Progress in Mathematics
285

Alexander Soifer
Editor

# Ramsey Theory

## Yesterday, Today, and Tomorrow

Birkhäuser

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ISBN 978-0-8176-8091-6 e-ISBN 978-0-8176-8092-3
DOI 10.1007/978-0-8176-8092-3
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2010938367



Front cover utilizes the photograph of Frank Plumpton Ramsey (1903–1930), aged 18. Reproduced by kind permission of the Provost and the Scholars of King's College, Cambridge.

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*Dedicated to Paul Erdős, who envisioned Ramsey Theory, shaped much of it by his problems, conjectures and results, and inspired us all to dive into its deep and beautiful waters.*

# What This Book Is About and How It Came into Being

*Ramsey theory* is a fascinating, approximately 100-year-old field of mathematics that has a non-empty intersection with combinatorics, number theory, geometry, ergodic theory, topology, combinatorial geometry, set theory, measure theory, and so on. Ramsey theory possesses its own unifying ideas, and some of its results are among the most beautiful theorems of mathematics. The main mathematical idea of Ramsey theory is this: no matter how large and elaborate a system $S$ is, and how large a positive integer $n$ is, we can choose a large enough super system $Q$ containing $S$, so that no matter how $Q$ is colored in $n$ colors, $Q$ contains a monochromatic copy of $S$. Thus one can say that Ramsey theory studies *mathematics of coloring*.

In 2008 the director, Fred Roberts, and the executive committee of DIMACS[1] invited me to organize a three-day workshop on Ramsey theory. In response to Dr. Roberts' desire to host a nongeneric original view of the field, I proposed *Ramsey Theory: Yesterday, Today, and Tomorrow.* This was approved, and the workshop took place on May 27–29, 2009 at the Busch Campus of Rutgers University in Piscataway, New Jersey. The workshop looked at the emergence and history of Ramsey theory (*Yesterday*), its results (*Today*), and its future (*Tomorrow*) through its open problems, conjectures, and aspirations. In addition to mathematical and historical research, we also looked at how Ramsey theory can harness the power of computing in discovering mathematical results.

The workshop turned out to be an international event. It attracted researchers from the United States, England, Czech Republic, Hungary, and Germany. The speakers included world-renowned leaders of the field, such as Ronald L. Graham, Joel H. Spencer, and Jaroslav Nešetřil. It also included some of the most promising young researchers such as Jacob Fox of Princeton University, Andrzej Dudek of Carnegie Mellon University, Lynn Scow of the University of California Berkeley, and Dmytro Karabash of the Courant Institute of Mathematical Sciences.

[1] The Center for Discrete Mathematics and Theoretical Computer Science, a collaborative project of Rutgers and Princeton Universities, AT&T Labs – Research, Alcatel–Lucent Bell Labs, Cancer Institute of New Jersey (CINJ), NEC Laboratories America, and Telcordia Technologies.

The workshop inspired Ann Kostant, the executive editor of mathematics at Birkhäuser, to propose that I organize and edit this volume of surveys authored or coauthored by workshop participants under the title of the workshop *Ramsey Theory: Yesterday, Today, and Tomorrow* for its “Progress in Mathematics” Birkhäuser series.

This volume opens with “Yesterday”, surveys of the prehistory and early history of Ramsey theory. They are followed by surveys of progress that has been made in Ramsey theory and in areas that arose from Ramsey theory, the descendants of Ramsey theory; these surveys point out directions in which Ramsey theory and its descendants may move in the future. The last three surveys address Euclidean Ramsey theory and related coloring problems. The survey on open problems is coauthored by Ronald L. Graham, one of the authors of Euclidean Ramsey theorems I, II, and III, 1973–1975, which constitute a major portion of the foundation of the subject. This survey is followed by a history of the mysterious problem of the chromatic number of the plane, and the final survey is on similar problems for the rational points in real Euclidean spaces.

In addition to invited and contributed talks, the workshop featured a “Problem Posing Session.” Accordingly, this volume includes a section of open problems proposed at the workshop.

On behalf of all contributors to this volume, I thank Fred Roberts and the DIMACS Executive Committee for inviting and supporting the workshop, and the National Science Foundation for financial support. I thank the entire DIMACS’ staff for their wonderful help, especially Nicole Clark, Linda Casals, and Mel Janowitz. I thank Ann Kostant of Birkhäuser for offering us such a fine vehicle for spreading the Ramseyan word, and the editors of the “Progress in Mathematics” series Hyman Bass, Joseph Oesterlé, and Alan Weinstein for accepting this volume in their enlightened series.

May 2010 Alexander Soifer

Some of the plenary speakers of the workshop, from the right: Ronald L. Graham, Jaroslav Nešetřil, Joel H. Spencer, and Alexander Soifer

# Contents

# Ramsey Theory Before Ramsey, Prehistory and Early History: An Essay in 13 Parts[1]

Alexander Soifer

*What amazes us today is, of course, that no one in Hamburg (including Schreier and Artin) had known about Schur's work [1916]. In that connection we must realize that the kind of mathematics involved in the [Baudet–Schur] conjecture was not mainstream, and that combinatorics was not a recognized field of mathematics at all.*

– Nicolaas G. de Bruijn[2]

*It takes a long time to become young.*

– Pablo Picasso

## 1 Overture

How does a new theory emerge? It usually manifests itself in the older and established areas of mathematics. Gradually a critical mass of results appears, prompting a realization that what we have is a new identifiable field of mathematical thought, with its own set of problems and methods. As a fetus in a womb, the new theory eventually does not fit in the existing classification of mathematical thought. That is when the child is born. Ramsey theory has not been an exception.

In retrospect we all put history in a neat chronological order. In reality, older results may not have been known in time to provide the influence that our chronological order would suggest. We plug historical holes in the emergence of Ramsey theory not unlike filling holes in human evolution. History teaches that it is not enough to be right. In order to influence the evolution of one's field, one has to be lucky to be right at the right time. Aristarchus of Samos (Ἀρισταρχος, 310 BC–ca. 230 BC) was the first to explicitly conjecture a heliocentric model of the solar system. However, he did not influence the evolution of astronomy. Almost 1,800 years later, Nicolaus Copernicus (1473–1543), presented (more comprehensively) the conjecture of the heliocentric cosmology again, and did influence the evolution of astronomy in a major way.

[1] Much of this material is contained in the author's monograph [Soi], however, this text contains new facts and observations that were not known to the author in 2008 when [Soi] was published. Also, the emphasis here is quite different from [Soi].
[2] E-mail to A. Soifer, January 5, 2004.

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A. Soifer (ed.), *Ramsey Theory: Yesterday, Today, and Tomorrow*,
Progress in Mathematics 285, DOI 10.1007/978-0-8176-8092-3_1,

Let me summarize in one paragraph the fetal development preceding Ramsey theory's birth. As we believe now, David Hilbert's cube lemma was the first Ramseyan result, but it did not influence anyone at the time and thus did not give birth to Ramsey theory. Issai Schur's 1916 theorem could have remained unnoticed too, but Schur was first to realize that he had run into something new and striking. And so Schur continued by conjecturing the result on monochromatic arithmetic progressions. However, Schur's conjecture did not reach Bartel Leendert van der Waerden, and the Ramseyan train of thought risked running out of fuel. The unborn Ramseyan mathematics was very lucky that another person, Pierre Joseph Henry Baudet, independently of Schur posed the same conjecture. Baudet passed away at the tender age of 30, but his conjecture impressed his friend and mentor Frederik Schuh. Schuh or Schuh's circle at the University of Amsterdam was the source of the conjecture for the 23-year old Van der Waerden. Having proved the conjecture, Van der Waerden walked away from Ramseyan prehistory. Issai Schur, on the other hand, continued to produce Ramseyan mathematics, and moreover directed and inspired his PhD students Richard Rado, Hildegard Ille and Alfred Brauer to do the same. Then came Frank Plumpton Ramsey who delivered the Two Commandments, the principles of the theory later named in his honor. Ramsey died at the age of 26, before his publication announced to the world the birth of the new theory.

In what follows, we take a more detailed look at the emergence of Ramsey theory, and trace how the Ramseyan baton was passed. A much more detailed exposition of Ramsey theory's prehistory and early history required a whole book, the book [Soi] that now exists. I have continued historical investigations, and this survey contains some new facts.

## 2 David Hilbert's 1892 Cube Lemma

As far as we know today, the first Ramseyan-type result appeared in 1892 as a little-noticed assertion in [Hil]. Its author was the great David Hilbert. In this work Hilbert proved the theorem of our interest merely as a tool for his study of irreducibility of rational functions with integral coefficients.

A set $Q_n(a, x_1, x_2, \ldots, x_n)$ of integers is called an *n-dimensional affine cube* if there exist $n + 1$ positive integers $a, x_1, \ldots, x_n$ such that

$$Q_n(a, x_1, x_2, \ldots, x_n) = \left\{ a + \sum_{i \in F} x_i : \varnothing \neq F \subseteq \{1, 2, \ldots, n\} \right\}.$$

It is convenient to use the symbol $[n]$ for the starting segment of positive integers:

$$[n] = \{1, 2, \ldots, n\}.$$

The theorem, which preceded the Schur and the Baudet–Schur–Van der Waerden theorems, reads as follows.

**The Hilbert Cube Lemma 1.** *For every pair of positive integers $r$, $n$, there exists a least positive integer $m = H(r, n)$ such that in every $r$-coloring of $[m]$ there exists a monochromatic $n$-dimensional affine cube.*

It seems that David Hilbert's monochromatic cube lemma was the first example of Ramseyan mathematics. Apparently nobody – including Hilbert – appreciated the lemma much. Hilbert did not continue research in the direction the lemma showed. The field did not learn about Hilbert's lemma until much later. The lemma was added as the first instance of Ramseyan thought, not due to its influence, which was nonexistent, but due to its respectable birth year of 1892.

## 3 The Issai Schur 1916 Theorem

Probably nobody remembered the 1892 Hilbert lemma by the time the second Ramseyan-type result appeared in 1916 as a little-noticed assertion in number theory. Its author was Issai Schur. Our interest here lies in the result he obtained during 1913–1916 when he worked at the University of Bonn as the successor to the famed topologist Felix Hausdorff. There Schur wrote his pioneering paper [Sch]: *Über die Kongruenz* $x^m + y^m \equiv z^m$ (mod.$p$). In it Schur offered another proof of the 1908 theorem by Leonard Eugene Dickson [Dic], who was trying to prove Fermat's Last Theorem.

For use in his proof, Schur created, as he put it, "a very simple lemma, which belongs more to combinatorics than to number theory." Its setting is positive integers, colored in finitely many colors. The beautiful proof I present here utilizes coloring as well. Paul Erdős received this proof from Vera T. Sos, and included it in his talk at the 1970 International Congress of Mathematicians in Nice, France [E71.13]. We use the following natural lemma, which can be proven by a straightforward induction.

Issai Schur, courtesy of his daughter Hilde Abelin-Schur.

**Lemma 2 (R. E. Greenwood and A. M. Gleason, 1955, [GG]).** *For any positive integer $n$ there is a positive integer $S(n)$ such that any $n$ coloring of edges of the complete graph $K_{S(n)}$ contains a monochromatic triangle $K_3$.*

**The Schur 1916 Theorem 3 ([Sch]).** *For any positive integer $n$ there is an integer $S(n)$ such that any $n$-coloring of the set $[S(n)]$ contains integers $a$, $b$, $c$ of the same color such that $a + b = c$.*

In this case we call $a, b, c$ a *monochromatic solution* of the equation $x + y = z$. In fact, Schur proved by induction that $S(n) = n!e$ would work.

*Proof of Schur's Theorem.* Let all positive integers be colored in $n$ colors $c_1, c_2, \ldots, c_n$. Due to Lemma 2, there is $S(n)$ such that any $n$-coloring of edges of the complete graph $K_{S(n)}$ contains a monochromatic triangle $K_3$.

Construct a complete graph $K_{S(n)}$ with its vertices labeled with integers from the initial integer array $[S(n)] = \{1, 2, \ldots, S(n)\}$. Now color the edges of $K_{S(n)}$ in $n$ colors as follows: let $i$ and $j$, $i > j$, be two vertices of $K_{S(n)}$, color the edge *ij* in the color of the integer $i - j$ (remember, all positive integers were colored in $n$ colors!). We get a complete graph $K_{S(n)}$ whose edges are colored in $n$ colors. By Lemma 2, $K_{S(n)}$ contains a triangle *ijk*, $i > j > k$, whose edges *ij*, *jk*, and *ik* are colored in the same color (Fig. 2).

Denote $a = i - j$; $b = j - k$; $c = i - k$. Since all three edges of the triangle *ijk* are colored in the same color, the integers $a$, $b$, and $c$ are colored in the same color in the original coloring of the integers (this is how we colored the edges of $K_{S(n)}$). In addition, we have the following equality:

$$a + b = (i - j) + (j - k) = i - k = c.$$

We are done! ■

The result of the Schur theorem 3 can be strengthened by an additional clever trick in the proof.

**Strong Version of Schur's Theorem 4.** *For any positive integer $n$ there is an integer $S^*(n)$ such that any $n$-coloring of the initial positive integer array $[S^*(n)]$ contains distinct integers $a$, $b$, $c$ of the same color such that $a + b = c$.*

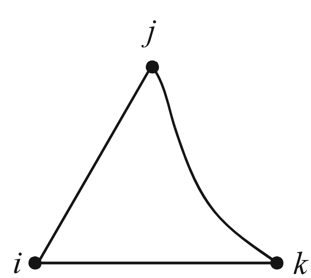



**Fig. 2**

*Proof.* Let all positive integers be colored in $n$ colors $c_1, c_2, \ldots, c_n$. We add $n$ more colors $c_{1'}, c_{2'}, \ldots, c_{n'}$ different from the original $n$ colors and construct a complete graph $K_{S(2n)}$ with the set of positive integers $\{1, 2, \ldots, S(2n)\}$ labeling its vertices (see the definition of $S(2n)$ in the proof of Theorem 3). Now we are going to color the edges of $K_{S(2n)}$ in $2n$ colors.

Let $i$ and $j$, $(i > j)$, be two vertices of $K_{S(2n)}$, and $c_p$ be the color in which the integer $i - j$ is colored, $1 \leq p \leq n$ (remember, all positive integers are colored in $n$ colors $c_1, c_2, \ldots, c_n$). Then we color the edge *ij* in color $c_p$ if the number $\lfloor i/(i-j) \rfloor$ is even, and in color $c_{p'}$ if the number $\lfloor i/(i-j) \rfloor$ is odd (for a real number $r$, the symbol $\lfloor r \rfloor$, as usual, denotes the largest integer not exceeding $r$).

We get a complete graph $K_{S(2n)}$ whose edges are colored in $2n$ colors. By Lemma 2, $K_{S(2n)}$ contains a triangle *ijk*, $i > j > k$, whose edges *ij*, *jk*, and *ik* are colored in the same color (see Fig. 2).

Denote $a = i - j$; $b = j - k$; $c = i - k$. Since all three edges of the triangle *ijk* are colored in the same color, from the definition of coloring of edges of $K_{S(2n)}$ it follows that in the original coloring of positive integers, the integers $a$, $b$, and $c$ were colored in the same color. In addition we have

$$a + b = (i - j) + (j - k) = i - k = c.$$

We are almost done. We only need to show (our additional pledge!) that the numbers $a, b, c$ are all distinct. In fact, it suffices to show that $a \neq b$. Assume the opposite: $a = b$ and $c_p$ is the color in which the number $a = b = i - j = j - k$ is colored. But then

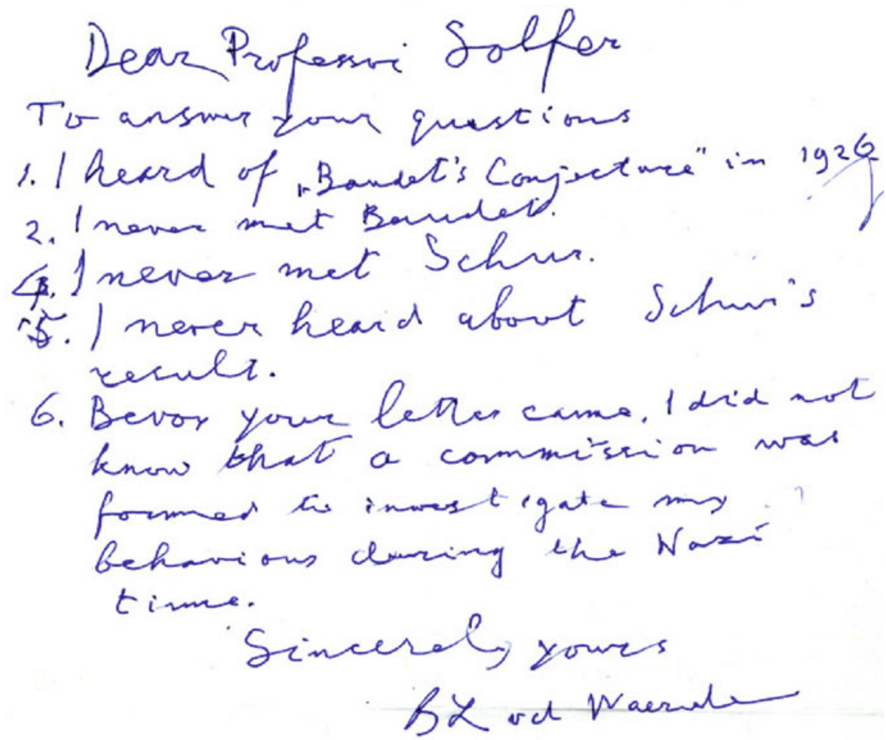
$$\left\lfloor \frac{i}{i-j} \right\rfloor = \left\lfloor 1 + \frac{j}{i-j} \right\rfloor = 1 + \left\lfloor \frac{j}{i-j} \right\rfloor = 1 + \left\lfloor \frac{j}{j-k} \right\rfloor,$$

i.e., the numbers $\lfloor i/(i-j) \rfloor$ and $\lfloor j/(j-k) \rfloor$ have different parity, thus the edges *ij* and *jk* of the triangle *ijk* must have been colored in different colors. This contradiction to the fact that all three edges of the triangle *ijk* have the same color proves that $a \neq b$. Theorem 4 is proven. ■

Nobody then asked questions of the kind Issai Schur posed and solved in this 1916 paper. Consequently, nobody appreciated this result much when it was published. Even Van der Waerden himself learned about the Schur theorem from me in 1995! See for yourselves Van der Waerden's April 24, 1995 letter to me:

Now Schur's theorem shines as one of the most beautiful, classic theorems of mathematics. Leon Mirsky loved this theorem, and wrote [Mir] on the occasion of the centenary of the birth of Schur:

> We have here a statement of the type: "if a system is partitioned arbitrarily into a finite number of subsystems, then at least one subsystem possesses a certain specified property." To the best of my knowledge, there is no earlier result which bears even a remote resemblance to Schur's theorem. It is this element of novelty that impresses itself so forcibly on the mind of the reader.



Dear Professor Soifer
To answer your questions
1. I heard of „Baudet's Conjecture" in 1926
2. I never met Baudet.
4. I never met Schur.
5. I never heard about Schur's result.
6. Bevor your letter came, I did not know that a commission was formed to investigate my behaviour during the Nazi time.
Sincerely yours
BL v d Waerden

Van der Waerden, April 24, 1995 letter to Alexander Soifer.

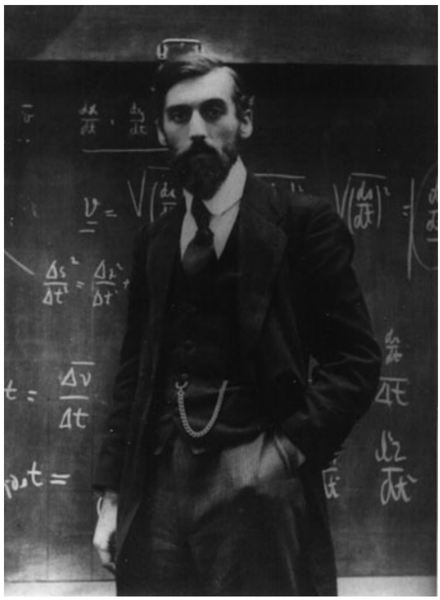
However, Mirsky continued [Mir]:

> After writing his paper, Schur never again touched on the problem discussed there; and this is in itself something of a mystery. For the strongest impression one receives on scanning his publications is the almost compulsive striving for comprehensiveness. There are few isolated investigations; in algebra, in analysis, in the theory of numbers, Schur reverts again and again to his original questions and pursues them to the point of where one feels that the last word has been spoken.... Why, then, did he not investigate any of the numerous questions to which his Theorem points so compellingly? There is no evidence to enable us to solve the riddle. (Footnote: As will emerge from the discussion below, Professor Rado, if anyone, should be able to throw light on the mystery – and he tells me that he cannot.)

Mirsky's statement, apparently backed by Richard Rado, was echoed in the authoritative book on Ramsey theory [GRS2, p. 70], thus becoming a universal view on this matter: "Schur never again touched on this problem." In fact, the new Ramseyan mathematics, discovered by Issai Schur in his 1916 paper, remained dear to his heart for years to come. Even though at the time, nobody was interested in the new direction the Schur 1916 theorem had shown, Issai Schur himself was: read on!

## 4 The Baudet–Schur–Van der Waerden 1927 Theorem

In trying to prove his own conjectures about quadratic residues and nonresidues, Issai Schur realized the need to conjecture another "very simple lemma, which belongs more to combinatorics than to number theory."[3] My historical research showed that the young 20-something Dutch mathematician Pierre Joseph Henry Baudet created this conjecture independently of Schur (read Chap. 34 of [Soi] dedicated to the historical research into the authorship of this conjecture).



Pierre Joseph Henry Baudet (1891–1921), courtesy of his son Henry Baudet.

Another Dutch youngster, the 23-year-old Bartel Leendert van der Waerden[4] published the proof of the conjecture, thus giving us a classic theorem that became a root of the wonderful tree of Ramseyan mathematics.

**The Baudet–Schur–Van der Waerden Theorem 5 (B. L. Van der Waerden, 1927, [Wae]).** *For any $k$, $l$, there is $W = W(k,l)$ such that any $k$-coloring of the set $[W]$ contains an $l$-term monochromatic arithmetic progression.*

B. L. van der Waerden, assisted by Emil Artin and Otto Schreier, proved this pioneering result while at Hamburg University and presented it the following year at the 1927 meeting of *D.M.V.*, *Deutsche Mathematiker Vereinigung* (German Mathematical Society) in Berlin.

3 [Sch].

4 Since Van der Waerden was Dutch, I strictly adhere to the Dutch rules in determining where to use "van" and where "Van." In fact, in Dutch "van" is used only if preceded by the given name(s) or initials.

Observe that the Baudet–Schur–Van der Waerden Theorem 5 implies and strengthens the Hilbert cube lemma 1. Baudet and Van der Waerden have not contributed anything else to Ramseyan mathematics. Sadly, Baudet had a valid reason: he passed away on Christmas Day of 1921. He was only 30. Van der Waerden did not have such a reason: he lived a 93-year-long productive life. Van der Waerden simply did not realize how important was the result he proved: he submitted his algebraic geometry papers to the most prestigious journal, *Mathematische Annalen*, yet sent this proof to an "obscure"[5] journal, *Nieuw Archief voor Wiskunde* of the Dutch Mathematical Society.

This leaves only Isaai Schur standing. The new Ramseyan mathematics, discovered by Issai Schur in his 1916 paper, remained dear to his heart for years to come. He cocreated the Baudet–Schur conjecture, and conjectured and proved the next Ramseyan result we study.

However, before we move on, I wish to emphasize how critical a contribution P. J. H. Baudet made to Ramsey theory. I have established fairly certainly [Soi, Chap. 34] that Issai Schur was not the source of the conjecture for Van der Waerden. Thus, without Baudet creating the conjecture on his own, Van der Waerden would have had nothing to prove about monochromatic arithmetic progressions, and the prehistory of Ramsey theory would have been quite different.

## 5 The Generalized 1928 Schur Theorem

One evening in the year 1927, Issai Schur and his two former doctoral students, Alfred and Richard Brauer learned from Johnny von Neumann, who was fresh from the D.M.V. meeting, that the young Dutchman Van der Waerden had proved what they knew as Schur's conjecture about monochromatic arithmetic sequences [Bra3]. A few days later, Alfred Brauer proved Schur's conjecture about quadratic residues by applying the new Baudet–Schur–Van der Waerden theorem. Schur then noticed that Brauer's method of proof could be used for obtaining a result about sequences of $n$th power residues. Soon Issai Schur found a short Olympiad-like way to prove the following result that generalized at once both Schur's 1916 theorem and Baudet–Schur–Van der Waerden's theorem.

**The Generalized Schur Theorem 6 (Schur, [Bra1, Bra2]).** *For any positive integers $k$ and $l$ there is a positive integer $S(k,l)$ such that any $k$-coloring of the initial segment of positive integers $[S(k,l)]$ contains a monochromatic arithmetic progression of length $l$ together with its difference.*

*Proof of Schur.* For 1 color we define $S(1,l) = l$, and the statement is true.

Assume the theorem is true for $k$ colors. We define

$$S(k+1, l) = W(k+1, (l-1)S(k,l)+1),$$

[5] I defer to the celebrated Dutch mathematician N. G. de Bruijn, whose characterization I am quoting here.

where $W(k,l)$ is as defined in Theorem 5. Let the set of integers $[S\,(k+1,l)]$ be colored in $k+1$ colors. Then by Theorem 5 (see the right side of the equality above), there is a $(l\ -\ 1)\,S\,(k,l) + 1$ term monochromatic arithmetic progression

$$a, a+d, \ldots, a+(l-1)\,S\,(k,l)\,d.$$

For every $x\ =\ 1, 2, \ldots, S(k,l)$, this long monochromatic arithmetic progression contains the following $l$-term arithmetic progression,

$$a, a+xd, \ldots, a+(l-1)\,xd.$$

If for one of the values of $x$, the difference $xd$ is colored the same color as the progression above, we have concluded the proof of the inductive step. Otherwise, the sequence

$$d, 2d, \ldots, S\,(k,l)\,d$$

is colored in only $k$ colors, and we can apply to it the inductive assumption to draw the required conclusion. ■

Schur wanted Alfred Brauer to include this theorem (as well as the one about $n$th power residues) in Brauer's paper because Schur believed he had used Brauer's method in these proofs. Schur did not want to take away any credit from his student. The student had to oblige but he "always called it Schur's result"[6] and gave Schur credit everywhere it was due in his paper [Bra1] that appeared in 1928. A few weeks later Brauer also proved Schur's conjecture about quadratic nonresidues, which appeared in the same wonderful, yet mostly overlooked paper [Bra1]. Yes, I say "overlooked" because even the authors of the standard text [GRS2], apparently were not familiar with this paper, for they included an almost identical result (Theorem 2, p. 70) without reference or credit to Schur.

The results presented here thus far represent what I named [Soi] *Ramsey Theory before Ramsey*. Now it is time to meet Himself: Ladies and Gentlemen, Dr. F. P. Ramsey!

## 6 The Frank Plumpton Ramsey Principle

Frank Plumpton Ramsey was the pride and hope of King's College, Cambridge. Having lived not quite 27 years, he made major contributions to philosophy, mathematical logic, economics and mathematics. In 1928, aged 25, he submitted a paper that was posthumously published in 1930 [Ram]. The paper contained infinite and finite versions of what has since appeared under the name of "the Ramsey theorem." I have always felt that something was wrong with the title "Ramsey theorem." To see that, it suffices to read the leader of the field, Ronald L. Graham, who in 1983 wrote [Gra1]: "The generic result in Ramsey Theory is due (not surprisingl) to

[6] [Bra2].

Frank Plumpton Ramsey (1903–1930), aged 18. Reproduced by kind permission of the Provost and Scholars of King's College, Cambridge.

F. P. Ramsey." Exactly: "the generic result," compared to much more specific typical examples, such as Schur's theorem and Baudet–Schur–Van der Waerden's theorem. The Ramsey theorem occupies a unique place in the Ramsey theory. It is a powerful tool. It is also a philosophical principle stating, as Theodore S. Motzkin put it, that a "complete disorder is an impossibility. Any structure will necessarily contain an orderly substructure"[7]. It is, therefore, imperative to call the Ramsey theorem by a much better fitting name: *the Ramsey principle*. We have two principles in Ramsey's paper [Ram]:

**Infinite Ramsey Principle 7.** For any positive integers $k$ and $r$, if the collection of all $r$-element subsets of an infinite set $S$ is colored in $k$ colors, then $S$ contains an infinite subset $S_1$ such that all $r$-element subsets of $S_1$ are assigned the same color.

**Finite Ramsey Principle 8.** For any positive integers $r$, $n$, and $k$ there is an integer $m_0 = R(r,n,k)$ such that if $m \geq m_0$ and the collection of all $r$-element subsets of an $m$-element set $S_m$ is colored in $k$ colors, then $S_m$ contains an $n$-element subset $S_n$ such that all $r$-element subsets of $S_n$ are assigned the same color.

It is amazing to me how quickly the news of the Ramsey principle traveled in the times that can hardly be called the Information Age. Ramsey's paper appeared in 1930. Already in 1933 the great Norwegian logician Thoralf Albert Skolem (1887–1963) published his own proof [Sko] of the Ramsey principle (with a reference

[7] Quoted from [GRS2].

Grave of Frank Plumpton Ramsey and his parents, Parish of the Ascension Burial Ground, Cambridge, photo by Mark S. Soifer. The tombstone reads: "In loving memory of Mary Agnes Ramsey 8 Jan. 1876–15 Aug. 1927. Also of Frank Plumpton Ramsey 22 Feb. 1903–19 Jan. 1930. Also of Arthur Stanley Ramsey 12 Sept. 1869–31 Dec. 1954."

to Ramsey's 1930 publication!). In 1935 yet another proof (for the graph-theoretic setting) appeared in the paper [ES] by the two young Hungarians, Paul Erdős and Gjörgy (George) Szekeres. We look at this remarkable paper next.

## 7 The Paul, Gjörgy, and Esther Happy End Problem

During the winter of 1932–1933, two young friends, the mathematics student Paul Erdős, aged 19, and the chemistry student György (later known as George) Szekeres, 21, solved the problem posed by their youthful ladyfriend Esther Klein, 22, but did not submit it to a journal for a year and a half. When Erdős finally sent this joint paper for publication, he chose J. E. L. Brouwer's journal, *Compositio Mathematica*, where it appeared in 1935 [ES].

Erdős and Szekeres were first to demonstrate the power and striking beauty of the Ramsey principle when they solved the problem. In the process of working with Erdős on the problem, Szekeres actually rediscovered the finite Ramsey principle before the authors ran into the 1930 Ramsey publication [Ram]. Erdős found an alternative proof with much better bounds for $ES(n)$.

**The Klein–Erdős–Szekeres Theorem 9 ([ES]).** *For any positive integer $n \geq 3$ there is an integer $ES(n)$ such that any set of at least $ES(n)$ points in the plane in general position*[8] *contains n points that form a convex polygon.*

Erdős and Szekeres found the bounds for the Erdős–Szekeres function $ES(n)$.

**The First Bounds 10 (Erdős and Szekeres [ES]).** For any positive integer $n \geq 3$,

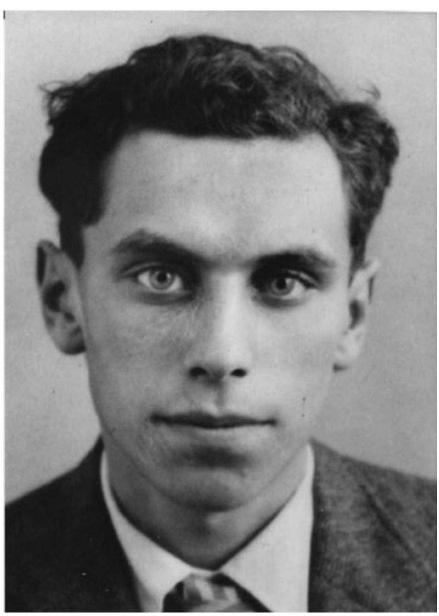
$$2^{n-2} < ES(n) \leq \binom{2n-4}{n-2} + 1.$$

The upper bound had withstood all attempts at improvement for 62 years, until 1997 when Fan Chung and Ronald L. Graham [CG] willed it down by one point. In the process, Fan and Ron offered a fresh approach which started an explosion of improvements found by Daniel J. Kleitman and Lior Pachter, then Géza Tòth and Pavel Valtr, and again in 2005 by Tòth and Valtr [TV] to the upper bound best-known today,

$$ES(n) \leq \binom{2n-5}{n-2} + 1,$$

but this is not prehistory, and so I must stop. It suffices to say that the Erdős–Szekeres 1935 conjecture is still alive:

**The Erdős–Szekeres–Klein Happy End \$500 Conjecture 11.** For any positive integer $n \geq 3$,



Paul Erdős in the early 1930s, courtesy of Paul Erdős.

[8] That is, no three points lie on a line.

$$ES(n) = 2^{n-2} + 1.$$

I must show you a beautiful alternative proof of Erdős–Szekeres's Theorem 9, especially since it was found by an undergraduate student, Michael Tarsy of Israel. He missed the class when the Erdős–Szekeres solution was presented, and had to come up with his own proof under the gun of the exam! Tarsy recalls (e-mail to me of December 12, 2006):

> Back in 1972, I took the written final exam of an undergraduate Combinatorics course at the Technion—Israel Institute of Technology, Haifa, Israel. Due to personal circumstances, I had barely attended school during that year and missed most lectures of that particular course. The so-called Erdős–Szekeres theorem was presented and proved in class, and we have been asked to repeat the proof as part of the exam. Having seen the statement for the first time, I was forced to develop my own little proof.
>
> Our teacher in that course, the late Professor Mordechai Levin, had published the story as an article, I cannot recall the journal's name, the word 'Gazette' was there and it dealt with Mathematical Education.[9]
>
> I was born in Prague (Czechoslovakia at that time) in 1948, but was raised and grew up in Israel since 1949. Currently I am a professor of Computer Science at Tel-Aviv University, Israel.

*Proof of Theorem 9 by Michael Tarsy.* Let $n \geq 3$ be a positive integer. By the Ramsey principle 8 ($r = 3$ and $k = 2$) there is an integer $m_0 = R(3, n, 2)$ such that, if $m > m_0$ and the collection of all 3-element subsets of an $m$-element subset $S_m$ are colored in two colors, then $S_m$ contains an $n$-element subset $S_n$ such that all 3-element subsets of $S_n$ are assigned the same color.

Let now $S_m$ be a set of $m$ points in the plane in general position labeled with integers $1, 2, \ldots, m$.

We color a 3-element set $\{i, j, k\}$, where $i < j < k$, red if we travel from $i$ to $j$ to $k$ in a clockwise direction, and blue if counterclockwise. By the above, $S_m$ contains an $n$-element subset $S_n$ such that all 3-element subsets of $S_n$ are assigned the same color, that is, have the same orientation. But this means precisely that $S_n$ forms a convex $n$-gon! ■

In their celebrated paper [ES], P. Erdős and G. Szekeres also discovered the monotone subsequence theorem. A sequence $a_1, a_2, \ldots, a_k$ of real numbers is called *monotone* if it is increasing, i.e., $a_1 \leq a_2 \leq \cdots \leq a_k$, or decreasing, i.e., $a_1 \geq a_2 \geq \cdots \geq a_k$ (we use weak versions of these definitions that allow equalities of consecutive terms).

**The Erdős–Szekeres Monotone Subsequence Theorem 12.** *Any sequence $S$ : $a_1, a_2, \ldots, a_r$ of $r > mn$ real numbers contains a decreasing subsequence of more than m terms or an increasing subsequence of more than n terms.*

A quarter of a century later, in 1959, A. Seidenberg of the University of California, Berkeley, found a brilliant "one-line" proof of Theorem 12, thus giving it a true Olympiad-like appeal.

[9] *The Mathematical Gazette*, 1976.

*Proof by A. Seidenberg [Sei].* Assume that the sequence $S$: $a_1, a_2, \ldots, a_r$ of $r > mn$ real numbers has no decreasing subsequence of more than $m$ terms. To each $a_i$ assign a pair of numbers $(m_i, n_i)$, where $m_i$ is the largest number of terms of a decreasing subsequence beginning with $a_i$ and $n_i$ the largest number of terms of an increasing subsequence beginning with $a_i$. This correspondence is an injection; that is, distinct pairs correspond to distinct terms $a_i, a_j, i < j$. Indeed, if $a_i \leq a_j$ then $n_i \geq n_j + 1$, and if $a_i > a_j$ then $m_i \geq m_j + 1$.

We get $r > mn$ distinct pairs $(m_i, n_i)$, (they are our pigeons) and $m$ possible values (they are our pigeonholes) for $m_i$, since $1 \leq m_i \leq m$. By the pigeonhole principle, there are at least $n+1$ pairs $(m_0, n_i)$ with the same first coordinate $m_0$. Terms $a_i$ corresponding to these pairs $(m_0, n_i)$ form an increasing subsequence! ■

Erdős and Szekeres note that the result of their Theorem 10 is best possible.

**Problem 13 ([ES]).** Construct a sequence of $mn$ real numbers such that it has no decreasing subsequence of more than $m$ terms and no increasing subsequence of more than $n$ terms.

*Solution.* Here is a sequence of $mn$ terms that does the job:

$$m, m-1, \ldots, 1; 2m, 2m-1, \ldots, m+1; \ldots; nm, nm-1, \ldots, (n-1)m+1. \blacksquare$$

H. Burkill and Leon Mirsky in their 1973 paper [BM] observe that the monotone subsequence theorem holds for countable sequences as well.

**Monotone Subsequence Theorem 14 ([BM]).** *Any sequence $S$: $a_1, a_2, \ldots, a_r$, $\ldots$ of real numbers contains an infinite increasing subsequence or an infinite strictly decreasing subsequence.*

*Hint.* Color the 2-element subsets of $S$ in two colors. ■

The authors "note in passing that the same type of argument enables us to show" the following cute result (without a proof).

**Curvature Preserving Subsequence Theorem 15 ([BM]).** *Any sequence $S$ : $a_1$, $a_2, \ldots, a_r, \ldots$ of real numbers possesses an infinite subsequence which is convex or concave.*

*Hint.* Recall Michael Tarsy's proof of Erdős–Szekeres theorem above, and color the 3-element subsets of $S$ in two colors! ■

## 8 Richard Rado's Regularity

It is fitting that the Schur theorem was generalized by one of Schur's best PhD students, Richard Rado. Rado calls a linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{$*$}$$

*regular*, if for any positive integer $r$, no matter how many all positive integers are colored in $r$ colors, there is a monochromatic solution of the equation (*). As before, we say that a solution $x_1, x_2, \ldots, x_n$ is *monochromatic*, if all numbers $x_1, x_2, \ldots, x_n$ are colored in the same color.

For example, the Schur 1916 theorem proves precisely that the equation $x + y - z = 0$ is regular. In 1933 Richard Rado, among other results, found the following criterion.

