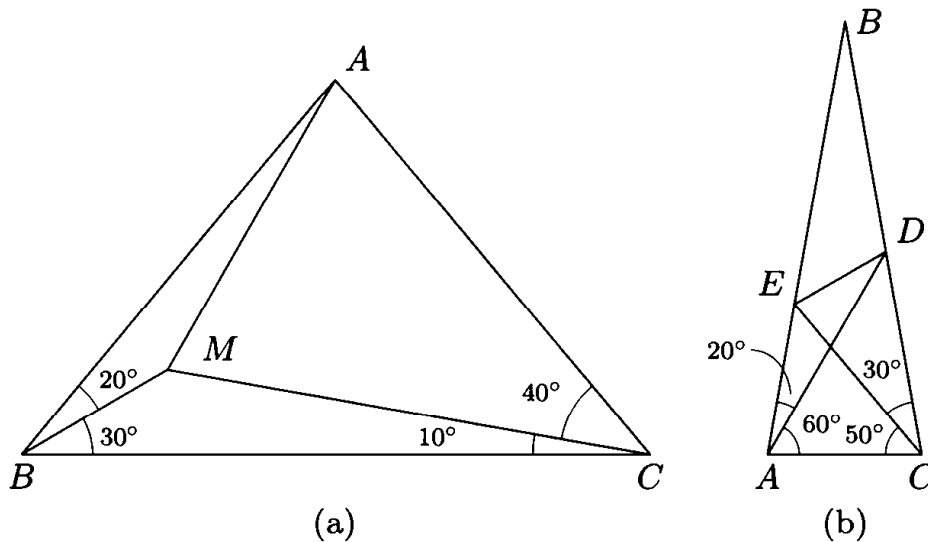


FIGURE 1

Problems of this type are usually related with intersection points of triples of diagonals of a regular polygon, in the present case, of a regular 18-gon.

Let us look at Figure 2. This figure shows that Problem 1 is equivalent to the following statement: *in a regular polygon of 18 sides the diagonals A_1A_{13} , A_3A_{14} , and A_6A_{15} intersect at one point.* Indeed, if these diagonals intersect at some point M , then

$$\angle A_1MA_6 = \frac{1}{2} (\sphericalangle A_1A_6 + \sphericalangle A_{13}A_{15}) = 50^\circ + 20^\circ = 70^\circ.$$

It is also clear that the angles of triangle $A_1A_6A_{14}$ are $80^\circ, 40^\circ, 40^\circ$ and $\angle MA_{14}A_6 = 30^\circ$, $\angle MA_6A_{14} = 10^\circ$.

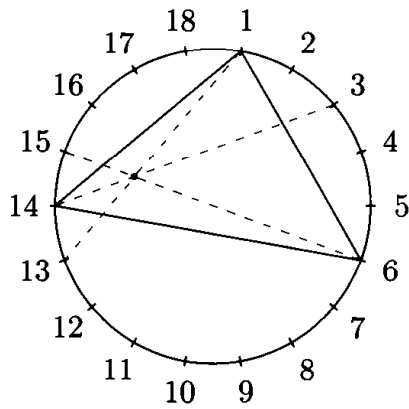


FIGURE 2

As to Problem 2, it is equivalent to the following statement: in a regular 18-gon the diagonals A_1A_{14} , A_7A_{16} and $A_{11}A_{17}$ intersect at one point (Figure 3).

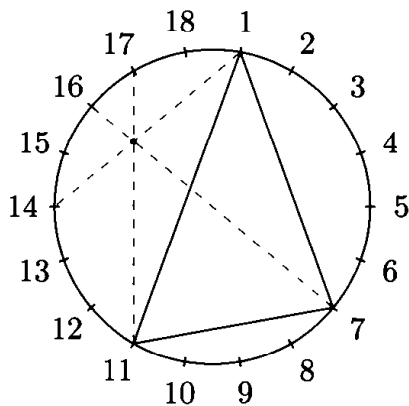


FIGURE 3

But Problem 2 can be solved by using a different triple of intersecting diagonals, namely A_1A_{13} , A_3A_{14} and A_6A_{15} (Figure 4). In the role of triangle ABC we take $A_{14}OA_{15}$. The diagonals A_1A_{13} and A_9A_{15} are symmetric with respect to the diagonal A_5A_{14} , hence both diagonals intersect the diameter at one point.

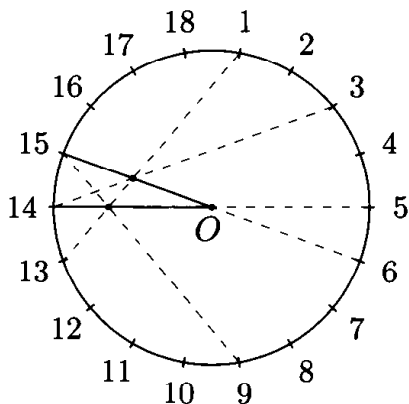


FIGURE 4

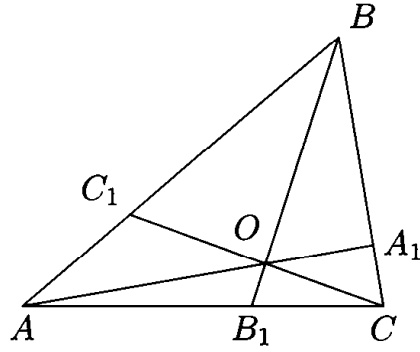


FIGURE 5

But we have not yet proved that the diagonals in Figures 2–4 do in fact intersect at one point. It is convenient to verify that triples of diagonals intersect at one point by using the following statement.

THEOREM. *The points A_1, B_1, C_1 are chosen on the sides of triangle ABC (A_1 on BC , etc.). The segments AA_1, BB_1 and CC_1 intersect at one point if and only if*

$$\frac{\sin BAA_1}{\sin CAA_1} \cdot \frac{\sin ACC_1}{\sin BCC_1} \cdot \frac{\sin CBB_1}{\sin ABB_1} = 1.$$

PROOF. First let us assume that the segments AA_1, BB_1 and CC_1 intersect at the point O (Figure 5). Then

$$2S_{AOB} : 2S_{AOC} = (AB \cdot AO \sin BAO) : (AC \cdot AO \sin CAO).$$

Therefore,

$$\begin{aligned} 1 &= \frac{S_{AOB}}{S_{AOC}} \cdot \frac{S_{COA}}{S_{COB}} \cdot \frac{S_{BOC}}{S_{BOA}} \\ &= \left(\frac{AB}{AC} \cdot \frac{CA}{CB} \cdot \frac{BC}{BA} \right) \frac{\sin BAO}{\sin CAO} \cdot \frac{\sin ACO}{\sin BCO} \cdot \frac{\sin CBO}{\sin ABO}, \end{aligned}$$

i.e.,

$$\frac{\sin BAA_1}{\sin CAA_1} \cdot \frac{\sin ACC_1}{\sin BCC_1} \cdot \frac{\sin CBB_1}{\sin ABB_1} = 1.$$

Now assume that the equation indicated in the theorem holds for the points A_1, B_1 and C_1 . Let O be the intersection point of the segments AA_1 and BB_1 . We must prove that the segment CC_1 passes through O as well. In other words, if C'_1 is the intersection point of the lines CO and AB , then $C'_1 = C_1$. The segments AA_1, BB_1 and CC'_1 intersect at one point, so, as we have just proved,

$$\frac{\sin BAA_1}{\sin CAA_1} \cdot \frac{\sin CBB_1}{\sin ABB_1} \cdot \frac{\sin ACC'_1}{\sin BCC'_1} = 1.$$

Comparing this formula with the condition of the theorem, we obtain

$$\sin ACC_1 : \sin BCC_1 = \sin ACC'_1 : \sin BCC'_1.$$

It remains to prove that when the point X moves along the segment AB , the value $\sin ACX : \sin BCX$ changes monotonically. The angles $\angle ACX$ and $\angle BCX$ themselves do change monotonically, but their sines are not necessarily monotone in the case of an obtuse angle C . But this is no tragedy. In any triangle there is an acute angle, and from the very beginning we could have taken such an angle for C . This completes the proof of the theorem. \square

Now the verification of the fact that the triple of diagonals shown in Figure 2 intersect at one point reduces to proving the following identity.

$$\frac{\sin 10^\circ}{\sin 70^\circ} \cdot \frac{\sin 30^\circ}{\sin 20^\circ} \cdot \frac{\sin 40^\circ}{\sin 10^\circ} = 1.$$

This is not hard to prove:

$$\sin 30^\circ \sin 40^\circ = \frac{1}{2} \sin 40^\circ = \sin 20^\circ \cos 20^\circ = \sin 20^\circ \sin 70^\circ.$$

The triple of diagonals shown in Figure 3 corresponds to the identity

$$\sin 20^\circ \sin 40^\circ \sin 20^\circ = \sin 30^\circ \sin 60^\circ \sin 10^\circ.$$

There are three more identities that yield triples of intersecting diagonals:

$$\begin{aligned} \sin 10^\circ \sin 20^\circ \sin 80^\circ &= \sin 20^\circ \sin 20^\circ \sin 30^\circ, \\ \sin 20^\circ \sin 30^\circ \sin 30^\circ &= \sin 10^\circ \sin 40^\circ \sin 50^\circ, \\ \sin 10^\circ \sin 20^\circ \sin 30^\circ &= \sin 10^\circ \sin 10^\circ \sin 100^\circ. \end{aligned}$$

Their verification is left to the reader.

Note that the change of order of the factors in these identities leads to completely different triples of intersecting diagonals.

Our interest in the regular polygon of 18 sides rather than some other regular polygon is related to the fact that triangles with angles that are multiples of 10° lead to that particular polygon. Among all the regular polygons with less than 18 vertices, interesting families of diagonals appear only in the 12-gon. For example, the diagonals A_1A_5 , A_2A_6 , A_3A_8 , and A_4A_{11} of the regular 12-gon intersect at one point (Figure 6). This statement is equivalent to the following well-known problem.

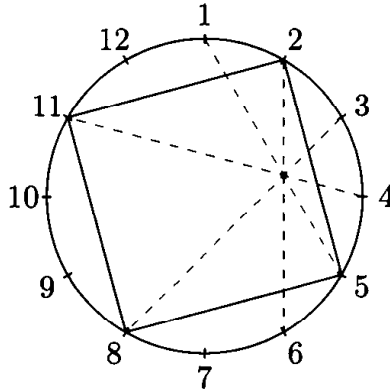


FIGURE 6

PROBLEM 3. The point P is chosen inside square $ABCD$ so that triangle ABP is equilateral. Prove that $\angle PCD = 15^\circ$.

Exercises. 1. Triangle ABC with angles $\angle A = 50^\circ$, $\angle B = 60^\circ$, $\angle C = 70^\circ$ is given.

a) Points D and E on sides BA and AC are chosen so that $\angle DCB = \angle EBC = 40^\circ$. Prove that $\angle AED = 30^\circ$.

b) Points D and E on sides BA and AC are chosen so that $\angle DCA = 50^\circ$ and $\angle EAC = 40^\circ$. Prove that $\angle AED = 30^\circ$.

2. In triangle ABC the angles $A, B,$ and C equal $14^\circ, 62^\circ,$ and 104° . The points D and E are chosen on sides AC and AB respectively so that $\angle DBC = 50^\circ$ and $\angle ECB = 94^\circ$. Prove that $\angle CED = 34^\circ$.