**The Rado Theorem 16 (A particular case of [Rad1]).** *Let E be a linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$, where all $a_1, a_2, \ldots, a_n$ are integers. Then E is regular if and only if some nonempty subset of the coefficients $a_i$ sums up to zero.*

Richard Rado found regularity criteria for systems of homogeneous equations as well. His fundamental contributions to and influence on Ramsey theory is hard to overestimate. I have just given you a taste of his theorems here. For more of Rado's results read his papers [Rad1, Rad2], and others, and the monograph [GRS2].

It is interesting to notice how differently people can see the same fact. For Richard Rado, Schur's theorem was about monochromatic solutions of a homogeneous linear equation $x + y - z = 0$, and so Rado generalized the Schur 1916 theorem to a vast class of homogeneous linear equations and systems of homogeneous linear equations [Rad1]. Three other mathematicians saw Schur's theorem quite differently. This group consisted of Jon Folkman, a young Rand Corporation scientist; Jon Henry Sanders, the last PhD student of the legendary Norwegian graph theorist Øystein Ore at Yale (B.A. 1964 Princeton University; PhD 1968, Yale University); and Vladimir I. Arnautov, at the time of his paper's submission a 30-year old Moldavian topological ring theorist. For the three, the Schur theorem spoke about monochromatic sets of symmetric sums

$$\{a_1, a_2, a_1 + a_2\} = \left\{\sum\nolimits_{i=1,2} \varepsilon_i a_i : \varepsilon_i = 0, 1; \varepsilon_1\varepsilon_2 \neq 0\right\}.$$

Consequently, the three proved a generalization of Schur's theorem different from Rado's kind, and paved the way for further important developments. I see therefore no choice at all but to name the following fine theorem for its three inventors. This may surprise readers accustomed to different attributions, and so I have addressed their concerns in [Soi] and here below.

**The Arnautov–Folkman–Sanders Theorem 17 ([Arn, San]).** *For any positive integers m and n there exists an integer AFS(m,n) such that for any m-coloring of the initial integers array [AFS(m,n)], there is an n-element subset $S \subset [AFS(m,n)]$ such that the set $\left\{\sum_{x \in F} x : \emptyset \neq F \subseteq S\right\}$ is monochromatic.*

On April 25, 2009, I received the following e-mail from Dr. Jon Henry Sanders:

> Dear Prof. Soifer: What has been referred to throughout the literature as the Graham–Rothschild conjecture (resolved by Hindman) was first posed by me (in the more general form for an arbitrary finite number of colors) in my dissertation, A Generalization of Schur's Theorem, Yale '68. Attached is a photocopy of pgs 9 and 10 of my dissertation – Theorem 2′

> is the conjecture. Since Rothschild was one of two readers of my dissertation (Plummer the other) it is strange that this misattribution has existed for so long.
>
> Jon Henry Sanders

While working on *The Mathematical Coloring Book* [Soi], I verified J. H. Sanders' proof of Theorem 17 in his dissertation (where it is called "Theorem 2"), but I failed to notice the conjecture. I regret my oversight, which I am correcting now. Looking at the 1968 dissertation [San] again, I see the conjecture listed as "Theorem 2′" and preceded by the words, "It is natural to ask whether either Lemma 1 or Theorem 2 generalize in the following way."

**The Sanders Conjecture 18 ([San], p. 9).** Let the positive integers be divided into $t$ classes $A_1, A_2, \ldots, A_t$, ($t$ a positive integer). Then there exists an infinite sequence $a_1, a_2, \ldots$ of positive integers and a number $l$, $1 \le l \le t$, such that $\sum_{i \in I} a_i \in A_l$ for all (nonempty) finite sets $I$ of positive integers.

In their important 1971 paper [GR] Ron Graham and Bruce Rothschild, having vastly generalized a number of Ramsey-type theorems, formulated this Conjecture 18 three years later than Jon H. Sanders (they did it for two classes), and thus credit for the conjecture ought to belong to Jon Henry Sanders. Of course, when the conjecture appeared in the important visible paper [GR], it won high praise from Paul Erdős, and thus attracted wide attention, including that of Neil Hindman.

In the paper submitted in 1972 and published in 1974 [Hin], Hindman proved the conjecture. As I have tried to do uniformly, I am giving credit for this result to both the author of the conjecture and the author of the proof.

**The Sanders–Hindman Theorem 18 (Hindman [Hin]).** *For any positive integer $n$ and any $n$-coloring of the set of positive integers $N$, there is an infinite subset $S \subseteq N$ such that the set $\left\{\sum_{x \in F} x : \emptyset \neq F \subset S; |F| < \aleph_0\right\}$ is monochromatic.*

Let us now go back and establish the most appropriate credit for Theorem 17. It is called the Folkman–Rado–Sanders' theorem in [GRS1], [Gra2], and [ES]; and Folkman's theorem in [Gra1] and [GRS2]. Most other authors have copied the attributions from one of these works. Which credit is most justified? In one publication only [Gra2], Ronald L. Graham gives the date of Jon Folkman's personal communication to Graham: 1965. In one publication only [Gra1], in 1981 Graham publishes Folkman's proof that uses Baudet–Schur–Van der Waerden's theorem. Thus, Folkman merits credit. In the standard text on Ramsey theory [GRS2], I find an argument for credit to Folkman alone, disagreeing with the first edition [GRS1] of the same book, "Although the result was proved independently by several mathematicians, we choose to honor the memory of our friend Jon Folkman by associating his name with the result."

Jon H. Folkman left this world tragically in 1969. He was 31. He was full of great promise. The sympathy and grief of his friends is understandable and noble. Yet, do we, mathematicians, have the liberty to award credits? In this case, how can we deny Jon Henry Sanders credit, when Sanders' independent authorship is absolutely clear and undisputed? (He could not have been privy to the above-mentioned personal communication). Sanders formulates and proves Theorem 17 in his 1968 Ph.D. dissertation [San]. Moreover, Sanders proves it in a different way from Folkman: he does not use Baudet–Schur–Van der Waerden's theorem, but instead generalizes Ramsey's theorem to what he calls in his dissertation "Iterated Ramsey Theorem" [San, pp. 3–4]. In addition, Sanders is the one who saw the train of thought the farthest by conjecturing the beautiful generalization that Hindman proved.

Vladimir Ivanovich Arnautov's discovery is striking. His paper is much closer in style to that of Schur's classic 1916 paper, where Schur's theorem appears as a useful tool, "a very simple lemma," and is immediately used for obtaining a number-theoretic result, related to Fermat's Last Theorem. Arnautov formulates and proves Theorem 17, but treats it as a useful tool and calls it simply "Lemma 2" (in the proof of Lemma 2, he uses Baudet–Schur–Van der Waerden's theorem). He then uses Lemma 2 and other Ramseyan tools (!) to prove that every (not necessarily associative) countable ring allows a nondiscrete topology. This brilliant paper was submitted to *Doklady Akademii Nauk USSR* on August 22, 1969, and on September 2, 1969 was recommended for publication by the celebrated topologist Pavel Sergeevich Aleksandrov.[10] We have no choice but to savor the pleasure of associating Aknautov's name with Theorem 17.

What about Rado, one may ask? As Graham–Rothschild–Spencer [GRS2] observe, Theorem 17 "may be derived as a corollary of Rado's theorem [Rad1]... by elementary, albeit nontrivial, methods."[11] In my opinion, this is an insufficient reason to attach Rado's name to Theorem 17. Arnautov, Folkman, and Sanders envisioned Schur's theorem generalization in a different direction from that of Rado, and paved the way for Sanders' conjecture proved by Hindman. In fact, Erdős came to the same conclusion in 1973 [E73.21] when he put Rado's name in parentheses (Erdős did not know about Arnautov's paper, or he would have likely added him to the authors of Theorem 17): "Sanders and Folkman proved the following result (which also follows from earlier results of Rado [Rad1])."

## 9 Density and Arithmetic Progressions

We start with the key definition from the Erdős–Turán 1936 paper [ET]. For a positive integer $N$, denote by $r_l(N)$ the maximum number of positive integers not exceeding $N$ such that no $l$ of them form an arithmetic progression. Paul Erdős and Paul Turán proved a number of results about $r_3(N)$ and conjectured that

$$r_3(N) = o(N).$$

[10] *Doklady Akademii Nauk USSR* published only papers by full and corresponding members of the Academy. A nonmember's paper had to be recommended for publication by a full member of the Academy.

[11] Theorem 17 also follows from Graham and Rothschild's results published in 1971 [GR1].

This conjecture was proven in 1953 by Klaus F. Roth [Rot]. The only conjecture about the general function $r_l(N)$ in the Erdős–Turán paper was attributed to their friend George Szekeres, and was later proven false. Sixteen years have passed before Endre Szemerédi in 1969 proved [Sz1] that

$$r_4(N) = o(N).$$

In a 1973 paper Paul Erdös [E73.21, pp. 118–119] remarked: "[this] very complicated proof is a masterpiece of combinatorial reasoning." A very surprising paragraph followed [ibid.]: "Recently, Roth [1970] obtained a more analytical proof of $r_4(n) = o(n)$. $r_5(n) = o(n)$ remains undecided. Very recently, Szemerédi proved $r_5(n) = o(n)$." Clearly, Erdős added the last sentence at the last moment, and should have removed the next to last sentence. The latter result has never been published, probably because Endre Szemerédi was already busy trying to finish the proof of the general case. On April 4, 2007, right after his talk at Princeton's Discrete Mathematics Seminar I asked Szemerédi whether he had that proof for 5-term arithmetic progressions, and what came of it. Endre replied: "Hmm, it was so close to finding the proof of the general case, maybe 2 months before, that I did not check all the details for 5. It was more difficult than the general case." Indeed, in 1974 he submitted, and in 1975 published [Sz2] a proof of the general case; that is, for any positive integer $l$,

$$r_l(N) = o(N).$$

This work in one stroke earned Szemerédi a reputation of a wizard of combinatorics. By then the terminology had changed, and I wish to present here the more contemporary formulation that is used in Szemerédi [Sz1]. We make use of the notion of "proportional length," known as *density*, in the sequence of positive integers $N = \{1, 2, \ldots, n, \ldots\}$. The density is one way to measure how large a subset of $N$ is. Its role is analogous to the one played by length in the case of the line $R$ of reals.

Let $A$ be a subset of $N$; define $A(n) = A \cap \{1, 2, \ldots, n\}$. Then *density* $d(A)$ of $A$ is naturally defined as the following limit if it exists,

$$d(A) = \lim_{n\to\infty} \frac{|A(n)|}{n}.$$

The *upper density* $\bar{d}(A)$ of $A$ is analogously defined as

$$\bar{d}(A) = \lim_{n\to\infty} \sup \frac{|A(n)|}{n}.$$

Now we are ready to look at a classically simple formulation of Szemerédi's result.

**The Szemerédi Theorem 19.** *Any subset of $N$ of positive upper density contains arbitrarily long arithmetic progressions.*

In various problem papers, Erdős gives the date of Szemerédi's accomplishment and Erdős's payment as 1972 (sometimes 1973, and once even 1974). The following statement appears most precise as Erdős made it very shortly after the discovery at the September 3–15, 1973 *International Colloquium* in Rome [E76.35] and places Szemerédi's proof around September 1972: "About a year ago Szemerédi proved $r_k(n) = o(n)$, his paper will appear in 'Acta Arithmetica.'…"

Erdős was delighted with Szemerédi's result and awarded him $1,000 in late 1972–1973 [E85.33]:

> In fact denote by $r_k(n)$ the smallest integer for which every sequence $1 \le a_1 < a_2 < \ldots < a_l \le n$, $l = r_k(n)$ contains an arithmetic progression of k terms. We conjectured
>
> $$lim\, r_k n)/n = 0. \qquad (15)$$
>
> I offered $1,000 for (15) and late in 1972 Szemerédi found a brilliant but very difficult proof of (15). I feel that never was a 1,000 dollars more deserved. In fact several colleagues remarked that my offer violated the minimum wage act.

On April 4, 2007, Szemerédi confirmed my historical deductions: "I proved [the] general case in fall 1972, and received Erdős's prize in 1973." I refer the interested reader to the original paper for the proof which is brilliant and hard.[12] Partial results are proven in [GRS2] (it is remarkable that even this standard text in the field did not include Szemerédi's complete proof!).

While Szemerédi's theorem is a very strong generalization of Baudet–Schur–Van der Waerden's theorem, Paul Erdős and Ronald L. Graham observe in their 1980 problem book [EG, p. 19] that the analogue of Szemerédi's theorem does not hold for Generalized Schur's Theorem 6. Can you think of a counterexample before reading the one below?

**Observation 21 (Erdős–Graham, 1980).** Szemerédi-like generalization does not hold for Generalized Schur's theorem.

*Proof.* The set of odd integers of density $1/2$ cannot contain even a 2-term arithmetic progression and its difference! ■

Now the following question naturally arises: who conjectured and when what Szemerédi proved?

No one would expect a mystery here: just look at Szemerédi's 1975 paper, in which he presents the history of advances in good detail. Szemerédi starts with giving credit for conjecturing his theorem to Paul Edrős and Paul Turán in their 1936 paper [ET]. And so I look at this short important paper, without finding the conjecture, except for the case of 3-term arithmetic progressions. This incorrect credit is then repeated in the standard Ramsey theory texts [GRS1] and [GRS2] in 1980 and 1990, respectively, and from there on everywhere else, until in 2002 Ronald L. Graham and Jaroslav Nešetil noticed the discrepancy, and explained it in the following way [GN, p. 356]. "Although they [Erdős and Turán] do not ask explicitly whether $r_l(N) = o(N)$ (as Erdős did many times since), this is clearly on their mind as they list consequences of a good upper bound for $r_l(N)$: long arithmetic progressions formed by primes and a better bound for the Van der Waerden

[12] As I learned in March 2009, the original exposition was even harder, nearly impenetrable. Ron Graham and Endre Szemerédi spent long hours looking over pages of the proof scattered around in Ron's house, while simplifying the proof.

numbers." Clearly, my friends Ron and Jarik and I agree that the conjecture does not appear in the 1936 [ET]. Their argument that the young Erdős and Turán had the conjecture "clearly on their mind" could be viewed more as an eloquent homage to the two great mathematicians rather than an historical truth. We therefore have to research further.

In his 1957 first-ever open-problem paper [E57.13], Paul Erdős indicates that before him and Turán, Issai Schur (!) called on studying longest arithmetic-progression-free opening segments of positive integers. Erdős writes: "The problem itself seems to be much older (it seems likely that Schur gave it to Hildegard Ille, in the 1920s)."

Erdős returns to Issai Schur's contribution in his 1961 second open-problem paper [E61.22], which in 1963 also appears in Russian [E61.22][13]: "The problem may be older but I can not definitely trace it. Schur gave it to Hildegard Ille around 1930." Paul told me that he "met Issai Schur once in mid 1930s," more precisely in 1936 in Berlin (amazingly, I found the eyewitness of this Erdős's visit: Hilde Brauer, the wife of Schur's PhD student and friend Alfred Brauer). Schur and Erdős shared a mutual admiration. Undoubtedly, they discussed prime numbers, but likely not arithmetic progressions. Erdős learned about Schur's interest in arithmetic progressions and early Ramsey-like conjectures and results from Hildegard Ille (1899–1942). Now this requires a bit of explanation, because they probably had never met!

Erich Rothe (1895–1988), Dr. phil. Universität Berlin 1926 under the eminent Erhard Schmidt and Richard Mises, married a fellow student Hildegard Ille, Dr. phil. Universität Berlin 1924 under Issai Schur. They taught at Universität Breslau, Germany (later and earlier Wroctaw, Poland) until, as Jews, they were forced to flee Nazi Germany in 1937, and came to the United States. Hildegard passed away at a young age. The accomplished mathematician Erich Rothe held a professorship at the University of Michigan from 1941 until his retirement in 1964. His eulogy (*Notices of Amer. Math. Soc.*, 1988, 544) quotes the Chair of the Department of Mathematics of the University of Michigan D. J. Lewis saying that "Rothe was a scholar of the old school. He was very broadly educated.... He was a wise and judicious man of much wit. His companionship was very much in demand."

Erich Rothe was Paul Erdős's source of reliable information on problems and conjectures in number theory that Issai Schur shared with Rothe's wife Hildegard (Ille) Rothe. From Rothe Erdős learned about Schur's authorship of the arithmetic progressions conjecture, proven by Van der Waerden. From Rothe Erdős learned that Issai Schur yet again contributed to number theory and Ramsey theory when he asked his graduate student Hildegard to investigate arithmetic progression-free arrays of positive integers. To my surprise, no one acknowledged the credit Erdős gave to Schur in his first open-problem papers [E57.13], [E61.22], and [E63.21].

I believe, however, that Erdős learned about Schur being first to investigate this subject after Erdős and Turán independently rediscovered it: their paper [ET] was published in 1936, while Erich and Hildegard Rothe came to the United States in 1937; moreover, Erdős–Rothe conversations took place after Hildegard's passing

[13] This Russian publication does not appear in any of Paul Erdős's bibliographies.

in 1942. Paul was certainly correct when in both his 1957 and again 1961 open-problem papers he wrote "The first publication on the function $r_k(n)$ is due to Turán and myself." This was an important paper, and Paul knew that. Yet, it contained the "density" conjecture only for 3-term arithmetic progressions. Graham and Nešetřil are correct when they write [GN] that "Erdős did [pose the general case conjecture] many times," but the real question is: when did he pose the conjecture for the first time?

I am reading again Erdős's first 1957 open-problem paper. Paul writes: "In [ET] we stated our conjecture that $\lim r_3(n)/n = 0\ldots$ Roth [Rot] proved that $r_3(n) = o(n)\ldots$ The true order of magnitude of $r_3(n)$ and, more generally, of $r_k(n)$, remains unknown." Paul discusses the general function $r_k(n)$, but the conjecture of the general case is not here. If the conjecture were to exist consciously in his mind, he would have included it in this open-problem article, I am almost certain of it. Paul had not, and this, in my opinion, is a reliable indicator that the general conjecture did not exist yet in 1957.

In the second 1961 open-problem paper, Paul publishes the general conjecture explicitly for the first time: "For $k > 3$ the plausible conjecture $r_k(n) = o(n)$ is still open." This "still open" indicates that the problem was created before Erdős submitted this paper, which was "Received October 5, 1960." This suggests the birth of the general conjecture in 1957–1959.

During his December 23, 1991 "favorite problems" lecture at the University of Colorado at Colorado Springs, Paul indicated when he first offered the high prize of \$1,000 for this conjecture: "Twenty-five years ago I offered \$1,000 for it." This places the \$1,000 offer in 1966 or so. In early January of 1992, in Colorado Springs Paul confirmed that this was the highest prize he has ever paid: "The maximum amount of money I paid [was] \$1,000 to Szemerédi in 1972. This was a conjecture of Turán and myself. If you have a sequence of positive density, then it contains arbitrary long arithmetic progression." Paul also told me then, "Turán and I posed this problem in the early 1930s." I hope, however, that my argument, presented here, indicates that it took time for the plot to thicken, that it was a long pregnancy, and from the early seeds in the 1930s the great conjecture had grown inside Paul Erdős' head and was born in 1957–1959.

Even after Szemerédi, Erdős was not quite happy with the state of knowledge in this field. In 1979 he offered an extravagant prize for the discovery of asymptotic behavior (published in 1981 [E81.16]): "It would be desirable to improve [lower and upper bounds] and if possible to obtain an asymptotic formula for $r_3(n)$ and more generally for $r_k(n)$. This problem is probably enormously difficult and I offer \$10,000 for such an asymptotic formula."

**Erdős's \$10,000 Open Problem 22.** Find an asymptotic formula for $r_3(n)$ and more generally for $r_k(n)$.

This train of thought, apparently started by Issai Schur, was his last contribution to Ramseyan mathematics. From there on Paul Erdős led Ramsey theory until his passing in 1996.

## 10 The Tibor Gallai Theorem

Gallai's theorem is one of my favorite results in all of mathematics. Surprisingly, it is not widely known even among mathematicians. Its creator was Tibor Gallai, born Tibor Grünwald, a member of the Hungarian Academy of Sciences, who passed away on January 2, 1992 at the age of 79. His lifelong close friend and co-author Paul Erdős was visiting me in Colorado Springs[14] when Prof. Vera T. Sós called from Budapest to give Paul the sad news of Gallai's passing. I asked Paul to write the eulogy for Gallai for *Geombinatorics*; it appeared in the very next issue [E92.14].

Gallai discovered a number of fabulous results, some of which were named after other mathematicians: he preferred not to publish even his greatest results. Why? On July 20, 1993 in Kesztely, Hungary during a dinner my (then) wife Maya, our baby Isabelle, and I shared with George Szekeres and Esther Klein, the legendary couple from the legendary circle of young Jewish mathematicians in early 1930s Budapest, I was able to ask them about the friend of their youth.

"Gallai was so terribly modest," explained George Szekeres. "He did not want to publish because it would show the world that he was clever, and he would be restless because of it."

"But he was very clever indeed," added Esther Klein-Szekeres. Esther continued: "Once I came to him and found him in bed. He said that he could not decide which foot to put down first."

"Gallai was Paul Erdős's best, closest friend," continued George. "I was very close with Turán. It was later that Paul Erdős and I became friends."

Paul Erdős told me that Tibor Gallai discovered the theorem of our prime interest in the late 1930s. He did not publish it either. It first appeared in the paper [Rad2] by Richard Rado (with a credit to "Dr. G. Grünwald", which was Gallai's last name then; the initial "G" should have been "T" and must be a typo). Rado submitted this paper on September 16, 1939; it is listed in bibliographies as a 1943 publication, but in fact came out only in 1945; World War II affected all facets of life, and made no exception for the great Gallai result. I hope you will enjoy it as much as I have, and try your wit and creativity in proving this beautiful and extremely general, classic result.

**The Gallai Theorem 23 [Rad2].** *Let $m$, $n$, $k$ be arbitrary positive integers. If the lattice points $Z^n$ (i.e., the points with integer coordinates) of the Euclidean space $E^n$ are colored in $k$ colors, and $A$ is an $m$-element subset of $Z^n$, then there is a monochromatic subset $A'$ of $Z^n$ that is homothetic (i.e., similar and parallel) to $A$.*

In fact, with not too much effort the Gallai theorem can be strengthened as follows.

**The Gallai Theorem, A Strong Version 24 [GRS2].** *Let $m$, $n$, $k$ be arbitrary positive integers. If the Euclidean space $E^n$ is colored in $k$ colors and $A$ is a $m$-element subset of $E^n$, then there is a monochromatic subset $A'$ of $E^n$ that is homothetic to $A$.*

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[14] We were working on our joint project, a book of Paul's open problems: *Problems of pgom Erdős*, which I hope to finish by 2009–2010.

## 11 De Bruijn–Erdős's 1951 Compactness Theorem

They were both young. On August 4, 1947 the 34-year-old Paul Erdős, in a letter to the 29-year-old Nicolaas Govert de Bruijn of Delft, The Netherlands, offered the following conjecture [E47/8/4]. "Let $G$ be an infinite graph. Any finite subset of it is the sum of $k$ independent sets (two vertices are independent if they are not connected). Then $G$ is the sum of $k$ independent sets." Paul added in parentheses "I can only prove it if $k = 2$". In his five-page August 18, 1947 reply [Bru1], De Bruijn reformulated the Erdős conjecture in a way that is very familiar to us today:

**Theorem.** *Let $G$ be an infinite graph, any finite subgraph of which can be $k$-coloured (that means that the nodes are coloured with $k$ different colours, such that two connected nodes have different colours). Then $G$ can be $k$-coloured.*

In his letter, Nicolaas then proceeded to prove the theorem. Paul later found a different proof, and included the latter in the joint paper, which appeared 4 years later, thus giving us the powerful tool and the celebrated result:

**De Bruijn-Erdős's Compactness Theorem 25 ([BE], 1951).** *An infinite graph $G$ is $k$-colorable if and only if every finite subgraph of $G$ is $k$-colorable.*[15]

More of the de Bruijn–Erdős story can be found in [Soi]. This theorem has played a very important role in a number of Ramseyan problems. For example, it converted the problem of finding the chromatic number of the plane into a problem about finite sets. Of course, it very essentially used the axiom of choice. With no choice things proved to be quite different, as was shown in [SS1] and [SS2].

## 12 Khinchin's Small Book of Big Impact

As we have seen earlier, B. L. van der Waerden proved his pioneering result in 1926 while at Hamburg University, but its publication [Wae] in a little-known Dutch journal hardly helped its popularity, and the popularity of Ramseyan ideas. Only Issai Schur and his two students Alfred Brauer and Richard Rado learned about and improved upon Van der Waerden's result almost immediately; and somewhat later, in 1936, Paul Erdős and Paul Turán commenced density considerations related to Van der Waerden's result [ET]. Schur's 1916 result and its generalizations were even less known: even in 1995 Van der Waerden wrote to me that he did not know it!

In 1928 the Russian visitor to Göttingen, the analyst Aleksandr Yakovlevich Khinchin (1894–1959) heard about Van der Waerden's proof and was impressed by it. OK, one Russian liked one Ramseyan result; you may be wondering, what is the big deal? Khinchin remembered his excitement and after World War II, in 1947 he included Van der Waerden's proof in his little book *Three Pearls of Number Theory*

[15] This theorem requires the axiom of choice or the equivalent.

as one of the three pearls [Khi1]. The booklet was an instant success, and a second edition came out in Russian in 1948 [Khi2]. It included a new, "much simpler and transparent," in the opinion of Khinchin, exposition of Van der Waerden's proof, proposed by the Russian mathematician M. A. Lukomsakja from Minsk. In 1951 this second edition of the book was translated into German [Khi3] and in 1952 into English [Khi4]. Each of these translations proved instrumental in bringing excitement about Ramseyan ideas around the world. It even inspired the emergence of two more independent proofs of Gallai's theorem. The 1951 German translation [Khi3] inspired Ernst Witt to discover his proof in 1951 ([Wit], submitted on September 21, 1951; published in 1952), while the 1952 English translation [Khi4] stimulated Adriano Garsia in finding his proof [Gar] in 1958. Khinchin writes [Khi3]: "It is not out of the question that Van der Waerden's theorem allows an even simpler proof, and all efforts in this direction can only be applauded."

Witt [Wit] quotes Khinchin's call to arms in his paper, and happily reports, "This was the occasion to strive for a new order of proof that then led directly to a more general grasp of the problem." The great success of this booklet not only made the Baudet–Schur–Van der Waerden theorem famous, it heralded to the wide mathematical world the arrival of the new Ramseyan ideas.

On May 27, 2009, during the *DIMACS Conference*, where I presented this paper, 9:15–10:45, I got additional confirmation of the influence of Khinchin's booklet. Ron Graham and Joel Spencer shared with me that this booklet introduced each of them for the first time to the name of Van der Waerden, his theorem, and Ramseyan ideas.

## 13 Long Live the Young Theory!

B. L. van der Waerden's words [Wae18] about his 1926 proof are quite applicable to the emergence of Ramsey theory: "It is like picking apples from a tree. If one has got an apple and another is hanging a little higher, it may happen that one knows: with a little more effort one can get that one too."

As Pablo Picasso put it, "It takes a long time to become young." And so the ideas we have surveyed here have become the young Ramsey theory. The growth in the 1970s manifested itself in a level of maturity that was summarized in the Graham–Rothschild–Spencer monograph [GRS1]. Fine mathematicians, some of whom have written surveys for this volume, joined Paul Erdős in creating the rich and dynamic Ramsey theory that we know today.

My gratitude goes to my son Mark S. Soifer for capturing the grave of Frank Plumpton Ramsey and his parents in Cambridge, England especially for this survey, and to Peter D. Johnson Jr., Stanisaw Radziszowski, and Pawel Radziszowski for valuable suggestions.

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# Eighty Years of Ramsey $R(3, k)$ ... and Counting!

Joel Spencer

How frequently does an intriguing problem come up over lunchtime, only to have it solved the next morning? How many mathematical problems are seemingly intractable? Decades go by without a hint of progress. What a delight when a problem is worked on over many many years with progress occurring incrementally until it finally succumbs. Fermat's Last Theorem is perhaps the best example. Hilbert's Tenth Problem is another marvellous story. In discrete mathematics, my vote is for the asymptotics of the Ramsey number $R(3, k)$. The story begins in 1931, is resolved in 1995, with a coda in 2008, and with the final story perhaps not yet told.

In this chapter we consider only the *asymptotics*, the behavior of $R(3, k)$ for $k$ large. There has been a great deal of work on the values $R(3, k)$ for $k$ small, with the exact values known for $3 \leq k \leq 9$. The frequently updated survey of Radziszowski [9], together with his paper appearing in this volume, gives these results and much much more.

## 1 Basics

But first, the problem. We deal throughout with graphs that are undirected and have neither loops nor multiple edges. We write $G = (V, E)$ where $V, E$ are the sets of vertices and edges of $G$, respectively.

**Definition 1.** A set $I \subseteq V$ is *independent* if for no $v, w \in I$ is $\{v, w\} \in E$. The *independence number* of a graph, denoted $\alpha(G)$, is the maximal size $|I|$ of an independent set in $G$.

The study of $R(3, k)$ splits into upper bounds and lower bounds so we define it in a slightly unusual way.

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A. Soifer (ed.), *Ramsey Theory: Yesterday, Today, and Tomorrow*,
Progress in Mathematics 285, DOI 10.1007/978-0-8176-8092-3_2,

**Definition 2.** $R(3,k) \leq n$ if for every triangle-free graph $G$ on $n$ vertices there *exists* an independent set $I$, $|I| \geq k$.

**Definition 3.** $R(3,k) > n$ if there *exists* a triangle-free graph $G$ on $n$ vertices which does not have an independent set $I$, $|I| \geq k$.

## 2 George, Esther, Paul

The story begins in late 1931. Three youngsters, full of mathematical promise, walk the beautiful hills around Budapest. George Szekeres had completed his studies in chemical engineering. His interest in mathematics was already very strong, but it would take 15 years of global uproar before he would take on that subject as his profession in Australia. Esther Klein was a talented mathematics student who had just returned from Gottingen with an intriguing problem. The youngest was only 18-years old, not yet *die Zauberer von Budapest*, but he was already well known in Hungarian circles for his mathematical abilities. This was Paul Erdős.

Some 50 years later, George Szekeres wrote about those times:

> The origins of the paper go back to the early thirties. We had a very close circle of young mathematicians, foremost among them Erdős, Turán and Gallai; friendships were forged which became the most lasting that I have ever known and which outlived the upheavals of the thirties, a vicious world war and our scattering to the four corners of the world. I [...] often joined the mathematicians at weekend excursions in the charming hill country around Budapest and (in the summer) at open air meetings on the benches of the city park.

Klein proposed a geometry problem that they all set out to solve. In short order, Szekeres gave a solution. But, in fact, he had rediscovered Ramsey's theorem. While Ramsey's paper had been published in 1927, Ramsey himself was interested in a problem in logic and none of the three had been aware of his work. Erdős presented an independent proof and their results appeared in a joint paper [7]. Erdős always called this paper the "Happy Ending Paper" as George Szekeres and Esther Klein were soon married. After spending the war years as refugees in Shanghai they emigrated to Australia where they were leaders in the development of Australian mathematics, particularly in introducing the Hungarian style of mathematical contests to generations of Australian students.

The Szekeres argument gave the existence of $R(3,k)$ (and much much more), but just how big is it? We give the basic upper bound.

**Theorem 2.1.** $R(3,k) \leq k^2$.

*Proof.* Here, and throughout, we aim for a computer science perspective on the proofs. These were certainly not the ways the original proofs were framed!

Let $G$ have $k^2$ vertices. Consider the program:

IF some $v \in V$ has degree $\geq k$
    Neighbors of $v$ form independent set $I$

ELSE
$I \leftarrow \emptyset$
WHILE $G$ is nonempty
Select any $v$.
Add $v$ to $I$
Delete $v$ and neighbors from $G$
End WHILE

The neighbors of any vertex $v$ of a triangle-free graph form an independent set. Thus in the IF case we find $I$ of the desired size. ELSE, for each $v$ added to $I$ at most $k$ vertices (it and its neighbors) are deleted. Having begun with $k^2$ vertices, the final $I$ has size at least $k$. $\square$

## 3 Erdős Magic

In April 1946 Erdős [4] made a conceptual breakthrough whose effects we are still feeling.

**Theorem 3.1.** *If*

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1$$

*there* exists *a graph $G$ on $n$ vertices with neither clique nor independent set of size $k$.*

*Proof.* Consider the *random graph* $G \sim G(n, \frac{1}{2})$. Technically, we have a probability space whose elements are the labelled graphs on $n$ vertices. Probabilities are determined by saying $\Pr[\{i,j\} \in E] = \frac{1}{2}$ and that these events are independent. For each set $S$ of $k$ vertices we have the "bad" event $B_S$ that $S$ is either complete or independent. Then $\Pr[B_S] = 2^{1-\binom{k}{2}}$. The probability of a disjunction is at most the sum of the probabilities so that ($\vee$ over all $S \subset V$, $|S| = k$)

$$\Pr[\bigvee B_S] \leq \binom{n}{k} 2^{1-\binom{k}{2}} < 1 \quad \square$$

Let *GOOD* denote the completement event $\wedge \overline{B_S}$. Then *GOOD* has positive probability. That is, there is positive probability that the random graph has the desired property. The probabilistic method, or Erdős Magic, is now born. As the event is nonempty there must be a point in the probability space for which it holds. That is, there absolutely positively must exist a graph $G$ with the desired properties. Our book [1] is one of many to cover the many applications of this methodology.

Let $G(n,p)$, as usual, denote the random graph with $n$ vertices with $p$ the probability of adjacency. In studying $R(3,k)$ one is led to the study of sparse random graphs. The probability that $G(n,p)$ has an independent set of size $k$ is at most $\binom{n}{k}$, the number of such sets, times $(1-p)^{\binom{k}{2}}$, the probability of no internal edges. We bound the first term from above by $n^k$ (it being rather remarkable how effective such gross bounds can be) and the second by $e^{-pk^2/2}$. When $k = 2.01((\ln n)/p)$ the second term dominates and so $G(n, p)$ almost surely has no such independent set.

Following Definition 2, attempts to find a lower bound on $R(3,k)$ start with a random graph $G(n, p)$. Immediately there is a problem. To avoid triangles one needs $p = O(n^{-1})$, but in this range there are independent sets of size $\Omega(n)$ and this would yield only a linear lower bound on $R(3,k)$. All successful approaches use a larger $p$, one for which $G(n, p)$ will have triangles, and then somehow fix the triangles. We begin with a rather weak result.

**Theorem 3.2.** $R(3,k) = \Omega((k/\ln k)^{3/2})$. *That is, there* exists *a graph $G$ on $n$ vertices with no triangle and no independent set of size $cn^{2/3}\ln n$.*

Here we begin with $2n$ vertices and consider a random graph with edge probability $p = n^{-2/3}$. There will be an expected number $\sim n^3 p^3/6 = n/6$ triangles. Setting $k = 2.01((\ln n)/p)$, the expected number of independent sets of size $k$ is less than one. By a problem, let us mean either a triangle or an independent set of size $k$. Thus the expected number of problems is around $n/6$. From Erdős Magic, there is a graph $G$ for which the number of problems is less than around $n/6$, and so certainly there is one where the number of problems is less than $n$. Take that graph $G$ and eliminate one vertex from each triangle and one vertex from each independent set of size $k$. The remaining graph, call it $G^*$, has no problems, and it has at least $n$ vertices.

The bulk of lower-bound arguments for $R(3,k)$ examine $G(n, p)$ with $p = cn^{-1/2}$ with $c$ an appropriate small constant. Here the expected number of triangles is roughly $c^3n^{3/2}/6$. The expected number of edges is roughly $c^2n^{3/2}/2$ which will be considerably more. One wants to make the graph triangle-free by somehow eliminating the relatively small number of edges in triangles, but doing this in a way that keeps the size of the independent set around $K((\ln n)/p)$. It is not so easy!

## 4 An Erdős Gem

Erdős was one of, possibly the, most prolific mathematicians in history. With the passage of time we can look at certain of his papers and recognize their depth and importance. In that light, the following 1961 Erdős gem [3] is a personal favorite of this author. It would be an outstanding paper in any time period, but that it was done when the probabilistic method was still in its infancy is truly a testament to Erdős's genius.

**Theorem 4.1.** $R(3,k) = \Omega((k/\ln k)^2)$. *That is, there* exists *a graph $G$ on $n$ vertices with no triangle and no independent set of size $cn^{1/2}\ln n$.*

Erdős considers $G(n, p)$ with $p = \epsilon n^{-1/2}$, $\epsilon$ a small constant. Set $x = cn^{1/2}\ln n$, $c$ a large constant. Call an $x$-set $I$ a failure if every edge $\{u, v\} \in G$ with $u, v \in I$ can be extended to a triangle $\{u, v, z\}$ where the third vertex $z$ lies outside of $I$. He shows that with high probability there are no failures. This takes quite some doing, but we can give a heuristic explanation. An edge in $I$ has, on average, $np^2 \sim \epsilon^2$ extensions to a triangle outside of $I$ thus probability less than one half (taking $\epsilon$ small) of being so extendable. Each pair $u, v \in I$ would then have probability at least $p/2$ of being an edge and not being so extendable. Now suppose these events were independent over the pairs. Then the chance that $I$ is a failure, that is, that no pair had this property, would be around $1-(p/2)$ to the power $\binom{x}{2}$. This is basically $\exp(-px^2/4)$ which is smaller than $n^{-x}$, smaller than one over the total number of $x$-sets. Then the expected number of failures $I$ would be much less than one and almost surely there would be none of them. The actual proof is more complicated as these events are very definitely not independent.

Let's assume the claim. Now we can put on our computer science hats (definitely not the original Erdős style!) and complete the proof. Take a $G$ with no failures. We apply a greedy algorithm to find a triangle-free subgraph of $G$. Order the edges of $G$ arbitrarily and consider them in that order. Accept an edge if it does not create a triangle along with the edges previously accepted. Let $H$ denote the final graph created. We tautologically do not have a triangle in $H$. Now consider any $x$-set $I$. As $I$ is not a failure for $G$ it has an edge $\{u, v\}$ which is not extendable to a triangle outside of $I$. When we reached this edge in determining $H$ it was either accepted or rejected. If it was accepted then $I$ was not independent in $H$. If it was rejected it was only because it would have created a triangle in $H$. But that triangle $\{u, v, w\}$ must lie entirely inside $I$ since the edge is not extendable to a triangle outside of $I$. That would mean that edges $\{u, w\}$ and $\{v, w\}$ would already be in $H$. So in this case too $I$ would not be independent. That is, $H$ does not contain any independent $x$-sets $I$.

## 5 Upper Bounds

We return to the upper bound and improvements on Theorem 2.1. In 1968, Graver and Yackel improved this result to

$$R(3, k) = O\left(k^2 \frac{\ln \ln k}{\ln k}\right) \tag{1}$$

That result held for 12 years, until it was supplanted in 1980 by Ajtai, Komlós and Szemerédi.

**Theorem 5.1.**

$$R(3, k) = O\left(\frac{k^2}{\ln k}\right) \tag{2}$$

At the time, the improvement was not considered so significant, but events proved otherwise. Ajtai, Komlós and Szemerédi actually proved a general theorem about independent sets in triangle-free graphs.

**Theorem 5.2.** *Let $G$ be a triangle-free graph on $n$ vertices in which the average degree is at most $k$. Then there exists an independent set $I$ with*

$$|I| \geq c\frac{n}{k}\ln k \tag{3}$$

Here $c$ is an absolute positive constant. Turán's theorem gives that a graph with $n$ vertices and average degree at most $k$ (and hence at most $nk/2$ edges) has independence number at least $n/(k+1)$, the extreme case occurring when $G$ is the union of disjoint cliques of size $k+1$. In this context Theorem 5.2 can be understood as saying that $\alpha(G)$ is increased when one requires that $G$ is triangle-free. The Ramsey bound Theorem 5.1 follows immediately from Theorem 5.2. For let $G$ be any triangle free graph on $n = c^{-1}(k^2/\ln k)$ vertices. If any vertex has degree at least $k$ its neighbors form an independent set of size at least $k$. Otherwise all vertices have degree less than $k$, hence the average degree is less than $k$, hence there is an independent set of size at least $c(n/k)\ln k$, which is $k$. In either case there is an independent set of size at least $k$.

We give a rough idea of the argument for Theorem 3. The key is a lemma which we do not prove. Let $G$ be a triangle-free graph with average degree $u$. The lemma states, roughly, that there is a vertex $v$ of degree about $u$ such that removing it and its neighbors yields a graph $G^-$ whose edge density is not more than that of $G$. Now begin with a triangle-free $G$ on $n$ vertices with average degree $k$ or less and consider a process where at each step we select a vertex $v$ as above, add it to the independent set, and remove $v$ and all of its neighbors. We parametrize time $t$ saying that at time $t$ the number of vertices $v$ so selected is $(n/k)t$. Let $S(t)n$ be the number of vertices remaining in the graph at that time. Under the assumption that density has not increased, at time $t$ the average degree would then be at most $S(t)k$. When $v$ is now selected $1 + S(t)k \sim S(t)k$ vertices are removed. Parametrized time $t$ has increased by $k/n$. This gives a difference equation

$$S\left(t+\frac{k}{n}\right) - S(t) \sim -\frac{S(t)k}{n} \tag{4}$$

which turns into a differential equation

$$S'(t) = -S(t) \tag{5}$$

with the simple solution

$$S(t) = ke^{-t} \tag{6}$$

The procedure continues until $t \sim \ln k$, giving an independent set of size $\sim (n/k)\ln k$.

## 6 The Lovász Local Lemma

One of the great advances in the probabilistic method was the Lovász local lemma, which first appeared in [5]. We give a formulation here that is not the most general, but will suffice for our application and indeed for almost all known applications. We are given a set $\Omega$ and for each $e \in \Omega$ a random variable $X_e$. We assume that the $X_e$ are mutually independent. We let $\Gamma$ index a set of events. For each $\alpha \in \Gamma$ we have a set $A_\alpha \subset \Omega$ and a "bad" event $B_\alpha$. The event $B_\alpha$ can depend only on the values $X_e$ with $e \in A_\alpha$.

In our example, the $X_e$ describe the random graph $G(n,p)$. We let $\Omega$ be the set of potential edges $e$ (that is, two element sets of vertices) on a vertex set $\{1,\ldots,n\}$. For each $e$ we let $X_e$ have values zero and one with $\Pr[X_e = 1] = p$ and the $X_e$ mutually independent. Then the edge set of $G(n,p)$ is those $e$ for which $X_e = 1$. Now the bad events will be of two types. $\Gamma$ is indexed by the three element subsets $S$ of vertices and the $k$ element subsets $T$ of vertices. For each triple $S = \{i,j,h\}$ of vertices we have the event $B_S$ that $S$ is a triangle. That is, $X_{ij} = X_{jh} = X_{ih} = 1$. For each $k$-set $T$ of vertices we have the event $B_T$ that $T$ is an independent set. That is, $X_{ij} = 0$ for all $i,j \in T$.

We write $\alpha \sim \beta$ if $\alpha \neq \beta$ (a technicality) and, critically, $A_\alpha \cap A_\beta \neq \emptyset$. Note that when a family of $\alpha$ have no $\alpha \sim \alpha'$ the corresponding events $A_\alpha$ are mutually independent. In our example, two events $B_S, B_{S'}$ are $S \sim S'$ if $S \neq S'$ and $S, S'$ overlap in at least two vertices, and hence in at least one edge $e$.

**Theorem 6.1.** *Let $B_\alpha$, $\alpha \in \Gamma$ be events as described above. Suppose there exist real numbers $x_\alpha$, $\alpha \in \Gamma$, with $0 \le x_\alpha < 1$ and*

$$\Pr(B_\alpha) \le x_\alpha \prod_{\beta \sim \alpha} (1 - x_\beta) \tag{7}$$

*Then*

$$\Pr(\wedge_{\alpha \in \Gamma} \overline{B_\alpha}) \ge \prod_{\alpha \in \Gamma} (1 - x_\alpha) \tag{8}$$

*In particular, with positive probability no event $B_\alpha$ holds.*

Our object now is to show $R(3,k) > n$ for $n$ as large as possible. We look at $G(n,p)$. If the conditions of Theorem 6.1 hold then with positive (albeit small!) probability $G(n,p)$ will have neither triangle nor independent set of size $k$. Erdős Magic then implies that there exists a specific $G$ on $n$ vertices with this property, so that $R(3,k) > n$. Suppose that for each 3-set $S$ we select the same value for $x_S$; call it $y$. Suppose that for each $k$-set $T$ we select the same value for $x_T$; call it $z$. Let's put an upper bound, for $\alpha$ of each type, of the number of $\beta$ of each type with $\alpha \sim \beta$. For each 3-set $S$ there are $3(n-3) \le 3n$ other $S'$ with $S \sim S'$. For each $k$-set $T$ there are $\binom{k}{2}(n-k) + \binom{k}{3} \le k^2 n/2$ $k$-sets $T$ with $S \sim T$. For each $k$-set $T$ there are at most $\binom{n}{k}$ (that is, all) $k$-sets $T'$ with $T \sim T'$. For each 3-set $S$ there are at most $\binom{n}{k}$ (that is, all) $k$-sets $T$ with $S \sim T$.

In application, Theorem 6.1 becomes:

**Theorem 6.2.** *If there exist* $p \in [0,1]$ *and* $y, z \in [0,1)$ *with*

$$p^3 \le y(1-y)^{3n}(1-z)^{\binom{n}{k}} \tag{9}$$

*and*

$$(1-p)^{\binom{k}{2}} \le z(1-y)^{k^2 n/2}(1-z)^{\binom{n}{k}} \tag{10}$$

*then* $R(3,k) > n$.

Theorem 6.2 leads to a problem in what we like to call asymptotic calculus. What is the largest $n$, as an asymptotic function of $k$ such that there exist $p \in [0,1]$, $y, z \in [0,1)$ satisfying 9 and 10. This is not an easy problem but it is an elementary problem.

Here this author can add a personal note. Some three decades ago I was able to show that the largest such $n$ was of the order $\Theta(k^2/\ln^2 k)$. This gave an alternate proof to Theorem 4.1, the gem of Erdős. One needed merely the analytic skills of a reasonable graduate student; one did not need the brilliance of Erdős, nor of Lovász, to find the bound. Sometimes in mathematics one's deepest work, even when successful, receives little attention. In this case the opposite was true and my applications [10] of Lovász local lemma to improve Ramsey bounds (also on $R(k,k)$ and on $R(l,k)$ for $l \ge 3$ fixed and $k \to \infty$) have been frequently quoted.

## 7 Random Greedy Triangle-Free

In 1995 [6] Paul Erdős, along with coauthors Peter Winkler and Stephen Suen, returned to the asymptotics of $R(3,k)$, a problem he had first considered some 63 or 64 years before. Erdős had a great faith, albeit unspoken, in his nose for interesting beautiful mathematics. In 1946 he "invented" the probabilistic method. For the next quarter century he published many papers in that area. While others appreciated the beauty of the results, he had few followers during that time. But he continued, convinced of the intrinsic interest in that area, and his convictions were borne out. Today (thanks, in part, to the development of probabilistic algorithms in computer science), the use of the probabilistic techniques he developed is widespread and is, in this author's opinion, one of his enduring legacies. He had an equal faith in Ramsey theory, to which he returned again and again, always coming up with new questions, new conjectures, new theorems, and new methodologies. It would be a perfect end to this narrative to say that in 1995 Erdős resolved the asymptotics of $R(3,k)$. Alas, that was not the case.

But Erdős did open the door to the final assault.

Erdős, with Winkler and Suen, examined the *random greedy triangle-free algorithm*. The algorithm itself is trivial. We begin with the empty graph $G$ on vertex set $\{1,\dots,n\}$. Order the $\binom{n}{2}$ potential edges randomly. Now consider these potential edges sequentially. When considering edge $e$, add $e$ to $G$ (we say *accept* $e$)

if doing so will not create a triangle in $G$. Otherwise we *reject* $e$ and $G$ stays the same. Continue until all of the potential edges are considered. (Equivalently, we may begin with the empty graph and at each stage add an edge selected uniformly from those that would not create a triangle.) This yields a graph $G^{final}$ which tautologically has no triangle. Erdős and his coauthors looked at its independent sets. They were able to give a partial analysis until the time when $Kn^{3/2}$ potential edges had been considered, $K$ a particular absolute constant. With that analysis, they could show that there was no independent set of size $c_1\sqrt{n}\ln n$. By Erdős Magic there therefore existed such a triangle-free graph. Setting $k = c_1\sqrt{n}\ln n$ this yielded $R(3,k) > n = c_2k^2\ln^{-2}k$, giving yet another new proof of Erdős 1961 gem. Indeed, they found a better constant $c_2$ than previously known.

This author was able to push their methods and analyze the random greedy triangle-free algorithm until $Kn^{3/2}$ edges had been examined, where $K$ could be an arbitrarily large constant. With that analysis, I could show that there was no independent set of size $c_1\sqrt{n}\ln n$ where $c_1$ could be made arbitrarily small. By Erdős Magic there therefore existed such a triangle-free graph. Setting $k = c_1\sqrt{n}\ln n$ this yielded

$$R(3,k) \gg k^2\ln^{-2}k \tag{11}$$

This argument was never published. For in a matter of months there was a stunning breakthrough.

## 8 *R(3, k)* Resolved!

When the asymptotics of $R(3,k)$ were finally resolved there was (so the joke goes) a great surprise. The mathematician finding the solution was not a Hungarian! Rather it was the Korean Jeong-Han Kim [8]. Kim had recently received his Ph.D. from Jeff Kahn at Rutgers and was one of the stars of the new generation, using more advanced and sophisticated probabilistic methods.

**Theorem 8.1.**

$$R(3,k) = \Theta\left(\frac{k^2}{\ln^2 k}\right) \tag{12}$$

Kim's proof was an extension in spirit of the methods of Erdős, Winkler and Suen. Rather than the pure random greedy triangle-free algorithm he used a nibble method. At each stage a small but carefully chosen number of edges were added to the graph. Sophisticated use of martingales played a key role. There were underlying differential equations with careful error analysis. And there was a lot of just plain cleverness. It was a masterwork, resolving a 64-year-old problem. Kim was awarded the Fulkerson Prize for this achievement.

Let me add a personal note, what was for this author a most memorable moment. In January to April 1998 I was in Australia, working with Nick Wormald at

University of Melbourne. On March 6 I gave an invited talk at the University of Sydney. My title was "60 years of Ramsey $R(3,k)$," covering much of the material in this paper. In the front row were George Szekeres and Esther Klein Szekeres. Though in their late 80s they were enjoying an active retirement. They had enjoyed a life full of mathematics and good cheer. They lived several more years, both passing away on the same day, August 28, 2005.

## 9 Random Greedy Triangle-Free Redux

This author thought at that time that the story of $R(3,k)$ was completed. (The value of a constant $c$ so that $R(3,k) \sim ck^2 \ln^{-2} k$ remains open to this day, but this problem seems beyond our reach.) But a coda, or perhaps a new beginning would be a more appropriate term, was added in April 2008. This author received an email from Tom Bohman [2]. He had been able to analyze the random greedy triangle-free algorithm. (By coincidence, I was at ETH (Zurich) and just the day before had been speaking about the algorithm with Angelica Steger and how an analysis had proven so elusive.) He was able to show that algorithm gave a final $G$ with $\Theta(n^{3/2}\sqrt{\ln n})$ edges and that the largest independent set would have size $\Theta(\sqrt{n}/\sqrt{\ln n})$. This gave another and, at least to this author, more natural proof of Kim's result.

We can give a natural heuristic for these results. Suppose that after $un^{3/2}$ potential edges have been considered $xn^{3/2}$ have been accepted and think of $x = f(u)$. What about the next potential edge $e = \{i,j\}$? Suppose we think of the $xn^{3/2}$ accepted edges as a random graph on that many edges, or $G(n,p)$ where $p = 2xn^{-1/2}$. (Not only is this supposition a stretch but it is clearly false as $G(n,p)$ would have many triangles but the algorithm tautologically gives a graph with no triangles. Nonetheless, this is a most useful heuristic.) We would accept $e$ if for no $k \neq i,j$ are both $\{i,k\},\{j,k\}$ in $G(n,p)$. They are both in with probability $p^2$ and so the probability of acceptance would be $(1-p^2)^{n-2} \sim e^{-4x^2}$. Under this heuristic the rate of acceptance is now $e^{-4x^2}$. This leads to the differential equation

$$f'(u) = \frac{dx}{du} = e^{-4x^2} \tag{13}$$

We have an initial condition $f(0) = 0$ as initially the graph is empty. We solve this to get

$$u = \int_0^x e^{4t^2}\,dt \tag{14}$$

While this integral does not have a closed form, for large $x$ we would have

$$u = e^{4x^2(1+o(1))}dt \tag{15}$$

If we can continue this heuristic to the end of the process we would have $u = n^{1/2+o(1)}$ and therefore $x = \Theta(\sqrt{\ln n})$. This would argue that the process ends with $\Theta(n^{3/2}\sqrt{\ln n})$ edges. Making another stretch and thinking of the final $G$ as a random graph on this many edges it would have $\alpha(G) = \Theta(\sqrt{n \ln n})$ as desired.

Well, it's not so easy. Bohman parametrizes time $t$ as above, when $tn^{3/2}$ edges have been accepted. The pairs $e = \{i, j\}$ are now in three categories. Some $e$ are `in`, meaning they are already in $G$. Some $e$ are `open`, meaning that there is no $k \neq i, j$ with $\{i, k\}, \{j, k\}$ already in the graph. Open edges can be added to $G$. (Tautologically they have not already been considered, for then they would be in $G$.) The other $e$ are `closed`, meaning their addition would cause a triangle, and so they cannot ever be added.

Bohman sets $Q$ equal the number of open edges and parametrizes $Q = q(t)n^2$. (Already there is a notion that with high probability $Q$ will be concentrated and this requires substantiation.) For each pair $i, j$ of vertices with $\{i, j\}$ not in $G$ he sets $X_{ij}$ equal the number of $k \neq i, j$ with both $\{i, k\}$ and $\{j, k\}$ open. He parametrizes $X_{ij} = x(t)n$. (Now there is a further notion that with high probability all the $X_{ij}$ are asymptotically the same.) He further lets $Y_{ij}$ equal the number of $k \neq i, j$ such that one of $\{i, k\}, \{j, k\}$ is open and the other is in. He parametrizes $Y_{ij} = y(t)n^{1/2}$.

Bohman now looks at the expected change in $X_{uv}$ when a random open edge is added to the graph. The main picture is the following. Consider a $w$ for which $\{u, w\}$ and $\{v, w\}$ are open. Now pick either $u$ or $v$, say $u$. Now consider a $z$ with one of $\{z, u\}, \{z, w\}$ open and the other in. Say $\{z, u\}$ is open and suppose it is selected and added to $G$. As $\{z, u\}$ and $\{z, w\}$ are now in, $\{u, w\}$ changes from open to closed. This decrements $X_{uv}$ be one. There are $x(t)n$ choices of $w$, two choices of $u$ or $v$ and then $y(t)n^{1/2}$ choices of $z$, giving $2x(t)y(t)n^{3/2}$ choices that decrement $X_{uv}$. As there are $q(t)n^2$ open edges in total and the next edge is chosen uniformly from them, the expected decrease in $X_{uv}$ is $[2x(t)y(t)/q(t)]n^{-1/2}$. Selection of one edge increased parametrized time by $n^{-1/2}$. This leads to the difference equation

$$x(t + n^{-1/2}) - x(t) = -[2x(t)y(t)/q(t)]n^{-1/2} \tag{16}$$

which in turn leads to the differential equation

$$x'(t) = -2x(t)y(t)/q(t) \tag{17}$$

By similar arguments one gets differential equations for $q$ and $y$:

$$y'(t) = -\frac{y^2(t) + 2x(t)}{q(t)} \tag{18}$$

$$q'(t) = -y(t) \tag{19}$$

This system of differential equations has the very clean solution:

$$x(t) = e^{-8t^2} \quad y(t) = 4te^{-4t^2} \quad q(t) = \frac{1}{2}e^{-4t^2} \tag{20}$$

It is intriguing to note that these values are the same, appropriately interpreted, as if $G$ were a random graph with $tn^{3/2}$ edges, or for $G(n, p)$ with $p = 2tn^{-1/2}$.

For example, the probability that a pair $\{i, j\}$ is not joined by a path of length two (effective, that it is open) would be $(1-p^2)^{n-2} \sim e^{-4t^2}$ and so the number of such pairs would be $\sim e^{-4t^2}\binom{n}{2} \sim q(t)n^2$. A direct argument, even a strong heuristic, that the values of $x, y, q$ mirror those of the random graph remains elusive.

The difficult part of Bohman's argument is to show that in the actual random process the values of $X_{uv}, Y_{uv}, Q$ remain close to the expected values $x(t)n, y(t)n^{1/2}, q(t)n^2$. Some careful analysis with strong use of martingales (actually super and submartingales) is used. The larger $t$ gets the more difficult this becomes as an instability in the random process could conceivably lead to further instability. If one needed only to get this result for a large constant $t$ there would be results on approximating a random process by a differential equation over a compact space. Of particular difficulty is that to get the full result Bohman carries the analysis out to $t = \epsilon\sqrt{\ln n}$ for some small positive absolute constant $\epsilon$. This corresponds to the consideration of $n^{3/2+\epsilon_1}$ potential edges for some small absolute constant $\epsilon_1$. (He doesn't take the analysis out to when $n^{2-o(1)}$ edges have been considered as the instabilities overwhelm the analysis before that. For that reason the constants he achieves are subject to improvement.) To do this he gives somewhat ad hoc error bounds on the variables at $t$ as a function of $t$. These bounds increase as a function of $t$ but remain small through $t = \epsilon\sqrt{\ln n}$. Last, but not least, this analysis gives the number of edges accepted and one needs further study to show that $\alpha(G)$ is appropriately small.

## 10 Epilogue

Is the story of $R(3, k)$ over? I think not. I think there is plenty of room for a consolidation of the results. My dream is a ten-page paper which gives $R(3, k) = \Theta(k^2/\ln k)$. The upper bound Theorem 2 can be nicely described with proof in a few pages. The Kim–Bohman lower bound is not quite there yet. But it seems that the Bohman approach could be greatly simplified, making use of some known stability results for random processes. When I speak on this topic I keep waiting for someone in the audience to say "Of course, this follows easily from the well-known results of XYZ." It hasn't happened yet.

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# Ramsey Numbers Involving Cycles

Stanisław P. Radziszowski

## 1 Scope and Notation

There is a vast amount of literature on Ramsey-type problems starting in 1930 with the original paper of Ramsey [Ram]. Graham, Rothschild and Spencer in their book, *Ramsey Theory* [GRS] and Soifer in the 2009 *The Mathematical Coloring Book, Mathematics of Coloring and the Colorful Life of Its Creators* [Soi] present exciting developments in the history, results and people of Ramsey theory. The subject has grown amazingly, in particular with regard to asymptotic bounds for various types of Ramsey numbers (for example, see the survey papers [GrRö; Neš; ChGra2]), but the progress on evaluating the basic numbers themselves has been very unsatisfactory for a long time.

Ramsey theory studies the conditions of when a combinatorial object necessarily contains some smaller given objects. The role of Ramsey numbers is to quantify some of the general existential theorems in Ramsey theory. In the case of the so-called generalized graph Ramsey numbers one studies partitions of the edges of the complete graph, under the condition that each of the parts avoid some prespecified arbitrary graph, in contrast to classical Ramsey numbers when the avoided graphs are complete.

This survey is a compilation of results on Ramsey numbers for the cases when one (or most, or all) of the avoided graphs is a cycle. The results commented on here are taken from a much broader general Ramsey numbers survey [Rad] by the author, which since 1994 has been updated periodically as a living article in the *Electronic Journal of Combinatorics*. Thus, while the results and data gathered here are subsumed by the August 2009 revision #12 of [Rad], the latter has only minimal comments associated with the results. This is remedied here. Furthermore, it

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A. Soifer (ed.), *Ramsey Theory: Yesterday, Today, and Tomorrow*,
Progress in Mathematics 285, DOI 10.1007/978-0-8176-8092-3_3,

seems that recent years have brought new vigorous attention of many researchers, especially to the cases involving cycles, to the extent that now it merits its own overview. For the ease of use, and to avoid potential confusion, we employ the same labels of references as in [Rad], even in cases when it forces us to use nonconsecutive labels, although doing so would seem in order. Similarly, the definitions and notation of this survey are entirely those from [Rad]. For deeper exposition of the basic concepts and intuition behind them consult the main surveys of this volume, and [GRS] or [Soi].

We do not attempt complete coverage of asymptotic results on Ramsey numbers with cycles, but rather concentrate on cases where exact formulas or concrete values have been obtained or significant work towards them was done. Hence, only the main facts on asymptotic behavior are presented, but with many pointers to further literature. The complete graph on three vertices and the cycle of length 3 is the same graph, $K_3 = C_3$. The study of $K_3$ in the context of Ramsey numbers is very rich in itself and is often considered separately from longer cycles. Here, we point to the results involving $C_3$ mainly in the context of longer cycles. Also, the bipartite graph $K_{2,2}$ is the same as quadrilateral $C_4$, and many papers discuss $C_4$ implicitly under the header of bipartite graphs. Similarly as for triangles, but now to a lesser extent, we decided to skip a number of results originating in bipartite graph theory. The surveys mentioned above give many more pointers to results on $K_3$ and $K_{2,2}$, and thus indirectly also for cycles.

Let $G_1, G_2, \ldots, G_m$ be graphs or $s$-uniform hypergraphs ($s$ is the number of vertices in each edge). $R(G_1, G_2, \ldots, G_m; s)$ denotes the $m$-color Ramsey number for $s$-uniform graphs/hypergraphs, avoiding $G_i$ in color $i$ for $1 \leq i \leq m$. It is defined as the least integer $n$ such that, in any coloring with $m$ colors of the $s$-subsets of a set of $n$ elements, for some $i$ the $s$-subsets of color $i$ contain a sub(hyper)graph isomorphic to $G_i$ (not necessarily induced). The value of $R(G_1, G_2, \ldots, G_m; s)$ is fixed under permutations of the first $m$ arguments.

If $s = 2$ (standard graphs) then $s$ can be omitted. The complete graph on $n$ vertices is denoted by $K_n$. If $G_i = K_k$, then we can write $k$ instead of $G_i$, and if $G_i = G$ for all $i$ we can use the abbreviation $R_m(G; s)$ or $R_m(G)$. For $s = 2$, $K_k - e$ denotes a $K_k$ without one edge. $P_i$ is a *path* on $i$ vertices, $C_i$ is a *cycle* of length $i$, and $W_i$ is a *wheel* with $i - 1$ spokes; that is, a graph formed by some vertex $x$, connected to all vertices of some cycle $C_{i-1}$, or $W_i = K_1 + C_{i-1}$. $K_{n,m}$ is a complete $n$ by $m$ bipartite graph, in particular $K_{1,n}$ is a *star* graph. The *book* graph $B_i = K_2 + \overline{K}_i = K_1 + K_{1,i}$ has $i + 2$ vertices, and can be seen as $i$ triangular pages attached to a single edge. Finally, for a graph $G$, let $nG$ stand for the graph formed by $n$ vertex disjoint copies of $G$.

## 2 Two-Color Numbers Involving Cycles

The history of knowledge of graph Ramsey numbers $R(G, H)$ seems to indicate that the difficulty of computing or estimating $R(G, H)$ increases with the density of edges in $G$. Thus, evaluation of the classical Ramsey numbers $R(k, l)$, when the avoided graphs are complete, is considered one of the hardest tasks, while we know significantly more when graphs $G$ and/or $H$ become sparse. An interesting famous case is formulated in the first theorem, which gives the exact value of $R(G, H)$ when $G$ and $H$ are such extremes. Of course, for all graphs $G$ and $H$, $R(G, H) = R(H, G)$.

**Theorem 1.** *(Chvátal [Chv], 1977)*
$R(T_n, K_m) = (n-1)(m-1) + 1$ *for any tree* $T_n$ *with* $n$ *vertices.*

Theorem 1 has a relatively easy proof. It holds in particular for $T_n = P_n$. Note that adding just one closing edge to $P_n$ forms a cycle $C_n$. However, perhaps surprisingly, the corresponding problem of Ramsey numbers $R(C_n, K_m)$ is far from being well understood. There has been remarkable progress in this area in the last 20 years, but still many of the basic questions remain open. We address them in more detail in Sect. 2.2.

## *2.1 Cycles*

Arguably the most widely known classical Ramsey number $R(3, 3)$ was mentioned implicitly by Bush [Bush] who reported that in the 1953 William Lowell Putnam Mathematical Competition, Question #2 in Part I asks for the proof of what can be denoted by $R(3, 3) \leq 6$. The 1955 paper by Greenwood and Gleason [GG] includes the result $R(3, 3) = 6$ with proofs. This is also the first reported case of cycle Ramsey numbers $R(C_3, C_3)$, since clearly $C_3 = K_3$. Chvátal and Harary [CH1] were the first to give the value $R(C_4, C_4) = 6$. The initial general result for cycles, $R(C_3, C_n) = 2n - 1$ for $n \geq 4$, was obtained by Chartrand and Schuster [ChaS] in 1971. The complete solution of the case $R(C_n, C_m)$ was obtained soon afterwards, independently by Faudree and Schelp [FS1] and Rosta [Ros1]. Both these proofs are somewhat complicated, however a new simpler proof by Károlyi and Rosta was published recently [KáRos], in 2001.

**Theorem 2.** *(Faudree, Schelp [FS1], 1974; Rosta [Ros1], 1973)*

$$R(C_n, C_m) = \begin{cases} 2n-1 & \text{for } 3 \leq m \leq n, m \textit{ odd}, (n, m) \neq (3, 3) \\ n - 1 + m/2 & \text{for } 4 \leq m \leq n, m \text{ and } n \textit{ even}, (n, m) \neq (4, 4) \\ \max\{n - 1 + m/2, 2m - 1\} & \text{for } 4 \leq m < n, m \textit{ even} \text{ and } n \textit{ odd} \end{cases}$$

Burr, Erdős, and Spencer [BES] in 1975 studied a variety of two color cases for multiple disjoint copies of several small graphs, and among those for short cycles. Their work includes a particularly elegant proof of $R(nC_3, mC_3) = 3n + 2m$ for $n \geq m \geq 1, n \geq 2$. This was extended by Li and Wang to $R(nC_4, mC_4) = 2n + 4m - 1$ for $m \geq n \geq 1$, $(n, m) \neq (1, 1)$ [LiWa1]. The same authors derived further formulas for $R(nC_4, mC_5)$ [LiWa2]. The general problem of $nC_m$, more formulas and bounds for various cases were studied also by Mizuno and Sato [MiSa], Denley [Den], Burr and Rosta [BuRo3], and Bielak [Biel1].

## 2.2 Cycles Versus Complete Graphs

The Ramsey numbers $R(C_n, K_m)$ pose different problems when different restricted relationships between $n$ and $m$ are assumed. For fixed $n = 3$ it becomes the study of the classical numbers $R(3, k)$, which attracted efforts of many researchers until in a 1995 breakthrough Kim proved that $R(3, k) = \Theta(k^2/\log k)$ [Kim] (for the history of this result see the chapter by Spencer [Spe3] in this volume). The exact values of $R(3, k)$ are known for $k \leq 9$ (see column $C_3$ of Table 1). Computation of the exact values for $k \geq 10$ is still elusive and well beyond known theoretical and computational methods. For more comments on the smallest open case $R(3,10)$ see the problem section of this volume. The other end of the problem seems to be much easier for fixed $m$. In particular, Theorem 2 gives $R(C_3, C_n) = 2n - 1$. Actually, this row of Table 1 seems to generalize to the following simple but apparently hard-to-prove conjecture.

*Conjecture 1.* (Faudree, Schelp [FS4], 1976)

$R(C_n, K_m) = (n-1)(m-1) + 1$ *for all* $n \geq m \geq 3$, *except* $n = m = 3$.

The authors of [EFRS2], while studying many similar problems, also restate the same conjecture. Over the last three decades there was a steady sequence of papers proving it for increasing sets of pairs of $n$ and $m$. The parts of Conjecture 1 were proved as follows.

First observe that the lower bound is easy, since the graph $(m-1)K_{n-1}$, formed by $m-1$ vertex-disjoint copies of $K_n$, clearly provides a witness for $R(C_n, K_m) > (n-1)(m-1)$, even without the exception $n = m = 3$. We note that this same construction similarly easily gives lower bounds for other cases in further sections of this survey.

**Table 1** Known Ramsey numbers $R(C_n, K_m)$ (Ch+ abbreviates ChenCZ1; see also comments on joint credits below)

| | $C_3$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ | $C_8$ | … | $C_n$ for $n \geq m$ |
|---|---|---|---|---|---|---|---|---|
| $K_3$ | 6<br>GG | 7<br>ChaS | 9<br>ChaS | 11<br>ChaS | 13<br>ChaS | 15<br>ChaS | … | $2n-1$<br>ChaS |
| $K_4$ | 9<br>GG | 10<br>CH2 | 13<br>He2/JR4 | 16<br>JR2 | 19<br>YHZ1 | 22<br>YHZ1 | … | $3n-2$<br>YHZ1 |
| $K_5$ | 14<br>GG | 14<br>Clan | 17<br>He2/JR4 | 21<br>JR2 | 25<br>YHZ2 | 29<br>BJYHRZ | … | $4n-3$<br>BJYHRZ |
| $K_6$ | 18<br>Kéry | 18<br>Ex2-RoJa1 | 21<br>JR5 | 26<br>Schi1 | 31<br>Schi1 | 36<br>Schi1 | … | $5n-4$<br>Schi1 |
| $K_7$ | 23<br>Ka2-GY | 22<br>RT-JR1 | 25<br>Schi2 | 31<br>CheCZN | 37<br>CheCZN | 43<br>JarBa/Ch+ | … | $6n-5$<br>ChenCZ1 |
| $K_8$ | 28<br>GR-MZ | 26<br>RT | | 36<br>ChenCX | 43<br>ChenCZ1 | 50<br>JarAl/ZZ3 | … | $7n-6$<br>conj. |
| $K_9$ | 36<br>Ka2-GR | 30–32<br>RT-XSR1 | | | | | … | $8n-7$<br>conj. |
| $K_{10}$ | 40–43<br>Ex5-RK2 | 34–39<br>RT-XSR1 | | | | | … | $9n-8$<br>conj. |

The hard part is to derive the upper bound. Bondy and Erdős proved it for $n \geq m^2 - 2$ [BoEr] in 1973; Chartrand and Schuster for $n > 3 = m$ [ChaS] in 1971; Yang, Huang, and Zhang for $n \geq 4 = m$ [YHZ1] in 1999; Bollobás et al. for $n \geq 5 = m$ [BJYHRZ] in 2000; Schiermeyer for $n \geq 6 = m$ and for $n \geq m \geq 7$ with $n \geq m(m-2)$ [Schi1] in 2003; Nikiforov for $n \geq 4m+2, m \geq 3$ [Nik] in 2005; and finally Chen, Cheng, and Zhang for $n \geq 7 = m$ [ChenCZ1] in 2008, All these developments, and a number of special small cases which had to be proved on the way, are summarized in Table 1. Still open conjectured cases are marked by "conj." The result $R(C_8, K_8) = 50$, which is a necessary starting point for confirming this conjecture for $m = 8$, was recently proved independently by Jaradat-Alzaleq [JarAl] and by Zhang-Zhang [ZZ3]. The proofs of the latter consist of quite intricate considerations of many subcases. Some new unifying approach to all rows of Table 1 would be very welcome. Let us also note that a stronger version of Conjecture 1 is likely true, since as one can now see in further rows of Table 1, the general formula holds even for some $m$ slightly larger than $n$.

Joint credit [He2/JR4] in Table 1 refers to two cases in which Hendry [He2] announced the values without presenting the proofs, which later were given in [JR4]. The special cases of $R(C_6, K_5) = 21$ [JR2] and $R(C_7, K_5) = 25$ were solved independently in [YHZ2] and [BJYHRZ]. The double pointer [JarBa/ChenCZ1] refers to two independent papers, similarly as [JarAl/ZZ3]. For joint credits marked in Table 1 with "-", the first reference is for the lower bound and the second for the upper bound.

Erdős et al. [EFRS2] asked what is the minimum value of $R(C_n, K_m)$ for fixed $m$. Interestingly, even without knowledge of most of the data gathered in Table 1, the authors suggested that it might be possible that $R(C_n, K_m)$ first decreases monotonically, then attains a unique minimum, then increases monotonically with $n$. What we now know, more than 30 years later, provides some strong evidence confirming their intuition.

For the columnwise (with fixed $n$) asymptotic behavior, beyond Kim's impressive result $R(3, k) = \Theta(k^2/\log k)$ [Kim] mentioned earlier and discussed in other chapters of this volume, we present in the next theorem the known bounds for $n = 4$.

**Theorem 3.** *([Spe2] 1977; 1980, [CLRZ] 2000)*

*There exist positive constants $c_1$ and $c_2$ such that*

$$c_1(m/\log m)^{3/2} \leq R(C_4, K_m) \leq c_2(m/\log m)^2.$$

The lower bound was obtained by Spencer [Spe2] using the probabilistic method. The upper bound was presented in a paper by Caro et al. [CRLZ], who in turn gave the credit to an unpublished work by Szemerédi from 1980. Erdős, in 1981, in the Ramsey problems section of the paper [Erd2] formulated a challenge by asking for a proof of $R(C_4, K_m) < m^{2-\varepsilon}$, for some $\varepsilon > 0$. Erdős placed this problem among the problems on which he "spent lots of time." No proof of this bound is known to date. The asymptotics of the general and odd $n$ cases of $R(C_n, K_m)$ were studied by several authors including [BoEr; FS4; EFRS2; Sud1; LiZa2; AlRö].

## 2.3 Cycles Versus Wheels

We remind the reader that in this survey the wheel graph $W_n = K_1 + C_{n-1}$ has $n$ vertices. This is different from some authors who use the definition $W_n = K_1 + C_n$ with $n + 1$ vertices. While in Table 2 of known small values of $R(W_n, C_m)$ this difference just shifts the values between adjacent rows, the general formulas are affected a bit more.

Both wheel and cycle graphs are sparse, so we expect that the corresponding Ramsey numbers $R(W_n, C_m)$ will be smaller and easier to compute. Indeed, the linear functions for all fixed $n$ and for fixed odd $m$, while the other parameter is large enough, are known. Yet proving the ranges for which these linear functions hold, and finding the concrete values for small cases, seem to be quite independent and challenging tasks. We gather what is known in Table 2 below, and then comment on some of the results therein as well as point to some open problems.

Since $W_4 = K_4$, the first data row in Table 2 is the same as the second data row of Table 1 for $K_4$. Similarly, the first row of Table 1 for $K_3 = W_3$ could be prepended to Table 2 as is, but we didn't do it for the sake of brevity. As in Table 1, the rows of Table 2 are easier to deal with for large $m$. Similarly, the full solutions for $n = 3$, 4 are the same as for $R(K_n, C_m)$, which were given in the previous section. The almost complete general rowwise solution is presented in Theorem 4.

**Theorem 4.** *([SuBT1, ZhaCC, ChenCN])*

*(a)* $R(W_n, C_m) = 3m - 2$ *for even* $n \geq 4$ *with* $m \geq n - 1$, $m \neq 3$,
*(b)* $R(W_n, C_m) = 2m - 1$ *for odd* $n \geq 3$ *with* $2m \geq 5n - 7$.

Theorem 4(a) was conjectured in a few papers by Surahmat et al. [SuBT1; SuBT2; Sur]. Parts of this conjecture were proved in [SuBT1; ZhaCC], and the

**Table 2** Ramsey numbers $R(W_n, C_m)$, for $n \leq 9$, $m \leq 8$ (results from unpublished manuscript are marked*)

| | $C_3$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ | $C_8$ | $C_m$ | for |
|---|---|---|---|---|---|---|---|---|
| $W_4$ | 9<br>GG | 10<br>CH2 | 13<br>He2 | 16<br>JR2 | 19<br>YHZ1 | 22<br>... | $3m-2$<br>... | $m \geq 4$<br>YHZ1 |
| $W_5$ | 11<br>Clan | 9<br>Clan | 9<br>Hc4 | 11<br>JR2 | 13<br>SuBB2 | 15<br>... | $2m-1$<br>... | $m \geq 5$<br>SuBB2 |
| $W_6$ | 11<br>BE3 | 10<br>JR3 | 13<br>ChvS | 16<br>SuBB2 | 19<br>... | 22<br>... | $3m-2$<br>... | $m \geq 4$<br>SuBB2 |
| $W_7$ | 13<br>BE3 | 9<br>Tse1 | | | | | $2m-1$ | $m \geq 14$<br>SuBT1 |
| $W_8$ | 15<br>BE3 | 11<br>Tse1 | | | 19*<br>ChenCN | 22*<br>... | $3m-2$*<br>... | $m \geq 7$<br>ChenCN |
| $W_9$ | 17<br>BE3<br>... | 12<br>Tse1 | | | | | $2m-1$ | $m \geq 19$<br>SuBT1 |
| $W_n$<br>for | $2n-1$<br>$n \geq 6$<br>BE3 | | $2n-1$<br>$n \geq 19$<br>Zhou2 | | $2n-1$<br>$n \geq 29$<br>Zhou2 | | Large<br>Wheels | |

proof was completed by Chen, Cheng, and Ng [ChenCN] in 2009. Theorem 4(b) was proved in 2006 by Surahmat, Baskoro and Tomescu [SuBT1], but Surahmat conjectured that it also holds for odd $n \geq 3$ with $m \geq 5$ and $m > n$ [Sur]. The latter stronger version of (b) remains open.

With fixed $m$ the analysis seems harder. We give only one result for odd $m$.