3. Prove that the diagonals $A_1A_{n+2}, A_{2n-1}A_3,$ and $A_{2n}A_5$ of a regular $2n$ -gon intersect at one point.

4. Prove that the diagonals $A_1A_7, A_3A_{11},$ and A_5A_{21} of a regular 24-gon intersect at a point lying on the diameter A_4A_{16} .

5. Prove that in a regular 30-gon the seven diagonals

$$A_1A_{13}, A_2A_{17}, A_3A_{21}, A_4A_{24}, A_5A_{26}, A_8A_{29}, A_{10}A_{30}$$

intersect at one point.

The Chromatic Polynomial of a Graph

In space consider a system of several points and segments joining some of these points. We are only interested in what pairs of points are joined by segments. Moreover, these segments are not necessarily rectilinear, they may be curved. Such a system is called a *graph*; the points are called *vertices* of the graph, while the segments are the *edges* of the graph.

A graph is called *planar* if it can be placed in the plane so that its edges are pairwise nonintersecting (except at endpoints). For example, the graph shown in Figure 1 is not planar. This follows from the solution of the well-known problem of the three houses and three wells: “Can each of the houses be joined by a path to each of the three wells so that the paths do not intersect?”

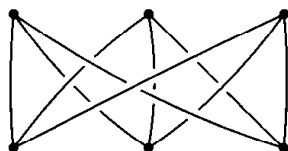


FIGURE 1

In 1890 it was proved that the vertices of any planar graph can be painted in five colors so that the ends of any edge will have different colors. For a long period after that the following “four color problem” was popular: “Can any planar graph be painted in four colors?” (By a coloring of a graph here and further we mean an assignment of colors to its vertices so that the ends of each edge are assigned different colors.) Only in 1977 was it proved that any planar graph can be painted in four colors. The proof was based on the verification of a huge number of possibilities, carried out by computer.

Colorings of graphs are related to a very interesting phenomenon that we shall now discuss. Let G be a graph and $f(G, x)$ be the number of colorings of this graph in x colors. Surprisingly, it turns out that $f(G, x)$ is a polynomial of degree n , where n is the number of vertices of G . Before proving this statement, let us consider two simple examples.

The simplest case is when the graph G has no edges at all, only n vertices. In that case any vertex may be colored in any way, independently of the other vertices. Therefore $f(G, x) = x^n$.

The other extreme case is that of a graph G with n vertices any two of which are joined by an edge. In that case all the vertices must be painted in different colors, so $f(G, x) = x(x - 1) \dots (x - n + 1)$.

In both cases $f(G, x)$ is a polynomial of degree n . As we shall soon see, this readily implies that $f(G, x)$ is a polynomial of degree n for any graph G with n vertices.

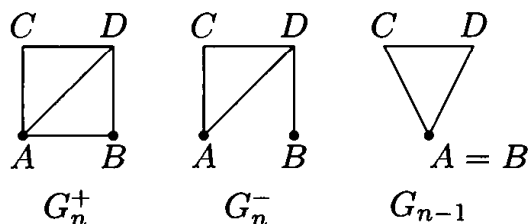


FIGURE 2

Consider the graph G_n^+ with n vertices in which the two vertices A and B are joined by an edge. To this graph let us assign the graph G_n^- with the same vertices and edges, except that it does not have an edge joining A and B . Now in turn to the graph G_n^+ assign the graph G_{n-1} with the same vertices and edges as the graphs G_n^+ and G_n^- , except that the vertices A and B are identified (if double edges arise under this identification, they are replaced by ordinary edges). Examples of the graphs G_n^- , G_n^+ , and G_{n-1} are shown on Figure 2. Note that to construct G_{n-1} the double edge joining the vertices $A = B$ and D has been replaced by an ordinary (single) edge.

Any coloring of the graph G_n^- with vertices A and B of the same color corresponds to a well-defined coloring of the graph G_{n-1} , while any coloring of the same graph with vertices A and B of different colors corresponds to a coloring of the graph G_n^+ . Therefore

$$(1) \quad f(G_n^-, x) = f(G_n^+, x) + f(G_{n-1}, x).$$

By using formula (1), it is easy to prove that if G_n is a graph with n vertices, then $f(G_n, x)$ is a polynomial of degree n . To this end, let us use induction on n . For $n = 1$ the graph consists of a single point, hence $f(G_1, x) = x$. Assume that for any graph G_{n-1} with $n - 1$ vertices the function $f(G_{n-1}, x)$ is a polynomial of degree n . Take an arbitrary graph G_n^- with n vertices. Applying formula (1) several times, we can pass from G_n^- to a graph with n vertices, any two of which are joined by an edge. The chromatic polynomial of that graph is $x(x - 1) \dots (x - n + 1)$, therefore

$$f(G_n^-, x) = x(x - 1) \dots (x - n + 1) + f_1 + \dots + f_k,$$

where f_1, \dots, f_k are chromatic polynomials of graphs with $n - 1$ vertices, i.e., polynomials of degree $n - 1$.

In the proof we used the fact that the chromatic polynomial of a graph all of whose vertices are joined by edges equals $x(x - 1) \dots (x - n + 1)$. But we can also use the fact that the chromatic polynomial of a graph without any edges is x^n . To this end we must rewrite formula (1) in the form

$$(2) \quad f(G_n^+, x) = f(G_n^-, x) - f(G_{n-1}, x).$$

Applying this formula several times, we can pass from the graph G_n^+ to the graph with n vertices and no edges. Therefore,

$$(3) \quad f(G_n^+, x) = x^n - g_1 - \dots - g_l,$$

where g_1, \dots, g_l are the chromatic polynomials of graphs with $n - 1$ vertices.

The second method of proof has certain advantages. For example, by writing the chromatic polynomial of a graph in the form (3), we readily prove the following theorem.

THEOREM. *The chromatic polynomial of a graph with n vertices has the form*

$$x^n - a_1x^{n-1} + a_2x^{n-2} - a_3x^{n-3} + \dots, \quad \text{where } a_i \geq 0.$$

This theorem was stated and proved in 1932 by the American mathematician Hassler Whitney.

Brocard Points

Besides such well known points in the triangle as the centers of the inscribed and circumscribed circles, the intersection point of the altitudes, the intersection point of the medians, other remarkable points are associated with every triangle. Probably those with the most interesting properties are the Brocard points.

The point P inside triangle ABC is said to be the *first Brocard point* if

$$\angle PAC = \angle PCB = \angle PBA$$

(Figure 1(a)). For the second Brocard point we must have

$$\angle QAB = \angle QCA = \angle QBC$$

(Figure 1(b)).

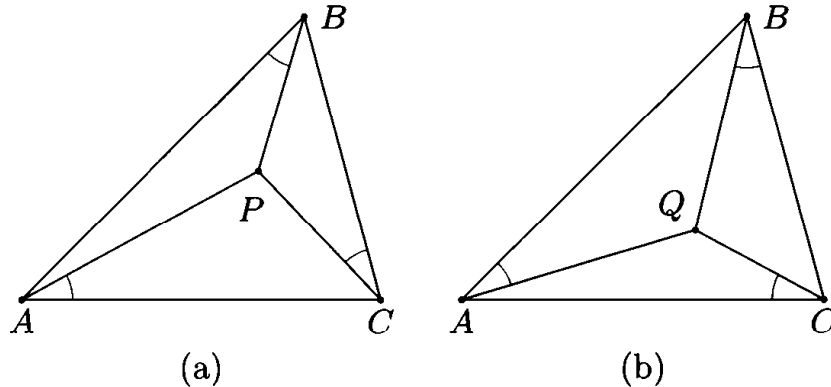


FIGURE 1

Before discussing the properties of these points, let us prove that for any triangle there exists exactly one first Brocard point and exactly one second Brocard point. We begin with the first Brocard point. On the sides of triangle ABC construct triangles A_1BC , AB_1C , and ABC_1 similar to it, as shown on Figure 2.

Since $\angle PCB = \angle C - \angle PCA$, the relations $\angle PAC = \angle PCB$ and $\angle PAC = \angle C - \angle PCA$ are equivalent. The last one may be rewritten as

$$\angle C = \angle PAC + \angle PCA = 180^\circ - \angle APC.$$

For a point P lying inside triangle ABC , this equation is equivalent to the point being located on the circle circumscribed to triangle AB_1C . A similar argument for the other angles shows that P is a Brocard point if and only if it belongs to the circumscribed circles to all three of the triangles A_1BC , AB_1C , and ABC_1 . Let P_1 be the intersection point of the circumscribed circles of the triangles A_1BC and AB_1C that differs from C . Then

$$\angle AP_1B = 360^\circ - \angle AP_1C - \angle CP_1B = \gamma + \beta = 180^\circ - \angle AC_1B,$$

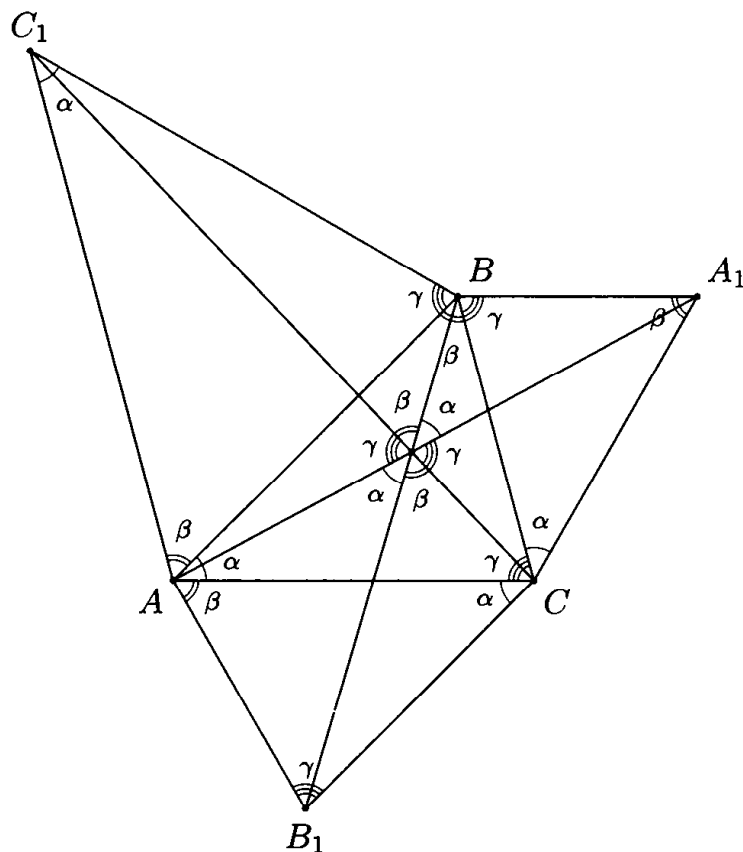


FIGURE 2

and so the point P_1 also lies on the circumscribed circle to triangle ABC_1 , i.e., $P_1 = P$ is a Brocard point. Let us join it to all the vertices of the triangles under consideration. From the congruence of angles subtending the same arc, the angles are precisely as shown in Figure 2. But since $\alpha + \beta + \gamma = 180^\circ$, the segments AA_1 , BB_1 , CC_1 pass through the point P .

Note that the circumscribed circles of triangles A_1BC , AB_1C , and ABC_1 are tangent to lines AB , BC , and CA respectively; this follows from the relations

$$\angle ABC = \angle BA_1C, \quad \angle BCA = \angle CB_1A, \quad \angle CAB = \angle AC_1B.$$

Hence the point P is also the intersection point of the three circles tangent to the lines AB , BC , and CA at the points B , C , and A and passing through the points C , A , and B respectively.