**Theorem 5.** *(Zhou [Zhou2], 1995)*

$R(W_n, C_m) = 2n - 1$ *for odd* $m$ *with* $n \geq 5m - 6$.

The special case of Theorem 5 with $m = 3$ was obtained by Burr and Erdős [BE3] in 1983. For these Ramsey numbers even a general formula for the number of critical graphs has been derived; in particular the critical graphs for $R(W_n, C_3)$ are unique for $n = 3, 5$, and for no other $n$ [RaJi]. The next column for $m = 4$ already poses open questions, both regarding concrete small values and the behavior for large $n$. Only an upper bound $R(C_4, W_n) \leq n + \lceil (n-1)/3 \rceil$ for $n \geq 7$ was obtained in [SuBUB]. Furthermore, besides the values recorded in Table 2, it is known that $R(C_4, W_n) = 13, 14, 16, 17$ for $n = 10, 11, 12, 13$, respectively [Tse1].

Finally, we note that the formula for Ramsey numbers involving $C_m$ again depends on the parity of $m$. Since $C_m$ is a subgraph forming much of the wheel, it should be no surprise that in the case of $R(W_n, C_m)$ we need to consider four distinct situations with respect to parity of $n$ and $m$. We present more such dependencies on parity in further sections as well.

## *2.4 Cycles Versus Books*

We recall that the book graph of $n$ triangular pages is defined as $B_n = K_2 + \overline{K}_n$. The book-complete and book-book Ramsey numbers have been studied extensively, and we direct the reader to the survey [Rad] for related results. The somewhat less overall studied case of book-cycle numbers, however, has attracted much recent attention. In this section we overview known results about the Ramsey numbers $R(B_n, C_m)$.

Since $B_1 = K_3$, the cases of Ramsey numbers for $B_1$ versus $C_m$ are the same as those for $R(K_3, C_m)$ presented in Sect. 2.2. The case of $B_2 = K_4 - e$ versus $C_m$ is completely solved: small cases were given by different authors as marked in the first row of Table 3, and the general case was solved in the 1978 and 1991 papers by Faudree, Rousseau, and Sheehan [FRS6] and [FRS8] (abbreviated in Table 3 as F*al*6 and F*al*8). Actually, in [FRS8], extending the results of [FRS6], the authors proved some theorems shedding much light on other more general cases. The main results are presented as Theorems 6 and 7. Note that now we have distinct cases only with respect to the parity of $m$.

**Theorem 6.** *(Faudree, Rousseau, Sheehan [FRS8], 1991)*

*(a)* $R(B_n, C_m) = 2m - 1$ *for* $n \geq 1$, $m \geq 2n + 2$,
*(b)* $R(B_n, C_m) = 2n + 3$ *for odd* $m \geq 5$ *with* $n \geq 4m - 13$.

**Table 3** Ramsey numbers $R(B_n, C_m)$ for $n, m \leq 11$ (using et al. abbreviations, *Fal* for FRS and *Cal* for CRSPS)

| | $C_3$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ | $C_8$ | $C_9$ | $C_{10}$ | $C_{11}$ | $C_m$ | for |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $B_2$ | 7<br>RS1 | 7<br>F*al*6 | 9<br>*Cal* | 11<br>F*al*8 | 13<br>... | 15 | 17 | 19 | 21 | $2m-1$<br>... | $m \geq 2$<br>F*al*8 |
| $B_3$ | 9<br>RS1 | 9<br>F*al*6 | 10<br>F*al*8 | 11<br>JR2 | *13* | 15<br>F*al*8 | 17<br>... | 19 | 21 | $2m-1$<br>... | $m \geq 8$<br>F*al*8 |
| $B_4$ | 11<br>RS1 | 11<br>F*al*6 | 11<br>F*al*8 | *12* | *13* | *15* | *17* | 19<br>F*al*8 | 21<br>... | $2m-1$<br>... | $m \geq 10$<br>F*al*8 |
| $B_5$ | 13<br>RS1 | 12<br>F*al*6 | 13<br>F*al*8 | *14* | *15* | *15* | *17* | *19* | *21* | $2m-1$ | $m \geq 12$<br>F*al*8 |
| $B_6$ | 15<br>RS1 | 13<br>F*al*6 | 15<br>F*al*8 | *16* | *17* | *18* | *18* | | *21* | $2m-1$ | $m \geq 14$<br>F*al*8 |
| $B_7$ | 17<br>RS1 | 16<br>F*al*6 | 17<br>F*al*8 | *16* | *19* | *20* | *21* | | | $2m-1$ | $m \geq 16$<br>F*al*8 |
| $B_8$ | 19<br>RS1 | 17<br>Tse1 | 19<br>F*al*8 | *17* | *19* | *22* | $\geq$ *23* | | | $2m-1$ | $m \geq 18$<br>F*al*8 |
| $B_9$ | 21<br>RS1 | 18<br>Tse1 | 21<br>F*al*8 | *18* | | | $\geq$ *25* | $\geq$ *26* | | $2m-1$ | $m \geq 20$<br>F*al*8 |
| $B_{10}$ | 23<br>RS1 | 19<br>Tse1 | 23<br>F*al*8 | *19* | | | | $\geq$ *28* | | $2m-1$ | $m \geq 22$<br>F*al*8 |
| $B_{11}$ | 25<br>RS1 | 20<br>Tse1 | 25<br>F*al*8 | | | | | | | $2m-1$ | $m \geq 24$<br>F*al*8 |
| | ... | | ... | | | | | | | | |
| $B_n$<br>for | $2n+3$<br>$n \geq 2$<br>RS1 | $\approx n$<br>Some | $2n+3$<br>$n \geq 4$<br>F*al*8 | | $2n+3$<br>$n \geq 15$<br>F*al*8 | | $2n+3$<br>$n \geq 23$<br>F*al*8 | | $2n+3$<br>$n \geq 31$<br>F*al*8 | Large<br>books | |

The centered entries in italics in the middle of Table 3 are from personal communication and manuscripts by Shao. The latter also include proofs of inequalities $R(B_n, C_n) \geq 3n-2$, $R(B_{n-1}, C_n) \geq 3n-4$ for $n \geq 3$, and an improvement to the bound on $m$ in Theorem 6(a) to $m \geq 2n-1 \geq 7$ [Zehui Shao, 2008, personal communication].

The columnwise situation is more difficult. Theorem 6(b) gives the values for odd $m$ and $n$ large enough, but likely the range of $n$ for which it holds can be extended. The special case for $m = 3$ was solved completely by Rousseau and Sheehan [RS1], and that for $m = 5$ is included in [FRS8]. For even $m$, already the smallest case of $C_4$ is very difficult, since it is related to the existence of certain combinatorial designs. In particular, Faudree, Rousseau, and Sheehan in 1978 proved the following Theorem 7. More theorems about asymptotics and bounds on $R(B_n, C_m)$ can be found in the papers [NiRo4, Zhou1].

**Theorem 7.** *(Faudree, Rousseau, Sheehan [FRS6], 1978)*

*For any prime power* $q$, $q^2+q+2 \leq R(C_4, B_{q^2-q+1}) \leq q^2+q+4$.

The authors of [FRS6] characterize the special conditions under which the so-called locally friendly graphs, whose existence is in question for larger $q$, are witnesses that the upper bound of Theorem 7 holds exactly. Since $B_n$ is a subgraph of $B_{n+1}$, hence likely $R(C_4, B_n) = n + O(\sqrt{n})$. This would be similar to the behavior of $R(C_4, K_{2,n})$ (see Sect. 3.2 of [Rad]). Finally we note that, besides the values recorded in Table 3 for $m = 4$, Tse obtained the exact values $R(C_4, B_{12}) = 21$ [Tse1], $R(C_4, B_{13}) = 22$, and $R(C_4, B_{14}) = 24$ [Tse2], using computer algorithms.

## *2.5 Cycles Versus Other Graphs*

Technically $C_3$ and $C_4$ are cycle graphs, yet in graph theory, and in particular in Ramsey theory, they are very often seen as $K_3$ and a special bipartite graph $K_{2,2}$, respectively. For example, numerous papers whose references are gathered in Sects. 3.2 and 4.8 of the survey [Rad] consider the Ramsey numbers involving them in the context of $K_3$ and complete bipartite graphs. First we present a sample result from this area concerning quadrilateral-star numbers $R(K_{2,2} = C_4, K_{1,n})$. The value of the latter, in other words, is 1 plus the order of the largest $C_4$-free graph whose complement has the maximum degree less than $n$.

**Theorem 8.** *(Parsons [Par3], 1975; Burr et al. [BEFRS5], 1989)*

*(a)* $n + \sqrt{n} - 6n^{11/40} \le R(C_4, K_{1,n}) = f(n) \le n + \sqrt{n} + 1$, *and*
*(b) For every prime power* $q$, $f(q^2) = q^2 + q + 1$ *and* $f(q^2 + 1) = q^2 + q + 2$.

While Theorem 8 gives pretty good bounds on $R(K_{2,2}, K_{1,n})$, many concrete cases are still evasive. For more bounds and values of $f(n)$ see [Par4; Par5; Chen; ChenJ; GoMC; MoCa; HaMe4]. For the results on $C_4$ versus trees consult [EFRS4; Bu7; BEFRS5; Chen], and for many other results involving bipartite graphs refer to Sect. 3.2 of [Rad], in particular to several papers by Lortz et al. referenced there. For general cases of cycles versus stars consult [Clark; Par6], cycles versus trees [BEFRS2; FSS1], and cycles versus $K_{n,m}$ and multipartite complete graphs [BoEr].

Next we present a solution to the basic problem of cycles versus paths. Note similarity of the formula of Theorem 9 to that in the cycle–cycle problem of Theorem 2. Some small specific subcases derived earlier by other authors are listed in [FLPS].

**Theorem 9.** *(Faudree, Lawrence, Parsons, and Schelp [FLPS], 1974)*

$$R(P_n, C_m) = \begin{cases} 2n-1 & \textit{for} \quad 3 \le m \le n, m \textit{ odd}, \\ n-1+m/2 & \textit{for} \quad 4 \le m \le n, m \textit{ even}, \\ \max\{m-1+\lfloor n/2 \rfloor, 2n-1\} & \textit{for} \quad 2 \le n \le m, m \textit{ odd}, \\ m-1+\lfloor n/2 \rfloor & \textit{for} \quad 2 \le n \le m, m \textit{ even}. \end{cases}$$

The classical result by Gerencsér and Gyárfás [GeGy] gives a formula for path numbers $R(P_n, P_m) = m + \lfloor n/2 \rfloor - 1$, for all $m \geq n \geq 2$. It is tempting to compare it in detail to Theorems 2 and 9. Merging together the conditions in the three formulas is routine but somewhat tedious. Obviously, for all $n$ and $m$ it holds that $R(P_n, P_m) \leq R(P_n, C_m) \leq R(C_n, C_m)$. Each of the two inequalities can become an equality, and, as derived in [FLPS], all four possible combinations of $<$ and $=$ hold for an infinite number of pairs $(n, m)$. For example, if both $n$ and $m$ are even, and at least one of them is greater than 4, then $R(P_n, P_m) = R(P_n, C_m) = R(C_n, C_m)$. The full specification of four cases would require several more lines of details.

Between 1997 and 2004, Rousseau and Jayawardene wrote several papers concerning Ramsey numbers for short cycles versus other graphs, where besides general theoretical results they computed many new exact values of $R(C_m, G)$ for specific $G$s (some of them were pointed to in Tables 1–2). They collected a large set of data which can give insights into new general claims (or refute them); namely, they found the values of Ramsey numbers for $C_4$ versus all graphs on six vertices [JR3], $C_5$ versus all graphs on six vertices [JR4], and $C_6$ versus all graphs on five vertices [JR2]. In addition, unfortunately only in an unpublished manuscript [RoJa2], the authors gave interesting upper bounds: $R(C_4, G) \leq 2q + 1$ for any isolate-free graph $G$ with $q$ edges, and $R(C_4, G) \leq p + q - 1$ for any connected graph $G$ on $p$ vertices and $q$ edges. In a similar direction, Burr et al. [BEFRS2] proved the equality $R(C_{2m+1}, G) = 2n - 1$ for sufficiently large sparse graphs $G$ on $n$ vertices, in particular $R(C_{2m+1}, T_n) = 2n - 1$ for all $n > 1512m + 756$, for $n$ vertex trees $T_n$.

## 3 Multicolor Numbers for Cycles

### *3.1 Three Colors*

The first larger paper in this area by Erdős, Faudree, Rousseau, and Schelp [EFRS1] appeared in 1976. It gives some formulas and bounds for multicolor Ramsey numbers of several simple graphs, including those for $R(C_m, C_n, C_k)$ and $R(C_m, C_n, C_k, C_l)$ for large $m$. The case of three colors is presented in Theorem 10.

**Theorem 10.** *(Erdős et al. [EFRS1], 1976)*
*For $m$ large enough all of the following hold.*

*(a)* $R(C_m, C_{2p+1}, C_{2q+1}) = 4m - 3$ *for* $p \geq 2$, $q \geq 1$,
*(b)* $R(C_m, C_{2p}, C_{2q+1}) = 2(m + p) - 3$,
*(c)* $R(C_m, C_{2p}, C_{2q}) = m + p + q - 2$ *for* $p, q \geq 1$.

The three-color case is thus clear when one of the cycles is sufficiently long. The situation gets harder when we are closer to the diagonal. Several such cases which were solved for concrete small parameters, mostly with the help of computer algorithms, are listed in Table 4. The diagonal itself (i.e., the cases of $R_3(C_m)$) were studied more and thus we know more there. The papers referenced in Theorems 11 and 12, and [GyRSS], used the powerful Szemerédi's regularity lemma [Szem] to prove the upper bounds. We present the main results in this direction in the sequel.

**Theorem 11.** *(Triple even cycles)*

*(a) Figaj and Łuczak [FiŁu1], 2007.*

$$R(C_{2\lfloor\alpha_1 n\rfloor}, C_{2\lfloor\alpha_2 n\rfloor}, C_{2\lfloor\alpha_3 n\rfloor})$$
$$= (\alpha_1 + \alpha_2 + \alpha_3 + \max\{\alpha_1, \alpha_2, \alpha_3\} + o(1))n, \ for\ all\ \alpha_1, \alpha_2, \alpha_3 > 0.$$

*In particular, for even n, we have $R(C_n, C_n, C_n) = (2 + o(1))n$.*
*(b) Benevides and Skokan [BenSk], 2009.*

$$R(C_n, C_n, C_n) = 2n \quad \textit{for all sufficiently large } n.$$

Observe that (b) is an improvement of the second part of (a). We also note that Theorem 11(a) implies (cf. Corollary 2 in [FiŁu1]) a solution to the related long-standing open problem for paths, namely that $R(P_m, P_n, P_k) = m + (n + k)/2 + o(m)$ for $m \geq n, k$. By now, we know even more. In a recent large paper Gyárfás et al. [GyRSS] were able to prove the exact diagonal result for long triple paths. They proved that an amazingly simple formula $R(P_n, P_n, P_n) = 2n - 2 + n \bmod 2$ holds for all sufficiently large $n$. In a not yet published paper Figaj and Łuczak [FiŁu2] extend their result to triples of cycles of mixed parity, obtaining asymptotic values similar in form to the formula of Theorem 11(a).

A lower bound $R_3(C_{2m}) \geq 4m$ for all $m \geq 2$ follows from a more general construction by Dzido, Nowik, and Szuca [DzNS], which is valid for any number of colors (see Sect. 3.2). For small $n$, only the case $R_3(C_4) = 11$, solved by Bialostocki and Schönheim [BS] in 1984 by using elegant edge counting reasoning, seems to be special. The other two known exact values, $R_3(C_6) = 12$ obtained by Yang and Rowlinson [YR2] in 1993 and $R_3(C_8) = 16$ by Sun [Sun] in 2006, already required an intensive use of computations. These two cases follow the pattern proved for large $n$, so it seems reasonable to pose the following Conjecture 2, which was actually done by Dzido [Dzi1]. The first currently open case is that of $R_3(C_{10})$. In order to settle it (as for all other open cases) one only needs to prove the upper bound $R_3(C_{10}) < 21$, since from the construction in [DzNS] we know that $R_3(C_{10}) \geq 20$.

*Conjecture 2.* (Triple even cycles, Dzido [Dzi1], 2005)
$R(C_n, C_n, C_n) = 2n$ for all even $n \geq 6$.

For the case of three odd cycles we begin with the well-known conjecture by Bondy and Erdős.

*Conjecture 3.* (Triple odd cycles, Bondy and Erdős, cf. [Erd2], 1981)
$R(C_n, C_n, C_n) = 4n - 3$ for all odd $n \geq 5$.

Now the situation is somewhat different, although still Szemerédi's regularity lemma (RL) played a critical role in establishing the upper bound. The following Theorem 12 confirms Conjecture 3 for large $n$, however, the derivation of really how large $n$ needs to be is difficult because of the use of RL.

**Theorem 12.** *(Triple odd cycles, Kohayakawa, Simonovits, Skokan [KoSS], 2005, 2009) $R(C_n, C_n, C_n) = 4n - 3$ for all sufficiently large odd n.*

We note that an equivalent formulation of the last theorem could be $R_3(C_{2m+1}) = 8m+1$ for all sufficiently large $m$. Theorem 12 improves a well-known Łuczak's result stating that $R(C_n, C_n, C_n) \leq (4+o(1))n$, with equality for odd $n$ [Łuc]. As observed by Erdős [Erd2] we really only need to prove the upper bound of Conjecture 3 (as in Conjecture 2), since the lower bound is easy. A classical case of $R_3(C_3) = 17$ [GG] is special, but the other two known exact initial values follow the pattern of Conjecture 3: $R_3(C_5) = 17$ obtained with computations by Yang and Rowlinson [YR1] in 1992, and an equality $R_3(C_7) = 25$ proved by Faudree,

**Table 4** Ramsey numbers $R(C_m, C_n, C_k)$ for $m$, $n$, $k \leq 7$ and $m = n = k = 8$ (Sun1+ abbreviates SunYWLX; Sun2+ abbreviates SunYLZ2; the work in [SunYWLX] and [SunYLZ2] is independent from [Tse3])

| $mnk$ | $R(C_m, C_n, C_k)$ | References | General results |
|---|---|---|---|
| **3 3 3** | **17** | GG | See [Rad] page 29 |
| 3 3 4 | 17 | ExRe | |
| 3 3 5 | 21 | Sun1+/Tse3 | $5k-4$ for $k \geq 5$, $m = n = 3$ [Sun1+] |
| 3 3 6 | 26 | Sun1+ | |
| 3 3 7 | 31 | Sun1+ | |
| 3 4 4 | 12 | Schu | |
| 3 4 5 | 13 | Sun1+/Rao/Tse3 | |
| 3 4 6 | 13 | Sun1+/Tse3 | |
| 3 4 7 | 15 | Sun1+/Tse3 | |
| 3 5 5 | ≥17 | Tse3 | |
| 3 5 6 | 21 | Sun1+ | |
| 3 5 7 | 25 | Sun1+ | |
| 3 6 6 | | | |
| 3 6 7 | 21 | Sun1+ | |
| 3 7 7 | | | |
| **4 4 4** | **11** | BS | |
| 4 4 5 | 12 | Sun2+/Tse3 | |
| 4 4 6 | 12 | Sun2+/Tse3 | $k+2$ for $k \geq 11$, $m = n = 4$ [Sun2+] |
| 4 4 7 | 12 | Sun2+/Tse3 | 12, 13, 13 for k = 8, 9, 10 [Sun2+] |
| 4 5 5 | 13 | Tse3 | |
| 4 5 6 | 13 | Sun1+ | |
| 4 5 7 | 15 | Sun1+ | Compare Theorem 10(b) |
| 4 6 6 | 11 | Tse3 | |
| 4 6 7 | 13 | Sun1+/Tse3 | |
| 4 7 7 | | | |
| **5 5 5** | **17** | YR1 | |
| 5 5 6 | 21 | Sun1+ | |
| 5 5 7 | 25 | Sun1+ | |
| 5 6 6 | | | |
| 5 6 7 | 21 | Sun1+ | |
| 5 7 7 | | | |
| **6 6 6** | **12** | YR2 | $R_3(C_{2q}) \geq 4q$ for $q \geq 2$ [DzNS] |
| 6 6 7 | 15 | Sun1+ | Compare Theorem 10(c) |
| 6 7 7 | | | Compare Theorem 10(a) |
| **7 7 7** | **25** | FSS2 | $R_3(C_{2q+1}) = 8q+1$ for large $q$ [KoSS] |
| **8 8 8** | **16** | Sun | $R_3(C_{2q}) = 4q$ for large $q$ [BenSk] |

Schelten, and Schiermeyer [FSS2] in 2003. The latter did not require any computer-supported computations, however, the proof is long and complicated. The first currently open case is that of $R_3(C_9)$. As in the even case, to solve it one only needs to show $R_3(C_9) \leq 33$.

Two interesting exact results for triple cycles were obtained by Sun et al., namely $R(C_3, C_3, C_k) = 5k - 4$ for $k \geq 5$ [SunYWLX], and $R(C_4, C_4, C_k) = k + 2$ for $k \geq 11$ [SunYLZ2]. All exceptions to these formulas for small $k$ are listed in Table 4. Such results are definitely important steps towards a very difficult, but perhaps achievable, goal of exact knowledge of all three-color Ramsey numbers for cycles. Almost all of the off-diagonal cases in Table 4 required the use of computer algorithms.

### 3.2 More Colors

For more than three colors, we first present all known nontrivial concrete results, except those for $R_k(C_3) = R_k(K_3)$ which typically belong to the study of cases involving triangle $K_3$ (see Sects. 5.1 and 5.2 of [Rad]). Two of the cases listed below, namely those of four-color $C_4$ and five-color $C_6$, required large-scale computations to prove the upper bound.

| | |
|---|---|
| $R_4(C_4) = 18$ | [Ex2] [SunYLZ1] |
| $18 \leq R_4(C_6)$ | [SunYJLS] |
| $27 \leq R_5(C_4) \leq 29$ | [LaWo1] |
| $R_5(C_6) = 26$ | [SunYJLS] [SunYW] |
| $21 \leq R(C_3, C_4, C_4, C_4) \leq 27$ | [XuR2] |
| $28 \leq R(C_3, C_3, C_4, C_4) \leq 36$ | [XuR2] |
| $49 \leq R(C_3, C_3, C_3, C_4)$ | see 5.6.n in [Rad] |

General formulas for $R(C_m, C_n, C_k, C_l)$, for large $m$ [EFRS1], were obtained in the same paper as the results listed in Theorem 10 for three colors. The results there for four colors are also quite similar in form, but significantly more complicated. For three colors there was steady follow-up work for the off-diagonal cases reported in the previous section, but it has yet to be done for more colors. However, the diagonal did attract attention: the study of $R_k(C_m)$, and in particular of the special case $R_k(C_4)$.

In the mid-seventies, Irving [Ir], Chung [Chu2], and Chung-Graham [ChGral] established that $R_k(C_4) \leq k^2 + k + 1$ for all $k \geq 1$, and $R_k(C_4) \geq k^2 - k + 2$ for all $k - 1$ which is a prime power. In 2000, Lazebnik and Woldar [LaWo1] improved the lower bound to $R_k(C_4) \geq k^2 + 2$ for odd prime power $k$, and finally the latter was extended to any prime power $k$ by Ling [Ling] and Lazebnik-Mubayi [LaMu].

We summarize the bounds for $k$-color diagonal cases of even cycles in the following Theorem 13. Interestingly, all six claims below cover different situations and each is best in some respect (say, each of (b), (c), (d) is better than the other two for some $k$ and $m$).

**Theorem 13.** *(Multicolor even cycles)*
*All of the following hold.*

*(a)* $R_k(C_{2m}) \geq (k+1)m$ *for odd k and* $m \geq 2$ [DzNS],
*(b)* $R_k(C_{2m}) \geq (k+1)m - 1$ *for even k and* $m \geq 2$ [DzNS],
*(c)* $R_k(C_{2m}) \geq 2(k-1)(m-1) + 2$ [SunYXL],
*(d)* $R_k(C_{2m}) \geq k^2 + 2m - k$ *for* $2m \geq k+1$ *and prime power k* [SunYJLS],
*(e)* $R_k(C_{2m}) = \Theta(k^{m/(m-1)})$ *for fixed* $m = 2, 3$ *and 5* [LiLih],
*(f)* $R_k(C_{2m}) \leq 201km$ *for* $k \leq 10^m/201m$ [EG].

For multicolor odd cycle diagonal Ramsey numbers the most commonly cited bounds are those from the 1973 paper by Bondy and Erdős [BoEr]:

$$2^k m < R_k(C_{2m+1}) \leq (k+2)!(2m+1).$$

The lower bound follows again from natural canonical colorings. A somewhat better upper bound $R_k(C_{2m+1}) < 2(k+2)!m$ was obtained by Erdős and Graham [EG], but there is likely much more room for further improvements. This has been accomplished for the special case of $C_5$ by Li [Li3] in a recent exciting development.

**Theorem 14.** *(Li [Li3], 2009)*

$$R_k(C_5) \leq \sqrt{18^k k!}/10 \quad \textit{for all } k \geq 3.$$

More discussion of asymptotic bounds for $R_k(C_n)$ can be found in the papers [Bu1; GRS; ChGra2; LiLih]. There is still much to do. In particular, we know very little about upper bounds on $R_k(C_n)$. We also recommend the 2008 survey paper of multicolor cycle cases by Li [Li2], which nicely complements the discussion of this paper, in particular with respect to asymptotic bounds.

## *3.3 Cycles Versus Other Graphs*

Similarly as in previous sections, it seems easier to proceed when the length of one cycle parameter is large enough. Erdős et al. [EFRS1] (this paper also contains the proof of Theorem 10) in 1976 studied the cases of $R(C_n, K_{t_1}, \ldots, K_{t_k})$ and $R(C_n, K_{t_1,s_1}, \ldots, K_{t_k,s_k})$ for large $n$. When the cycle lengths are kept fixed, the techniques needed are different. Alon and Rödl [AlRö] in 2005 obtained a surprising asymptotic result that for more colors involving $C_4$, and in general even cycles, the problem is more manageable. Some similar results in this direction were obtained in [ShiuLL]. For the numbers $R(C_4, K_n)$ the bounds of Theorem 3 are quite far apart, while the next theorem settles the exact asymptotics for more colors. The paper [AlRö] includes several other asymptotic results, including those for $K_3$ and other even cycles instead of $C_4$.

**Theorem 15.** *(Alon, Rödl [AlRö], 2005)*
*For three colors* $R(C_4, C_4, K_n) = \Theta(n^2 \text{poly} - \log n)$,
*and for more colors* $R\,(C_4, C_4, \ldots, C_4, K_n) = \Theta(n^2/\log^2 n)$.

Despite known exact asymptotics, we have a rather poor understanding of small cases for this type of numbers. Below we list the bounds established in [XSR1] for the mixed cases involving $C_3$, $C_4$, and $K_4$ (see also the bounds from [XuR2] given in Sect. 3.2). The lower bounds were obtained by a few different constructions, in contrast to several other numbers involving cycles for which the natural canonical colorings are normally used. The upper bounds follow from known bounds on the maximum number of edges in $C_4$-free graphs and known bounds for smaller Ramsey numbers. That's almost all of what we know for this type of concrete numbers. We challenge the reader to improve any of the following bounds.

$$19 \le R(C_4,\ C_4,\ K_4) \le 22, \qquad 31 \le R(C_4,\ C_4,\ K_4) \le 50,$$
$$25 \le R(C_3,\ C_4,\ K_4) \le 32, \qquad 42 \le R(C_3,\ C_4,\ K_4) \le 76,$$
$$52 \le R(C_4,\ K_4,\ K_4) \le 72, \qquad 87 \le R(C_4,\ K_4,\ K_4) \le 179,$$

We end the section on multicolor cycle numbers with a compilation of some promising initial exact results for three colors concerning a mixture of cycles and paths. For two paths and a cycle it is known that $R(P_3, P_3, C_m) = m$ and $R(P_3, P_4, C_m) = m + 1$ for $m \ge 6$ [Dzi2], $R(P_4, P_4, C_m) = m + 2$ for $m \ge 6$ and $R(P_3, P_5, C_m) = m + 1$ for $m \ge 8$ [DzKP], $R(P_4, P_5, C_m) = m + 2$ for $m \ge 23$ and $R(P_4, P_6, C_m) = m + 3$ for $m \ge 18$ [ShaXSP], and $R(P_m, P_n, C_k) = 2n + 2\lfloor m/2 \rfloor - 3$ for large $n$ and odd $m \ge 3$ [DzFi2]. For two cycles and a path we know that $R(P_3, C_m, C_m) = R(C_m, C_m) = 2m - 1$ for odd $m \ge 5$ [DzKP]. Most of the small cases not covered by the above formulas are listed in [Rad]. Also, Dzido and Fidytek [DzFi2] presented a table of exact values of $R(P_3, P_k, C_m)$ for all $3 \le k \le 8$ and $3 \le m \le 9$. All this may help in launching new conjectures for three-color numbers involving these most basic mixed parameters.

## 4 Hypergraph Numbers for Cycles

We close this survey with some interesting recent results on hypergraph Ramsey numbers for so-called loose and tight cycles. A *loose* three-uniform cycle $C_n$ on the set $[n] = \{1, 2, \ldots, n\}$ is the set of triples $\{123, 345, 567, \ldots, (n-1)n1\}$, forming a cycle with an overlap of consecutive edges of exactly one point. Note that for loose cycles $n$ must be even. A three-uniform cycle $C_n$ formed by $\{123, 234, 345, \ldots (n-1)n1, n12\}$, in which consecutive edges share two points, is called *tight*. Loose and tight paths are defined similarly.

For such loose cycles, Haxell et al. [HaŁP1+] proved that $R(C_{4k}, C_{4k}; 3) > 5k - 2$ and $R(C_{4k+2}, C_{4k+2}; 3) > 5k + 1$. Furthermore, asymptotically these lower bounds are tight. Generalizations to $r$-uniform hypergraphs and graphs other than cycles were studied in [GySS].

For tight cycles, Haxell et al. [HaŁP2+] proved that $R(C_{3k}, C_{3k}; 3) \approx 4k$ and $R(C_{3k+i}, C_{3k+i}; 3) \approx 6k$ for $i = 1$ or 2. For tight paths the same paper establishes $R(P_k, P_k; 3) \approx 4k/3$. We finally note that the tetrahedron, or four triples on the set of four points, is a tight three-uniform hypergraph cycle $C_4$. The corresponding Ramsey number, $R(4, 4; 3) = 13$ [MR1], is the only nontrivial classical Ramsey number for hypergraphs whose exact value is known.

Papers containing results obtained with the help of computer algorithms have been marked with stars. The references are ordered alphabetically by the last name of the first author, and where multiple papers have the same first author they are ordered by the last name of the second author, etc. References' labels are the same as in [Rad] if they appear in this much more extensive survey, and thus numerical labels for some entries can be nonconsecutive.

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# On the Function of Erdős and Rogers

Andrzej Dudek and Vojtěch Rödl*

## 1 Introduction

In 1930, Frank Ramsey published a seminal paper "On a problem of formal logic" [13] beginning a new area of research known today as Ramsey theory (for a comprehensive introduction to Ramsey theory see, e.g., [9]). In particular, Ramsey proved that for given integers $t$ and $u$ there is an integer $n$ such that any blue–red coloring of the edges of the complete graph $K_n$ yields either a blue copy of $K_t$ or a red copy of $K_u$. Such smallest integer $n$ is denoted by $R(t,u)$. In other words, $R(t,u)$ is the smallest integer $n$ such that every $K_t$-free graph of order $n$ contains an independent set of size $u$, or equivalently, it contains a $u$-subset of vertices with no $K_2$. One can consider a more general problem replacing $K_2$ by $K_s$ for some $2 \leq s < t$. Following this approach in 1962 Erdős and Rogers [7] introduced the following function. For fixed integers $2 \leq s < t$ let

$$f_{s,t}(n) = \min\big\{\max\{|S| : S \subseteq V(H) \text{ and } H[S] \text{ contains no } K_s\}\big\},$$

where the minimum is taken over all $K_t$-free graphs $H$ of order $n$. (As a matter of fact a variation of this function was already considered by Erdős and Gallai [6].) We comment first on the meaning of $l \leq f_{s,t}(n) < u$. To prove the lower bound it means to show that every $K_t$-free graph of order $n$ contains a subset of $l$ vertices with no copy of $K_s$. To prove the upper bound it requires the construction of a $K_t$-free graph of order $n$ such that every subset of $u$ vertices contains a copy of $K_s$.

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* The author's research was partially supported by NSF grant DMS 0800070.

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A. Soifer (ed.), *Ramsey Theory: Yesterday, Today, and Tomorrow*,
Progress in Mathematics 285, DOI 10.1007/978-0-8176-8092-3_4,

As we have just seen, the problem of determining $f_{s,t}(n)$ extends that of determining Ramsey numbers. More precisely,

$$R(t,u) = \min\{n : f_{2,t}(n) \geq u\}.$$

Therefore, the problem of determining the precise value of $f_{s,t}(n)$ for $2 \leq s < t$ is rather hopeless. In addition, this function has attracted a considerable amount of attention and has been studied by several researchers.

In this paper, we review both old and recent results on function $f_{s,t}(n)$.

## 2 The Most Restrictive Case

In this section we consider the case when $t = s + 1$. Erdős and Rogers [7] proved using a probabilistic argument that

$$f_{s,s+1}(n) \leq O\left(n^{1-1/O(s^4 \log s)}\right). \tag{1}$$

Their proof is based on the concentration of measure phenomenon in the high-dimensional sphere. A similar idea was also used by Alon and Krivelevich [1] who gave an elegant construction of a $K_{s+1}$-free graph such that every subset of $O\left(n^{1-1/O(s^4 \log s)}\right)$ vertices contains a copy of $K_s$ (see Sect. 2.1 for details).

Bollobás and Hind [2] improved (1) showing that for any $\varepsilon > 0$ and $s \geq 3$,

$$f_{s,s+1}(n) \leq O\left(n^{1-((s+3)/((s-2)(s+1)))+\varepsilon}\right). \tag{2}$$

Moreover, they gave the first lower bound showing that for $s \geq 2$,

$$\Omega\left(n^{1/2}\right) \leq f_{s,s+1}(n). \tag{3}$$

Subsequently, Krivelevich [10, 11] improved the previous bounds and showed that

$$\Omega\left(n^{1/2}(\log\log n)^{1/2}\right) \leq f_{s,s+1}(n) \leq O\left(n^{1-(2/(s+2))}(\log n)^{1/(s-1)}\right). \tag{4}$$

Summarizing the previous results one can see that the best bounds have essentially been of the form

$$\Omega\left(n^{1/2+o(1)}\right) \leq f_{s,s+1}(n) \leq O\left(n^{1-\epsilon(s)}\right),$$

where $\epsilon(s)$ tends to zero as $s$ goes to infinity. This raised the following question asked by Krivelevich [10] and later by Sudakov [14, 15]. Is it true that for every $0 < \delta < 1$ and $s$ sufficiently large $f_{s,s+1}(n)$ is greater than $n^{1-\delta}$? We showed in [4] that this is not the case proving that for every fixed integer $s \geq 2$,

$$f_{s,s+1}(n) \leq O\left(n^{2/3}\right). \tag{5}$$

In the next sections we present proofs of (1), (3), and (5).

## 2.1 Proof of $f_{s,s+1}(n) \le O\big(n^{1-1/O(s^4 \log s)}\big)$ [1]

We start with notation. For two vectors $\bar{x} = (x_1, \dots, x_k)$ and $\bar{y} = (y_1, \dots, y_k)$ of integers we define the *Hamming distance* between $\bar{x}$ and $\bar{y}$ as,

$$d(\bar{x}, \bar{y}) = \big|\{i : x_i \neq y_i \text{ and } 1 \le i \le k\}\big|.$$

Moreover, for a nonempty subset $U \subseteq [s]^k$ and a vector $\bar{x} \in [s]^k$ denote the distance between $\bar{x}$ and $U$ as

$$d(\bar{x}, U) = \min\{d(\bar{x}, \bar{y}) : \bar{y} \in U\}.$$

Finally, for a given $\delta > 0$ define the $\delta$-*neighborhood* $U_{(\delta)}$ of a nonempty subset $U \subseteq [s]^k$ as

$$U_{(\delta)} = \{\bar{x} \in [s]^k : d(\bar{x}, U) \le \delta\}.$$

Now we construct a graph $H = (V, E)$ as follows. Let $s \ge 2$ be a fixed integer and let $k$ be a sufficiently large positive integer (in particular $k \ge s$). Let

$$V = [s]^k \quad \text{and} \quad E = \left\{ \{\bar{x}, \bar{y}\} \in V^2 : d(\bar{x}, \bar{y}) > k \left(1 - 1\Big/\binom{s+1}{2}\right)\right\}.$$

First we show that $H$ is a $K_{s+1}$-free graph. Assume by contradiction that there are $\bar{x}^1, \dots, \bar{x}^{s+1} \in V$ such that $H[\{\bar{x}^1, \dots, \bar{x}^{s+1}\}] = K_{s+1}$; i.e.,

$$d(\bar{x}^i, \bar{x}^j) > k \left(1 - 1\Big/\binom{s+1}{2}\right) \tag{6}$$

for every $1 \le i \neq j \le s+1$. Since every coordinate can attain only $s$ distinct values we infer that for every coordinate $h$, $1 \le h \le k$, there is a pair of vertices in $K_{s+1}$ having the same value of the $h$th coordinate. Thus,

$$\sum_{1 \le i < j \le s+1} d(\bar{x}^i, \bar{x}^j) \le k\binom{s+1}{2} - k = \binom{s+1}{2}\left(k - k\Big/\binom{s+1}{2}\right),$$

and consequently, there is a pair $\bar{x}^i$, $\bar{x}^j$ of vertices of $K_{s+1}$ such that $d(\bar{x}^i, \bar{x}^j) \le k - k/\binom{s+1}{2}$ which contradicts (6).

It remains to show that every sufficiently large subset of vertices of $H$ contains a copy of $K_s$. First we show that every sufficiently large subset of vertices of $H$ contains an $s$-simplex defined as follows. An $s$-*simplex* is an $s$-subset $\{\bar{x}^1, \dots, \bar{x}^s\}$ of vertices $V$ so that

$$d(\bar{x}^i, \bar{x}^j) = k \quad \text{for every } 1 \le i \neq j \le s. \tag{7}$$

For simplicity we also endow $V$ with the normalized counting measure $P$ (which can be also viewed as the probability measure) defined as $P(A) = |A|/|V|$ for every $A \subseteq V$.

**Proposition 2.1.** *If $V_0 \subseteq V$ and $P(V_0) > (s-1)/s$, then $V_0$ contains an $s$-simplex.*

*Proof.* By symmetry, every vertex of $V$ belongs to the same number of $s$-simplices. Clearly, if an $s$-simplex $S$ is not in $V_0$, then at least one vertex of $S$ is not in $V_0$. Hence, choosing randomly and uniformly a simplex $S$ among all $s$-simplices in $V$ we infer that

$$P(S \not\subseteq V_0) \leq s\frac{|V|-|V_0|}{|V|} = s(1-P(V_0)) < s\left(1-\frac{s-1}{s}\right) = 1,$$

and thus, there is an $s$-simplex $S$ in $V_0$. □

In order to finish the proof we need the following isoperimetric inequality. For the proof see, e.g., Lemma 3.3 in [1] or Proposition 7.12 in [12].

**Proposition 2.2.** *For $c > 0$ define $\delta(c) = \lceil\sqrt{k}(\sqrt{(\log s)/2}+c)\rceil$. If $U \subseteq V$ and $P(U) \geq \exp(-2c^2)$, then $P(U_{(\delta)}) > (s-1)/s$.*

Let

$$c = \frac{\sqrt{k}}{2\binom{s+1}{2}} - \sqrt{(\log s)/2} - 1 = \frac{\sqrt{k}}{2\binom{s+1}{2}}(1+o(1)), \tag{8}$$

where $o(1)$ tends to zero as $k$ tends to infinity. Note that

$$\delta(c) = \left\lceil\sqrt{k}(\sqrt{(\log s)/2}+c)\right\rceil = \left\lceil k/2\tbinom{s+1}{2} - \sqrt{k}\right\rceil < \frac{k}{2\binom{s+1}{2}} - \sqrt{k} + 1. \tag{9}$$

Recall that $|V| = s^k$ and so by (8)

$$\begin{aligned}\exp(-2c^2) &= |V|^{-2c^2\log_{|V|}\mathrm{e}} = |V|^{-(2c^2/(\log|V|))} = |V|^{-(2c^2/(k\log s))}\\ &= |V|^{-((2+o(1))/(s^2(s+1)^2\log s))}.\end{aligned}$$

We show that every subset $U \subseteq V$ of size

$$|U| = |V|\exp(-2c^2) = |V|^{1-((2+o(1))/(s^2(s+1)^2\log s))}$$

contains a copy of $K_s$. Indeed, if $P(U) \geq \exp(-2c^2)$, then by Proposition 2.2 $P(U_{(\delta)}) > (s-1)/s$, where $\delta = \lceil\sqrt{k}(\sqrt{(\log s)/2}+c)\rceil$. Hence, by Proposition 2.1 (applied to $V_0 = U_{(\delta)}$) the set $U_{(\delta)}$ contains an $s$-simplex $S$ (*cf.* (7)). Let $\bar{x}^1,\dots,\bar{x}^s \in U_{(\delta)}$ be its vertices. It remains to show that the set $U$ induces a copy of $K_s$.

It follows from the definition of $U_{(\delta)}$ that for every $\bar{x}^i$ there is $\bar{y}^i \in U$ such that

$$d(\bar{x}^i,\bar{y}^i) \leq \delta, \tag{10}$$

where $i = 1,\dots,s$. Let $1 \leq i \neq j \leq s$. Then, by the triangle inequality

$$d(\bar{x}^i,\bar{x}^j) \leq d(\bar{x}^i,\bar{y}^i) + d(\bar{y}^i,\bar{x}^j) \leq d(\bar{x}^i,\bar{y}^i) + d(\bar{y}^i,\bar{y}^j) + d(\bar{y}^j,\bar{x}^j),$$

and consequently by (10),

$$d(\bar{x}^i, \bar{x}^j) \leq d(\bar{y}^i, \bar{y}^j) + 2\delta.$$

Hence, since $d(\bar{x}^i, \bar{x}^j) = k$ and by (9) $2\delta < k/\binom{s+1}{2} - 2\sqrt{k} + 2 < k/\binom{s+1}{2}$,

$$d(\bar{y}^i, \bar{y}^j) \geq d(\bar{x}^i, \bar{x}^j) - 2\delta > k - k/\binom{s+1}{2}.$$

That means that the vertices $\bar{y}^1, \dots, \bar{y}^s \in U$ form a copy of $K_s$ in $H$.

This completes the proof of (1).

### 2.2 Proof of $\Omega\left(n^{\frac{1}{2}}\right) \leq f_{s,s+1}(n)$ for $s \geq 2$ [2]

We show that for any $s \geq 2$ and $n$ large enough

$$\lfloor ((s-1)n)^{1/2} \rfloor \leq f_{s,s+1}(n). \tag{11}$$

Let $G$ be a $K_{s+1}$-free graph of order $n$. We are going to show that $G$ contains a set of $\lfloor ((s-1)n)^{1/2} \rfloor$ vertices with no copy of $K_s$. Let $v$ be a vertex of maximal degree and let $W$ be the set of neighbors of $v$. Clearly, $G[W]$ is $K_s$-free, and hence, if $|W| \geq \lfloor ((s-1)n)^{1/2} \rfloor$ then we are done. Therefore, we may assume that $|W| < \lfloor ((s-1)n)^{1/2} \rfloor$. Consequently, the chromatic number of $G$ satisfies

$$\chi(G) \leq |W| + 1 \leq \lfloor ((s-1)n)^{1/2} \rfloor. \tag{12}$$

Let $W_1, \dots, W_{s-1}$ be color classes in a $\chi(G)$-vertex-coloring of $G$ such that $|W_1 \cup \dots \cup W_{s-1}|$ is maximal. Clearly $G[W_1 \cup \dots \cup W_{s-1}]$ is $K_s$-free. Moreover, by (12),

$$|W_1 \cup \dots \cup W_{s-1}| \geq (s-1)\frac{n}{\chi(G)} \geq \lfloor ((s-1)n)^{1/2} \rfloor,$$

and hence, (11) holds, as required.

### 2.3 Proof of $f_{s,s+1}(n) \leq O\left(n^{2/3}\right)$ for $s \geq 2$ [4]

The main idea we employ here is similar to an approach taken in [3].

First we recall some basic properties of generalized quadrangles. A *generalized quadrangle* $Q(4,q)$ is an incidence structure on a set $P$ of points and a set $\mathcal{L}$ of lines such that:

(Q1) Any two points lie in at most one line.
(Q2) If $u$ is a point not on a line $L$, then there is a unique point $w \in L$ collinear with $u$, and hence, no three lines form a triangle.
(Q3) Every line contains $q+1$ points, and every point lies on $q+1$ lines.

It is known that for every prime power $q$ such an incidence structure $Q(4,q)$ exists with $|P| = |\mathcal{L}| = q^3 + q^2 + q + 1$. For more information about generalized quadrangles see [8, 16].

Fix an integer $s \geq 2$. For every $m$ we are going to construct a $K_{s+1}$-free graph $H$ of order $\Theta(m^3)$ such that any induced subgraph of order $\lfloor |V(H)|/m \rfloor$ contains a copy of $K_s$. Thus, setting $n = (cm)^3$, $c = c(s)$, implies

$$\lfloor |V(H)|/m \rfloor \leq c^3 m^2 = cn^{2/3},$$

and consequently, (5) holds.

By Bertrand's postulate there is a prime number $q$ such that

$$s^2 m \leq q + 1 \leq 2s^2 m.$$

Let $Q(4,q)$ be a generalized quadrangle with a set $P$ of points and a set $\mathcal{L}$ of lines. We construct a "random graph" $H$ with vertex set $P$. Clearly,

$$|V(H)| = |P| = q^3 + q^2 + q + 1 = \Theta(m^3).$$

First, we partition every line $L \in \mathcal{L}$ into $s$ sets of the same size $\ell$ (for simplicity, we assume that $s$ divides $q+1$), hence

$$\ell = \frac{q+1}{s} \geq sm. \tag{13}$$

More precisely, for each line $L$ we choose one ordered partition

$$L = \bigcup_{i=1}^{s} L_i$$

satisfying $|L_1| = \cdots = |L_s| = \ell$ randomly and uniformly from the set of all such partitions. Now we join every $u \in L_i$ and $w \in L_j$, $1 \leq i < j \leq s$, by an edge obtaining a complete $s$-partite graph of order $s\ell$. Note that by (Q1) every edge is determined by a unique line. Moreover, condition (Q2) yields that $H$ contains no clique of size $s+1$. We show that in some such graph $H$ (randomly chosen from the space of such graphs) every set $U \subseteq V(H)$, $|U| = \lfloor |P|/m \rfloor = \Theta(m^2)$, contains a copy of $K_s$.

For $U \subseteq V(H)$ with cardinality $|U| = \lfloor |P|/m \rfloor$ let $\mathcal{A}(U)$ be the event that $K_s$ is not a subgraph of $H[U]$. Clearly, $\mathcal{A}(U)$ implies $\mathcal{A}(L \cap U)$ for each $L \in \mathcal{L}$. Consequently,

$$\mathcal{A}(U) \subseteq \bigcap_{L \in \mathcal{L}} \mathcal{A}(L \cap U),$$

and since all events $\mathcal{A}(L \cap U)$ are independent,

$$\Pr(\mathcal{A}(U)) \leq \prod_{L \in \mathcal{L}} \Pr(\mathcal{A}(L \cap U)). \tag{14}$$

For a fixed line $L \in \mathcal{L}$ we bound from above the probability that $\mathcal{A}(L \cap U)$ occurs. Let $L = \bigcup_{i=1}^{s} L_i$ be a partition of $L$. Note that if $\mathcal{A}(L \cap U)$ happens then for some $i$, $1 \le i \le s$, $L_i \cap U = \emptyset$; i.e., $L_i$ and $U$ are disjoint. Let $|U \cap L| = u_L$. The probability that for a fixed $i$, $1 \le i \le s$, $U \cap L_i = \emptyset$ equals the probability that for a fixed partition $L = \bigcup_{i=1}^{s} L_i$ a randomly chosen subset $T$ with $|T| = u_L$ satisfies $T \cap L_i = \emptyset$. Hence,

$$\Pr(\mathcal{A}(L \cap U)) \le s \frac{\binom{q+1-\ell}{u_L}}{\binom{q+1}{u_L}} \le s \left( \frac{q+1-\ell}{q+1} \right)^{u_L} \le s \exp \left( -\frac{\ell u_L}{q+1} \right).$$

Consequently, (14) yields,

$$\Pr(\mathcal{A}(U)) \le s^{|\mathcal{L}|} \exp \left( -\frac{\ell}{q+1} \sum_{L \in \mathcal{L}} u_L \right).$$

Moreover, since every point in $U$ belongs to exactly $q+1$ lines,

$$\sum_{L \in \mathcal{L}} u_L = \sum_{L \in \mathcal{L}} |U \cap L| = |U|(q+1).$$

Hence,

$$\Pr(\mathcal{A}(U)) \le s^{|\mathcal{L}|} \exp(-\ell|U|) = s^{|P|} \exp(-\ell|U|).$$

This implies that

$$\Pr\left( \bigcup_U \mathcal{A}(U) \right) \le \binom{|P|}{\lfloor |P|/m \rfloor} s^{|P|} \exp(-\ell|U|) \le (\mathrm{e}m)^{|P|/m} s^{|P|} \exp(-\ell|U|),$$

where the union is taken over all subsets $U \subseteq V(H)$ with cardinality $\lfloor |P|/m \rfloor$. Finally, note that by (13) $\ell|U| \ge sm\lfloor |P|/m \rfloor \ge (99/100)s|P|$, and hence,

$$\Pr\left( \bigcup_U \mathcal{A}(U) \right) \le \exp\left( |P| \left( \frac{1}{m} + \frac{\log m}{m} + \log s - \frac{99}{100} s \right) \right),$$

which tends to zero as $m$ tends to infinity.

This completes the proof of (5).

## 3 General Bounds

Here we discuss general bounds on $f_{s,t}(n)$ for any $t \ge s+1$. First observe that for any $t \ge s+1$,

$$f_{s,t}(n) \le f_{s,s+1}(n).$$

Consequently, (1), (2), (4), and (5) trivially bound from above $f_{s,t}(n)$. In particular, for every $t \geq s+1$,

$$f_{s,t}(n) \leq O\big(n^{2/3}\big).$$

Moreover, some of the previous results can be easily generalized. For instance, Bollobás and Hind [2] showed

$$\Omega\big(n^{1/(t-s+1)}\big) \leq f_{s,t}(n).$$

Subsequently, Krivelevich [10, 11] slightly improved this lower bound and also gave a new general upper bound,

$$\Omega\big(n^{1/(t-s+1)}(\log\log n)^{1-(1/(t-s+1))}\big) \leq f_{s,t}(n) \leq O\big(n^{s/(t+1)}(\log n)^{1/(s-1)}\big). \tag{15}$$

Recently, Sudakov [14, 15] improved the lower bound showing that

$$\Omega\big(n^{(s/2t)+O(1/t^2)}\big) \leq f_{s,t}(n).$$

In fact, he showed a more general result. For a fixed $s \geq 3$ consider a sequence $\{a_i\}_{i=3-s}^{\infty}$ defined as $a_i = 1$ for every $3-s \leq i \leq 0, a_1 = \frac{3s-4}{5s-6}$, and $\frac{1}{a_i} = 1 + \frac{1}{s-1}\sum_{j=i-(s-1)}^{i-1}\frac{1}{a_j}$ for every $i \geq 2$. Then, for any $s \geq 3$ and $k \geq 2$,

$$\Omega\left(n^{a_k(s)}\right) \leq f_{s,s+k}(n). \tag{16}$$

We found in [3] an explicit formula for $a_k(s)$ in a special case when $s \geq k \geq 2$ showing that

$$\frac{1}{a_k(s)} = 1 + \frac{3s-2}{3s-4}\left(\frac{s}{s-1}\right)^{k-2}.$$

Consequently, for any $s \geq k \geq 2$,

$$\Omega\left(n^{1/\left(1+((3s-2)/(3s-4))(s/(s-1))^{k-2}\right)}\right) \leq f_{s,s+k}(n).$$

Furthermore, we showed [3] that for any $\varepsilon > 0$, $k \geq 1$, and sufficiently large $s \geq s_0 = s_0(\varepsilon, k)$,

$$f_{s,s+k}(n) \leq O\big(n^{((k+1)/(2k+1))+\varepsilon}\big). \tag{17}$$

Consequently, for every $\varepsilon > 0$ and sufficiently large integers $k \geq k_0 = k_0(\varepsilon)$ and $s \geq s_0 = s_0(\varepsilon, k)$,

$$\Omega\big(n^{1/2-\varepsilon}\big) \leq f_{s,s+k}(n) \leq O\big(n^{1/2+\varepsilon}\big). \tag{18}$$

Below we present the main ideas from the proofs of (16) and (17).

### 3.1 Proof of $\Omega\left(n^{a_k(s)}\right) \leq f_{s,s+k}(n)$ [14, 15]

For simplicity we show a slightly weaker result which was also proved by Sudakov [14]. We define a new sequence of integers $\{b_i\}$ which is easier to handle. Let $s \geq 4$ be a fixed integer and $\{b_i\}_{i=2-s}^{\infty}$ be defined as $b_i = 1$ for every $2 - s \leq i \leq 0$ and

$$\frac{1}{b_i} = 1 + \frac{1}{s-1} \sum\nolimits_{j=i-(s-1)}^{i-1} \frac{1}{b_j}$$

for every $i \geq 1$. It was shown in [14] that for any $k \geq 0$,

$$\Omega\left(n^{b_k}\right) \leq f_{s,s+k}(n). \tag{19}$$

Since for all $i \geq 1$, $b_i < a_i$, (19) is weaker than (16).

In order to prove (19) we need two auxiliary results which we state without proofs (for details see Lemmas 3.1 and 3.2 in [15]). The first one is the well-known estimate on the size of maximum independent size in $s$-uniform hypergraph.

**Proposition 3.1.** *Let $\mathcal{H}$ be an $s$-uniform hypergraph on $n$ vertices with $m \geq \Omega(n)$ edges. Then, $\mathcal{H}$ contains an independent set of size*

$$\alpha(\mathcal{H}) \geq \Omega\left(\frac{n^{s/(s-1)}}{m^{1/(s-1)}}\right).$$

The next proposition gives an estimate on the number of edges in an $s$-uniform hypergraph. Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and $U \subseteq V$. We denote by $N_{\mathcal{H}}(U)$ the neighborhood of $U$ in $\mathcal{H}$, i.e.,

$$N_{\mathcal{H}}(U) = \left\{\bigcup E : U \subseteq E \text{ and } E \in \mathcal{E}\right\} \setminus U.$$

**Proposition 3.2.** *Let $\mathcal{H} = (V, \mathcal{E})$ be an $s$-uniform hypergraph of order $n$. Then, the number of edges is bounded by*

$$|\mathcal{E}| \leq O\left(n \prod_{u=1}^{s-1} \max_{U \subseteq V,\ |U|=u} |N_{\mathcal{H}}(U)|\right).$$

Note that if $H$ is a graph (two-uniform hypergraph) of order $n$ and with $m$ edges, then Proposition 3.2 suggests the obvious bound $m \leq n\Delta(H)$. Moreover, observe that Proposition 3.2 is tight for complete $s$-uniform hypergraphs.

The heart of the proof of (19) lies in the following proposition. Denote by $N_G^*(U)$ the set of all common neighbors in $G = (V, E)$ of vertices from $U \subseteq V$; that is,

$$N_G^*(U) = \{v \in V : \{v, u\} \in E \text{ for all } u \in U\} \setminus U.$$

**Proposition 3.3.** *Let $G = (V, E)$ be a graph of order $n$ such that each of its cliques induced by $U \subseteq V$ of size $u$, $1 \le u \le s-1$, satisfies*

$$|N_G^*(U)| < n^{b_{k+1}/(b_{k+1-u})}. \tag{20}$$

*Then $G$ contains a $K_s$-free subgraph of order $\Omega(n^{b_{k+1}})$.*

*Proof.* Let $\mathcal{H}$ be an $s$-uniform hypergraph whose vertices are the vertices of $G$ and whose edges are all copies of $K_s$ in $G$. Clearly, an independent set in $\mathcal{H}$ corresponds to a $K_s$-free subgraph in $G$. Therefore, in order to finish the proof it is enough to show that

$$\alpha(\mathcal{H}) \ge \Omega(n^{b_{k+1}}). \tag{21}$$

Also observe that

$$|N_{\mathcal{H}}(U)| \le |N_G^*(U)| \tag{22}$$

for any subset of vertices $U$ of size at most $s-1$ inducing a clique in $G$. Denote by $m$ the number of edges of $\mathcal{H}$. Then, Proposition 3.2, (22) and (20) yield,

$$\begin{aligned} m &\le O\left(n \prod_{u=1}^{s-1} \max_{U \subseteq V,\ |U|=u} |N_{\mathcal{H}}(U)|\right) \le O\left(n \prod_{u=1}^{s-1} \max_{G[U] \text{ is a clique},\ |U|=u} |N_G^*(U)|\right) \\ &\le O\left(n \prod_{u=1}^{s-1} n^{b_{k+1}/b_{(k+1-u)}}\right) = O\left(n^{1+\sum_{u=1}^{s-1}(b_{k+1}/(b_{k+1-u}))}\right) \\ &= O\left(n^{1+b_{k+1}\sum_{i=0}^{s-2}(1/(b_{k-i}))}\right). \end{aligned} \tag{23}$$

We may also assume that $m \ge \Omega(n)$. For otherwise, if $m \le o(n)$, then $\mathcal{H}$ contains an independent set of size $n - m \ge n - o(n) \ge \Omega(n)$. Thus, Proposition 3.1 and (23) imply that $\mathcal{H}$ contains an independent set of size

$$\alpha(\mathcal{H}) \ge \Omega\left(\frac{n^{s/(s-1)}}{m^{1/(s-1)}}\right) \ge \Omega\left(n^{1-((b_{k+1})/(s-1))\sum_{i=0}^{s-2}(1/(b_{k-i}))}\right).$$

Finally note that the recurrence relation for sequence $\{b_i\}$ yields

$$b_{k+1} = 1 - \frac{b_{k+1}}{s-1} \sum_{i=0}^{s-2} \frac{1}{b_{k-i}}.$$

This completes the proof of (21) and so the proof of Proposition 3.3. □

Now we are going to prove (19). For a fixed $s \ge 4$ we show by induction on $k \ge 0$ that any $K_{s+k}$-free graph $G$ of order $n$ contains a $K_s$-free subgraph of order $\Omega(n^{b_k})$.

If $k = 0$, then $G$ itself is $K_s$-free, and hence, $f_{s,s}(n) = n^{b_0}$ for $b_0 = 1$. Next suppose that the statement holds for all $k' \le k$; i.e., $f_{s,s+k'}(n) \ge \Omega(n^{b_{k'}(s)})$. We show that it also holds for $k+1$. Let $G$ be a $K_{s+(k+1)}$-free graph. We need to show that $G$ contains a subset of $\Omega(n^{b_{k+1}(s)})$ vertices with no copy of $K_s$.

First let us assume that $k \geq s-2$. Let $U$ be a subset of vertices of $G$ which forms a clique of size $1 \leq u \leq s-1$. Then, $k+1-u \geq 0$ and the subgraph of $G$ induced by $N_G^*(U)$ contains no clique of size $s+(k+1-u)$. Hence, if $|N_G^*(U)| \geq n^{b_{k+1}/b_{k+1-u}}$, then the inductive hypothesis applied to $N_G^*(U)$ yields that $G[N_G^*(U)]$ contains a $K_s$-free subgraph of size $\Omega(|N_G^*(U)|^{b_{k+1-u}}) \geq \Omega\left(n^{b_{k+1}}\right)$. Thus, we may assume that (20) holds, and consequently, Proposition 3.3 yields (19).

Now let us suppose that $0 \leq k < s-2$. Similarly, as in the previous case one can assume by the inductive hypothesis that $|N_G^*(U)| < n^{b_{k+1}/b_{k+1-u}}$ for every subset $U$ inducing a clique in $G$, but now only of size $1 \leq u \leq k+1$. We show that this is also true for any $k+1 < u \leq s-1$. Clearly, every clique induced by $U$ of size larger than $k+1$ contains a subclique induced by $U' \subseteq U$ of size $k+1$. Since $N_G^*(U) \subseteq N_G^*(U')$ and $b_{-(s-2)} = \cdots = b_0 = 1$, we obtain for $|U| = u$ that nevertheless

$$|N_G^*(U)| \leq |N_G^*(U')| \leq n^{((b_{k+1})/b_0)} = n^{b_{k+1}/(b_{k+1-u})},$$

for all $k+1 < u \leq s-1$. Consequently, we may assume that the assumption of Proposition 3.3 is satisfied and so (19) holds, as required.

## 3.2 *Sketch of the Proof of $f_{s,s+k}(n) \leq O\left(n^{((k+1)/(2k+1))+\varepsilon}\right)$ for $s \geq s_0 = s_0(\varepsilon, k)$ [4]*

First we recall some basic properties of projective planes (for more information see [8, 16]). A *projective plane* $PG(2,q)$ is an incidence structure on a set $P$ of points and a set $\mathcal{L}$ of lines such that:

(P1) Any two points lie in a unique line.
(P2) Any two lines meet in a unique point.
(P3) Every line contains $q+1$ points, and every point lies on $q+1$ lines.

It is known that for every prime power $q$ such an incidence structure $PG(2,q)$ exists with $|P| = |\mathcal{L}| = q^2+q+1$.

Fix an arbitrarily small $\varepsilon > 0$ and an integer $k \geq 1$. Let $s$ be a fixed and sufficiently large integer depending on $\varepsilon$ and $k$ only. We show that for every integer $m$ there exists a graph $H$ of order $\lfloor (cm)^{2+(1/k)+2\varepsilon} \rfloor$, $c = c(s)$, such that $H$ is $K_{s+k}$-free and any subgraph of $H$ induced by a set of cardinality $\lfloor |V(H)|/m \rfloor$ contains a copy of $K_s$. Thus, setting $n = (cm)^{2+(1/k)+2\varepsilon}$ implies

$$\begin{aligned}\lfloor |V(H)|/m \rfloor &\leq c^{2+(1/k)+2\varepsilon} m^{1+(1/k)+2\varepsilon} = cn^{(k+1+2\varepsilon k)/(2k+1+2\varepsilon k)} \\ &\leq cn^{(k+1/2k+1)+\varepsilon},\end{aligned}$$

and consequently, (17) will hold.

By Bertrand's postulate there is a prime number $q$ such that

$$m^{1+(1/2k)+\varepsilon} \le q+1 \le 2m^{1+(1/2k)+\varepsilon}.$$

Let $PG(2,q)$ be a projective plane with a set $P$ of points and a set $\mathcal{L}$ of lines. We construct a "random graph" $H$ with the vertex set $P$. Clearly,

$$|V(H)| = |P| = q^2+q+1 = \Theta(m^{2+(1/k)+2\varepsilon}).$$

We partition every line $L \in \mathcal{L}$ into $s+1$ sets

$$L = \bigcup_{i=0}^{s} L_i, \tag{24}$$

randomly and uniformly from the set of all partitions (24) satisfying

$$|L_1| = \cdots = |L_s| = \ell = sm \quad \text{and} \quad |L_0| = q+1-s\ell.$$

Observe that $|L_0| \gg \bigcup_{i=1}^{s} |L_i|$ as $m$ tends to infinity. The edges of $H$ are all pairs $\{u,w\}$, where $u \in L_i$, $w \in L_j$ and $1 \le i < j \le s$. In other words, $\bigcup_{i=1}^{s} L_i$ induces a complete $s$-partite graph of order $s\ell$, which we denote by $K_s(L)$. The edge set of $H$ then equals

$$E(H) = \bigcup_{L\in\mathcal{L}} K_s(L).$$

Note that since every two points from $P$ lie in a unique line the graph $H$ is well-defined.

Denote by $\mathcal{H} = \mathcal{H}(\varepsilon,k,s,q)$ the space of all such graphs $H$. Note that

$$\begin{aligned}|\mathcal{H}| &= \left(\binom{q+1}{\ell}\binom{q+1-\ell}{\ell}\cdots\binom{q+1-(s-1)\ell}{\ell}\right)^{|\mathcal{L}|}\\ &= \left(\frac{(q+1)!}{(\ell!)^s(q+1-s\ell)!}\right)^{|\mathcal{L}|}.\end{aligned}$$

One can show that a graph $H$ randomly chosen from the space $\mathcal{H}$ has the following properties:

(i) Every set $U \subseteq V(H)$, $|U| = \lfloor |P|/m \rfloor = \Theta(m^{1+(1/k)+2\varepsilon})$, induces in $H$ a subgraph containing a copy of $K_s$.
(ii) $H$ is $K_{s+k}$-free.

The proof of property (i) is similar to the proof from Sect. 2.3. The proof of (ii) is more technical and so we also skip it (for details see Theorem 1.2 in [4]).

## 4 Concluding Remarks

In this paper we presented the current stage of research on function $f_{s,t}(n)$. In [5] Erdős asked to estimate $f_{s,t}(n)$ as accurately as possible. In particular, he conjectured the following.

*Conjecture 4.1 (Erdős [5]).* For $s + 1 < t$,

$$\lim_{n\to\infty} \frac{f_{s+1,t}(n)}{f_{s,t}(n)} = \infty.$$

Sudakov [14] showed that this conjecture holds for

$$(s, t) \in \{(2, 4), (2, 5), (2, 6), (2, 7), (2, 8), (3, 6)\}.$$

Indeed, by (3) we obtain that $f_{3,4}(n) \geq \Omega(n^{1/2})$ and by (16) we get $f_{3,5}(n) \geq \Omega(n^{5/12})$, $f_{3,6}(n) \geq \Omega(n^{10/31})$, $f_{3,7}(n) \geq \Omega(n^{4/15})$, $f_{3,8}(n) \geq \Omega(n^{40/177})$, and $f_{4,6} \geq \Omega(n^{4/9})$. On the other hand, (15) implies $f_{2,4}(n) \leq O(n^{2/5} \log n)$, $f_{2,5}(n) \leq O(n^{2/6} \log n)$, $f_{2,6}(n) \leq O(n^{2/7} \log n)$, $f_{2,7}(n) \leq O(n^{2/8} \log n)$, $f_{2,8}(n) \leq O(n^{2/9} \log n)$, and $f_{3,6}(n) \leq O(n^{3/7}(\log n)^{1/2})$, which imply Sudakov's result.

Motivated by (3), (5), and (18) we state another conjecture.

*Conjecture 4.2.* For every $\varepsilon > 0$ and $s$ large enough,

$$f_{s,s+1}(n) \leq O(n^{(1/2)+\varepsilon}).$$

In view of (4), if Conjecture 4.2 holds then it is best possible.

**Acknowledgment** The authors would like to thank Alexander Soifer for organizing the successful DIMACS Workshop "Ramsey Theory Yesterday, Today and Tomorrow."

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# Large Monochromatic Components in Edge Colorings of Graphs: A Survey

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## 1 Introduction

The aim of this survey is to summarize an area of combinatorics that lies on the border of several areas: Ramsey theory, resolvable block designs, factorizations, fractional matchings and coverings, and partition covers. Unless stated otherwise, coloring means *edge colorings* of graphs; an $r$-coloring is an assignment of elements of $\{1, 2, \ldots, r\}$ to the edges.

### *1.1 A Remark of Erdős and Rado and Its Extension*

A very simple statement – the leitmotif of the survey – is a remark of Erdős and Rado. It can be phrased in different ways.

**Proposition 1.1.** *The following statements are equivalent:*

- *Either a graph or its complement is connected.*
- *Every* 2*-colored complete graph has a monochromatic spanning tree.*
- *If two partitions are given on a ground set such that each pair of elements is covered by some block of the partitions then one of the partitions is trivial, i.e., has only one block.*
- *Pairwise intersecting edges of a bipartite multigraph have a common vertex.*

The first two statements are clearly equivalent. The equivalence of the third and fourth follows through duality: the blocks of the two partitions (through duality) become the two partite sets of the bipartite graph and the vertices become

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A. Soifer (ed.), *Ramsey Theory: Yesterday, Today, and Tomorrow*,
Progress in Mathematics 285, DOI 10.1007/978-0-8176-8092-3_5,

(possibly multi-)edges. The equivalence of the second and third statements comes from considering the correspondence of blocks of the two partitions with the components of the two colored subgraphs in the 2-coloring of the edges of a complete graph.

Any of the (equivalent) statements formulated in Proposition 1.1 can be proved immediately; in Sect. 2 we overview many extensions of it. Several natural questions arise: can one say more about the monochromatic spanning tree guaranteed by Proposition 1.1; may connectivity be replaced by stronger properties, such as small diameter, higher connectivity (or both). These are discussed in Sect. 2.1. Another important extension is surveyed in Sect. 2.2 when 2-colorings are replaced by Gallai-colorings; these are colorings where the number of colors is not restricted but the requirement is that there is no multicolored (rainbow) triangle in the colorings. It turned out that many results that hold for 2-colorings have extensions, or even "black-box" extensions, to Gallai-colorings as well.

A separate section, Sect. 3, is devoted to $r$-colorings. The problem was suggested in [24] and the case $r = 3$ was solved there; a minor inaccuracy was corrected in [1]. The problem was rediscovered in [5]. The general result for $r$ colors was proved in [25]. It extends Proposition 1.1 as follows.

**Theorem 1.2 ([25]).** *The following equivalent statements are true:*

- *In every $r$-coloring of $K_n$ there is a monochromatic component with at least $n/(r-1)$ vertices.*
- *If $r$ partitions are given on a ground set of $n$ elements such that each pair of elements is covered by some block of the partitions then one of the partitions has a block of size at least $n/(r-1)$.*
- *If an intersecting $r$-partite (multi)hypergraph has $n$ edges then it has a vertex of degree at least $n/(r-1)$ (intersecting means that any two edges have a vertex in common).*

The equivalence of statements in Theorem 1.2 follows the same way as in Proposition 1.1 and can be proved by two different proof techniques shown in Sects. 3.1 and 3.2. The next subsection gives an important construction showing that Theorem 1.2 is close to best possible.

## *1.2 Colorings from Affine Planes*

Consider an affine plane of order $r-1$ that is $r$ partitions of a ground set $V, |V| = (r-1)^2$ into blocks of size $r-1$ so that each pair of elements of $V$ is covered by a unique block. (If $r-1$ is a prime power, affine planes indeed exist.) There is a natural way to color the edges of a complete graph with vertex set $V$: for $i = 1, 2, \dots, r$ color the pairs within the blocks of the $i$th partition class with color $i$. For example, for $r = 3$ we obtain the 3-coloring of $K_4$ (a factorization), for $r = 4$ we obtain the 4-coloring of $K_9$ where each color class is the union of three vertex disjoint triangles. In general, this coloring is an example showing that Theorem 1.2 holds with equality: every monochromatic connected component has $|V|/(r-1) = r-1$ vertices. Further cases of equality are discussed in the next subsection.

### *1.3 Extending Colorings by Substitutions*

A useful way of extending a coloring of a complete graph is to substitute colored complete graphs to its vertices so that the edges between the substituted parts retain their original colors.

In the $r$-coloring described above, the cardinality of the vertex set is fixed: $|V| = (r-1)^2$. One can easily extend it by substituting complete graphs – usually with arbitrary colorings – into all vertices. For example, to see that Theorem 1.2 is sharp for $n = k(r-1)^2$ (and when affine plane of order $r-1$ exists) just substitute arbitrarily $r$-colored complete graphs on $k$ vertices into the coloring defined in the previous subsection. If $n \neq k(r-1)^2$ then more subtle substitutions still can be used, these problems are explored in Sect. 3.3.

The colorings defined here and in the previous subsection work only when affine planes exist. On the other hand, if they do not exist then a result of Füredi [21] immediately implies that Theorem 1.2 can be improved (see Sect. 3.2 for more details).

**Theorem 1.3.** *Suppose that affine planes of order* $r-1$ *do not exist. Then in every* $r$*-coloring of* $K_n$ *there is a monochromatic component with at least* $n(r-1)/r(r-2)$ *vertices.*

The first case when Theorem 1.3 applies is $r = 7$.

**Problem 1.4.** Let $f(n)$ be the cardinality of the largest monochromatic component that must occur in every 7-coloring of $K_n$. Then, from the previous results, the asymptotic of $f(n)$ is between $6n/35$ and $7n/35$. Improve these bounds!

The asymptotic of $f(n)$ in Problem 1.4 would follow from Füredi's problem ([22], Problem 4.6): to find $\alpha$ where

$$\alpha = \max\{\tau^*(\mathcal{H}) : \mathcal{H} \text{ is intersecting 7-partite hypergraph}\}.$$

In fact, $f(n) \sim n/\alpha$; see Sect. 3.2.

## 2 2-Colorings

### *2.1 Type of Spanning Trees, Connectivity, Diameter*

Looking at the first form of Proposition 1.1, it is natural to ask what kind of monochromatic spanning trees can be found in every 2-coloring of a complete graph. Bialostocki, Dierker, and Voxman [3] suggested three types: trees of height at most two; trees obtained by subdividing the edges of a star with $k$ edges (a $k$-octopu s); and trees obtained by identifying an endpoint of a path with the center of a star (a broom).

**Theorem 2.1 ([3]).** *In every* 2*-coloring of* $K_n$ *there exists a monochromatic spanning* $k$*-octopus with* $k \geq \lceil (n-1)/2 \rceil$ *and also a monochromatic spanning tree of height at most two.*

The third suggested type, the broom, remained a conjecture until Burr found a proof. Unfortunately Burr's manuscript [11] was not published (although generalizations [16,31] appeared), so it is doubly justified to reproduce Burr's "book-proof" here.

**Theorem 2.2 ([11]).** *In every* 2*-coloring of* $K_n$ *there exists a monochromatic spanning broom.*

*Proof.* Assume w.l.o.g. that in a red–blue coloring of a complete graph, the red graph is $k$-connected and the blue graph is at most $k$ connected. Then the blue graph becomes disconnected after the deletion of a set $X$ of at most $k$ vertices. Since the red graph is $k$-connected, by a theorem of Dirac (see [43], Exercise 6.66) $X$ can be covered by a red cycle (an edge if $k = 1$). Thus the vertex set of $K_n$ can be covered by a red cycle $C$ and a red complete bipartite graph $G = [A, B]$. Observe that a complete bipartite graph always has a spanning broom such that its starting point is arbitrary. Therefore covering $C$ with a red path then continuing in the complete bipartite graph $[A \setminus C, B \setminus C]$ we can find a red broom. □

Concerning the diameter of a monochromatic connected spanning subgraph, the following result is folkloristic (forgive me if I missed further references).

**Theorem 2.3 ([3, 45, 49]).** *In every* 2*-coloring of a complete graph there is a monochromatic spanning subgraph of diameter at most three.*

*Proof.* If vertices $u, v$ are at a distance at least three in red then $uv$ is blue and every other vertex $w$ is adjacent to at least one of $u, v$ in blue. Thus there is a spanning double star in blue. □

How large is the largest monochromatic piece of diameter two? The following coloring shows that one cannot expect more than $3n/4$. Start with the 2-coloring of $K_4$ where both color classes form a $P_4$. Substitute nearly equal vertex sets into this coloring to get a total of $n$ vertices. Erdős and Fowler [14] proved that this example is best possible.

**Theorem 2.4 ([14]).** *In every* 2*-coloring of* $K_n$ *there is a monochromatic subgraph of diameter at most two with at least* $3n/4$ *vertices.*

The proof of Theorem 2.4 is difficult. A weaker result (also best possible) with a very simple proof is the following.

**Theorem 2.5 ([26]).** *In every 2-coloring of $K_n$ there is a color, say red, and a set $W$ of at least $3n/4$ vertices such that any pair of points in $W$ can be connected by a red path of length at most two.*

Another natural question is the maximum order of a monochromatic $k$-connected subgraph in 2-colorings of $K_n$. This question was introduced in [9] and further elaborated in [41, 42].

*Example.* Let $B$ be the 2-colored complete graph on vertex set [5] with red edges $12, 23, 34, 25, 35$ and with the other edges blue. (Both color classes form a "bull".) Assuming that $n > 4(k-1), k \geq 2$, let $B(n,k)$ be a 2-colored complete graph with $n$ vertices obtained by replacing vertices $1, 2, 3, 4$ in $B$ by arbitrary 2-colored complete graphs of $k-1$ vertices and replacing vertex 5 in $B$ by a 2-colored complete graph of $n-4(k-1)$ vertices. All edges between the replaced parts retain their original colors from $B$. Note that $B(n,k)$ denotes a member of a rather large family of graphs. The definition of $B(n,k)$ is used in the case $n = 4(k-1)$ as well, but in this case vertices 1, 4 (2, 3) of $B$ are replaced by red (blue) complete subgraphs (and vertex 5 is deleted). Thus in this case we have just one graph for each $k$, which we denote by $B(k)$. Observe that the color classes of $B(k)$ form isomorphic graphs and there is no monochromatic $k$-connected subgraph in $B(k)$.

It is easy to check that in $B(n,k)$ the maximal order of a $k$-connected monochromatic subgraph is $n-2(k-1)$. It is conceivable that each $B(n,k)$ is an optimal example for every $k$; i.e., the following assertion is true.

*Conjecture 2.6 ([9]).* For $n > 4(k-1)$, every 2-colored $K_n$ has a $k$-connected monochromatic subgraph with at least $n-2(k-1)$ vertices.