PROBLEM 1. On the sides of triangle ABC construct triangles A_1BC , AB_1C , and ABC_1 similar to it so that the segments AA_1 , BB_1 , and CC_1 intersect at the second Brocard point.

Now let us pass to the study of properties of Brocard points.

THEOREM 1. Let P be the first Brocard point, and suppose that the lines AP , BP , and CP intersect the circumscribed circle to triangle ABC at the points A_1 , B_1 , and C_1 . Then $\triangle ABC = \triangle B_1C_1A_1$. (For the second Brocard point $\triangle ABC = \triangle C_1A_1B_1$.)

PROOF. Let us prove, for example, that the segments AB and B_1C_1 are congruent. To this end it suffices to establish that the arcs AB and B_1C_1 are congruent.

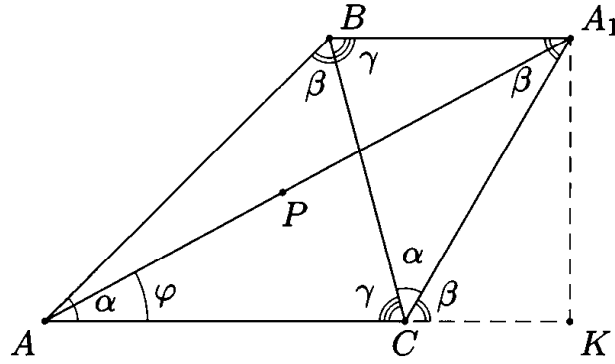


FIGURE 3

These arcs have the common arc AC_1 , while the complementary arcs C_1B and A_1B are congruent, since they are subtended by equal angles. Similarly one proves the congruence of the other corresponding sides of triangles ABC and $B_1C_1A_1$. \square

THEOREM 2. *From the first Brocard point P drop the perpendiculars PA' , PB' , and PC' on sides BC , CA , and AB . Then triangles ABC and $B'C'A'$ are similar, the coefficient of similitude being $\sin \varphi$, where*

$$\varphi = \angle PAC = \angle PCB = \angle PBA.$$

PROOF. Points B' and C' lie on the circle with diameter AP , therefore

$$\angle B'C'P = \angle B'AP = \varphi \quad \text{and} \quad \angle C'B'P = \angle C'AP = \alpha - \varphi.$$

Similarly,

$$\begin{aligned} \angle C'A'P &= \varphi \quad \text{and} \quad \angle A'C'P = \beta - \varphi, \\ \angle A'B'P &= \varphi \quad \text{and} \quad \angle B'A'P = \gamma - \varphi. \end{aligned}$$

Hence the angles of triangle $B'C'A'$ are equal to the angles of ABC , and so these triangles are similar. Moreover, the angles of triangles ABP and $B'PC'$ are equal, so

$$B'C' : AB = B'P : AP = \sin \varphi.$$

\square

Now let us consider the angle φ . It can be expressed in terms of the angles of triangle ABC . To this end let us erase all the unneeded parts of Figure 2 (see Figure 3) and from the point A_1 drop the perpendicular A_1K to line AC . Then

$$\text{ctg } \varphi = \frac{AK}{A_1K} = \frac{AC}{A_1K} + \frac{CK}{A_1K} = (\text{ctg } \alpha + \text{ctg } \gamma) + \text{ctg } \beta.$$

For the second Brocard point we can obtain the same expression for the angle φ . This angle is called the *Brocard angle*.

For the Brocard angle φ , one can obtain the following implicit expression:

$$\sin^3 \varphi = \sin(\alpha - \varphi) \sin(\beta - \varphi) \sin(\gamma - \varphi).$$

Indeed, by the sine theorem

$$\frac{AP}{BP} = \frac{\sin \varphi}{\sin(\alpha - \varphi)}, \quad \frac{BP}{CP} = \frac{\sin \varphi}{\sin(\beta - \varphi)}, \quad \frac{CP}{AP} = \frac{\sin \varphi}{\sin(\gamma - \varphi)}.$$

Multiplying these equations, we obtain the required expression.

THEOREM 3. *For the Brocard angle φ we have the following inequalities:*

- (a) $\varphi \leq 30^\circ$;
- (b) $\varphi^3 \leq (\alpha - \varphi)(\beta - \varphi)(\gamma - \varphi)$;
- (c) $8\varphi^3 \leq \alpha\beta\gamma$ (the Yff inequality).

PROOF. (a) Using the relation $\text{ctg } \varphi = \text{ctg } \alpha + \text{ctg } \beta + \text{ctg } \gamma$, after some simple transformations we obtain

$$\frac{\sin(\alpha + \varphi)}{\sin \varphi} = \frac{\sin(\alpha + \beta)}{\sin \beta} + \frac{\sin(\alpha + \gamma)}{\sin \gamma} = \frac{\sin \gamma}{\sin \beta} + \frac{\sin \beta}{\sin \gamma} \geq 2.$$

Therefore, $2\sin \varphi \leq \sin(\alpha + \varphi) \leq 1$, and so $\varphi \leq 30^\circ$. (The inequality $\varphi \leq 90^\circ$ is obvious from geometric considerations.)

(b) Consider the function $f(x) = \ln(x/\sin x) = \ln x - \ln \sin x$. It is clear that the functions

$$f'(x) = \frac{1}{x} - \text{ctg } x \quad \text{and} \quad f''(x) = \frac{1}{\sin^2 x} - \frac{1}{x^2}$$

are positive for $0 < x < \pi$. Therefore, the function $f(x)$ monotonically increases as x increases from 0 to π and, moreover, it is convex on this interval, i. e.,

$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n)$$

for $0 \leq x_i \leq \pi$, $0 \leq \lambda_i$, $\lambda_1 + \cdots + \lambda_n = 1$. In particular, $f(\varphi) \leq f(\pi/6)$, since $\varphi \leq \pi/6$, and

$$\begin{aligned} f(\pi/6) &= f\left(\frac{\varphi + (\alpha - \varphi) + \varphi + (\beta - \varphi) + \varphi + (\gamma - \varphi)}{6}\right) \\ &\leq \frac{1}{6}(f(\varphi) + f(\alpha - \varphi) + f(\varphi) + f(\beta - \varphi) + f(\varphi) + f(\gamma - \varphi)). \end{aligned}$$

Using the monotonicity of the logarithm, we can rewrite this in the form

$$\left(\frac{\varphi}{\sin \varphi}\right)^6 \leq \left(\frac{\pi/6}{\sin(\pi/6)}\right)^6 \leq \frac{\varphi^3(\alpha - \varphi)(\beta - \varphi)(\gamma - \varphi)}{\sin^3 \varphi \sin(\alpha - \varphi) \sin(\beta - \varphi) \sin(\gamma - \varphi)}.$$

Using the fact that $\sin(\alpha - \varphi) \sin(\beta - \varphi) \sin(\gamma - \varphi) = \sin^3 \varphi$, we obtain

$$\varphi^3 \leq (\alpha - \varphi)(\beta - \varphi)(\gamma - \varphi).$$

(c) The inequality $\varphi^3 \leq (\alpha - \varphi)(\beta - \varphi)(\gamma - \varphi)$ implies that

$$64\varphi^6 \leq 4^3 \varphi(\alpha - \varphi)\varphi(\beta - \varphi)\varphi(\gamma - \varphi).$$

It is also clear that

$$4\varphi(\alpha - \varphi) \leq \alpha^2, \quad 4\varphi(\beta - \varphi) \leq \beta^2, \quad 4\varphi(\gamma - \varphi) \leq \gamma^2.$$

□

Let P be the first Brocard point of triangle ABC . On the sides CA , AB , and BC choose points A_1 , B_1 , and C_1 so that

$$\angle APA_1 = \angle BPB_1 = \angle CPC_1 = \alpha$$

(Figure 4). Then triangles APA_1 , BPB_1 , and CPC_1 are similar. This means that by a rotational homothety centered at P with angle α and coefficient of similitude $\sin \varphi / \sin(\alpha + \varphi)$ triangle ABC will be taken to triangle $A_1B_1C_1$.

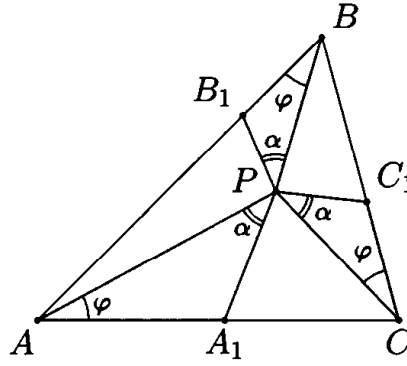


FIGURE 4

THEOREM 4. *Suppose the vertices B and C of a triangle are fixed, while the vertex A moves so that the Brocard angle φ of triangle ABC remains constant. Then the point A moves along a circle of radius $(a/2)\sqrt{\text{ctg}^2 \varphi - 3}$, where $a = BC$ (the Neuberg circle).*

PROOF. First let us show that

$$\text{ctg } \varphi = \frac{a^2 + b^2 + c^2}{4S},$$

where S is the area of the triangle. By the cosine theorem,

$$a^2 = b^2 + c^2 - 2bc \cos \alpha = b^2 + c^2 - 4S \text{ctg } \alpha.$$

Adding three analogous equations, we obtain

$$4S(\text{ctg } \alpha + \text{ctg } \beta + \text{ctg } \gamma) = a^2 + b^2 + c^2,$$

i.e., $4S \text{ctg } \varphi = a^2 + b^2 + c^2$.

Thus for a triangle with vertices at the points with coordinates $(\pm a/2, 0)$ and (x, y) , the Brocard angle φ is determined by the equation

$$\text{ctg } \varphi = \frac{a^2 + (a/2 + x)^2 + y^2 + (-a/2 + x)^2 + y^2}{2ay},$$

i.e., $2x^2 + 2y^2 + 3a^2/2 = 2ay \text{ctg } \varphi$. The last equation defines a circle of radius $(a/2)\sqrt{\text{ctg}^2 \varphi - 3}$ centered at $(0, (a/2) \text{ctg } \varphi)$. \square

THEOREM 5. *From the point M perpendiculars MA_1 , MB_1 , and MC_1 are dropped to the lines BC , CA , and AB . For a fixed triangle ABC , the set of points M for which the Brocard angle of triangle $A_1B_1C_1$ has a given value consists of two circles, one of which lies inside the circumscribed circle of triangle ABC , while the other lies outside it (the Schoute circles).*

PROOF. Let a_1, b_1, c_1 be the lengths of the sides of triangle $A_1B_1C_1$ and S_1 be its area. The theorem deals with the set of points M satisfying the equation

$$(1) \quad 4S_1 \text{ctg } \varphi = a_1^2 + b_1^2 + c_1^2.$$

The points B_1 and C_1 lie on the circle of diameter AM , therefore

$$a_1 = B_1C_1 = AM \sin B_1AC_1 = \frac{aAM}{2R},$$

where R is the radius of the circumscribed circle of triangle ABC . Thus

$$a_1^2 + b_1^2 + c_1^2 = \frac{a^2 AM^2 + b^2 BM^2 + c^2 CM^2}{4R^2}.$$

Therefore, if (x, y) are the coordinates of M in some Cartesian coordinate frame, then

$$(2) \quad a_1^2 + b_1^2 + c_1^2 = \frac{a^2 + b^2 + c^2}{4R^2}(x^2 + y^2) + px + qy + r,$$

where p, q, r are constants.