For $k = 2$ it is easy to prove the conjecture.

**Theorem 2.7 ([9]).** *For $n \geq 5$ there is a monochromatic 2-connected subgraph with at least $n-2$ vertices in every 2-coloring of $K_n$.*

*Proof.* Every 2-coloring of $K_5$ contains a monochromatic cycle. Proceeding by induction, let (w.l.o.g.) $H$ be a 2-connected red subgraph with $n-3$ vertices in a 2-coloring of $K_n$. If some vertex of $W = V(K_n) \setminus V(H)$ sends at least two red edges to $H$ then we have a 2-connected red subgraph with $n-2$ vertices. Otherwise the blue edges between $V(H)$ and $W$ determine a 2-connected blue subgraph of at least $n-2$ vertices (either a blue $K_{2,n-4}$ or a blue $K_{3,n-3}$ from which three pairwise disjoint edges are removed). $\square$

Conjecture 2.6 was answered positively in [42] for $k = 3$ and for $n \geq 13k$. In [9] it was remarked that it is enough to prove the conjecture for $4(k-1) < n < 7k-5$. Another related conjecture – the graph $B(k)$ shows that it is sharp if true – is the following.

*Conjecture 2.8 ([9]).* Every 2-colored $K_n$ contains a monochromatic subgraph that is at least $(n/4)$-connected.

The following result was needed as a lemma in [36]. It shows that high connectivity and small diameter can be simultaneously required for monochromatic subgraphs with order close to $n$.

**Theorem 2.9 ([36]).** *For every $k$ and for every 2-colored $K_n$ there exists $W \subset V(K_n)$ and a color such that $|W| \geq n - 28k$ and any two vertices in $W$ can be connected in that color by $k$ internally vertex disjoint paths, each with length at most three.*

Notice that the paths connecting vertices of $W$ in Theorem 2.9 may leave $W$, as in Theorem 2.5. Probably Theorem 2.9 can be strengthened, as Theorem 2.4 strengthens Theorem 2.5.

**Problem 2.10.** Is it possible to strengthen Theorem 2.9 by requiring that the monochromatic paths connecting the pairs of $W$ are completely within $W$?

## *2.2 Gallai-Colorings: Substitutions to 2-Colorings*

Edge colorings of complete graphs in which no triangles are colored with three distinct colors are called Gallai-colorings in [31]. These colorings are very close to 2-colorings as the following decomposition theorem shows. This result is implicit in Gallai's seminal paper [23] and was refined in [12]. The form below is from [31].

**Theorem 2.11.** *Every Gallai-coloring can be obtained from a 2-colored complete graph with at least two vertices by substituting Gallai-colored complete graphs into its vertices.*

Theorem 2.11 is a natural tool to extend results from 2-colorings to Gallai-colorings. In [31] several results were extended, most notably Burr's theorem (see Theorem 2.16). Certain properties are not extendible though; there is obviously a monochromatic star with at least $((n-1)/2) + 1$ vertices in every 2-coloring of $K_n$ but this does not extend to Gallai-colorings. Substituting almost equal green complete graphs into the vertices of a 2-colored $K_5$ in which the red and blue colors form pentagons, we get a Gallai-coloring that shows that the following result is almost sharp (for $n = 5k + 2$ one can be added).

**Theorem 2.12 ([31]).** *In every Gallai-coloring of $K_n$ there is a monochromatic star with at least $2n/5$ edges.*

In [35] a method was devised that can extend a result from 2-colorings to Gallai-colorings. It works for certain classes of graphs and when it works it provides a "black-box" extension; i.e., one does not need to know the (occasionally very difficult) proof of the 2-coloring result. To define those classes, a family $\mathcal{F}$ of finite connected graphs was called *Gallai-extendible* in [35] if contains all stars and if for all $F \in \mathcal{F}$ and for all proper nonempty $U \subset V(F)$ the graph $F' = F'(U)$ defined as follows is also in $\mathcal{F}$:

- $V(F') = V(F)$.
- $E(F') = E(F) \setminus \{uv : u, v \in U\} \cup \{ux : u \in U, x \notin U, vx \in E(F)$ for some $v \in U\}$.

**Theorem 2.13 ([35]).** *Suppose that $\mathcal{F}$ is a Gallai-extendible family, and that there exists a function $f : \mathbf{N} \to \mathbf{N}$ such that for every $n$ and for every 2-coloring of $K_n$ there is a monochromatic $F \in \mathcal{F}$ with $|V(F)| \geq f(n)$.*

*Then, for every $n$ and every Gallai-coloring of $K_n$ there exists a monochromatic $F' \in \mathcal{F}$ such that $|V(F')| \geq f(n)$ – with the same function f.*

As shown in [35], graphs with spanning trees of height at most $h \geq 2$, graphs of diameter at most $d$ for each $d > 1$, and graphs having a spanning double star are all Gallai-extendible. Therefore Theorems 2.3, 2.4, and Corollary 4.6 have black-box extensions to Gallai-colorings.

**Theorem 2.14 ([35]).** *In every Gallai-coloring of $K_n$ one can find monochromatic spanning trees of height at most two, monochromatic double stars and monochromatic diameter two subgraphs with at least $3n/4$ vertices.*

Graphs having a spanning complete bipartite subgraphs are also Gallai-extendible, therefore we have the following.

**Theorem 2.15 ([35]).** *Every Gallai-colored $K_n$ contains a monochromatic complete bipartite subgraph with at least $\lceil (n+1)/2 \rceil$ vertices.*

There are cases when Theorem 2.13 is not applicable (at least directly): brooms (or graphs having spanning brooms) are not Gallai-extendible, however, Theorem 2.2 remains true for Gallai-colorings as shown in [31] (conjectured by Bialostocki in [3]).

**Theorem 2.16 ([31]).** *In every Gallai-coloring of $K_n$ there exists a monochromatic spanning broom.*

# 3 Multicolorings: Basic Results and Proof Methods

## *3.1 Complete Bipartite Graphs: Counting Double Stars*

Usually Ramsey numbers are larger than the lower bound coming from the corresponding Turán numbers of the graph in the majority color. However, the following lemma is an exception.

**Lemma 3.1 ([25]).** *In every $r$-coloring of a complete bipartite graph on $n$ vertices there is a monochromatic subtree with at least $n/r$ vertices.*

This lemma was obtained in [25] by proving that a majority color class (a color class with the largest number of edges) always has a subtree with at least $n/r$ vertices. A short proof of this is due to Mubayi [45] and Liu, Morris, and Prince [41]. In fact they prove the following stronger statement: if the edges of the complete bipartite graph with $n$ vertices are colored with $r$ colors, there is a monochromatic double star with at least $n/r$ vertices. A *double star* is a tree obtained by joining the centers of two disjoint stars by an edge.

**Lemma 3.2 ([41, 45]).** *In every r-coloring of a complete bipartite graph on n vertices there is a monochromatic double star with at least n/r vertices.*

*Proof.* Suppose that $G = [A, B]$ is an $r$-colored complete bipartite graph, let $d_i(v)$ denote the degree of $v$ in color $i$. For any edge $ab$ of color $i$, $a \in A, b \in B$, set $c(a,b) = d_i(a) + d_i(b)$. Using the Cauchy–Schwartz inequality, we get

$$\begin{aligned}\sum_{ab\in E(G)} c(a,b) &= \sum_{a\in A}\sum_{i=1}^{r} d_i^2(a) + \sum_{b\in B}\sum_{i=1}^{r} d_i^2(b)\\ &\geq |A|r\left(\frac{\sum_{a\in A}\sum_{i=1}^{r} d_i(a)}{|A|r}\right)^2 + |B|r\left(\frac{\sum_{b\in B}\sum_{i=1}^{r} d_i(b)}{|B|r}\right)^2\\ &= |A||B|\left(\frac{|A|+|B|}{r}\right),\end{aligned}$$

therefore for some $a \in A, b \in B$, $c(a,b) \geq (|A|+|B|)/r$; i.e., there is a monochromatic double star of the required size. □

Lemma 3.2 implies Theorem 1.2 in a stronger form.

**Corollary 3.3.** *Suppose that the edges of $K_n$ are colored with r colors. Then either all color classes have monochromatic spanning trees or there is a monochromatic double star with at least $n/(r-1)$ vertices.*

*Proof.* Indeed, if a color class does not have a spanning tree, there is a complete bipartite subgraph colored with $r-1$ colors and Lemma 3.2 concludes the proof. □

It is possible that for $r \geq 3$ the second conclusion of Corollary 3.3 is always true. This problem and some results in this direction can be found in Sect. 4.2.

A possible improvement of Lemma 3.1 is suggested in [6].

*Conjecture 3.4.* If the edges of a complete bipartite graph $[A, B]$ are colored with $r$ colors then there exists a monochromatic subtree with at least $\lceil |A|/r\rceil + \lceil |B|/r\rceil$ vertices.

For $2 \leq r \leq 4$ Conjecture 3.4 was proved in [6] with an example that shows that, unlike the case of Lemma 3.1, for $r = 5$ the conjectured large monochromatic subgraph is not in the majority color.

### 3.2 Fractional Transversals: Füredi's Method

To present a very powerful method introduced by Füredi, the notion of fractional covers and matchings is summarized. A *fractional transversal* of a hypergraph is a nonnegative weighting on the vertices such that the sum of the weights over any edge is at least 1. The *value* of a fractional transversal is the sum of the weights over all vertices of the hypergraph. Then $\tau^*(\mathcal{H})$ is the minimum of the values over all fractional transversals of $\mathcal{H}$. A *fractional matching* of a hypergraph is a nonnegative weighting on the edges such that the sum of weights over the edges containing any fixed vertex is at most 1. The *value* of a fractional matching is the sum of the weights over all edges of the hypergraph. Then $\nu^*(\mathcal{H})$ is the maximum of the values over all fractional matchings of $\mathcal{H}$. By LP duality, $\tau^*(\mathcal{H}) = \nu^*(\mathcal{H})$ holds for every hypergraph $\mathcal{H}$.

Assume that the edges of $K_n$ are $r$-colored. By Theorem 1.2, to find a monochromatic component with at least $n/(r-1)$ vertices is equivalent to finding a vertex of degree at least $n/(r-1)$ in an intersecting $r$-partite multihypergraph $\mathcal{H}$ with $n$ edges. Füredi proved [20] that in such hypergraphs $\tau^*(\mathcal{H}) \leq (r-1)$. Using the observation that weighting all edges by the reciprocal of the maximum degree of the hypergraph is a fractional matching with value $|E(\mathcal{H})|/D(\mathcal{H})$, we get

$$\frac{|E(\mathcal{H})|}{D(\mathcal{H})} \leq \nu^*(\mathcal{H}) = \tau^*(\mathcal{H}) \leq r-1, \tag{1}$$

where $D$ is the maximum degree of $\mathcal{H}$. Thus we have $n/(r-1) = |E(\mathcal{H})|/(r-1) \leq D(\mathcal{H})$.

Notice that the above proof uses the LP duality theorem and this is applicable in other variants of the problem; see, for example, Sect. 3.5. Moreover, whenever the nonexistence of affine planes of order $r-1$ is known, Füredi [21] improved his upper bound $\tau^* \leq (r-1)$ by $1/(r-1)$ and this leads to Theorem 1.3.

### 3.3 Fine Tuning

Theorem 1.2 says that in any $r$-coloring of $K_n$ there is a monochromatic component with at least $n/(r-1)$ vertices. We have already seen that this is sharp if $r-1$ is a prime power and $n$ is divisible by $(r-1)^2$. The first case when one can improve on this (by one) occurs for $r = 3$ and $n = 4k+2$ ([1]). In [6] the order of the largest monochromatic connected subgraph of $K_n$ has been found for $r = 4, 5$ and for all values of $n$. It turned out that these values depend on the smallest multicover of affine planes. An *$i$-cover* of a hypergraph is a nonnegative integer weight assignment to the vertices such that the sum of weights on every edge is at least $i$. The minimum total weight over all $i$-covers is the $i$-cover number of the hypergraph. Let $w(i,q)$ be defined as the minimum of the $i$-cover numbers over all affine planes of order $q$. (The $i$-covers of affine planes are also called affine blocking sets.) For example, a fundamental result of Jamison [10,37] says that $2q-1$ points (points on the union of two intersecting lines) is the smallest 1-cover of the Desarguesian affine plane. However, $w(1,9) < 17$ because the Hughes plane of order 9 has a transversal of 16 points [10].

The most general sharp result is obtained by Füredi's method in [21] (similarly as explained in Sect. 3.2). In terms of the parameter $w(i,q)$, it gives a sharp result whenever the number of colors is one less than a power of prime. The result confirms a conjecture of Bierbrauer [7]. It is more convenient to use inverse notation here: let $f(D,r)$ be the maximum $n$ such that there exists an $r$-coloring of the edges of $K_n$ for which the largest monochromatic connected subgraph has no more than $D$ vertices.

**Theorem 3.5 ([21]).** *Assume that an affine plane of order $q$ exists. Define $i$ for every $D$ by $i = q\lceil D/q\rceil - D$ where $0 \le i < q$. Then, for every $D \ge q^2 - q$,*

$$f(D, q+1) = q^2 \left\lceil \frac{D}{q} \right\rceil - w(i,q).$$

## 3.4 When Both Methods Work: Local Colorings

The analogue of Theorem 1.2 for local $r$-colorings was obtained in [32]. A *local $r$-coloring* of a complete graph is a coloring where the number of colors incident to each vertex is at most $r$. How large is the largest monochromatic connected subgraph in local $r$-colorings of $K_n$?

Let $f(n,r)$ denote the largest $m$ such that in every local $r$-coloring of the edges of $K_n$ there is a monochromatic connected subgraph with $m$ vertices. Clearly $f(n,r) \le n/r - 1$ whenever Theorem 1.2 is sharp, because $r$-colorings are special local $r$-colorings. This function has been also defined implicitly in [3], in connection with mixed Ramsey numbers. In particular, $RM(\mathcal{T}_n, G)$ was defined as the minimum $m$ such that in any edge coloring of $K_m$ there is either a monochromatic tree on $n$ vertices or a totally multicolored copy of $G$. The special case when $G$ is a star was treated in [4]. Since the requirement of forbidding a multicolored star $K_{1,\,r+1}$ is equivalent to local $r$-colorings, the next result implies the asymptotic value of $RM(\mathcal{T}_n, K_{1,!r+1})$ (extending the special case $r = 2$ in [4]).

**Theorem 3.6 ([32]).** $f(n,r) \ge rn/(r^2 - r + 1)$ *with equality if a finite plane of order $r - 1$ exists and $r^2 - r + 1$ divides $n$.*

The construction for showing that Theorem 3.6 is sharp when indicated is as follows. Consider the points of a finite plane of order $r - 1$ as the vertices of a complete graph, label the lines, and color each pair of vertices by the label of the line going through it. Then substitute each vertex $i$ by a $k$-element set $A_i$ so that the $A_i$s are pairwise disjoint. The coloring is extended naturally with the proviso that the edges within $A_i$s are colored with some color among the colors that were incident to vertex $i$. The result is a locally $r$-colored $K_n$ where $n = k(r^2 - r + 1)$ and the largest monochromatic connected subgraph has $kr = nr/(r^2 - r + 1)$ vertices.

Both methods discussed in Sects. 3.1, and 3.2 can be used to prove Theorem 3.6. The method of counting double stars can be applied through the following theorem.

**Theorem 3.7 ([32]).** *Assume that the edges of a complete bipartite graph $G = [A, B]$ are colored so that the edges incident to any vertex of $A$ are colored with at most $p$ colors and the edges incident to any vertex of $B$ are colored with at most $q$ colors. Then there exists a monochromatic double star with at least $|A|/q + |B|/p$ vertices.*

A corollary of Theorem 3.7 is an extension of Lemmas 3.1 and 3.2.

**Corollary 3.8 ([32]).** *In every local $r$-coloring of a complete bipartite graph $G$ there exists a monochromatic double star with at least $|V(G)|/r$ vertices.*

*Proof of Theorem 3.6.* If any monochromatic, say red component $C$ satisfies $|C| \geq rn/(r^2 - r + 1)$, we have nothing to prove. Otherwise apply Theorem 3.7 for the complete bipartite graph $[A, B] = [V(C), V(G) \setminus V(C)]$. The edges incident to any $v \in A$ are colored with at most $p = r - 1$ colors and the edges incident to any $v \in B$ are colored with at most $q = r$ colors. Thus, using Theorem 3.7 and $|A| < rn/(r^2 - r + 1)$, there is a monochromatic component of size at least

$$\begin{aligned} |A|/q + |B|/p &= \frac{|A|}{r} + \frac{n - |A|}{r-1} = \frac{n}{r-1} - |A|\left(\frac{1}{r-1} - \frac{1}{r}\right) \\ &> n\left(\frac{1}{r-1} - \frac{r}{r^2 - r + 1}\left(\frac{1}{r(r-1)}\right)\right) = \frac{rn}{r^2 - r + 1}. \end{aligned}$$ □

*Proof of Theorem 3.7.* Let $d_i(v)$ denote the degree of $v$ in color $i$. For any edge $ab$ of color $i$, $a \in A, b \in B$, set $c(a, b) = d_i(a) + d_i(b)$. Let $I(v)$ denote the set of colors on the edges incident to $v \in V(G)$. Then, by using the Cauchy–Swartz inequality and the local coloring conditions, we get

$$\begin{aligned} \sum_{ab \in E(G)} c(a, b) &= \sum_{a \in A} \sum_{i \in I(a)} d_i^2(a) + \sum_{b \in B} \sum_{i \in I(b)} d_i^2(b) \\ &\geq |A|p\left(\frac{\sum_{a \in A}\sum_{i \in I(a)} d_i(a)}{|A|p}\right)^2 + |B|q\left(\frac{\sum_{b \in B}\sum_{i \in I(b)} d_i(b)}{|B|q}\right)^2 \\ &= |A||B|\left(\frac{|B|}{p} + \frac{|A|}{q}\right), \end{aligned}$$

therefore for some $a \in A, b \in B$, $c(a, b) \geq |A|/q + |B|/p$. Since the edges incident to $a$ or $b$ in the color of $ab$ span a monochromatic connected double star with $c(a, b)$ vertices, Theorem 3.7 follows. □

The second proof of Theorem 3.6 follows the argument shown in Sect. 3.2. Assume that the edges of $K_n$ are locally $r$-colored. Consider the hypergraph $H$ whose vertices are the vertices of $K_n$ and whose edges are the vertex sets of the connected monochromatic components. In the dual of $H$, $H^*$, every edge has at most $r$

vertices and each pair of edges has a nonempty intersection. Füredi proved [20] that in such hypergraphs the fractional transversal number, $\tau^*(H^*) \le r - 1 + (1/r)$. Then, as (1) in Sect. 3.2, we have

$$\frac{|E(H^*)|}{D(H^*)} \le \nu^*(H^*) = \tau^*(H^*) \le r - 1 + \frac{1}{r} \tag{2}$$

where $D$ is the maximum degree of $H^*$. Thus $((r|E(H^*)|)/(r^2 - r + 1)) \le D(H^*)$. Noting that $|E(H^*)| = n$ and $D(H^*)$ equals the maximum size of an edge in $H$ (i.e., the maximum size of a connected component in the local $r$-coloring), the inequality of Theorem 3.6 follows.

## 3.5 Hypergraphs

Theorem 1.2 was extended to hypergraphs in [17] as follows. We note here that for hypergraphs there are several notions of connectivity. Unless stated otherwise we consider a hypergraph connected if its *cover graph* – the pairs of vertices that are covered by at least one edge of the hypergraph – spans a connected graph.

**Theorem 3.9 ([17]).** *In every $r$-coloring of the edges of the complete $t$-uniform hypergraph on $n$ vertices, there is a connected monochromatic subhypergraph on at least $n/q$ vertices, where $q$ is the smallest integer satisfying $r \le \sum_{i=0}^{t-1} q^i$. The result is best possible if $q$ is a prime power and $n$ is divisible by $q^t$.*

The lower bound of Theorem 3.9 comes from Füredi's method. Let $f(n, r, t)$ be defined as the minimum size of a monochromatic component that must be present in any $r$-coloring of the $t$-sets of an $n$-element set. Since here hypergraphs are colored instead of graphs, the equivalent formulations of Theorem 1.2 have to be modified accordingly. Instead of intersecting $r$-partite (multi)hypergraphs we have *$t$-wise intersecting* (multi)hypergraphs (i.e., hypergraphs in which any $t$ edges have a common vertex). Then – similarly to the arguments leading to (1) and (2) – one can estimate $f(n, r, t)$ as follows.

**Lemma 3.10 ([17]).** $f(n, r, t) \ge n/\tau^*(r, t)$ *where*

$$\tau^*(r, t) = \max\{\tau^*(\mathcal{H}) : \mathcal{H} \textit{ is } r\textit{-partite, } t\textit{-wise intersecting hypergraph}\}.$$

The example showing that Theorem 3.9 is sharp when stated is a natural extension of the construction in Sect. 1.2 from affine planes to affine spaces of dimension $t$. Consider $A(t, q)$, the affine space of dimension $t$ and order $q$, define color class $i$ by the $t$-element subsets of points that are within some hyperplane of the $i$th parallel class of hyperplanes. This coloring can be extended by substituting sets for points of $A(t, q)$ as in Sect. 1.3; in particular, if $n = q^t m$, one can substitute $m$ vertices to all points of $A(t, q)$.

It is worth noting that for $t = 2$ we have $r = q + 1$ and Theorem 3.9 becomes Theorem 1.2. For $t \ge 3$ there are big gaps in the values of $r$ for which Theorem 3.9

provides a sharp answer. For example, if $t = 3$, we get from Theorem 3.9 that for $r \leq 3$ we have a spanning monochromatic connected subhypergraph (i.e., one spanning all vertices) and for $r = 7$ we have one spanning at least $n/2$ vertices. For four five, and six colors Theorem 3.9 provides the same lower bound ($n/2$). The value of $\tau^*(4,3)$ was determined in [25] and the values $\tau^*(5,3), \tau^*(6,3)$ in [29]. Through Lemma 3.10 it follows that

**Theorem 3.11.**

$$f(n,4,3) \geq \frac{3n}{4}[25], \quad f(n,5,3) \geq \frac{5n}{7}, \quad f(n,6,3) \geq \frac{2n}{3}[29].$$

In fact, Theorem 3.11 is sharp for infinitely many $n$ (when the fractions in the lower bounds are integers).

# 4 Multicolorings: Type of Components

It would be interesting to know more about the structure of the largest monochromatic components. In the basic extremal colorings (Sect. 1.2) the components are complete graphs and after substitutions (Sect. 1.3) the components are balanced complete partite graphs. Thus it is expected that extremal colorings have strong connectivity properties.

## *4.1 Components with Large Matching*

In Ramsey-type applications, for example, in [19, 30], and others, it turned out that the problem of finding a large matching in a monochromatic component can be applied to Ramsey problems concerning paths and cycles. Let $g(n,r)$ be the maximum $m$ such that in every $r$-coloring of $K_n$ there is a monochromatic component with a matching that covers at least $m$ vertices. There are two natural upper bounds for $g(n,r)$. From the constructions showing that Theorem 1.2 is sharp (at least asymptotically) it follows that $g(n,r) \leq n/(r-1)$ for infinitely many $n$ and $r$. Since the Ramsey number of matchings was determined long ago by Cockayne and Lorimer [13], it follows that $g(n,r) \leq 2n/(r+1)$. The two bounds coincide for $r = 3$ and in [30] it was proved that indeed, $g(n,3)$ is asymptotic to $n/2$ and this was a very important step to determine exactly the three color Ramsey number of paths. The following is probably a difficult problem (even for $r = 4$).

**Problem 4.1.** Is $g(n,r)$ asymptotic to $n/(r-1)$?

The affirmative answer would imply (through the regularity lemma, applying a principle introduced by Łuczak in [44]) that the $r$-color Ramsey number of $P_n$ is asymptotic to $(r-1)n$ and would probably be useful in many other applications as well.

### 4.2 Double Stars

Another type of component that emerged from the first proof of Theorem 1.2 is the double star. Corollary 3.3 states that in $r$-colorings of $K_n$ where at least one color class is disconnected, there is a monochromatic double star with at least $n/(r-1)$ vertices. Perhaps this statement remains true for all colorings.

**Problem 4.2.** For $r \geq 3$, is there a monochromatic double star of size asymptotic to $n/(r-1)$ in every $r$-coloring of $K_n$?

However, even the following problem is open.

**Problem 4.3.** Is there a constant $d$ (perhaps $d = 3$) such that in every $r$-coloring of $K_n$ there is a monochromatic subgraph of diameter at most $d$ with at least $n/(r-1)$ vertices?

For $r = 3$ the affirmative answer to Problem 4.3 follows from a result of Mubayi.

**Theorem 4.4 ([45]).** *In every* 3*-coloring of* $K_n$ *there is a monochromatic subgraph of diameter at most four with at least* $n/2$ *vertices.*

The best known estimate for double stars is the following.

**Theorem 4.5 ([33]).** *For* $r \geq 2$ *there is a monochromatic double star with at least* $(n(r+1)+r-1)/r^2$ *vertices in any* $r$*-coloring of the edges of* $K_n$.

**Corollary 4.6 ([33]).** *In every* 2*-coloring of* $K_n$ *there is a monochromatic double star with at least* $(3n+1)/4$ *vertices.*

Corollary 4.6 is close to best possible, 2-colorings of $K_n$ where the largest monochromatic double star is asymptotic to $3n/4$ and can be obtained from random graphs or from Paley graphs. In [15] the existence of such a 2-coloring was proved by the random method. However, for $r \geq 3$ the random method does not provide a good upper bound for $f(n,r)$.

Observing that a double star has diameter at most three, the bound in Theorem 4.5 provides a slight improvement (for $r \geq 3$) of the following result of Mubayi.

**Theorem 4.7 ([45]).** *There is a monochromatic subgraph of diameter at most three with at least* $n/(r-1+1/r)$ *vertices in every* $r$*-coloring of* $K_n$.

## 5 Variations

We finish the survey by showing some variations of the basic theme in chronological order.

### 5.1 Vertex-Coverings by Components

A well-known conjecture, frequently cited as the Lovász–Ryser conjecture, states the following extension of Theorem 1.2. It is stated in three forms to parallel Theorem 1.2. $\tau(\mathcal{H})$ denotes the transversal number, the minimum number of vertices needed to intersect all edges of $\mathcal{H}$.

*Conjecture 5.1.* The following equivalent statements are true:

- In every $r$-coloring of $K_n$, $V(K_n)$ can be covered by the vertex sets of at most $r-1$ monochromatic components.
- If $r$ partitions are given on a ground set of $n$ elements such that each pair of elements is covered by some block of the partitions then the ground set can be covered by at most $r-1$ blocks.
- For every intersecting $r$-partite (multi)hypergraph $\mathcal{H}$, $\tau(\mathcal{H}) \leq r-1$.

Conjecture 5.1 is proved for $r \leq 4$ in [25] and for $r = 5$ in [48]. Related problems can be found in a recent survey by Kano and Li [38].

### 5.2 Coloring by Group Elements

Bialostocki and Dierker conjectured that Proposition 1.1 can be generalized as follows. In every coloring of the edges of $K_{n+1}$ with colors in $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ there is a spanning tree with color sum zero modulo $n$ (to get Proposition 1.1, use 0, 1 as two colors). The conjecture is proved for $n$ prime in [2] and for general $n$ in [18], [47]. In fact, the proof of Schriver and Seymour in [47] works for hypergraphs as well. An $r$-uniform hypertree $\mathcal{T}$ is a connected $r$-uniform hypergraph with $p$ edges on $p(r-1)+1$ vertices. Notice that for $r = 2$ we get the usual definition of a tree in graphs.

**Theorem 5.2 ([47]).** *Suppose that $\mathcal{K}$ is the complete $r$-uniform hypergraph on $p(r-1)+1$ vertices and the edges of $\mathcal{K}$ are labeled with an Abelian group of order $p$. Then $\mathcal{K}$ has a spanning hypertree with total weight zero.*

### 5.3 Coloring Geometric Graphs

Following [46], a *geometric graph* is a graph whose vertices are in the plane in general position and whose edges are straight-line segments joining the vertices. A geometric graph is *convex* if its vertices form a convex polygon. A subgraph of a geometric graph is *noncrossing* if no two edges have a common interior point. Ramsey-type problems for geometric graphs were first studied in [39] and [40]. The following result of Károlyi, Pach, and Tóth (a geometric generalization of Proposition 1.1) was conjectured by Bialostocki and Erdős (see [3] with a proof for convex geometric graphs).

**Theorem 5.3 ([39]).** *In every 2-coloring of a geometric complete graph there is a non-crossing monochromatic spanning tree.*

*Proof.* The nice inductive proof of Theorem 5.3 from [39] is as follows. We may assume that vertices $P_1, \ldots, P_n$ of the geometric $K_n$ have strictly increasing $x$-coordinates. Set $L(i) = \{P_j : 1 \leq j \leq i\}$, $R(i) = \{P_j : i < j \leq n\}$. We may also assume that the edges along the convex hull (meaning, really, the perimeter of the convex hull) of $K_n$ have the same color, say red, otherwise induction works by removing a point $P_j$ of the convex hull where two colors meet. Induction also works if for any $i$, $2 \leq i \leq n-1$, the monochromatic spanning trees in $L(i), R(i)$ have the same color. Thus these spanning trees switch colors at each $i$, moreover for $i = 2$ the switch is from red to blue and for $i = n-1$ the switch is from blue to red, otherwise a red edge along the convex hull from $P_1$ or from $P_n$ would define red noncrossing spanning trees. The conclusion is that for some $i$, $2 \leq i \leq n-2$, there is a red–blue switch at $i$ and blue–red switch at $i+1$. Taking a (red) edge along the convex hull that joins the left (red) tree at $i$ with the right (red) tree at $i+1$ results in a red noncrossing spanning tree. □

One can ask whether Lemmas 3.1 and 3.2 have geometric versions as well. The simplest case is when the complete bipartite graph is balanced and drawn with partite sets $A = \{(1,0), (2,0), \ldots, (n,0)\}$ and $B = \{(1,1), (2,1), \ldots, (n,1)\}$ (and the edge $ab$ is the line segment joining $a \in A$ and $b \in B$ in $R^2$). Call this representation a *simple geometric* $K_{n,n}$.

It is possible that (for two colors) Lemma 3.1 extends to simple geometric $K_{n,n}$ (perhaps even for arbitrary drawings of $K_{n,n}$).

**Problem 5.4 ([27, 28]).** In every 2-coloring of a simple geometric $K_{n,n}$ there is a noncrossing monochromatic subtree (a caterpillar) with at least $n$ vertices.

However, the stronger result, Lemma 3.2 does not extend but has the following geometric version.

**Theorem 5.5 ([27, 28]).** *In every 2-coloring of a simple geometric $K_{n,n}$ there is a noncrossing monochromatic double star with at least $4n/5$ vertices. This bound is asymptotically best possible.*

## 5.4 Coloring Noncomplete Graphs

Can one extend some of the results above from complete graphs to arbitrary graphs? Somewhat surprisingly, the answer is yes. Theorem 1.2 can be extended to arbitrary graphs as follows. Let $\alpha(G)$ denote the cardinality of a largest independent set of $G$.

**Theorem 5.6.** *The following equivalent statements are true:*

- *In every $r$-coloring of a graph $G$ with $n$ vertices there is a monochromatic component with at least $n/((r-1)\,\alpha\,(G))$ vertices.*
- *If $r$ partitions are given on a ground set of $n$ elements such that among any $\alpha+1$ elements at least one pair is covered by some block of the partitions then one of the partitions has a block of size at least $n/((r-1)\alpha)$.*
- *If an $r$-partite hypergraph has $n$ edges and among them at most $\alpha$ are pairwise disjoint then it has a vertex of degree at least $n/((r-1)\alpha)$.*

*Proof.* The equivalence of the statements can be proved by the same translation process as in Theorem 1.2. Their proof is again by Füredi's method, using his result in a form that is more general than in the previous applications. Let $\nu(\mathcal{H})$ denote the maximum number of pairwise disjoint edges in a hypergraph $\mathcal{H}$. □

**Theorem 5.7 ([20]).** *If an $r$-uniform hypergraph $\mathcal{H}$ does not contain a projective plane of order $r-1$ than $\tau^*(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$.*

To see that the third statement of Theorem 5.6 holds, let $\mathcal{H}$ be an $r$-partite hypergraph with $n$ edges and $\nu(\mathcal{H}) \leq \alpha$. Since a finite plane of order $r-1$ is obviously not $r$-partite, Theorem 5.7 applies and – as in previous applications – (1) in Sect. 3.2 and (2) in Sect. 3.4,

$$\frac{|E(\mathcal{H})|}{D(\mathcal{H})} \leq \nu^*(\mathcal{H}) = \tau^*(\mathcal{H}) \leq (r-1)\alpha,$$

where $D$ is the maximum degree of $\mathcal{H}$. Thus we have

$$\frac{n}{(r-1)\alpha} = \frac{|E(\mathcal{H})|}{(r-1)\alpha} \leq D(\mathcal{H}).$$

□

Theorem 5.6 may give hope that results mentioned so far for coloring complete graphs can have (hopefully nice) extensions or at least analogues for coloring graphs with fixed independence number. It looks as if this area is rather unexplored; almost all previous results can be the subjects of investigation. The test cases can very well be graphs with $\alpha(G) = 2$.

One particular attempt is started in [34] to extend Gallai-colorings to arbitrary graphs as edge colorings without multicolored triangles. Suppose that we have a Gallai-coloring of a graph $G$ with $\alpha(G) = 2$. Let $f(n)$ be the minimum order of the largest monochromatic connected subgraph over all such colorings of graphs with $n$ vertices. Clearly, by looking at the union of two disjoint complete graphs, $f(n) \leq n/2$. At first sight it is not clear that $f(n)$ is linear; it turns out [34] that it is,

$$\frac{n}{5} \leq f(n) \leq \frac{3n}{8}$$

but not with coefficient $\frac{1}{2}$. In general, $f(n,\alpha)$ is within reasonable limits.

**Theorem 5.8 ([34]).** $(\alpha^2+\alpha-1)^{-1}n \le f(n,\alpha) \le (cn\log\alpha)/\alpha^2$

The following quick proof of the linearity of $f(n)$ (with a coefficient weaker than in Theorem 5.8) points to a far-reaching generalization. Let $G$ be a graph with $n$ vertices with a Gallai-coloring. By Ramsey's theorem every set of $k = R(3,\alpha(G)+1)$ vertices contains a triangle. By easy counting this implies that $G$ has at least $cn^3$ triangles, where $c$ depends only on $\alpha$. To each triangle $T$ assign an edge of $T$ whose color is repeated in $T$. By the pigeonhole principle, some $xy \in E(G)$ is assigned to $cn^3/\binom{n}{2} \ge 2cn$ triangles $T_i = xyz_i$. Since in each $T_i$ there are two edges in the color of $xy$, say in red, the graph spanned by the red edges in the union of the $\{x,y,z_i\}$ is connected, and has at least $2cn+2$ vertices.

With the idea of the proof above, Theorem 5.8 can be extended to hypergraphs and also to colorings that do not contain any multicolored copy of a fixed hypergraph $F$ (in Gallai-colorings $F = K_3$). As for graphs, for a hypergraph $\mathcal{H}$, $\alpha(\mathcal{H})$ denotes the maximum cardinality of $S \subset V(\mathcal{H})$ such that no edges of $\mathcal{H}$ are completely in $S$.

**Theorem 5.9 ([34]).** *Suppose that the edges of an $r$-uniform hypergraph $\mathcal{H}$ are colored so that $\mathcal{H}$ does not contain multicolored copies of an $r$-uniform hypergraph $F$. Then there is a monochromatic connected subhypergraph $\mathcal{H}_1 \subseteq \mathcal{H}$ such that $|V(\mathcal{H}_1)| \ge c|V(\mathcal{H})|$, where $c$ depends only on $\mathcal{F}$, $r$, and $\alpha(\mathcal{H})$ (thus does not depend on $\mathcal{H}$).*

**Acknowledgement** Thanks to Arie Bialostocki for his inspiring conjectures and for his remarks on this survey. The careful reading of a referee is also appreciated. Research supported in part by OTKA Grant No. K68322.

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# Szlam's Lemma: Mutant Offspring of a Euclidean Ramsey Problem from 1973, with Numerous Applications

Jeffrey Burkert and Peter Johnson

## 1 1973: A Volcano Erupts

If you treasure semantic precision, you might name the volcano that erupted (figuratively speaking!) in 1973: Coloring Problems in Geometrically Defined Hypergraphs. We prefer: Euclidean Coloring Problems.

The main lava spout of the volcano was a paper: "Euclidean Ramsey Theorems I" [2]. As the name of the paper indicates, the focus there was on coloring problems in Euclidean spaces that can be given a Ramsey form. For instance, in one class of problems, one fixes a finite subset $F$ of some Euclidean space, and a number $r$ of colors one is allowed to use, and then one asks: is there a positive integer $N$ such that for all $n \geq N$, for every coloring of $n$-dimensional Euclidean space with $r$ colors, some copy of $F$ in the space must be monochromatic? And, if so, what is the smallest value of $N = N(F, r)$? (There are different meanings of "copy" available, but in this context "copy of" usually means "set congruent to." See the next section for definitions.) What makes these problems "Ramsey"? We do not attempt an answer. We think it wisest to leave Ramseyness as an informal concept; with some experience, you will know Ramseyness when you see it.

It seems to us that the great legacy of the 1973 eruption is simply the problems, whether Ramsey-inspired, or part of a Ramsey tower of problems. Here is the 1973 problem of the title of this paper, as announced in the abstract:

> Is it possible to color the Euclidean plane with two colors, say red and blue, so that no two blue points are a distance 1 apart and no four red points are the vertices of a unit square, a square of side length 1?

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A. Soifer (ed.), *Ramsey Theory: Yesterday, Today, and Tomorrow*,
Progress in Mathematics 285, DOI 10.1007/978-0-8176-8092-3_6,

This problem did not appear in [2], but in [3], in 1975. We are calling it a "problem from 1973" because the second author heard about it in 1973, from Don Greenwell, which goes to show that the lava was flowing from more than one spout in those days. Where Greenwell got the problem from he does not remember, although he had just finished his PhD in graph theory at Vanderbilt, at that time a noted hot spot for graph theory and combinatorics, and had spent some time as the officemate of a young visiting Hungarian mathematician, Lázlo Lovász.

## 2 Some Definitions and More Background

As in [2, 3], $\mathbb{E}^n$ stands for $n$-dimensional Euclidean space, meaning the vector space of $n$-tuples of real numbers equipped with the usual Euclidean distance. The Euclidean distance between $n$-tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{E}^n$ is $\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$, sometimes abbreviated $|x - y|$. We may revert to denoting the real numbers, which is the field of scalars for the vector spaces $\mathbb{E}^n$, by $\mathbb{R}$. Therefore the set of real numbers has two names, $\mathbb{R}$ and $\mathbb{E}^1$.