We can also express S_1 in terms of the coordinates (x, y) of the point M . To this end it is convenient to choose the origin of coordinates at the center O of the circle circumscribed to triangle ABC . In that case, as we shall presently prove,

$$(3) \quad S_1 = \frac{S_{ABC}}{4R^2} |R^2 - x^2 - y^2|;$$

in other words,

$$S_1 = \frac{S_{ABC}}{4R^2} |R^2 - d^2|,$$

where d is the distance from the point M to the center O of the circumscribed circle. To establish this formula, consider the auxilliary points A_2, B_2, C_2 , at which the lines AM, BM, CM intersect the circumscribed circle; let S_2 be the area of triangle $A_2B_2C_2$.

Let us show that triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar. Obviously,

$$\angle C_1B_1M = \angle C_1AM = \angle BAA_2 = \angle BB_2A_2 = \angle MB_2A_2.$$

Similarly, $\angle A_1B_1M = \angle MB_2C_2$, hence $\angle C_1B_1A_1 = \angle C_2B_2A_2$. The congruence of the other corresponding angles of triangles $A_1B_1C_1$ and $A_2B_2C_2$ is proved in the same way. From the similitude of these triangles it follows that

$$\frac{S_1}{S_2} = k^2, \quad \text{where} \quad k = \frac{a_1}{a_2} = \frac{AM \cdot CM}{2R \cdot B_2M}.$$

But $B_2M \cdot BM = |d^2 - R^2|$, and it follows that

$$\frac{S_1}{S_2} = \frac{(AM \cdot BM \cdot CM)^2}{4R^2(d^2 - R^2)^2}.$$

Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are inscribed in a circle of radius R , hence $S = \frac{abc}{4R}$ and $S_2 = \frac{a_2b_2c_2}{4R}$, so that

$$\frac{S_2}{S} = \frac{a_2b_2c_2}{abc}.$$

From the similitude of triangles B_2MC_2 and CMB it follows that $a_2 = \frac{a \cdot B_2M}{CM}$.

Besides, as we have seen, $a_1 = \frac{a \cdot AM}{2R}$. Hence,

$$\frac{a_2}{a} = \frac{B_2M}{CM} = \frac{|d^2 - R^2|}{BM \cdot CM}.$$

Therefore,

$$\frac{S_2}{S} = \frac{|d^2 - R^2|^3}{(AM \cdot BM \cdot CM)^2}.$$

Now we can write,

$$\frac{S_1}{S} = \frac{S_1 S_2}{S_2 S} = \frac{(AM \cdot BM \cdot CM)^2}{4R^2(d^2 - R^2)^2} \cdot \frac{|d^2 - R^2|^3}{(AM \cdot BM \cdot CM)^2} = \frac{|d^2 - R^2|}{4R^2},$$

as required.

The equation $S_1 = 0$ determines the circumscribed circle of the triangle ABC . This set corresponds to the zero Brocard angle. Equations (1), (2), and (3) imply that to the Brocard angle φ corresponds the set determined by the equation

$$\pm \operatorname{ctg} \varphi \frac{S_{ABC}}{R^2} (R^2 - x^2 - y^2) = \frac{a^2 + b^2 + c^2}{4R^2} (x^2 + y^2) + px + qy + r.$$

Here the plus sign is chosen for the points inside the circumcircle, the minus sign, for those outside it. It is easy to see that each of the obtained equations is that of a circle. This is because if $f = 0$ and $g = 0$ are equations of circles, then so is $\lambda f = g$. Moreover, the center of the circle $\lambda f = g$ lies on the line joining the centers of the circles $f = 0$ and $g = 0$. In our case the center of one circle will be the center of the circle circumscribed to triangle ABC , while the center of the second is the point for which the value of $a^2 AM^2 + b^2 BM^2 + c^2 CM^2$ is minimal (the *Lemoine point*). \square

It follows from Theorem 2 that both Brocard points of triangle ABC and its circumcenter lie on the same Schoute circle. It can be proved that the Lemoine point also lies on this circle.

Diophantine Equations for Polynomials

Polynomials possess many properties of natural numbers. For example, a polynomial can be decomposed into factors, for a pair of polynomials the greatest common divisor is defined, etc. In this connection one can formulate problems about polynomials similar to well-known exercises and problems about natural numbers. As a rule, the solution for polynomials is considerably simpler. For example, the famous conjecture of Pierre Fermat asserting that for $n \geq 3$ the equation $x^n + y^n = z^n$ has no solutions in natural numbers was proved only just recently. But its analog, the unsolvability of the equation $f^n + g^n = h^n$ for polynomials, as we shall soon see, has a relatively simple proof.

The description of all triples of natural numbers α, β, γ , for which the equation $f^\alpha + g^\beta = h^\gamma$ for polynomials f, g, h has a nontrivial solution, was obtained at the end of the 19th century. We shall present a more modern version of this classification.

For the proof of the unsolvability of Diophantine equations for polynomials, an effective approach is based on the following statement.

THEOREM 1 (Mason). *Let $a(x)$, $b(x)$, and $c(x)$ be pairwise coprime polynomials satisfying the equation $a + b + c = 0$. Then the degree of each of these polynomials is no greater than $n_0(abc) - 1$, where $n_0(abc)$ is the number of distinct roots of the polynomial abc .*

PROOF. Put $f = a/c$ and $g = b/c$. Then f and g are rational functions connected by the relation $f + g + 1 = 0$. Differentiating this equation, we obtain $f' = -g'$. Hence

$$\frac{b}{a} = \frac{g}{f} = -\frac{f'/f}{g'/g}.$$

The rational functions f and g have the special form $\prod (x - \rho_i)^{r_i}$, $r_i \in \mathbb{Z}$. For the function $R(x) = \prod (x - \rho_i)^{r_i}$, we have the relation

$$\frac{R'}{R} = \sum \frac{r_i}{x - \rho_i}.$$

Let

$$a(x) = \prod (x - \alpha_i)^{a_i}, \quad b(x) = \prod (x - \beta_j)^{b_j}, \quad c(x) = \prod (x - \gamma_k)^{c_k}.$$

Then

$$\begin{aligned} f'/f &= \sum \frac{a_i}{x - \alpha_i} - \sum \frac{c_k}{x - \gamma_k}, \\ g'/g &= \sum \frac{b_j}{x - \beta_j} - \sum \frac{c_k}{x - \gamma_k}. \end{aligned}$$

Hence, after multiplying by the polynomial

$$N_0 = \prod (x - \alpha_i)(x - \beta_j)(x - \gamma_k)$$

of degree $n_0(abc)$, the rational functions f'/f and g'/g become polynomials of degree no higher than $n_0(abc) - 1$. Since the polynomials $a(x)$ and $b(x)$ are coprime, the relation

$$\frac{b}{a} = -\frac{N_0 f/f'}{N_0 g/g'}$$

implies that the degree of each of the polynomials $a(x)$ and $b(x)$ is no greater than $n_0(abc) - 1$. For the polynomial $c(x)$ the proof is similar. \square

Theorem 1 has interesting corollaries that we shall state as Theorems 2–4. The degree of a polynomial f will be denoted by $\deg f$.

THEOREM 2 (Davenport). *Let f and g be coprime polynomials of positive degree. Then*

$$\deg(f^3 - g^2) \geq \frac{1}{2} \deg f + 1.$$

PROOF. If $\deg f^3 \neq \deg g^2$, then

$$\deg(f^3 - g^2) \geq \deg f^3 = 3 \deg f \geq \frac{1}{2} \deg f + 1.$$

We can therefore assume that $\deg f^3 = \deg g^2 = 6k$.

Consider the polynomials $F = f^3$, $G = g^2$, and $H = F - G = f^3 - g^2$. Clearly, $\deg H \leq 6k$. According to Theorem 1,

$$\max(\deg F, \deg G, \deg H) \leq n_0(FGH) - 1 \leq \deg f + \deg g + \deg H - 1,$$

i.e., $6k \leq 2k + 3k + \deg H - 1$. Thus, $\deg H \geq k + 1 = \frac{1}{2} \deg f + 1$. \square

THEOREM 3. *Suppose f , g , and h are coprime polynomials, and at least one of them is not a constant. Then the relation*

$$f^n + g^n = h^n$$

cannot hold for $n \geq 3$.

PROOF. According to Theorem 1, the degree of each of the polynomials f^n , g^n , and h^n is not greater than $\deg f + \deg g + \deg h - 1$, i.e.,

$$n \deg f, n \deg g, n \deg h \leq \deg f + \deg g + \deg h - 1.$$

Adding these three inequalities, we obtain

$$n(\deg f + \deg g + \deg h) \leq 3(\deg f + \deg g + \deg h - 1) < 3(\deg f + \deg g + \deg h).$$

Therefore, $n < 3$. \square

The Diophantine equation $f^\alpha + g^\beta = h^\gamma$ for the polynomials f, g, h has an obvious solution if one of the numbers α, β, γ equals 1. So we shall assume that $\alpha, \beta, \gamma \geq 2$.

THEOREM 4. *Let α, β, γ be natural numbers satisfying $2 \leq \alpha \leq \beta \leq \gamma$. Then the equation*

$$f^\alpha + g^\beta = h^\gamma$$

has coprime solutions for the following triples (α, β, γ) : $(2, 2, \gamma)$, $(2, 3, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$.

PROOF. Let a , b , and c be the degrees of the polynomials f , g , and h . Then according to Theorem 1,

$$\begin{aligned} (1) \quad & \alpha a \leq a + b + c - 1, \\ (2) \quad & \beta b \leq a + b + c - 1, \\ (3) \quad & \gamma c \leq a + b + c - 1. \end{aligned}$$

Therefore,

$$\alpha(a + b + c) \leq \alpha a + \beta b + \gamma c \leq 3(a + b + c) - 3,$$

and so $\alpha < 3$. By assumption, $\alpha \geq 2$, hence $\alpha = 2$. For $\alpha = 2$, inequality (1) takes the form

$$(4) \quad a \leq b + c - 1.$$

Adding inequalities (4), (2), and (3), we obtain

$$\beta b + \gamma c \leq 3(b + c) + a - 3.$$

Having in mind that $\beta \leq \gamma$, and applying inequality (4) once more, we obtain

$$\beta(b + c) \leq 4(b + c) - 4,$$

and so $\beta < 4$, i.e., $\beta = 2$ or 3 .

It remains to prove that if $\beta = 3$, then $\gamma \leq 5$. For $\beta = 3$ inequality (2) takes the form

$$(5) \quad 2b \leq a + c - 1.$$

Adding inequalities (4) and (5), we obtain

$$b \leq 2c - 2.$$

In that case inequality (4) implies that

$$a \leq 3c - 3.$$

From the two last inequalities together with inequality (3) it follows that

$$\gamma c \leq 6c - 6,$$

and so $\gamma \leq 5$. □

Polynomials satisfying the relation $f^\alpha + g^\beta = h^\gamma$ are closely connected with regular polyhedra. In detail this connection is described in Felix Klein's famous book "Lectures on the icosahedron and solutions of fifth degree equations"; in particular, the book indicates how to construct these polynomials. We only state the end result.

The case $\alpha = \beta = 2$, $\gamma = n$ is related to the degenerate regular polyhedron, namely the plane polygon of n sides. The required relation is of the form

$$\left(\frac{x^n + 1}{2}\right)^2 - \left(\frac{x^n - 1}{2}\right)^2 = x^n.$$

The case $\alpha = 2$, $\beta = 3$, $\gamma = 3$ is related to the regular tetrahedron. The relation has the form

$$12i\sqrt{3}(x^5 - x)^2 + (x^4 - 2i\sqrt{3}x^2 + 1)^3 = (x^4 + 2i\sqrt{3}x^2 + 1)^3.$$

The case $\alpha = 2$, $\beta = 3$, $\gamma = 4$ is related to the cube and the regular octahedron. The relation has the form

$$(x^{12} - 33x^8 - 33x^4 + 1)^2 + 108(x^5 - x)^4 = (x^8 + 14x^4 + 1)^3.$$

The case $\alpha = 2$, $\beta = 3$, $\gamma = 5$ is related to the dodecahedron and the icosahedron. The relation has the form $T^2 + h^3 = 1728f^5$, where

$$\begin{aligned} T &= x^{30} + 1 + 522(x^{25} - x^5) - 10005(x^{20} + x^{10}), \\ H &= -(x^{20} + 1) + 228(x^{15} - x^5) - 494x^{10}, \\ f &= x(x^{10} + 11x^5 - 1). \end{aligned}$$

Theorem 3 shows that the equation $x^n + y^n = z^n$, where x, y, z are natural numbers, has a solution if and only if exactly the same equation for polynomials has a solution. The following question naturally arises: Is it true that the equation $x^\alpha + y^\beta = z^\gamma$ will have polynomial solutions if and only if it has solutions in natural numbers?

The first example gives some hope: the equation $x^2 + y^3 = z^4$ has polynomial as well as natural solutions. Indeed, an infinite series of solutions of this equation in natural numbers may be obtained as follows. Put $x = n(n-1)/2$, $y = n$ and $z^2 = n(n+1)/2$. We are required to find a number n so that z will be an integer. The equation $2z^2 = n(n+1)$ may be rewritten in the form

$$(2n+1)^2 - 2(2z)^2 = 1.$$

This is the famous Fermat–Pelle equation, which has infinitely many solutions. For example, if $n = 8$, we obtain $x = 28$, $y = 8$, $z = 6$. Besides this infinite series of solutions, others may be indicated.

But equation $x^2 + y^4 = z^6$ dashes our hopes. This equation has no polynomial solutions, but has natural solutions. One of them is $x = 3^7 \cdot 5^9 \cdot 7 \cdot 29^8$, $y = 2 \cdot 3^3 \cdot 5^5 \cdot 29^4$, $z = 3^2 \cdot 5^3 \cdot 29^3$.

The Pascal Lines

In 1640 Blaise Pascal, who was then 16 years old, printed, in the form of a poster, 50 copies of his work “Essai sur les coniques”. These posters were hung on walls and distributed among scientists. This essay on conic sections contained several definitions and lemmas; there were no proofs. The main result of this treatise was the theorem about the inscribed hexagon. This theorem is now called Pascal’s theorem, and Pascal himself, as witnessed by Leibnitz, called it the “mystical hexagon” (*Hexagramma mysticum*). The theorem asserts the following. If the hexagon $ABCDEF$ is inscribed in a circle, then the intersections of the extensions of its opposite sides (i.e., of the lines AB and DE , BC and EF , CD and AF) lie on a straight line. Pascal also noted that by using central projection, one can derive from this theorem a similar statement about hexagons inscribed in any conic section (an ellipse, a parabola, or a hyperbola).

In Pascal’s theorem, the hexagon $ABCDEF$ is not necessarily convex; in the given case the hexagon may be any closed polygonal curve with six edges. For six fixed points on the circle, there exists exactly 60 closed hexagonal lines with vertices at these points. Indeed, choose any of the points; it can be joined to any of the five other points, the second point can be joined to the remaining four, and so on. As the result we obtain $5 \cdot 4 \cdot 3 \cdot 2 = 120$ different versions, but in our method of counting, we count each hexagonal line twice (because we take the orientation of the line into account). Thus the number of different hexagonal lines with vertices at six given points is $120/2 = 60$. To these 60 hexagonal lines correspond 60 Pascal straight lines. Jacob Steiner established in 1827 that these 60 Pascal lines may be split into 20 triples of lines intersecting at one point. Later Kirkman studied the configuration of the 60 Pascal lines in more detail and established (in 1850) that besides the Steiner points these lines have triple intersections at 60 more points.

Many proofs of Pascal’s theorem are known, but all of them are not too easy. We shall present one of the possible proofs that also allows us to obtain proofs of the results of Steiner and Kirkman.

For convenience we introduce the following notation. We shall suppose that straight line AB is given by the equation $l_{AB} = 0$; this equation is determined up to proportionality. In the coordinates x, y , the function l_{AB} is of the form $l_{AB}(x, y) = ax + by + c$, and l_{AB} vanishes at the points A and B .

LEMMA. *Let the points A, B, C , and D lie on the circle given by a second degree equation $f = 0$. Then*

$$f = \lambda l_{AB}l_{CD} + \mu l_{BC}l_{AD},$$

where λ and μ are some numbers.

FIRST PROOF. Let X be a point of the given circle distinct from the points A, B, C , and D . Choose numbers λ_1 and μ_1 so that

$$\lambda_1 l_{AB}(X)l_{CD}(X) + \mu_1 l_{BC}(X)l_{AD}(X) = 0,$$

and consider the curve given by the equation $f_1 = 0$, where

$$f_1 = \lambda_1 l_{AB}l_{CD} + \mu_1 l_{BC}l_{AD}.$$

This curve is determined by a second degree equation and passes through the points A, B, C, D , and X . But if a second degree curve passes through five distinct points of a circle, then it coincides with this circle and so $f = \alpha f_1$, where α is some number. \square

SECOND PROOF. Let us introduce the (not necessarily Cartesian) frame with axes AB and AD . Then the lines AB and AD are determined by the equations $y = 0$ and $x = 0$ respectively, while the equation $f = 0$ defining the circle is a second degree equation with respect to x and y .

The restriction of the functions f and

$$\lambda l_{AB}l_{CD} + \mu l_{BC}l_{AD} = \lambda y l_{CD} + \mu x l_{BC}$$

to any of the coordinate axes are quadratic trinomials with two common roots (A and B , or A and D). Thus the numbers λ and μ may be chosen so that the polynomial

$$P(x, y) = f(x, y) - \lambda y l_{CD}(x, y) - \mu x l_{BC}(x, y)$$

vanishes both for $x = 0$ and for $y = 0$. This means that it is divisible by xy , i.e., $P(x, y) = qxy$, where q is a constant. At the point C the polynomial P vanishes while $xy \neq 0$. Hence $q = 0$, i.e.,

$$f = \lambda l_{AB}l_{CD} + \mu l_{BC}l_{AD}.$$

\square

By means of this lemma we shall prove the theorems of Pascal, Steiner, and Kirkman. Consider the hexagon $ABCDEF$ whose vertices lie on the circle $f = 0$. The quadrilaterals $ABCD$, $AFED$, and $BEFC$ are inscribed in this circle, so f may be represented in any of the following forms:

- (1) $f = \lambda_1 l_{AB}l_{CD} + \mu_1 l_{AD}l_{BC},$
- (2) $f = \lambda_2 l_{AF}l_{ED} + \mu_2 l_{AD}l_{EF},$
- (3) $f = \lambda_3 l_{BE}l_{CF} + \mu_3 l_{BC}l_{EF}.$

Equating the expressions (1) and (2), we obtain

$$\lambda_1 l_{AB}l_{CD} - \lambda_2 l_{AF}l_{ED} = (\mu_1 l_{BC} - \mu_2 l_{EF})l_{AD}.$$

Let X be the intersection point of AB and ED . The functions $l_{AB}l_{CD}$ and $l_{AF}l_{ED}$ vanish at the point X , while l_{AD} does not vanish at this point. Therefore, the function $\mu_1 l_{BC} - \mu_2 l_{EF}$ vanishes at the point X , i.e., the point X lies on the line $\mu_1 l_{BC} = \mu_2 l_{EF}$. Similarly it can be proved that the intersection point of the lines CD and AF lies on the line $\mu_1 l_{BC} = \mu_2 l_{EF}$. It is also obvious that the intersection point of BC and EF lies on the line $\mu_1 l_{BC} = \mu_2 l_{EF}$. As the result, we obtain the following statement.

PASCAL'S THEOREM. *If the points $A, B, C, D, E,$ and F lie on one circle, then the intersection points of the lines AB and DE, BC and EF, CD and FA lie on one straight line.*

But let us continue our arguments. Equating (2) and (3), we see that the intersection points of the lines AF and BE, ED and CF, AD and BC lie on the line $\mu_2 l_{AD} = \mu_3 l_{BC}$. Finally, if we equate (1) and (3), we establish that the intersection points of the lines AB and CF, CD and BE, AD and EF lie on the line $\mu_1 l_{AD} = \mu_3 l_{EF}$. It is easy to check that the three lines just obtained,

$$\mu_1 l_{BC} = \mu_2 l_{EF}, \quad \mu_2 l_{AD} = \mu_3 l_{BC}, \quad \mu_1 l_{AD} = \mu_3 l_{EF},$$

intersect at one point. Indeed, if X is the intersection point of the first two of these lines, then

$$\mu_1 \mu_2 l_{BC}(X) l_{AD}(X) = \mu_2 \mu_3 l_{EF}(X) l_{BC}(X).$$

Cancelling $\mu_2 l_{BC}(X)$, we obtain $\mu_1 l_{AD} = \mu_3 l_{EF}$ (we shall not dwell on the degenerate case in which $\mu_2 l_{BC}(X) = 0$).

The *Pascal line* of a hexagon inscribed in a circle is defined as the straight line containing the intersection points of the three pairs of its opposite sides. Here, as we have already pointed out, we may think that the hexagon is a closed self-intersecting polygonal line. The statement proved above may be formulated as follows.

STEINER'S THEOREM. *Let the points $A, B, C, D, E,$ and F lie on the same circle. Then the Pascal lines of the hexagons $ABCDEF, ADEBCF,$ and $ADCFEB$ intersect at one point.*

Recall that in the proof of this theorem the initial quadrangles were $ABCD, AFED$ and $BEFC$. One can also start with the quadrangles $ABFE, ABDC$ and $CDFE$. Then we obtain the following statement.

KIRKMAN'S THEOREM. *Let the points $A, B, C, D, E,$ and F lie on the same circle. Then the Pascal lines of the hexagons $ABFDCE, AEFBDC,$ and $ABDFEC$ intersect at one point.*

It is easy to verify that each of the 60 Pascal lines corresponding to six given points of a circle belong to exactly one Steiner triple and exactly three Kirkman triples.

One Butterfly and Two Butterflies Theorems

The lemma that was used in the proof of the theorems of Pascal, Steiner, and Kirkman in our essay about Pascal lines is useful in proving many other geometric theorems. Let us recall its statement.

LEMMA. *Let the points $A, B, C,$ and D lie on the circle given by a second degree equation $f = 0$. Then*

$$f = \lambda l_{AB}l_{CD} + \mu l_{BC}l_{AD},$$

where λ and μ are some numbers, while l_{XY} denotes the (linear) equation of the line XY .

A self-intersecting quadrilateral resembles a butterfly, in a way. This explains the names of the next two statements.