A $k$-coloring of a set $A$ is a function from $A$ into a set of cardinality no greater than $k$. Alternatively, a $k$-coloring of $A$ can be thought of as a partition of $A$ into $k$ or fewer subsets.

If $S \subseteq \mathbb{E}^n$ for some $n$ and $d > 0$, a *rather red coloring* of $S$ with respect to $d$ is a 2-coloring of $S$, with red and blue, such that no two blue points are a distance $d$ apart. The term "rather red" is intended to indicate that the red set will predominate, and this will usually be the case if $S \neq \emptyset$ and $S$ is closed under multiplication by positive scalars. For instance, when $S = \mathbb{E}^2$, as in the problem from 1973 mentioned in Sect. 1, for every blue point in a rather red coloring with respect to 1 the entire unit circle of radius 1 centered at the blue point is red; and for every blue disc there is a red annulus around it. So one expects to see more red than blue, in a rather red coloring of $E^2$. Of course, if no points of $S$ are a distance $d$ apart then coloring $S$ all blue gives a rather red coloring of $S$ with respect to $d$, but we find the "rather red" terminology sufficiently appealing that we can endure such infelicities.

Incidentally, it would seem to be a nice problem in computational geometry to determine the supremum (maximum?) of the upper densities of the blue set in rather red colorings $\mathbb{E}^n$ in which the blue set (and therefore, also, its complement, the red set) is Lebesgue measurable. Letting $B$ be the blue set of such a coloring, $R(m)$ the $n$-dimensional box with $2^n$ vertices $(\pm m, \ldots, \pm m)$, and $\lambda$ denote Lebesgue measure, the upper density of $B$ is $\limsup_{m\to\infty} (\lambda(B \cap R(m)))/((2m)^n)$. When $n = 1$ it is clear that the maximum such upper density is $1/2$ (proof omitted), and we suspect that when $n = 2$ the maximum is $\pi/8\sqrt{3}$, achieved by arranging open blue disks of diameter 1, each a distance 1 from each of the nearest six others, in the pattern of the vertices of the usual triangular lattice in the plane.

We wonder if it makes a difference if the color sets are not required to be Lebesgue measurable, and Lebesgue measure is replaced by outer Lebesgue measure, in the definition of upper density. This problem may be more measure- or set-theoretic than geometric. We do not know the answer even when $n = 1$.

To return to the main point that we are driving toward in this section: rather red colorings are the main ingredient in a large and interesting family of Euclidean coloring problems, of the following form: for a given subset $S$ of some $\mathbb{E}^n$, and a distance $d$, one asks if there is a rather red coloring of $S$ with respect to $d$ which *forbids for red* each member of a family of geometric configurations in $S$; that is, no member of the family is to be all red. Usually $S = \mathbb{E}^n$ and $d = 1$ and the family of geometric configurations to be forbidden for red is the collection of sets *congruent* to some fixed finite set $F \subseteq S$. As usual, two subsets of $\mathbb{E}^n$ are congruent if one is the image of the other under the composition of a translation and a rotation (in either order). If either $X$ and $Y$ are congruent, or $X$ and some reflection of $Y$ are congruent, then we say that $X$ and $Y$ are weakly congruent. If $X$ and $Y$ are (weakly) congruent then we say that $Y$ is a (weakly) congruent copy of $X$. If $X$ is symmetric with respect to some reflection of $\mathbb{E}^n$, meaning that it is congruent to its own reflection, then weak congruence to $X$ is the same as congruence to $X$.

Thus the problem from 1973 resurrected in Sect. 1 may be restated: is there a rather red coloring of $\mathbb{E}^2$, with respect to the distance 1, which forbids for red congruent copies of the four vertices of a unit square?

To see why this problem was judged to be the next big thing among rather red coloring problems in the early 1970s, by those best able to judge, the authors of [2, 3], here is a proof of a proposition about forbidding 3-point sets for red with a rather red coloring of $\mathbb{E}^2$.

**Proposition 2.1.** *For every rather red coloring of* $\mathbb{E}^2$*, the red set contains congruent copies of every set of* 3 *points in* $\mathbb{E}^2$.

*Proof.* Suppose, to the contrary, that there is a rather red coloring, say with respect to the distance 1, of $\mathbb{E}^2$ and a 3-point set $T = \{u, v, w\}$ such that no congruent copy of $T$ is all red. Suppose that $d = |u - v| \le |u - w|, |v - w|$.

*Case 1*: $0 < d \le 2$. There must be a blue point, and thus there must be a red circle $C$ of radius 1, somewhere in the plane. Place a congruent copy $T' = \{u', v', w'\}$ of $T$ so that $u'$ and $v'$ are on $C$ and $w'$ is outside of $C$. (For machine-checking this proof, a bit of argument is necessary to see that such a $T'$ can be placed, but if the reader will try one or two instances of $T$, this obstacle will quickly disappear.) Now think of $T'$ moving so that $u'$ and $v'$ stay on the red circle. Then $w'$ will move around a blue circle of radius $> 1$; but then there are certainly two blue points a distance 1 apart. Therefore, we are in:

*Case 2*: $d > 2$. If no two blue points are a distance $d$ apart then our coloring is rather red with respect to $d$, as well as to 1, and the argument in Case 1 then shows that the red set must contain a congruent copy of $T$.

Therefore there are two blue points a distance $d > 2$ apart. Let $C_1, C_2$ be the two red circles of radius 1 about them. Place a congruent copy $T' = \{u', v', w'\}$ of $T$ so that $u', v'$ are on $C_1, C_2$, respectively, with the line segment between them parallel to the segment joining the blue centers of $C_1$ and $C_2$. See Fig. 1.

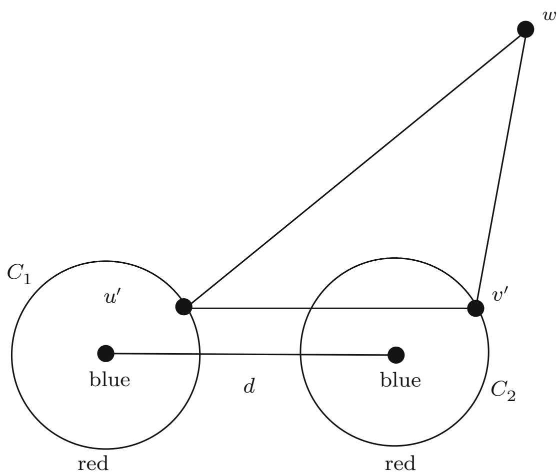


**Fig. 1** Part of the proof of Proposition 2.1

Let $T'$ move so that $u', v'$ stay on $C_1, C_2$, respectively. Then $w'$ describes a blue circle of radius 1, so, again, there are blue points a distance 1 apart. Consequently, the red set must contain a congruent copy of $T$ after all. □

On the grounds that 4 is the next integer after 3, and that if you do not know if any 4-point subset of $\mathbb{E}^2$ can be forbidden for red by a rather red coloring of $\mathbb{E}^2$, then you may as well focus on a famous 4-point set, the vertices of a square. Proposition 2.1 establishes why the problem of Sect. 1 was plausibly next in a queue (more exactly, next along a path from the root in a rooted tree) of Euclidean coloring problems. But the story is a bit more complicated. Here, from [3], is a stronger result than Proposition 2.1, with a shorter proof.

**Proposition 2.2 ([3]).** *For every rather red coloring of $\mathbb{E}^2$, the red set contains a translate of each set of three points in $\mathbb{E}^2$.*

*Proof.* Suppose that we have a rather red coloring of $\mathbb{E}^2$ with respect to the distance 1 and that $T = \{t_1, t_2, t_3\} \subseteq \mathbb{E}^2$. Let $V$ denote the set of seven vertices of the graph in Fig. 2. This graph drawn in the plane with the edges being line segments of equal length – say 1 – is known as the *Moser spindle*. It first appeared in 1961 [15], and it has had a role in the evolution of Euclidean coloring theory in many ways analogous to the role of the Petersen graph in graph theory as a whole. See [16] for more on this topic. Observe that the Moser spindle has vertex independence number 2. Therefore, for each $t \in T$ at most two of the points in $t+V = \{t+v | v \in V\}$ are blue. Therefore $|\{v \in V|$ for some $t \in T, t+v$ is blue$\}| \leq 6$. Since $|V| = 7$ it follows that for some $v \in V$, $T+v$ is red. □

In Sect. 5 we show another proof of Proposition 2.2, a proof that seems related to this one, but which was discovered quite independently.

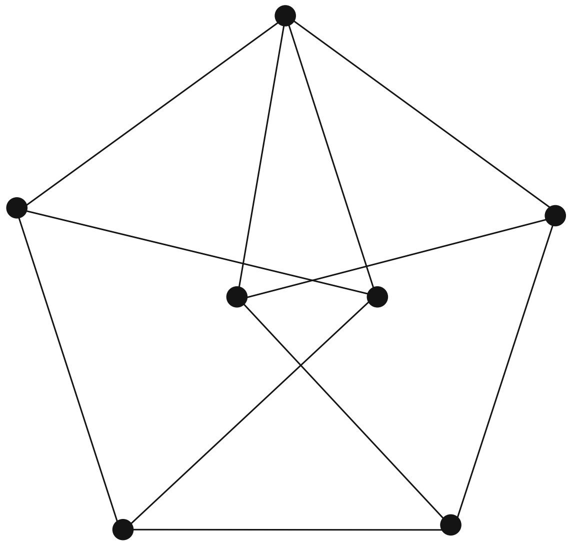

**Fig. 2** The Moser spindle

Proposition 2.2 points out additional nodes in the hierachy of problems concerning rather red colorings. For instance, should it turn out (as we see that it does, in the next section) that there is no rather red coloring of $\mathbb{E}^2$ with respect to 1 which forbids red congruent copies of the vertices of a unit square, then one can ask: what if "congruent copies" is replaced by "translates"? We show in Sect. 5 that this very question is both open and unexpectedly fraught with significance.

## 3 What Happened to the Rather Red Coloring Problem from 1973?

The second author and Don Greenwell, hungry young new PhDs at the time, got an attractive result on the aforementioned problem. In any such coloring, both the red set and the blue set would be dense in $\mathbb{E}^2$, and the red set would be so dense that the connected path components of the blue set would be single points. But we published nothing on the matter, outside of an abstract for a conference talk, partly because we felt strongly that our theorem would prove to be vacuously true; that is, we believed that there were no such rather red colorings of $\mathbb{E}^2$. From this distance in time we wonder what the ethics are on publishing a cute result about mathematical entities that you are pretty sure do not exist. Greenwell and Johnson didn't ponder the matter in that way, but they were in no hurry. Perhaps if the problem had remained unsolved for a while, they would have ventured forth with the density result.

But in 1976, although the paper didn't appear until 1979 [13], the murmur went around that Rozália Juhász had settled the matter. Not only was there no rather red coloring of $\mathbb{E}^2$ with respect to the distance 1 which would forbid for red congruent copies of the vertices of a unit square: in fact, for every rather red coloring of $\mathbb{E}^2$, the red set must contain congruent copies of *every* four-point subset of $\mathbb{E}^2$.

In the opinion of the second author, this result ranks in the history of Euclidean coloring theorems very much as Rod Laver ranks in the history of tennis; but there is another result in [13] that has a place in history as well. Juhász gives an example of a 12-point subset of $\mathbb{E}^2$ and a rather red coloring which forbids congruent copies of this subset for red. This was a considerable improvement over the previous record for this sort of thing: Erdös et al., in [3], had given a subset of $\mathbb{E}^2$ with $10^{12}$ points which could be forbidden for red by a rather red coloring of $\mathbb{E}^2$.

Let $m_c = m_c(\mathbb{E}^2) = \min[|F|; F \subseteq \mathbb{E}^2$ and congruent copies of $F$ are forbidden for red by some rather red coloring of $\mathbb{E}^2]$. By 1979, after the publication of [13], R. Juhász had single-handedly established that

$$5 \leq m_c \leq 12.$$

In [1] it was established that $m_c \leq 8$, so we have

$$5 \leq m_c \leq 8,$$

and there the matter rests. But we have something more to say on this matter in Sect. 5.

## 4 Distance Graphs

If $X$ is a nonempty set and $\rho : X \times X \to [0, \infty)$ satisfies: for all $x, y \in X$, $\rho(x, y) = \rho(y, x)$ and, $\rho(x, y) = 0$ if and only if $x = y$, then we say that $\rho$ is a *distance function on* $X$. A *metric* is a distance function which satisfies the triangle inequality: $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

If $\rho$ is a distance function on $X$ and $D \subseteq (0, \infty)$, the *distance graph* $G = G(X, D)$ (notice that mention of $\rho$ is suppressed in the notation) is the graph defined by $V(G) = X$ and $xy \in E(G)$ if and only if $\rho(x, y) \in D$. If $D = \{d\}$, we write $G(X, d)$ rather than $G(X, \{d\})$. If $P$ is a graph parameter we write $P(X, D)$ rather than $P(G(X, D))$. If $X \subseteq \mathbb{E}^n$ it is understood, unless otherwise specified, that $\rho$ is the usual Euclidean distance. Notice that if $X \subseteq \mathbb{E}^n$ is closed under multiplication by positive scalars (real numbers), then all the single-distance graphs $G(X, d)$, $d \in (0, \infty)$ are isomorphic; conventionally $G(X, 1)$ is taken as the representative of these graphs.

The problem of determining the chromatic number $\chi(\mathbb{E}^2, 1)$ of the unit distance graph "in the plane" (i.e., with $\mathbb{E}^2$ as vertex set and Euclidean distance) is a famous one. Posed by Edward Nelson in 1950 in his first year as a student at the University of Chicago, it made its way through the intellectual plasma by word of mouth before appearing in print in the 1960s in [5, 10, 15], and in numerous papers of Paul Erdös. See [16], Chap. 2, for the full story, or as near to the full story as will ever be known.

It seems likely that the growth of interest in this problem, helped in no small measure by a stream of mentions by Erdös, was a significant source of the energy that led to the Euclidean coloring eruption of 1973.

Yet not only has the problem resisted solution, in fact the range of possible values of $\chi(\mathbb{E}^2, 1)$ has not diminished since 1950:

$$4 \leq \chi(\mathbb{E}^2, 1) \leq 7.$$

(Hadwiger [10] attributes both bounds to John Isbell, a fellow student of Nelson's, but Isbell credits Nelson with the lower bound. See *The MCB* [16]!) For future reference we look briefly at the proofs of these bounds. Of the inequality $4 \leq \chi(\mathbb{E}^2, 1)$ there are two proofs, although some would say that they are two versions of the same proof. The proof due to the Mosers [15] involves simply the verification that the Moser spindle, in Fig. 2, is realizable as a subgraph of $G(\mathbb{E}^2, 1)$, and that its chromatic number is 4. The other proof, published by Hadwiger [10], is, according to John Isbell, the same as that discovered by Edward Nelson; we might call it "the big red circle" proof (see Fig. 3). It goes like this: because of the existence of equilateral triangles, $\chi(\mathbb{E}^2, 1) \geq 3$, so suppose that $\chi(\mathbb{E}^2, 1) = 3$; suppose that $\mathbb{E}^2$ is colored with red, blue, and green so that no two points a distance 1 apart are the same color. There must be a red point somewhere. Take one. Now, the circle of radius 1 around that red point is blue and green. Take two points on that circle that are a distance 1 apart. One is blue and one is green, and therefore the other point besides the center of the circle which is a distance 1 from both points is red. But this shows that the entire circle of radius $\sqrt{3}$ around the original red point is red, so there are certainly two red points a distance 1 apart, which is a contradiction.

The proof that $\chi(\mathbb{E}^2, 1) \leq 7$, due to Isbell in November 1958 [16], is constructive, using a device invented by Hadwiger in 1945 for use on a different problem. One forms a *Hadwiger tile* by taking a regular hexagon and the six regular hexagons surrounding it, as though in a tiling of the plane by regular hexagons; let the diameter of the hexagons be slightly less than 1 and color the seven hexagons in the tile with seven colors. The diameter of each hexagon could be exactly 1, but then you would have to take care in coloring the boundaries of the hexagons; the important thing is that no two points distance 1 apart within the same hexagon are to be the same color.

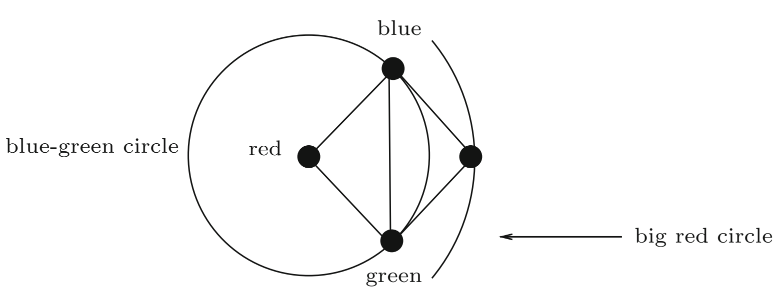


**Fig. 3** The big red circle proof that $\chi(\mathbb{E}^2, 1) > 3$

And then one tiles the plane with precolored translates of this Hadwiger tile. It turns out that for each constituent hexagon, the nearest hexagons of the same color are too far away for there to be two points of that color a distance 1 apart.

## 5 Szlam's Lemma, a Connection Between Rather Red Colorings and Chromatic Numbers

In 1999 Arthur Szlam, at the time a participant in a summer Research Experience for Undergraduates program at Auburn University, discovered the following result, which we take the liberty of stating somewhat differently from the main result in [17].

**Szlam's Lemma I.** *Suppose that $S \subseteq \mathbb{E}^n$ is closed under vector addition, $d > 0$, $F \subseteq S$, and there is a rather red coloring of $S$ with respect to $d$ that forbids for red all translates of $F$ in $S$. Then $\chi(S, d) \leq |F|$.*

*Proof.* For each $v \in S$, $v + F$ is not all red, so we can find $f = \varphi(v) \in F$ such that $v + f$ is blue. Thus we have a function $\varphi : S \to F$ such that $v + \varphi(v)$ is blue for every $v \in S$. (We are pretty sure that if $F$ is finite, then the existence of $\varphi$ does not require the axiom of choice!) We claim that the $|F|$-coloring $\varphi$ forbids the distance $d$, which would imply the conclusion. If $u, v \in S$ and $\varphi(u) = \varphi(v) = f \in F$, then $u + f, v + f$ are blue, so $d \neq |(u + f) - (v + f)| = |u - v|$. □

As a corollary, since $\chi(\mathbb{E}^2, d) = \chi(\mathbb{E}^2, 1) \geq 4$ for any $d > 0$, Proposition 2.2 follows; this is the alternative proof promised in Sect. 2. One senses a close cousinship of the two proofs, but they do not seem to be different versions of the same proof.

For one thing, we see immediately by Szlam's lemma, but not by the first proof of Proposition 2.2, that if there were a rather red coloring of $\mathbb{E}^2$ which forbade for red translates of a set $F \subseteq \mathbb{E}^2$ such that $|F| = 4, 5$, or $6$, then we would have a genuine breakthrough on the problem of determining $\chi(\mathbb{E}^2, 1)$; a solution, in fact, if $|F| = 4$. (But we are not betting on the existence of such a rather red coloring and such an $F$. It may be that $\chi(\mathbb{E}^2, 1) = 4$ and yet that for every rather red coloring of $\mathbb{E}^2$, the red set contains translates of every 4-point subset of $\mathbb{E}^2$.) Consequently, Szlam's Lemma I incentivizes the search for the value of $m_c(\mathbb{E}^2) \in \{5, 6, 7, 8\}$, discussed in Sect. 1.

A propos, let us define $m_t(\mathbb{E}^2) = \min[|F|; F \subseteq \mathbb{E}^2$ and there exists a rather red coloring of $\mathbb{E}^2$ such that no translate of $F$ is all red$]$. By Proposition 2.2, or by Szlam's Lemma I, $4 \leq m_t(\mathbb{E}^2)$. Intriguingly, we have $4 \leq m_t(\mathbb{E}^2) \leq 7$. To see that $m_t(\mathbb{E}^2) \leq 7$, let the plane be tiled by translates of the Hadwiger tile, shown in Fig. 4. Color the closed hexagon at the center of each tile blue, and color everything else red. Let $F$ be the set of centers of the seven regular hexagons that make up each tile. If $h$ is the diameter of each of the regular hexagons comprising the tiles then the

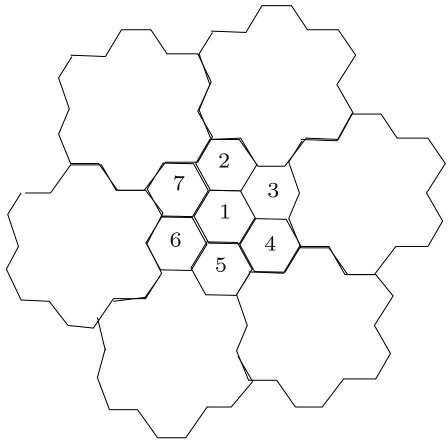


**Fig. 4** A Hadwiger tile, colored with seven colors, surrounded by other Hadwiger tiles

coloring is rather red with respect to each distance $d \in (h, h(\sqrt{7}/2))$, and at least one point, and sometimes two points, of every translate of $F$ is blue.

Szlam noticed another remarkable connection among $\chi(\mathbb{E}^2, 1)$, $m_c(\mathbb{E}^2)$, and $m_t(\mathbb{E}^2)$.

**Proposition 5.1 ([17]).** *Either* $\chi(\mathbb{E}^2, 1) > 4$ *or* $m_c(\mathbb{E}^2) \leq 7$.

*Proof.* Suppose $\chi(\mathbb{E}^2, 1) = 4$ and let $\mathbb{E}^2$ be colored with four colors so that no two points of the same color are a distance 1 apart. Let one of the colors be blue and consider points of any of the other three colors to be red. Since no two blue points are a distance 1 apart, this coloring of $\mathbb{E}^2$ with red and blue is a rather red coloring. Every congruent copy of the Moser spindle, Fig. 2, with edge length 1, is a subgraph of $G(\mathbb{E}^2, 1)$ with chromatic number 4, so all four of the colors in the original coloring must appear on its vertices. Therefore, no congruent copy of the Moser spindle is all red. Thus $m_c(\mathbb{E}^2) \leq 7$, if $\chi(\mathbb{E}^2, 1) = 4$. □

**Corollary 5.2.** *Either* $m_t(\mathbb{E}^2) > 4$ *or* $m_c(\mathbb{E}^2) \leq 7$.

Another distinction that Szlam's Lemma I has over the original proof of Proposition 2.2 is that it has a veritable menagerie of analogues and generalizations, whereas it is not evident how to use the original proof of Proposition 2.2 for anything much beyond proving Proposition 2.2. One very obvious improvement of Szlam's Lemma I is achieved simply by replacing $d \in (0, \infty)$ by a set $D \subseteq (0, \infty)$. By the very same proof one concludes that if $S$ is colored with red and blue so that no distance between blue points is in $D$ and, for some $F \subseteq S$, no translate of $F$ in $S$ is all red, then $\chi(S, D) \leq |F|$. But why stop there? Why not generalize the roles of vector addition and Euclidean distance?

**Szlam's Lemma II.** *Suppose that $\rho$ is a distance function on $X$, a nonempty set, and $*$ is a binary operation on $X$. Suppose that $\rho$ is invariant with respect to right $*$-translates, meaning that, for all $x, y, z \in X$, $\rho(x, y) = \rho(x * z, y * z)$. Suppose* *that* $D \subseteq (0, \infty)$*. Suppose that* $X$ *is colored with red and blue so that if* $x,\ y \in X$ *are both blue then* $\rho(x, y) \notin D$*. Finally, suppose that* $F \subseteq X$ *and that no left* $*$*-translate* $x * F = \{x * f \,|\, f \in F\}$ *of* $F$ *is all red. Then* $\chi(X, D) \leq |F|$*.*

The reader might enjoy the exercise of proving this result, taking the proof of Szlam's Lemma I as a model. Or, the proof can be inspected in [12], where the lemma is applied to the cases when $X = \mathbb{R}^2$, $* = +$, ordinary vector addition in $\mathbb{R}^2$, $\rho$ is a translation-invariant metric on $\mathbb{R}^2$, and $D = \{d\}$. We use the notation $\chi_\rho(\mathbb{R}^2, d)$ to indicate the dependence on $\rho$, in this case. An example is given in [12] in which $\chi_\rho(\mathbb{R}^2, d) = 3$, and $\rho$ determines the usual topology on $\mathbb{R}^2$. It is asked in [12], and the question remains open, whether there exists $\rho$, a translation-invariant metric on $\mathbb{R}^2$ determining the usual topology, such that $\chi_\rho(\mathbb{R}^2, 1) = 2$. And there is the question of how big $\chi_\rho(\mathbb{R}^2, d)$ can be: if $\rho$ is translation-invariant and induces the usual topology on $\mathbb{R}^2$, is it necessarily the case that $\chi_\rho(\mathbb{R}^2, d) \leq \chi(\mathbb{E}^2, 1)$ for all $d > 0$?

In [11] there appeared an attempt at the ultimate generalization of Szlam's lemma: no distances, no binary operations, just sheer naked sets. To grasp this result, know that a *hypergraph* is a pair $\mathcal{H} = (V, \mathcal{E})$ in which $\mathcal{E}$ is a collection of nonempty subsets of $V$. If $\mathcal{E}$ contains no singletons, the *chromatic number* of $\mathcal{H}$, denoted $\chi(\mathcal{H})$, is the smallest cardinal number of colors needed to color $V$ so that no $e \in \mathcal{E}$ is monochromatic.

**Szlam's Lemma III.** [11] *Suppose that* $U$ *and* $V$ *are nonempty sets, and* $R$ *and* $B$ *partition* $U \times V$*. Let* $\mathcal{E}_U = \{S \subseteq U;$ *for each* $v \in V$*,* $(S \times \{v\}) \cap R \neq \emptyset\}$ *and* $\mathcal{E}_V = \{S \subseteq V |$ *for each* $u \in U$*,* $(\{u\} \times S) \cap B \neq \emptyset\}$*. Let* $\mathcal{H}_U = (U, \mathcal{E}_U)$ *and* $\mathcal{H}_V = (V, \mathcal{E}_V)$*. Then either* $\mathcal{E}_V = \emptyset$ *or* $\chi(\mathcal{H}_U) \leq \min_{e \in \mathcal{E}_V} |e|$*.*

Any assertion involving $\chi(\mathcal{H}_u)$ tacitly includes the assertion that $\mathcal{H}_u$ contains no singletons.

The proof of SL3 is very short, but we do not give it here. Interested readers might have a go, using the proof of SLI as a model.

However, we show that SL2, and thus also SL1, is a corollary of SL3. Let $X, \rho, *, D$, and $F$ satisfy the hypothesis of SL2, with respect to some coloring of $X$ with red and blue. Let $U = V = X$ and let $R = \{(x, y) \in U \times V = X^2 \mid x * y$ is red$\}$ and $B = \{(x, y) \in X^2 \mid x * y$ is blue$\}$. Then $R$ and $B$ partition $X^2 = U \times V$ and, in the notation of SL3, $\mathcal{E}_U$ contains every doubleton $\{x, y\} \subseteq X$ such that $\rho(x, y) \in D$, and $F \in \mathcal{E}_V$. Therefore, by SL3, $\chi(X, D) \leq \chi(\mathcal{H}_U) \leq \min_{e \in \mathcal{E}_V} |e| \leq |F|$.

So SL3 is stronger than SL2; in fact, SL3 is the strongest version of Szlam's lemma that we know of. As a general informal rule, the sparser and weaker the hypotheses, the stronger and more applicable the theorem: who could disagree? Well, we do, especially about the "more applicable" claim. In the human world, applicability involves memorability, not just logical strength; if no one can remember a theorem, it is usually headed for the dustbin of history, where it can share space with, for instance, the Wiener Tauberian theorem, which not only went straight to the dustbin, but dragged there all the earlier Tauberian theorems with it.

SL3 [11] was intended to generate interest in Szlam's lemma, but we fear that it had the opposite effect of dampening any interest that [12] and [17] might have generated. (There was in [11] a lame attempt to apply the lemma in a nonstandard way, in which $U = V$ is the set of points on a sphere in $\mathbb{E}^3$, and there is a weird noncommutative binary operation involving rotations. The conclusion was at least not obvious, but the few readers yawned and the ghost of Edsel Ford smiled.

Szlam's lemma is most usefully viewed not as a result, but as a trick or method for obtaining upper bounds on the chromatic numbers of certain hypergraphs by looking for two-colorings of the vertex set satisfying certain requirements. In the last section we show another instance of Szlam's lemma, in a quite different setting, and an application resulting in a new lower bound on the van der Waerden numbers, a not particularly good lower bound, being a lower bound on the cyclic van der Waerden numbers, which are less than the van der Waerden numbers, but which does beat known lower bounds in some cases, and which is quite constructive. Hypergraph colorings can be exhibited! And the method has not been exhausted, in this area.

## 6 van der Waerden Numbers, Cyclic van der Waerden Numbers, and a Lower Bound on Them Both

Suppose $k \geq 3$ and $r \geq 2$ are integers. The van der Waerden number $W(k, r)$ is the smallest positive integer $N$ such that for any coloring of $\{0, \ldots, N-1\}$ (or of any other block of $N$ consecutive integers) with $r$ colors, there is a monochromatic $k$-term arithmetic progression in the block. That $W(k, r)$ is well-defined is a consequence of a celebrated result of van der Waerden [18].

If $N > 1$ and $n \in \mathbb{Z}$ (the set of integers), the congruence class of $n$ mod $N$ is denoted $\bar{n}$, suppressing mention of $N$. Thus the ring of integers mod $N$ is $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z} = \{\bar{0}, \bar{1}, \ldots, \overline{N-1}\}$. A *$k$-term arithmetic progression* mod $N$ is a $k$-subset of $\mathbb{Z}_N$ the elements of which can be listed as $\bar{a} + t\bar{d}$, $t = 0, \ldots, k-1$, for some $\bar{a}, \bar{d} \in \mathbb{Z}_N$. It is important to seize the requirement that the $k$ congruence classes on the list $\bar{a} + t\bar{d}$, $t = 0, \ldots, k-1$, must be distinct; for instance, $\bar{3} + t\bar{6}$, $t = 0, 1, 2$, does not constitute a 3-term arithmetic progression mod 12. Indeed, there are no $k$-term arithmetic progressions mod 12 with difference $\bar{d} = \bar{6}$ for any $k \geq 3$.

Sometimes arithmetic progressions mod $N$ do not look like arithmetic progressions; for instance $\{\bar{0}, \bar{3}, \bar{8}\}$ is a 3-term arithmetic progression mod 11, because $\bar{3} + \bar{8} = \bar{0}$ and $\bar{0} + \bar{8} = \bar{8}$ in $\mathbb{Z}_{11}$.

On the other hand, what looks like an ordinary arithmetic progression "is" an arithmetic progression mod $N$ for $N$ sufficiently large. That is, if $a \geq 0$, $d > 0$, and $k \geq 3$ are integers, and $a + (k-1)d < N$, then $\{\bar{a}, \bar{a} + \bar{d}, \ldots, \bar{a} + (k-1)\bar{d}\}$ is a $k$-term arithmetic progression mod $N$.

We define the cyclic van der Waerden number $W_c(k, r)$, for $k \geq 3$, $r \geq 2$, to be the smallest positive integer $M$ such that for all $N \geq M$, if $\mathbb{Z}_N$ is colored with $r$ colors there must be a monochromatic $k$-term arithmetic progression mod $N$.

Duty impels us to use the "smallest... $M$ such that for all $N \geq M$" formulation because we do not know if it is true that if, for every $r$-coloring of $\mathbb{Z}_N$, there must be a monochromatic $k$-term arithmetic progression mod $N$, then the same holds with $N$ replaced by $N+1$. If that were the case, then the definition of $W_c(k,r)$ could be simplified to resemble that of $W(k,r)$.

However, because ordinary arithmetic progressions within a block of $N$ consecutive integers convert to arithmetic progressions mod $N$, by van der Waerden's theorem $W_c(k,r)$ is well defined, and $W_c(k,r) \le W(k,r)$. Therefore upper bounds on the $W(k,r)$ are upper bounds on the $W_c(k,r)$, and lower bounds on the $W_c(k,r)$ are lower bounds on the $W(k,r)$.

The best upper bound on the $W(k,r)$ is of recent vintage, from an already famous paper of W. T. Gowers [6]:

$$W(k,r) \le 2^{2^{r^{2^{2^{k+9}}}}}.$$

We are not working on upper bounds here, and there is no clear road from upper bounds on the $W_c(k,r)$ to upper bounds on the $W(k,r)$, but we can venture to suggest that looking for upper bounds on the $W_c(k,r)$ may refresh the search for upper bounds on the $W(k,r)$. For one thing, it does not seem to be beyond human ingenuity to bound the ratios $W(k,r)/W_c(k,r)$ above, and that really would open a road in the right direction from the cyclic to the original upper bound problem.

The first general lower bound on the $W(k,r)$ was due to Erdös and Rado [4]:

$$\sqrt{2(k-1)r^{k-1}} < W(k,r) \tag{ER}$$

This was improved by Leo Moser [14]:

$$kr^{c\log r} < W(k,r),$$

for some constant $c$. This bound has the obvious defect that in order to use it to estimate $W(k,r)$ for smallish $k$ and $r$, you need to know what $c$ is, or perhaps an estimate of some $c$, for the range of $k$ and $r$ of interest. Gunderson and Rödl obtained such an improvement for the case $k=3$ [8]:

$$r^{lnr/9} < W(3,r) \tag{GR}$$

Without further ado, here is our result.

**Theorem 6.1.** *Suppose that $k \ge 3$ and $r \ge 2$ are integers, and $p$ is the largest prime not exceeding $k$. Then $p^{\lfloor \log_2 r \rfloor} < W_c(k,r)$.*

The proof, which comes along shortly, uses another instance of Szlam's lemma.

**Szlam's Lemma IV.** *Suppose that $N$ is a positive integer and $\mathcal{S}$ is a collection of subsets of $\mathbb{Z}_N$ closed under translation mod $N$. (That is, if $S \in \mathcal{S}$ and $\bar{a} \in \mathbb{Z}_N$ then $\bar{a} + S \in \mathcal{S}$.) Suppose that $R$ and $B$ partition $\mathbb{Z}_N$, and no set in $\mathcal{S}$ is contained in $B$. Suppose that $\emptyset \ne F \subseteq \mathbb{Z}_N$ and no translate of $F$ mod $N$ is contained in $R$. Then $\mathbb{Z}_N$ can be $|F|$-colored so that no set in $\mathcal{S}$ is monochromatic.*

In hypergraph lingo, if $\mathcal{S}, R, B$, and $F$ are as above, and $\mathcal{H} = (\mathbb{Z}_N, \mathcal{S})$, then $\mathcal{X}(\mathcal{H}) \le |F|$.

*Proof of SL4*, on the model of the proof of SL1.

Let $f = |F|$ and let $\bar{a}_1, \ldots, \bar{a}_f$ be the elements of $F$. We partition $\mathbb{Z}_N$ into sets $P_1, \ldots, P_f$, some of which may be empty, defined by: $\bar{z} \in \mathbb{Z}_N$ is in $P_i$ for the smallest $i$ such that $\bar{z} + \bar{a}_i \in B$. Since $(\bar{z} + F) \cap B \neq \emptyset$ for each $\bar{z} \in \mathbb{Z}_N$, the $P_i$ are well defined, clearly pairwise disjoint, and they cover $\mathbb{Z}_N$. It remains to be seen that no $P_i$ contains any $S \in \mathcal{S}$.

Suppose, to the contrary, that for some $i \in \{1, \ldots, f\}$ and $S \in \mathcal{S}$, $S \subseteq P_i$. Then, by the definition of $P_i$, $\bar{a}_i + \bar{z} \in B$ for each $\bar{z} \in S$, so $\bar{a}_i + S \subseteq B$. But then $\bar{a}_i + S$ is a set in $\mathcal{S}$ contained in $B$, a contradiction. □

**Corollary 6.2.** *Suppose that $k \geq 3$, $r \geq 2$, and $N > 0$ and that $\mathbb{Z}_N$ is partitioned into two sets, $A$ and $B$, such that no $k$-term arithmetic progression mod $N$ is contained in $B$. Suppose that for some $T \subseteq \mathbb{Z}_N$, $|T| = r$, no translate of $T$ mod $N$ is contained in $A$. Then*

$$W_c(k, r) > N$$

*Proof.* Apply Szlam's Lemma IV with $\mathcal{S}$ being the collection of $k$-term arithmetic progressions mod $N$. □

The change of notation between SL4 and its corollary was impelled by a clash of notational traditions: in rather red Euclidean colorings, the color names are red and blue, whence $R$ and $B$ in SL4, while in the van der Waerden world of colorings that forbid or fail to forbid monochromatic arithmetic progressions, the number of colors is $r$.

To see that SL4 is derivable from SL3, given $\mathcal{S}, R, B$, and $F$ and as SL4, let $U = V = \mathbb{Z}_N$ in SL3 and replace $R, B$ there with $\tilde{R} = \{(\bar{a}, \bar{b}) \in \mathbb{Z}_N^2 \mid \bar{a} + \bar{b} \in R\}$ and $\tilde{B} = \{(\bar{a}, \bar{b}) \in \mathbb{Z}_N^2 \mid \bar{a} + \bar{b} \in B\}$. Then $\mathcal{S} \subseteq \mathcal{E}_U$, so $\mathcal{H} = (\mathbb{Z}_N, \mathcal{S})$ is a subhypergraph of $\mathcal{H}_U$, and $F \in \mathcal{E}_V$. By SL3 it follows that

$$\chi(\mathcal{H}) \leq \chi(\mathcal{H}_U) \leq |F|.$$

*Proof of Theorem 6.1*. Let $n = \lfloor \log_2 r \rfloor$, so $2^n \leq r < 2^{n+1}$. Since $W_c(k, r) \geq W_c(p, r) \geq W_c(p, 2^n)$, to prove the theorem it suffices to show that $W_c(p, 2^n) > p^n$. We demonstrate the sufficient condition for $W_c(p, 2^n) > p^n$ given in Corollary 6.2 by induction on $n$: For each $n = 1, 2, \ldots, \mathbb{Z}_{p^n}$ is partitioned into sets $A(n)$, $B(n)$ such that $B(n)$ contains no $p$-term arithmetic progression mod $p^n$, and for some $T(n) \subseteq \mathbb{Z}_{p^n}$ with $|T(n)| = 2^n$, no translate of $T(n)$ mod $p^n$ is in $A(n)$.

In what follows we omit the overbar in the notation $\bar{z}$, to diminish clutter and also because in the induction we go from $\mathbb{Z}_{p^n}$ to $\mathbb{Z}_{p^{n-1}}$. Since the overbar notation provides no way of conveying the distinction, it is more economical to give it up entirely and to indicate which congruence system we are in by explicit mention.