ONE BUTTERFLY THEOREM. *Suppose the chords KL and MN pass through the midpoint O of the chord AB . Then the lines KN and ML intersect line AB at two points equidistant from O .*

TWO BUTTERFLIES THEOREM. *Suppose that the self-intersecting quadrilaterals $KLMN$ and $K'L'M'N'$, inscribed in the same circle, intersect the chord AB of this circle at the points P, Q, R, S and P', Q', R', S' respectively (the side KL at the point P , LM at the point Q , etc.). If three of the points P, Q, R, S coincide respectively with three of the points P', Q', R', S' , then the remaining two points also coincide. (It is assumed that the chord AB does not pass through the vertices of the quadrangles.)*

PROOF OF THE ONE BUTTERFLY THEOREM. Let $f = 0$ be the equation of the circle. According to the lemma,

$$f = \lambda l_{KL}l_{MN} + \mu l_{KN}l_{ML}.$$

This equation also holds for the restrictions of all the functions under consideration to the line AB . Introduce the coordinate x on the line AB , taking O for the origin. Then we can assume that $f = x^2 - a$ and $l_{KL}l_{MN} = x^2$, hence $l_{KN}l_{ML} = bx^2 - c$. Therefore, the roots of the equation $l_{KN}l_{ML} = 0$ are equidistant from O , as required. \square

PROOF OF THE TWO BUTTERFLIES THEOREM. To be definite, let $P = P', Q = Q',$ and $R = R'$. According to the lemma,

$$\lambda l_{KL}l_{MN} + \mu l_{KN}l_{ML} = f = \lambda' l_{K'L'}l_{M'N'} + \mu' l_{K'N'}l_{M'L'}.$$

Considering the restriction of this equation to AB , we obtain an equation of the form

$$(1) \quad \alpha(x-p)(x-r) + \beta(x-r)(x-s) = \alpha'(x-p)(x-r) + \beta'(x-q)(x-s').$$

Now we must prove that $s = s'$.

The equation (1) may be rewritten in the form

$$\alpha''(x-p)(x-r) = (x-q)[\beta(x-s) - \beta'(x-s')].$$

The point O can only coincide with S , hence $Q \neq P$ and $Q \neq R$, and so $(x-p)(x-r)$ is not divisible by $(x-q)$. Hence $\beta(x-s) - \beta'(x-s') = 0$. Therefore, $s = s'$. \square

The Van der Waerden Theorem on Arithmetical Progressions

The van der Waerden theorem on arithmetical progressions has an interesting history. The following conjecture was put forward by Baudet:

- (A) *If the natural numbers are split in some way into two classes, then for any natural number l one can find an arithmetical progression of length l in one of these classes (i.e., the chosen class contains the numbers*

$$a, a + d, a + 2d, \dots, a + (l - 1)d$$

for some a and d).

This conjecture soon became rather well known. Many mathematicians attempted to prove it. But this turned out to be no easy task. The first important results were obtained by the outstanding mathematicians E. Artin and O. Schreier.

First Schreier proved that Baudet's conjecture is equivalent to the following statement:

- (B) *For any natural number l there exists a natural number $N(l)$ such that if the numbers $1, 2, \dots, N(l)$ are split into two classes, then one of these classes contains an arithmetical progression of length l .*

Then Artin proved that (B) is equivalent to the following statement:

- (C) *For any two natural numbers l and k there exists a natural number $N(l, k)$ such that if the numbers $1, 2, \dots, N(l, k)$ are split into k classes, then one of these classes contains an arithmetical progression of length l .*

Statement (A) obviously follows from (C). But, as it turned out, it is more convenient to prove the more general (C) by using double induction on k and l . Such a fairly complicated proof was devised by van der Waerden in 1927.

The splitting of a set into k classes can be visualized by imagining that it is painted in k colors. In those terms (C) can be stated as follows:

- (C') *For any two natural numbers l and k there exists a natural number $N(l, k)$ such that if the numbers $1, 2, \dots, N(l, k)$ are painted in k colors, then one can find a one-color arithmetical progression of length l .*

Later new proofs of the van der Waerden theorem, as well as of its generalizations, appeared. We shall discuss a farguing generalization whose proof is relatively easy. This proof is due to P. G. Anderson.¹

¹P. G. Anderson, *A generalization of Baudet's conjecture (Van der Waerden's theorem)*, Amer Math. Monthly, **83** (1976), 359–361.

the cube $v + K_N$. This cube consists of N^n points, so there can be k^{N^n} different colorings of the lattice points in k colors. To these k^{N^n} colorings let us assign a family of k^{N^n} new colors and use them to paint each point v in accordance with the coloring of the cube $v + K_N$. Now there will be many more colors, but the new coloring has the following property: the new color of two points u and v is the same if and only if the old color of the points $u + x$ and $v + x$ is the same for all points x of the cube K_N , i.e., the cubes $u + K_N$ and $v + K_N$ were painted exactly in the same way in the old coloring. Let us call this new coloring *induced*; it depends on the initial coloring and on the size of the cube K_N .

Suppose both (A_S) and $(C_{S,w,p})$ hold. Then there exists a natural number N_p such that for any coloring of the cube K_{N_p} in k colors one can find natural numbers a_1, \dots, a_p and a vector v for which each of the sets T_0, T_1, \dots, T_p are painted in one color. Consider the induced coloring in $k' = k^{N_p^n}$ colors corresponding to the cube K_{N_p} . We can apply the assertion $(C_{S,w,1})$ proved above to this coloring. As the result, we see that there exists a cube $K_{N'}$ containing one-color sets (in the induced coloring) $a'w + v'$ and $a'S + v'$. The fact that the set $a'S + v'$ is one-color means that in the initial coloring the cubes $K_{N_p} + a's + v'$ are identically colored at all points s of the set S . Each of these identically colored cubes contain the following one-color sets:

$$T_q + a's + v', \quad q = 0, 1, \dots, p,$$

and the color of these sets does not depend on s . Thus we have obtained the one-color sets

$$T'_{q+1} = a'S + T_q + v'.$$

If we add the one-color set $T'_0 = a'w + T_0 + v'$ to these sets, we obtain the required family of the one-color sets

$$\begin{aligned} T'_0 &= (a' + a_1 + \dots + a_p)w + v'', \\ T'_1 &= a'S + (a_1 + a_2 + \dots + a_p)w + v'', \\ &\dots\dots\dots \\ T'_{p+1} &= (a' + a_1 + \dots + a_p)S + v'', \end{aligned}$$

where $v'' = v + v'$.

STEP 2. *If $(C_{S,w,p})$ is true for all natural numbers p , then so is $(A_{S \cup w})$.*

We shall only use the fact that $(C_{S,w,p})$ holds for $p = k$, where k is the number of colors. In that case we obtain one-color sets T_0, T_1, \dots, T_k . There are more of these sets than there are colors, hence two of them must have the same color. Assume that, say, the colors of T_r and T_q , where $r < q$, are the same. Recall that

$$\begin{aligned} T_r &= (a_1 + \dots + a_{r-1})S + (a_r + \dots + a_{q-1})w + (a_q + \dots + a_p)w + v, \\ T_q &= (a_1 + \dots + a_{r-1})S + (a_r + \dots + a_{q-1})S + (a_q + \dots + a_p)w + v. \end{aligned}$$

Put $a' = a_r + \dots + a_{q-1}$ and $v' = (a_1 + \dots + a_{r-1})s + (a_q + \dots + a_p)w + v$, where s is some point of S . Then $a'(S \cup w) + v'$ is a one-color set of the required form.

Isogonal Conjugate Points

To each triangle ABC there corresponds a very interesting transformation of the plane. This transformation may be described as follows. Let P be some point. Let us reflect the lines AP , BP , and CP in the bisectors of the angles A , B , and C , respectively. The obtained lines intersect at one point Q (Figure 1). To prove this statement and to study the properties of the transformation $P \mapsto Q$, it is convenient to use the so-called trilinear coordinates, which we shall now introduce.

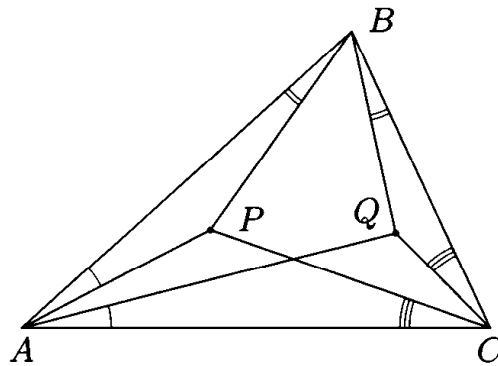


FIGURE 1

Let a , b , c be the distances from the point X lying inside triangle ABC to its sides BC , CA , AB . Then the number triple (a, b, c) is said to constitute the *trilinear coordinates* of the point X . Here the triple (a, b, c) is considered up to proportionality, i.e., the triple $(\lambda a, \lambda b, \lambda c)$, where $\lambda > 0$, corresponds to the same point as (a, b, c) . In that case any triple (a, b, c) , where $a, b, c > 0$, uniquely determines an inner point of triangle ABC . Indeed, the set of points with trilinear coordinates of the form (a, b, x) , where a and b are fixed, is the segment CK , where K is a point of side AB . As we move along this segment, the value a/x monotonically varies from 0 to ∞ .

Trilinear coordinates can be defined not only for the interior points of the triangle, but for all the points of the plane. We shall assume that a , b , c are the distances from X to the lines BC , CA , AB with sign, i.e., $a > 0$, if X and A are both on one side of BC and $a < 0$ if X and A are on different sides of BC ; the signs of b and c are defined similarly.

The set of points with trilinear coordinates (a, b, x) , where a and b are fixed numbers, is a straight line passing through the vertex C . Under the symmetry in the bisector of angle C , this line is taken to the line consisting of points with trilinear coordinates of the form (b, a, x) , or, which is the same, (a^{-1}, b^{-1}, y) . Thus if the point P has the trilinear coordinates (a, b, c) , then the desired point Q is well

defined as the point with trilinear coordinates (a^{-1}, b^{-1}, c^{-1}) . In that case P and Q are called *isogonally conjugate* with respect to triangle ABC .

Isogonal conjugation has exactly four fixed points (i.e., four points each of which is isogonally conjugate to itself). These points have the trilinear coordinates $(1, 1, 1)$, $(-1, 1, 1)$, $(1, -1, 1)$ and $(1, 1, -1)$. Clearly, the first of them is the center of the inscribed circle, and the other three are the centers of the exterior inscribed circles.

It is also obvious that the first Brocard point is isogonally conjugate to the second one (see the essay about the Brocard points).

Let us prove some less obvious, but more interesting properties of isogonal conjugation.

THEOREM 1. *The center of the circumscribed circle is isogonally conjugate to the intersection points of the altitudes.*

PROOF. Let α, β, γ be the angles of triangle ABC . Then the center of its circumcircle has the trilinear coordinates $(\cos \alpha, \cos \beta, \cos \gamma)$.

Now let us compute the trilinear coordinates of the intersection point H of the altitudes. For instance, let us compute the length of HA_1 , where A_1 is the base of the altitude dropped from A . Clearly $HA_1 = BH \cos \gamma$ and $BH \sin \alpha = BC \cos \beta = 2R \sin \alpha \cos \beta$, where R is the radius of the circumcircle. Hence $HA_1 = 2R \cos \beta \cos \gamma$. This means that the intersection point of the altitudes has the trilinear coordinates $(1/\cos \alpha, 1/\cos \beta, 1/\cos \gamma)$. \square

THEOREM 2. *The point for which the sum of squares of the distances to the vertices is minimal is isogonally conjugate to the point for which the sum of squares of distances to the sides is minimal.*

PROOF. First let us note that the first of these two points is the intersection point M of the medians. The proof of this is based on the relation

$$\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} = 0.$$

Indeed, this relation implies that for an arbitrary point X we have

$$\begin{aligned} XA^2 + XB^2 + XC^2 &= |\overrightarrow{XM} + \overrightarrow{MA}|^2 + |\overrightarrow{XM} + \overrightarrow{MB}|^2 + |\overrightarrow{XM} + \overrightarrow{MC}|^2 \\ &= 3XM^2 + 2(\overrightarrow{XM}, \overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC}) \\ &\quad + MA^2 + MB^2 + MC^2 \\ &= 3XM^2 + MA^2 + MB^2 + MC^2 \geq MA^2 + MB^2 + MC^2. \end{aligned}$$

It is easy to check that the intersection point of the medians has the trilinear coordinates (a^{-1}, b^{-1}, c^{-1}) , where a, b, c are the lengths of the triangle's sides. This immediately follows from the fact that the areas of the triangles AMB , BMC , and CMA are equal. Therefore the point isogonally conjugate to the intersection point of the medians has the trilinear coordinates (a, b, c) .

Consider an arbitrary point with trilinear coordinates (x, y, z) . The distances from this point to the sides of the triangle equal kx, ky, kz , where k is determined from the relation $k(ax + by + cz) = 2S$ (S is the area of triangle ABC). Thus the sum of squares of distances from the point to the sides equals

$$4S^2 \frac{x^2 + y^2 + z^2}{(ax + by + cz)^2}.$$

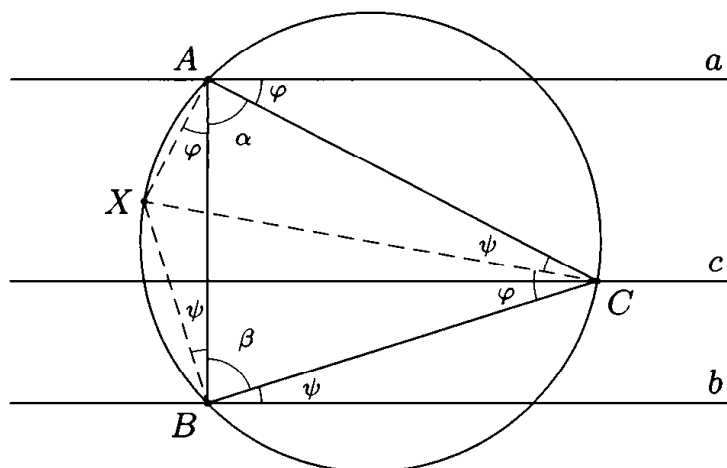


FIGURE 2

We must prove that this sum is minimal for the case in which $(x, y, z) = (a, b, c)$, i.e.,

$$\frac{x^2 + y^2 + z^2}{(ax + by + cz)^2} \geq \frac{a^2 + b^2 + c^2}{(a^2 + b^2 + c^2)^2}.$$

The proof of the last inequality is quite simple. \square

THEOREM 3. *The circumscribed circle to the triangle is isogonally conjugate to the line at infinity. In other words, if X is a point of the circumcircle, while the lines a, b, c are symmetric to AX, BX, CX in the bisectors of angles A, B, C , then the lines a, b, c are parallel (Figure 2).*

PROOF. Let us show, for instance, that the sum of the angles $\alpha + \varphi$ and $\beta + \psi$ that AB forms with the lines a and b is equal to π . Indeed, $\varphi + \psi = \gamma$, while $\alpha + \beta + \gamma = \pi$. \square

REMARK. It is not hard to prove that in trilinear coordinates the line at infinity and the circumcircle are determined by the equations

$$ax + by + cz = 0 \quad \text{and} \quad ayz + bzx + cxy = 0,$$

respectively, where a, b, c are the lengths of the sides of the triangle.

THEOREM 4. *Let all the angles of triangle ABC be less than 120° . Then the point from which all three sides of ABC are seen under an angle of 120° (the Torricelli point), is isogonally conjugate to the point whose projection on the sides of ABC form an equilateral triangle.*

PROOF. The Toricelli point T may be constructed as follows. On the sides of ABC let us construct (outwards) the triangles A_1BC, AB_1C , and ABC_1 . Then their circumcircles intersect at the point T . This construction implies that the segments $AC_1, C_1B, BA_1, A_1C, CB_1, B_1A$ are seen from T under angles of 60° , hence T is the intersection point of the segments AA_1, BB_1, CC_1 . Now it is easy to prove that T has the trilinear coordinates

$$\left(\frac{1}{\sin(60^\circ + \alpha)}, \frac{1}{\sin(60^\circ + \beta)}, \frac{1}{\sin(60^\circ + \gamma)} \right).$$

Indeed, the distances from the point C_1 to the lines BC and CA equal $c \sin(60^\circ + \beta)$ and $c \sin(60^\circ + \alpha)$. Hence the ratio of distances from T to sides BC and CA equals

$$\sin(60^\circ + \beta) : \sin(60^\circ + \alpha).$$

And so we must prove that the projections on the sides of ABC of the points with trilinear coordinates

$$(\sin(60^\circ + \alpha), \sin(60^\circ + \beta), \sin(60^\circ + \gamma))$$

form a regular triangle. To this aim we argue as follows. Consider a point P from which the sides of the triangle are seen under the angles $60^\circ + \alpha$, $60^\circ + \beta$, $60^\circ + \gamma$; let us show that:

- 1) the projections A_1B_1, C_1 of P on the sides of ABC form a regular triangle;
- 2) the point P has the trilinear coordinates

$$(\sin(60^\circ + \alpha), \sin(60^\circ + \beta), \sin(60^\circ + \gamma)).$$

The points A_1 and B_1 lie on the circle with diameter CP . Hence

$$\angle PA_1B_1 = \angle PCB_1 = \angle PCA.$$

Similarly, $\angle PA_1C_1 = \angle PBA$. Hence,

$$\begin{aligned} \angle B_1A_1C_1 &= \angle PCA + \angle PBA = (\gamma - \angle PCB) + (\beta - \angle PBC) \\ &= \beta + \gamma + \angle BPC - 180^\circ = 60^\circ. \end{aligned}$$

Similarly we can prove that the other angles of triangle $A_1B_1C_1$ equal 60° .

In order to compute the trilinear coordinates of P , first let us note that

$$AP : BP : CP = \frac{1}{\sin(60^\circ + \alpha)} : \frac{1}{\sin(60^\circ + \beta)} : \frac{1}{\sin(60^\circ + \gamma)}.$$

This is because, by the sine theorem,

$$B_1C_1 = AP \sin B_1AC_1 = AP \sin \alpha.$$

Moreover, as we already know, $B_1C_1 = C_1A_1 = A_1B_1$. It is also clear that

$$PA_1 \cdot BC = BP \cdot CP \sin(60^\circ + \alpha)$$

(because both expressions are equal to half the area of triangle BPC). Therefore

$$PA_1 : PB_1 : PC_1 = \sin(60^\circ + \alpha) : \sin(60^\circ + \beta) : \sin(60^\circ + \gamma),$$

as required. \square

THEOREM 5 (Morley). *Suppose that the vertices of triangle ABC lie in the complex plane on the unit circle $|z| = 1$; suppose, further, that a, b, c are the complex coordinates of these vertices. Then any two points p and q isogonally conjugate with respect to triangle ABC are related by the equation*

$$p + q + abc\bar{p}\bar{q} = a + b + c.$$

PROOF. On the complex plane the bisectors of the internal and external angles between the vectors z and w coincide with the bisectors of the internal and external angles between the vectors z' and w' if and only if

$$\frac{zw}{\bar{z}\bar{w}} = \frac{z'w'}{\bar{z}'\bar{w}'}$$

Indeed, let $z = |z|e^{i\varphi}$ and $w = |w|e^{i\psi}$. Then $zw/(\bar{z}\bar{w}) = e^{2i(\varphi+\psi)}$, while the bisectors of the internal and external angles between z and w correspond to the

angles $(\varphi + \psi)/2 + k\pi/2$. But the specification of the angle $2(\varphi + \psi)$ up to 2π is the same as the specification of the angle $(\varphi + \psi)/2$ up to $\pi/2$.

Thus, if the rays AP and AQ are symmetric in the bisector of angle A , then

$$\frac{(p-a)(q-a)}{(\bar{p}-\bar{a})(\bar{q}-\bar{a})} = \frac{(b-a)(c-a)}{(\bar{b}-\bar{a})(\bar{c}-\bar{a})}.$$

The expression in the right-hand side equals a^2bc since $ab(\bar{b}-\bar{a}) = a-b$ and $ac(\bar{c}-\bar{a}) = a-c$. Therefore,

$$pq - a(p+q) + a^2 = a^2bc\bar{p}\bar{q} - abc(\bar{p} + \bar{q}) + bc.$$

Similarly we obtain

$$pq - b(p+q) + b^2 = b^2ac\bar{p}\bar{q} - abc(\bar{p} + \bar{q}) + ac.$$

Let us take the difference of the last two equations and cancel $a-b$ in both sides. As the result we obtain the required equality. \square

THEOREM 6. *Let P_1 and P_2 be isogonally conjugate. From P_i drop the perpendiculars P_iA_i , P_iB_i , P_iC_i to the sides BC , CA , AB , respectively. Then the circumcircles of triangles $A_1B_1C_1$ and $A_2B_2C_2$ coincide.*

PROOF. Let us prove that the points B_1, B_2, C_1, C_2 lie on one circle. Indeed,

$$\angle P_1B_1C_1 = \angle P_1AC_1 = \angle P_2AB_2 = \angle P_2C_2B_2,$$

but since $\angle P_1B_1A = \angle P_2C_2A = 90^\circ$, it follows that $\angle C_1B_1A = \angle B_2C_2A$. The center of the circle that contains the four points is the intersection point of the perpendiculars to the midpoints of the segments B_1B_2 and C_1C_2 , and both these perpendiculars pass through the midpoint O of the segment P_1P_2 , i.e., O is the center of that circle. In particular, B_1 and C_1 are equidistant from O . Similarly, A_1 and B_1 are equidistant from O , i.e., O is the center of the circle circumscribed to $A_1B_1C_1$. Moreover, $OB_1 = OB_2$. \square

For the case in which P_1 is the intersection point of the altitudes, while P_2 is the center of the circumcircle, the coincidence of these two circles means that the basepoints of the altitudes and the midpoints of the sides lie on one circle (the *Euler circle*).

A more interesting statement is obtained for the case in which P_1 lies on the circumcircle of the triangle. In this case the point P_2 moves away to infinity. This means that the basepoints of the perpendiculars dropped to the sides of the triangle from an arbitrary point of the circumcircle lie on a circle of infinite radius, i.e., they lie on one straight line (often called the *Simson line*, although a historically more precise name for it is the *Wallace line*). The Wallace line for the point P_1 is perpendicular to the direction of the point at infinity P_2 , i.e., it is perpendicular to the lines symmetric to AP_1 , BP_1 , CP_1 with respect to the bisectors of angles A , B , C (this assertion is equivalent to the property of the circle of being perpendicular at each point to the radius passing through this point; of course, in our case we are dealing with the circle of infinite radius.)

In conclusion, let us present without proof one more property of isogonally conjugate points.

THEOREM 7. *The foci of any ellipse inscribed in a triangle are isogonally conjugate.*

Cubic Curves Related to the Triangle

There are different ways of assigning, to each triangle, a cubic curve, i.e., a curve given by an equation of the form

$$\sum_{i+j \leq 3} a_{ij} x^i y^j = 0.$$

Some of these curves have interesting geometric properties. These cubic curves, or *cubics*, usually bear the name of the geometers who first investigated them: the Darboux cubic, the Thomson cubic, the Neuberg cubic, and the McCay cubic.

The most interesting properties of cubics related to triangles are based in one way or another on isogonal conjugation with respect to the triangle. Therefore our exposition will be based on properties of isogonal conjugation, which we studied in the previous essay. We shall also make use of trilinear coordinates introduced there. It is easy to understand that in trilinear coordinates (x, y, z) any cubic curve is given by an equation of the form

$$\sum_{i+j+k=3} c_{ijk} x^i y^j z^k = 0.$$

Originally, the cubics related to triangles were defined by means of various geometric constructions. The most famous cubics among these can be obtained by a unified construction¹. This construction is based on the following statement.

THEOREM. *Let F be a point in the plane. For a given triangle ABC consider all possible pairs of isogonally conjugate points P and Q for which the line PQ passes through F . Then P and Q sweep a cubic curve that passes through the vertices of the triangle, through the centers of the inscribed circle and the three exterior inscribed circles, as well as through the point F itself.*

PROOF. Let F have the trilinear coordinates (f_1, f_2, f_3) . If P has the trilinear coordinates (x, y, z) , then the point Q isogonally conjugate to it has the trilinear coordinates (x^{-1}, y^{-1}, z^{-1}) , i.e., (yz, zx, xy) . Therefore the condition that the points P, Q, F lie on a single straight line can be written in the form

$$\begin{vmatrix} f_1 & f_2 & f_3 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = 0,$$

i.e.,

$$(1) \quad f_1 x(y^2 - z^2) + f_2 y(z^2 - x^2) + f_3 z(x^2 - y^2) = 0.$$

¹H M Cundy, C F Parry, *Some cubic curves associated with a triangle*, *Journal of Geometry*, **53** (1995), 41-66.

It is easy to check that

$$\begin{aligned} F &= (f_1, f_2, f_3), & A &= (1, 0, 0), & B &= (0, 1, 0), & C &= (0, 0, 1), \\ I &= (1, 1, 1), & I_a &= (-1, 1, 1), & I_b &= (1, -1, 1), & I_c &= (1, 1, -1) \end{aligned}$$

all lie on the curve given by equation (1), i.e., the coordinates of these points satisfy this equation. \square

One can directly see from the geometric definition of the curve (1) that it is transformed into itself under isogonal conjugation. Indeed, if P lies on the curve (1), then its isogonally conjugate point Q also lies on the curve (1).

The point F that we used to construct the cubic curve (1) will be called the *pivot* of this curve.

The Darboux cubic. The pivot of this curve is the point \mathcal{H} symmetric to the intersection point H of the altitudes with respect to the center O of the circumcircle. It is easy to verify that the point \mathcal{H} has the trilinear coordinates

$$(\cos \alpha - \cos \beta \cos \gamma, \cos \beta - \cos \gamma \cos \alpha, \cos \gamma - \cos \alpha \cos \beta),$$

where α, β, γ are the angles of our triangle.

In trilinear coordinates, the Darboux cubic is given by the equation

$$(\cos \alpha - \cos \beta \cos \gamma)x(y^2 - z^2) + \dots = 0.$$

(We have written out only the coefficient at $x(y^2 - z^2)$; the coefficients at $y(z^2 - x^2)$ and at $z(x^2 - y^2)$ can be written in the obvious way.)

The Darboux cubic passes through the following points: the orthocenter and the center of the circumcircle.

The Darboux cubic has the following geometric description.

THEOREM 1. *Let A_1, B_1, C_1 be the projections of the point D on the lines BC, CA, AB . The point D lies on the Darboux cubic if and only if the lines AA_1, BB_1, CC_1 intersect at one point.*

PROOF. According to Ceva's theorem, the lines AA_1, BB_1, CC_1 intersect at one point if and only if

$$AC_1 \cdot BA_1 \cdot CB_1 = C_1B \cdot A_1C \cdot B_1A,$$

where AC_1 , etc. are the oriented lengths of the segments (i.e., the numbers AC_1 and C_1B have the same sign if C_1 lies on the segment AB , while if C_1 lies outside AB , then these numbers have opposite signs).

Let (x, y, z) be the normed trilinear coordinates of the point D , i.e., x, y, z are the distances from D to BC, CA, AB with signs taken in consideration. It is easy to verify that

$$AC_1 = \frac{z \cos \alpha + y}{\sin \alpha},$$

etc. Therefore the lines AA_1, BB_1, CC_1 intersect at one point if and only if

$$(z \cos \alpha + y)(y \cos \gamma + x)(x \cos \beta + z) = (z \cos \beta + x)(x \cos \gamma + y)(y \cos \alpha + z).$$

This relation can readily be transformed into the equation for the Darboux cubic. \square

REMARK 1. If the equation

$$AC_1 \cdot BA_1 \cdot CB_1 = C_1B \cdot A_1C \cdot B_1A$$

holds for some point D , then the same equation holds for the point D' symmetric to D with respect to the center of the circumcircle. Therefore the Darboux cubic is symmetric with respect to the center of the circumcircle.

REMARK 2. It is not difficult to prove that the lines AA_1 , BB_1 , CC_1 intersect at one point if and only if there exists a second order curve tangent to the sides of the triangle (or their continuations) at the points A_1 , B_1 , C_1 .

The Thomson cubic. The pivot of this curve is the center of mass M . Recall that the center of mass of the triangle has the trilinear coordinates (bc, ca, ab) .

In trilinear coordinates, the Thomson cubic is given by the equation

$$bcx(y^2 - z^2) + cay(z^2 - x^2) + abz(x^2 - y^2) = 0.$$

This equation can be rewritten in the form

$$(\cos \alpha + \cos \beta \cos \gamma)x(y^2 - z^2) + \dots = 0.$$

The Thomson cubic passes through the following points: the orthocenter and the center of the circumcircle, the midpoints of the sides and the midpoints of the altitudes.

It is clear from Remark 2 to Theorem 1 that the Darboux cubic has the following geometric description. Consider all the second order curves that are tangent to the sides of the given triangle or their continuations. Among them let us distinguish those for which the perpendiculars to the sides of the triangle at the points of tangency intersect at one point. Then the intersection points of these perpendiculars sweep through the Darboux cubic. It can be proved that the centers of the distinguished second order curves sweep through the Thomson cubic.

The McCay cubic. The pivot of this cubic curve is the center O of the circumcircle. Recall that the center of the circumcircle has the trilinear coordinates $(\cos \alpha, \cos \beta, \cos \gamma)$.

In trilinear coordinates, the McCay cubic is given by the equation

$$\cos \alpha x(y^2 - z^2) + \cos \beta y(z^2 - x^2) + \cos \gamma z(x^2 - y^2) = 0.$$

The McCay cubic passes through the following points: the orthocenter and the center of the circumcircle.

THEOREM 2. *Let the vertices of a triangle be located at the points a, b, c of the unit circle of the complex plane. The point corresponding to the complex number z lies on the McCay cubic if and only if we have*

$$(z - a)(z - b)(z - c) = abc(a\bar{z} - 1)(b\bar{z} - 1)(c\bar{z} - 1).$$

PROOF. Let the points z and w be isogonally conjugate with respect to the given triangle. Then according to Morley's theorem (Theorem 5 from the previous essay) the points z and w are related by the equation

$$(2) \quad z + w + abc\bar{z}\bar{w} = a + b + c.$$

Therefore

$$(3) \quad \bar{z} + \bar{w} + \bar{a}\bar{b}\bar{c}zw = \bar{a} + \bar{b} + \bar{c}.$$

Multiplying both sides of equation (3) by $abc\bar{z}$ and subtracting equation (2) from the result, we obtain

$$(4) \quad w = \frac{a + b + c - z - (\bar{a} + \bar{b} + \bar{c} - \bar{z})abc\bar{z}}{1 - |abcz|^2}.$$

By the definition of the McCay cubic, the line zw passes through the center of the circumcircle, i.e., through the origin. This means that $w/\bar{w} = z/\bar{z}$. Expressing w/\bar{w} by using equation (4), after some manipulation we obtain the required equation. \square

COROLLARY. *The McCay cubic intersects the circle circumscribed to the corresponding triangle at three points forming the vertices of a regular triangle. (We only take into account the intersections distinct from the vertices of the original triangle.)*

PROOF. We shall assume that the vertices of the given triangle are located on the unit circle of the complex plane. Then for any point z lying on the circumcircle of the triangle, we have $\bar{z} = z^{-1}$. Therefore the intersection points of the McCay cubic with the circumcircle satisfy the equation

$$(z - a)(z - b)(z - c) = -z^{-3}abc(z - a)(z - b)(z - c).$$

If we exclude the vertices of the triangle, the remaining points will satisfy the relation $z^3 = -abc$. These points form a regular triangle. \square

We shall assume that $\angle PQR$ is the value of the angle by which we must rotate (counterclockwise) the vector \overrightarrow{QP} in order to make it codirected with the vector \overrightarrow{QR} .

THEOREM 3. *A point M lies on the McCay cubic if and only if*

$$\angle MAB + \angle MBC + \angle MCA = \frac{\pi}{2} + k\pi.$$

PROOF. Again we shall assume that the vertices of the triangle lie on the unit circle of the complex plane. We put $\alpha = \angle MAB$, $\beta = \angle MBC$, $\gamma = \angle MCA$. Let z be the complex number corresponding to the point M . Then

$$\frac{b - a}{z - a} \cdot \frac{\bar{z} - \bar{a}}{\bar{b} - \bar{a}} = e^{2i\alpha}, \quad \text{i.e.,} \quad e^{2i\alpha} = -b \frac{a\bar{z} - 1}{z - a}.$$

Therefore

$$e^{2i(\alpha+\beta+\gamma)} = -abc \frac{(a\bar{z} - 1)(b\bar{z} - 1)(c\bar{z} - 1)}{(z - a)(z - b)(z - c)}.$$

Thus the point z lies on the McCay cubic if and only if $e^{2i(\alpha+\beta+\gamma)} = -1$, i.e.,

$$\alpha + \beta + \gamma = \frac{\pi}{2} + k\pi.$$

\square

It is easy to check that

$$\angle MAB + \angle MBC + \angle MCA + \angle MAC + \angle MCB + \angle MBA = (2n + 1)\pi.$$

Hence M lies on the McCay cubic if and only if

$$\angle MAB + \angle MBC + \angle MCA = \angle MAC + \angle MCB + \angle MBA + 2l\pi.$$

The Neuberg cubic. The pivot of this curve is the point at infinity of the line OH . In other words, the Neuberg cubic consists of pairs of isogonally conjugate points P and Q for which PQ is parallel to OH .

In trilinear coordinates, the Neuberg cubic is given by the equation

$$(\cos \alpha - 2 \cos \beta \cos \gamma)x(y^2 - z^2) + \dots = 0.$$

The Neuberg cubic is the undisputed leader in the number of remarkable points of the triangle through which it passes. Indeed, this curve passes through the following points: center of the circumcircle; orthocenter; vertices of the regular triangles constructed on the sides of triangle ABC (both exterior and interior); points symmetric to the vertices of ABC with respect to its sides; two points from which the sides of ABC are seen under a 60° or a 120° angle (the isogonal centers of the triangle); two points that satisfy the relation $AX \cdot BC = BX \cdot CA = CX \cdot AB$ (isodynamical centers of the triangle).