If $n = 1$, $p^n = p$; set $B(1) = \{0, \ldots, p-2\}$, $A(1) = \{p-1\}$, and $T(1) = \{0, 1\}$. There is only one $p$-term arithmetic progression mod $p$, $\mathbb{Z}_p$ itself, and clearly $B(1)$ does not contain it. $T(1)$ has $2 = 2^1$ elements, so clearly no translate of it, mod $p$, could be contained in $A(1)$.

Now suppose that $n > 1$. For $j = 0, \ldots, p-1$, let $U_j = \{jp^{n-1}, jp^{n-1} + 1, \ldots, (j+1)p^{n-1} - 1\} = jp^{n-1} + U_0$. (The translation here is in $\mathbb{Z}_{p^n}$.) Then $\mathbb{Z}_{p^n}$ is the disjoint union of $U_0, \ldots, U_{p-1}$. We define, with all additions taking place in $\mathbb{Z}_{p^n}$,

$$\begin{aligned} B(n) &= \cup_{0 \le j \le p-2}(jp^{n-1} + B(n-1)), \\ A(n) &= \left(\cup_{0 \le j \le p-2}(jp^{n-1} + A(n-1))\right) \cup U_{p-1}, \\ \text{and } T(n) &= T(n-1) \cup (p^{n-1} + T(n-1)). \end{aligned}$$

Since $A(n-1)$, $B(n-1)$ partition $\mathbb{Z}_{p^{n-1}}$, by the induction hypothesis, it follows that $jp^{n-1} + B(n-1)$, $jp^{n-1} + A(n-1)$ partition $U_j$, $j = 0, \ldots, p-2$, and thus that $A(n)$, $B(n)$ partition $\mathbb{Z}_{p^n}$.

Since $T(n-1) \subseteq U_0 = \{0, \ldots, p^{n-1} - 1\}$, $T(n-1)$ and $p^{n-1} + T(n-1)$ are disjoint in $\mathbb{Z}_{p^n}$, so, applying the induction hypothesis, $|T(n)| = 2|T(n-1)| = 2 \cdot 2^{n-1} = 2^n$. Next we verify that no translate of $T(n)$ mod $p^n$ lies in $A(n)$. If $a \in U_j$ then $a + T(n) \subseteq U_j \cup U_{j+1} \cup U_{j+2}$ (reduce subscripts mod $p$), and $a + T(n-1) \subseteq U_j \cup U_{j+1}$. If, say, $a = a_0 + jp^{n-1}$, $a_9 \in U_0$, then $a + T(n-1)$, reduced mod $p^{n-1}$, is $a_0 + T(n-1) \subseteq \mathbb{Z}_{p^{n-1}}$. Suppose that $0 \le j \le p-3$. If $a+T(n) \subseteq A(n)$ then $a+T(n-1) \subseteq (jp^{n-1}+A(n-1)) \cup ((j+1)p^{n-1}+A(n-1))$ and so, reducing mod $p^{n-1}$, $a_0 + T(n-1) \subseteq A(n-1)$, contrary to the induction hypothesis. So $(a + T(n)) \cap B(n) \neq \emptyset$ if $a \in U_j$, $0 \le j \le p-3$.

If $a \in U_{p-2}$ then

$$C = [(a + T(n-1)) \cap U_{p-2}] \cup [(a + p^{n-1} + T(n-1)) \cap U_0],$$

a subset of $a + T(n)$, reduces mod $p^{n-1}$ to $a_0 + T(n-1)$, which is not contained in $A(n-1)$. Therefore $C$ intersects $B(n) \cap (U_{p-2} \cup U_0)$, which implies that $a + T(n)$ intersects $B(n)$.

Finally, if $a \in U_{p-1}$ then $a + p^{n-1} + T(n-1) \subseteq U_0 \cup U_1$ and reduces mod $p^{n-1}$ to $a_0 + T(n-1)$; since $(a_0 + T(n-1)) \cap B(n-1) \neq \emptyset$,

$$\begin{aligned} \emptyset &\neq [a + p^{n-1} + T(n-1)] \cap [B(n-1) \cup (p^{n-1} + B(n-1)] \\ &\subseteq (a + T(n)) \cap B(n), \text{ so } a + T(n) \nsubseteq A(n). \end{aligned}$$

What remains to be proved is that $B(n)$ contains no $p$-term arithmetic progression mod $p^n$. Suppose that $a, a+d, \ldots, a+(p-1)d$ is a $p$-term arithmetic progression mod $p^n$. If, for any $j \in \{0, \ldots, p-1\}$, $a + jd \in U_{p-1} \subseteq A(n)$, then we are done. Also, if the terms $a, a+d, \ldots, a+(p-1)d$ are distinct mod $p^{n-1}$, then, reducing mod $p^{n-1}$, in $\mathbb{Z}_{p^{n-1}}$ $a, a+d, \ldots, a+(p-1)d$, a $p$-term arithmetic progression mod $p^{n-1}$, intersects $A(n-1) \subseteq \mathbb{Z}_{p^{n-1}} = U_0$, which implies that the original $p$-term arithmetic progression mod $p^n$ intersects $A(n)$.

So suppose that $a, a+d, \ldots, a+(p-1)d$ are not distinct mod $p^{n-1}$. Therefore $a + id \equiv a + jd$ mod $p^{n-1}$ for some $0 \le i < j \le p-1$; then $p^{n-1} \mid (j-i)d$. Since $1 \le j - i \le p-1$ and $p$ is a prime, it follows that $p^{n-1} \mid d$. Let us suppose that $d = sp^{n-1}$ for some $s \in \{1, \ldots, p-1\}$.

Again appealing to the assumption that $p$ is a prime, as $j$ runs over $\{1, \ldots, p-1\}$, $js$ runs over $\{1, \ldots, p-1\}$, mod $p$. Suppose that $a = a_0 + cp^{n-1}$, $a_0 \in U_0$, $0 \leq c \leq p-1$. If $c = p-1$ then $a \in U_{p-1} \subseteq A(n)$, so we may assume that $0 \leq c \leq p-2$. Let $x = p-1-c$ and let $j \in \{1, \ldots, p-1\}$ satisfy $js \equiv x$ mod $p$; say $js = x + mp$.

Then

$$\begin{aligned} a + jd &= a_0 + cp^{n-1} + jsp^{n-1} \\ &= a_0 + cp^{n-1} + (x + mp)p^{n-1} \\ &= a_0 + (c + x)p^{n-1} + mp^n \\ &= a_0 + (p-1)p^{n-1} + mp^n \\ &\equiv a_0 + (p-1)p^{n-1} \bmod p^n \end{aligned}$$

Therefore, $a + jd \in U_{p-1} \subseteq A(n)$, so the $p$-term arithmetic progression $a, a + d, \ldots, a + (p-1)d$ mod $p^n$ is not contained in $B(n)$. □

How good is $p^{\lfloor \log_2 r \rfloor}$ as a lower bound of $W(k, r)$? As one might expect, it is not very good, for $k > 3$ and large $r$. But for $k = 3 = p$ it beats $2r$, the Erdös–Rado bound, for all $r \geq 16$ and for half the values of $r \in \{2, \ldots, 15\}$. It beats the Gunderson-Rödl bound, $r^{lnr/9}$, for all $r \leq 2^{19}$. These triumphs are minor, but are triumphs nonetheless, since it is somewhat surprising that a lower bound on the $W_c(k, r)$ is larger than known lower bounds on the $W(k, r)$ for any values of $k$ and $r$. Also, the comparison shows that the Theorem 6.1 is independent of the earlier results, inequalities (ER) and (GR).

In the quest for lower bounds on the $W(k, r)$, we may be giving away too much in focusing on the $W_c(k, r)$. Lower bounds on the $W(k, r)$ might be obtainable by applying SL4 with $\mathcal{S}$ being the set of all translates mod $N$ of $k$-term arithmetic progressions in $\{0, \ldots, N-1\}$; that is, we could take $\mathcal{S}$ to be the set of all $k$-term arithmetic progressions $\bar{a}, \bar{a} + \bar{d}, \ldots, \bar{a} + (k-1)\bar{d}$ mod $N$ in which $1 \leq d \leq (N-1)/(k-1)$. Let $\hat{\mathcal{S}}$ be so defined, and let $\mathcal{S}$ be the set of all $k$-term arithmetic progressions mod $N$. The following problems are certainly of interest in estimating $W_c(k, r)$ and $W(k, r)$, in view of SL4, but they seem also to possess their own charm, independent of other concerns:

1. For a given $k$ and $N$, what is the largest size $|B|$ of a set $B \subseteq \mathbb{Z}_N$ containing no set in $\mathcal{S}$ (or in $\hat{\mathcal{S}}$)?
2. For a given $k$ and $N$, what is the smallest $|T|$ for $T \subseteq \mathbb{Z}_N$ such that there exists $B \subseteq \mathbb{Z}_N$, containing no set in $\mathcal{S}$ (or $\hat{\mathcal{S}}$), which intersects every translate of $T$ mod $N$?

Clearly the more complicated second question is the more important of the two in the application of SL4 to the study of the $W(k, r)$, but the first question may be of use in attacking it, and has the virtue of simplicity. When $k$ is a prime and $N = k^n$ the inductive construction in the proof of Theorem 6.1 produces a set $B = B(n)$ containing no set in $\mathcal{S}$ and intersecting every translate mod $N$ of $T = T(n)$, with $|B| = (k-1)^n$ and $|T| = 2^n$.

We have experimented with small $k$ and $N < 100$, choosing $B$ *greedily*: put $\bar{0}, \ldots, \overline{k-2}$ in $B$ and after that, in going through $\mathbb{Z}_N$, test each candidate $\bar{c}$ for membership in $B$ by checking to see if adding $\bar{c}$ to $B$ will cause $B$ to contain a member of $\mathcal{S}$. (If so, $\bar{c}$ is rejected; we have not yet tried this with $\hat{\mathcal{S}}$.) These experiments led to the discovery of the construction in the proof of Theorem 6.1, so we cannot claim the experimental results as evidence that the construction produces a $B$ of maximum cardinality, or a $T$ of minimum cardinality. Still, it is the best we have when $k$ is a prime and $N = k^n$.

When $k$ is not a prime the construction may not work. For example, when $k = 6$ and $N = 36$ the construction gives $B = \{\bar{0}, \ldots, \bar{4}\} \cup \{\bar{6}, \ldots, \overline{10}\} \cup \{\overline{12}, \ldots, \overline{16}\} \cup \{\overline{18}, \ldots, \overline{22}\} \cup \{\overline{24}, \ldots, \overline{28}\}$; then $\bar{0}, \bar{3}, \bar{6}, \bar{9}, \overline{12}, \overline{15}$ is a 6-term arithmetic progression mod 36 contained in $B$. (Disturbing? Observe that this sequence is not a 6-term arithmetic progression mod 6. As shown in the proof of Theorem 6.1, when $k$ is prime this kind of thing can happen only if the difference in the arithmetic progression mod $k^n$ is divisible by $k^{n-1}$, which then ensures that the progression intersects $\{(k-1)k^{n-1}, (k-1)k^{n-1}+1, \ldots, k^n - 1\} \subseteq A(n) = \mathbb{Z}_{k^n} \backslash B$.)

Our thanks to Emma Friedman, who took an interest in the problem of finding $W(k, r)$, especially when $k = 3$, during a summer Research Experience for Undergraduates at Auburn University in 2005, causing the second author to bestir himself and eventually find out a bit about the background of the problem, mainly from [7], kindly provided in preprint form by David Gunderson. Thanks are also due to Jennifer Hurt, who explained some of [7] to us.

**Acknowledgment** Research supported partly by National Science Foundation grant no. 0353723.

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# Open Problems in Euclidean Ramsey Theory

Ron Graham and Eric Tressler

## 1 Introduction

Ramsey theory is the study of structure that must exist in a system, most typically after it has been partitioned. A good example is the well-known theorem of van der Waerden [28], which states that given any $k, r \in \mathbb{N}$, there exists a least integer $w(k, r)$ such that if $[w(k, r)] := \{1, \ldots, w(k, r)\}$ is partitioned into $r$ sets (or $r$-*colored*), then there exists a monochromatic arithmetic progression of length $k$ (that is, a $k$-term arithmetic progression in one of the sets).

Euclidean Ramsey theorems are similar in nature, but are concerned with geometric objects, most often $\mathbb{E}^n$ or partitions of graphs with geometric properties, such as the hypercube embedded in $\mathbb{E}^n$. Euclidean Ramsey theory abounds with open problems, nearly all of them elementary to state. First, though, we need some definitions.

For a finite set $X \subset \mathbb{E}^k$, let *Cong(X)* denote the set of all subsets of $\mathbb{E}^k$ which are *congruent* to X under some Euclidean motion. We say that $X$ is *Ramsey* if for every integer $r$, there is a least integer $N(X, r)$ such that if $N \geqslant N(X, r)$ then for any $r$-coloring of $\mathbb{E}^N$, there is a monochromatic $X' \in Cong(X)$. We denote this property by the usual "arrow" notation $\mathbb{E}^N \rightarrow X$. The negation of this statement is denoted by $\mathbb{E}^N \not\rightarrow X$.

It is not hard to see that any Ramsey set must be finite. Furthermore, it follows from compactness arguments (where we are using the Axiom of Choice) that if $X$ is Ramsey, then in fact there must be a *finite* set $S$ such that $S \rightarrow X$.

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A. Soifer (ed.), *Ramsey Theory: Yesterday, Today, and Tomorrow*, Progress in Mathematics 285, DOI 10.1007/978-0-8176-8092-3_7,

A more restricted notion is that of being $r$-Ramsey. This just means that a monochromatic copy of $X$ must occur whenever the underlying set $S$ is $r$-colored. In this case, we write $S \xrightarrow{r} X$. The negation of this statement is written as $S \stackrel{r}{\not\to} X$.

*Conjecture 1 ([9]).* For any nonequilateral triangle $T$ (i.e., the set of three vertices of $T$),

$$\mathbb{E}^2 \xrightarrow{2} T.$$

*Conjecture 2.* For any triangle $T$, there exists a 3-coloring of $\mathbb{E}^2$ without a monochromatic copy of $T$.

For any particular equilateral triangle $T$, one can color the plane with alternating half-open red and blue strips with height the altitude of $T$, avoiding a monochromatic copy of $T$. This was conjectured to be essentially the only possible 2-coloring that avoids any triangle, though recently V. Jelínek, J. Kynčl, R. Stolař, and T. Valla showed that there exist infinitely many such colorings [13]. They have also shown that Conjecture 1 is true if one color class is open and the other is closed.

Conjecture 1 is known to be true for many classes of triangles; a partial list can be found in [9], and a proof that it is true for right triangles due to L. Shader is in [22]. Less seems to be known about Conjecture 2; the 3-coloring by alternating half-open strips avoids a large class of triangles, but not all. Recently we have discovered a 3-coloring of $\mathbb{E}^2$ that avoids the degenerate triangle with sides $a, a, 2a$, shown in Fig. 1. This tiling extends to cover $\mathbb{E}^2$; each hexagon has diameter $2a$ and all of the hexagons are half-open as shown for the uppermost hexagon in Fig. 1.

For any collinear set $S$, it is known that with 16 colors one can avoid a monochromatic copy of $L$ in $\mathbb{E}^n$ for all $n$ [27], but it is an open question if this is the best possible.

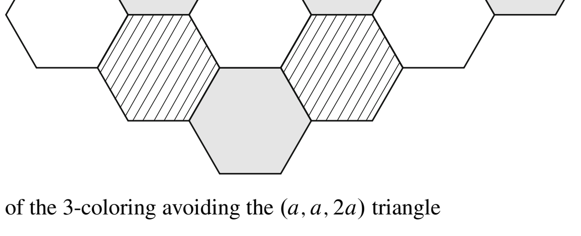


**Fig. 1** A sketch of the 3-coloring avoiding the $(a, a, 2a)$ triangle

## 2 Ramsey Sets

It is a long-standing problem to discover which sets are Ramsey, as defined above. Igor Křiž has some strong positive theorems to this end:

**Theorem 1 ([14]).** *Suppose $X \subseteq \mathbb{E}^N$ has a transitive group of isometries with a solvable subgroup with at most two orbits. Then $X$ is Ramsey.*

**Theorem 2 ([15]).** *If $X$ is the set of vertices of a trapezoid, $X$ is Ramsey.*

Frankl and Rödl [10] have shown that every nondegenerate simplex is Ramsey. It is shown in [8] that any Ramsey set must lie on the surface of a sphere (in some dimension); we call such sets *spherical*. In particular, collinear sets, as pointed out above, can always be avoided with at most 16 colors in any number of dimensions. It may turn out that the Ramsey sets are very easy to describe:

*Conjecture 3 ($1,000).* Every spherical set is Ramsey.

A weaker conjecture is:

*Conjecture 4 ($100).* Every 4-point subset of a circle is Ramsey.

Recall that given a Ramsey set $X$ and an integer $r$, we defined $N(X, r)$ to be the least integer such that if $N \geqslant N(X, r)$ then for any $r$-coloring of $\mathbb{E}^N$, there is a monochromatic $X' \in Cong(X)$. One might go further than asking which sets are Ramsey; given a Ramsey set and an integer $r$, what can we say about $N(X, r)$? As we show below, even for the simplest nontrivial case, a 2-point set, this is a major open question.

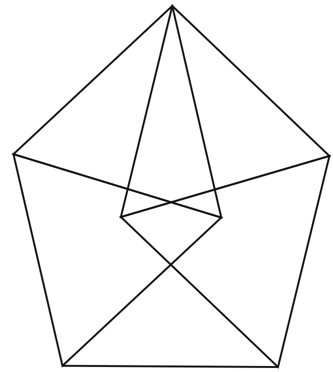
## 3 Unit Distance Graphs

A *unit distance graph* in a metric space $(X, \rho)$ is a graph $G = (X, E)$ with vertex set $X$ and edge set $\{x, y \in X : \rho(x, y) = 1\}$. $\|\cdot\|$ denotes the usual Euclidean norm. $\chi(X, \rho)$ denotes the chromatic number of $X$ under the metric $\rho$, though we omit $\rho$ when it is the Euclidean norm.

The most widely known problem in Euclidean Ramsey theory is probably that of determining the chromatic number of the plane, $\chi(\mathbb{E}^2)$. This question is attributed to Nelson (see [23–25] for a full account), and there is a wide literature surrounding it. Despite its broad interest, the best known bounds are $4 \leq \chi(\mathbb{E}^2) \leq 7$. Proof of the lower bound is in Fig. 2; it is a unit distance graph with chromatic number 4, usually known as the Moser spindle, after Leo Moser. The upper bound is given by a hexagonal tiling of the plane using hexagons of diameter slightly less than 1 (there is room for error), and 7-coloring them in a fairly obvious way, left to the reader to discover.

Given that the known bounds are so easy to prove, it may be surprising that the problem has proven to be so stubborn. In 1981, Falconer showed that if we assume

**Fig. 2** Proof that $4 \leq \chi(\mathbb{E}^2)$



the axiom that all subsets of $\mathbb{E}^n$ are Lebesgue measurable, then $\chi(\mathbb{E}^2) \geq 5$. For other conditional results about $\chi(\mathbb{E}^2)$, see [26]. Other results about the chromatic number of $\mathbb{E}^n$ include the bounds $6 \leq \chi(\mathbb{E}^3) \leq 15$ ([5, 17]) and $7 \leq \chi(\mathbb{E}^4) \leq 49$ ([11]).

In [24], a variant of this problem is discussed (and tentatively attributed to Erdős). Say that a set $S$ in $\mathbb{E}^2$ *realizes distance* $d$ if some two points $x, y \in S$ are distance $d$ apart. The *polychromatic number of the plane* is the least number of colors $\chi_p(\mathbb{E}^2)$ such that it is possible to color the plane with $\chi_p(\mathbb{E}^2)$ colors so that no color realizes all distances. Of course, $\chi_p(\mathbb{E}^2) \leq \chi(\mathbb{E}^2)$, since in the latter case no color realizes distance 1. The bounds $4 \leq \chi_p(\mathbb{E}^2) \leq 6$ are due to Raiskii and Stechkin, respectively, and appeared in [21], though as with $\chi(\mathbb{E}^2)$, the determination of the actual number $\chi_p(\mathbb{E}^2)$ is open.

The chromatic number of rational space has also been studied. In [2], M. Benda and M. Perles show that $\chi(\mathbb{Q}^2) = 2$, $\chi(\mathbb{Q}^3) = 2$, and $\chi(\mathbb{Q}^4) = 4$. The authors pose some problems in their conclusion: try to determine $\chi(\mathbb{Q}^n)$ for some $n \geq 5$ (in [4], K. B. Chilakamarri shows that $\chi(\mathbb{Q}^5) \geq 6$)), or try to find $\chi(X^2)$ where $X$ is some algebraic extension of $\mathbb{Q}$, for example, $\mathbb{Q}[\sqrt{2}]$.

The question of which graphs are unit distance graphs in $\mathbb{E}^2$ is also open. For instance, it is easy to see that the graph $K_4$ cannot be a unit distance graph in $\mathbb{E}^2$, but it is not known if any particular subgraphs are excluded. In [3] a problem is posed: must every bipartite graph that is not a unit distance graph contain $K_{2,3}$ as a subgraph? The answer is no: it is not very difficult to show that the five-dimensional hypercube, $Q_5$, with all 16 of its space diagonals attached, is a counterexample. Here is a sketch of the proof:

*Sketch of proof.* The two-dimensional hypercube, embedded in the plane as a unit distance graph, clearly has to have some two of its opposite vertices at least distance $\sqrt{2}$ apart (opposite in the sense that they are maximally distant pairs in $Q^2$). The same is true of $Q_5$. Now let $G$ be the graph $Q_5$ with all of its space diagonals added (connect two vertices by an edge if they are distance 5 apart in $Q_5$). Since some two of the vertices of $Q_5$ in a unit embedding are necessarily farther than unit distance apart, we cannot embed $G$ as a unit distance graph in $\mathbb{E}^2$. Moreover, $G$ is bipartite and contains no copy of $K_{2,3}$ as a subgraph.

Paul O'Donnell has shown in [18, 19] that there exist 4-chromatic unit distance graphs of arbitrary girth. Since a complete characterization of the unit distance graphs in the plane would immediately determine the value of $\chi(\mathbb{E}^2)$, this is probably a very difficult task; still, it would be interesting to know what can be said.

Finally, we ask a basic question about unit distances in the plane: how dense can a Lebesgue measurable set $S$ be in $\mathbb{E}^n$ if it avoids unit distance? A good first attempt in $\mathbb{E}^2$ is to tile the plane with hexagons whose centers are distance 2 apart, and let $S$ be the set of open circles of diameter 1 centered in the hexagons; this achieves density $(\pi/8\sqrt{3}) > 0.2267$. However, in 1967, Croft showed in [7] that by modifying this coloring slightly it is possible to achieve a density of more than 0.2294. Coulson and Payne examined the same problem in $\mathbb{E}^3$ [6], but there has been no improvement over Croft's result in the case of $\mathbb{E}^2$.

## 4 More General Distance Graphs

There are also many open problems about graphs more general than unit distance graphs; here we only consider Euclidean $n$-space. For $A \subseteq \mathbb{R}$, let $G^A(\mathbb{E}^n)$ be the graph in $\mathbb{E}^n$ with vertex set $\mathbb{E}^n$ and edge set $\{x, y \in \mathbb{E}^n : \|x - y\| \in A\}$. Very recently Ardal et al. have shown in [1] that if we let $X$ be the set of all odd integers, then $\chi(G^X(\mathbb{E}^2)) \geq 5$, but the current upper bound on this number is the trivial bound $\aleph_0$.

In related work, L. Ivanov has considered the chromatic number of $G^{[1,d]}(\mathbb{E}^n)$ for $d > 1$ [12]. For this generalization of the unit distance graph, very little is currently known. Similar questions are pursued in [20], in which the authors present new results about $\chi(G^A(\mathbb{R}^n))$ and $\chi(G^A(\mathbb{Q}^n))$ for $|A| \in \{2, 3, 4\}$.

These problems are interesting partly because so little is known. Kuratowski's theorem [16] beautifully classifies the planar graphs, but there is no such theorem for unit graphs. The unit distance graph in the plane (and we have no need to be more general here) is simple enough to describe to a nonmathematician, and so enigmatic that finding its chromatic number is a new four-color map problem for graph theorists.

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# Chromatic Number of the Plane & Its Relatives, History, Problems and Results: An Essay in 11 Parts[1]

Alexander Soifer

*[I] can't offer money for nice problems of other people because then I will really go broke... It is a very nice problem. If it were mine, I would offer $250 for it.*

– Paul Erdős
Boca Raton, February, 1992

## 1 The Problem

In August 1987 I attended an inspiring talk by Paul Halmos at Chapman College in Orange, California. It was entitled "*Some Problems You Can Solve, and Some You Cannot*." This problem was an example of a problem "you cannot solve."

"A fascinating problem... that combines ideas from set theory, combinatorics, measure theory, and distance geometry," write Hallard T. Croft, Kenneth J. Falconer, and Richard K. Guy in their book *Unsolved Problems in Geometry* [CFG].

"If Problem 8 takes that long to settle [as the celebrated Four-Color Conjecture], we should know the answer by the year 2084," write Victor Klee and Stan Wagon in their book *New and Old Unsolved Problems in Plane Geometry* [KW].

Are you ready? Here it is:

*What is the smallest number of colors $\chi$ sufficient for coloring the plane in such a way that no two points of the same color are unit distance apart?*

This number $\chi$ is called *the chromatic number of the plane*. *To color the plane* means to assign one color to every point of the plane.

A *segment* here stands for just a 2-point set. Similarly, a *polygon* will stand for a finite set of point. A *monochromatic* set is a set all of whose elements are assigned the same color. In this terminology, we can formulate the chromatic number of the plane problem (CNP) as follows. What is the smallest number of colors sufficient for coloring the plane in a way that forbids monochromatic segments of length 1?

[1] Much but not all of this text is contained in the author's monograph [Soi].

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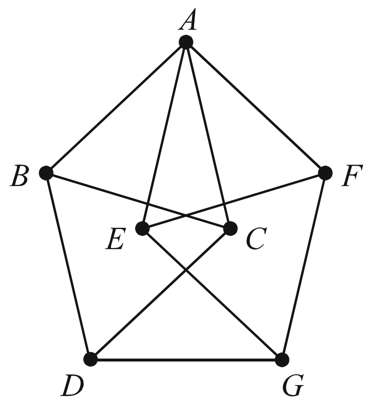



**Fig. 1** The Moser spindle

**Lower Bound 1.** $4 \leq \chi$.

*Solution by the Canadian geometers, brothers Leo and William Moser* (1961, [MM]). Toss on the given 3-colored plane what we now call *the Moser spindle* (Fig. 1). Every edge in the spindle has the length 1.

Assume that the seven vertices of the spindle do not contain a monochromatic segment of length 1. Call the colors used to color the plane red, white, and blue. The solution now faithfully follows the children's song: "*A B C D E F G*...".

Let the point $A$ be red; then $B$ and $C$ must be one white and one blue, therefore $D$ is red. Similarly $E$ and $F$ must be one white and one blue, therefore $G$ is red. We got a monochromatic segment $DG$ of length 1 in contradiction to our assumption. ■

*Observe:* The Mosers spindle with a vertex coloring has worked for us in solving Problem 1 precisely because any three points of the spindle contain two points distance 1 apart. This implies that *in a Moser spindle that forbids monochromatic distance 1, at most two points can be of the same color.*

Solomon W. Golomb found a substantially different construction proof of the lower bound. In a September 25, 1991 letter Golomb informed me that he likely found this example, which I naturally call the *Golomb graph*, in the time period 1960–1965.

*Second proof of the lower bound* . Just toss the Golomb graph on a 3-colored (red, white, and blue) plane (Fig. 2). Assume that in the graph there are no adjacent (i.e., connected by an edge) vertices of the same color. Let the center point be colored red, then since it is connected by unit edges to all vertices of the regular hexagon $H$, $H$ must be colored white and blue in alternating fashion. All vertices of the equilateral triangle $T$ are connected by unit edges to the three vertices of $H$ of the same color, say, white. But then white cannot be used in coloring $T$, and thus $T$ is colored red and blue. But this implies that two of the vertices of $T$ are assigned the same color. This contradiction proves that three colors are not enough to properly color the ten vertices of the Golomb graph, let alone the whole plane. ■

**Upper Bound 2.** $\chi \leq 7$.

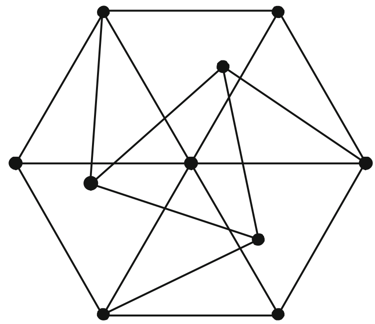


**Fig. 2** The Golomb graph

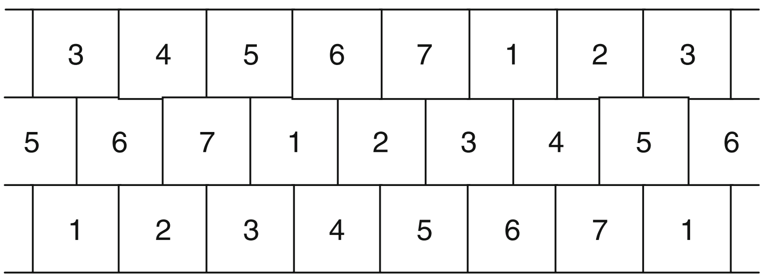



**Fig. 3**

*Proof.* This is László Székely's proof from [Sze1]. His original picture needs a small correction in its Fig. 1, and boundary coloring needs to be addressed, which I am doing here. We start with a row of squares of diagonal 1, with cyclically alternating colors of the squares 1, 2,. . . , 7 (Fig. 3). We then obtain consecutive rows of colored squares by shifting the preceding row to the right through 2.5 square sides.

Upper and right boundaries are included in the color of each square, except squares at the upper left and lower right corners. ■

In 1995 my former student and now a well-known puzzlist Edward Pegg, Jr. sent to me two distinct 7-colorings of the plane. In the one I am sharing with you (Fig. 4), Ed uses 7-gons for six of the colors, and tiny squares for the seventh color. Interestingly, the seventh color occupies only about $1/3$ of $1\%$ of the plane. In Fig. 4, all thick black bars have unit length. A unit of the tiling uses a heptagon and half a square.

The area of each square is 0.00412220518993071681 62. . .

The area of each heptagon is 0.62265127164647629646. . .

Area ratio thus is 302.0962048019455285300783627265828. . .

If one third of one percent of the plane is removed, the remainder can be 6-colored with this tiling!

It is amazing that the results shown above give us the best-known-to-mathematics bounds for the chromatic number of the plane $\chi$. They were published almost half a century ago. Still, all we know today is that

$$\chi = 4, \text{ or } 5, \text{ or } 6, \text{ or } 7.$$

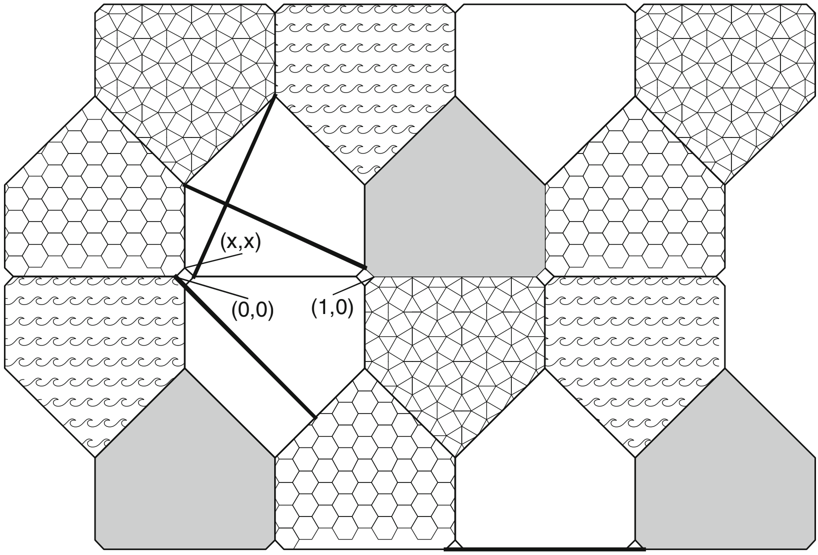



**Fig. 4**

A very broad spread! Which do you think is the exact value of $\chi$? Paul Erdős thought that $\chi \geq 5$.

Victor Klee shared with me in 1991 a very interesting story. In 1980 he lectured in Zürich, Switzerland. The 77-year-old celebrated algebraist Bartel L. van der Waerden was in attendance. When Vic presented the state of this problem, van der Waerden became so interested that he stopped listening to the lecture; he started working on the problem. He tried to prove that $\chi = 7$!

For many years I believed that $\chi = 7$, or else 6. Paul Erdős used to say that "God has a transfinite Book, which contains all theorems and their best proofs, and if He is well intentioned toward those, He shows them the Book for a moment." If I ever deserved the honor and had a choice, I would have asked to peek at the page with the chromatic number of the plane problem. Wouldn't you?

## 2 The History

*[This is] a long standing open problem of Erdős.*
–Hallard T. Croft, 1967

*[I] can not trace the origin of this problem*
– Paul Erdős, 1961

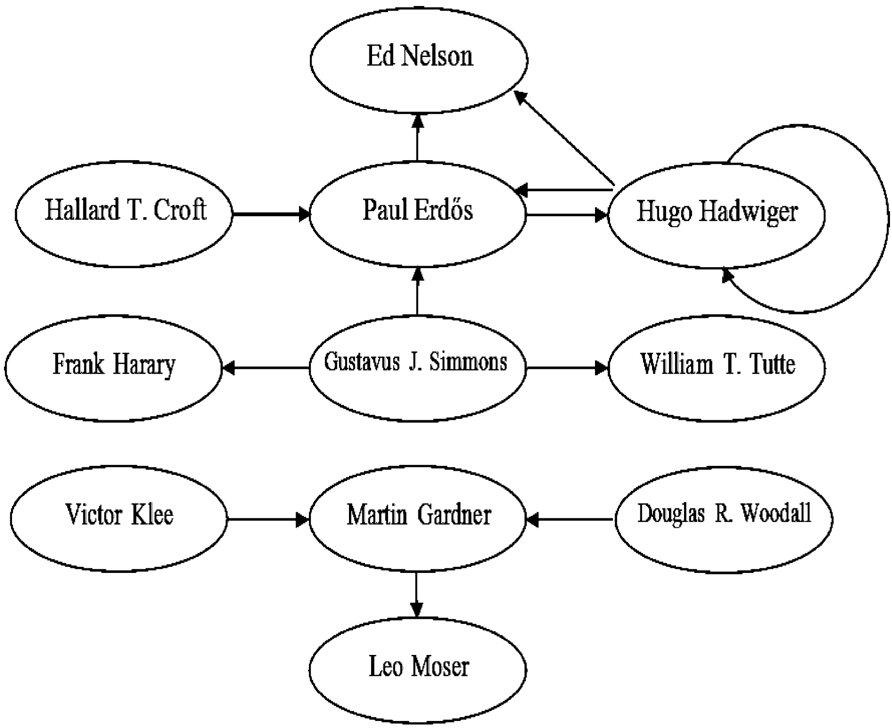



**Fig. 5** Credits for the creation of the chromatic number of the plane problem

*It is often easier to be precise about Ancient Egyptian history*
*than about what happened among our contemporaries.*
– Nicolaas Govert de Bruijn, 1995[2]

*It happened long ago and perhaps did not happen at all.*
– An Old Russian Joke

It is natural for one to inquire into the authorship of one's favorite problem. And so in 1991 I turned to countless articles and books. Some of the information I found appears here in Fig. 5 and Table 1; take a look. In Fig. 5, arrows are drawn from mathematicians giving credit to those who allegedly created the problem.

Are you confused? I was too!

As you can see in the table, Douglas R. Woodall credits Martin Gardner, who in turn refers to Leo Moser. Hallard T. Croft calls it "a long standing open problem of Erdős," Gustavus J. Simmons credits "Erdős, [Frank] Harary and [William Thomas] Tutte," while Paul Erdős himself "can not trace the origin of this problem"! Later Erdős credits "Hadwiger and Nelson," while Victor Klee and Stan Wagon state that the problem was "posed in 1960–61 by M. Gardner and Hadwiger." Croft comes

[2] [Bru6].

**Table 1** Who created the chromatic number of the plane problem?

| Publication | Year | Author(s) | Problem creator(s) or source named |
|---|---|---|---|
| [Gar2] | 1960 | Gardner | "**Leo Moser**... writes..." |
| [Had4] | 1961 | Hadwiger (after Klee) | **Nelson** |
| [E61.21] | 1961 | Erdős | "I can not trace the origin of this problem" |
| [Cro] | 1967 | Croft | "A long standing open problem of **Erdős**" |
| [Woo1] | 1973 | Woodall | **Gardner** |
| [Sim] | 1976 | Simmons | **Erdős**, **Harary** and **Tutte** |
| [E80.38] | 1980–1981 | Erdős | **Hadwiger** and **Nelson** |
| [E81.23] | | | |
| [E81.26] | | | |
| [CFG] | 1991 | Croft, Falconer, and Guy | "Apparently due to **E. Nelson**" |
| [KW] | 1991 | Klee and Wagon | "Posed in 1960–61 by **M. Gardner** and **Hadwiger**" |

again, this time with Kenneth J. Falconer and Richard K. Guy, to cautiously suggest that the problem is "apparently due to E. Nelson" [CFG]. Yet, Richard Guy did not know who "E. Nelson" was and why he and his coauthors "apparently" attributed the problem to him (our conversation in the back seat of a car in Keszthely, Hungary, when we both attended Paul Erdős 80th birthday conference in August of 1993).

Thus, at least seven mathematicians were credited with creating the problem: Paul Erdős, Martin Gardner, Hugo Hadwiger, Frank Harary, Leo Moser, Edward Nelson, and William T. Tutte: a great group of mathematicians to be sure. But it was hard for me to believe that they all created the problem, be it independently or all seven together.

I felt an urge, akin to that of a private investigator, a Sherlock Holmes, to untangle the web of conflicting accounts. It took 6 months to solve this historical puzzle. A good number of mathematicians, through conversations and e-mails, contributed their insight: Branko Grünbaum, Peter D. Johnson, Tony Hilton, and Klaus Fischer first come to mind. I am especially grateful to Paul Erdős, Victor Klee, Martin Gardner, Edward Nelson, and John Isbell for contributing their parts of the puzzle. Only their accounts, recollections, and congeniality made these findings possible.

I commenced my investigation on June 19, 1991 by mailing a letter to Paul Erdős. I informed Paul that "I am starting a new *Mathematical Coloring Book*, which will address problems where coloring is a part of a problem and/or a part of solution (a major part),"[3] and then asked the question: "There is a famous open problem of

[3] This seems to be my first mention of what has become an 18-year long project!

finding the chromatic number of the plane (minimal number of colors that prevents distance one between points of the same color). Is this your problem?"

On August 10, 1991, Paul shared his appreciation of the problem, for which he could not claim the authorship [E91/8/10]: "The problem about the chromatic number of the plane is unfortunately not mine." In a series of letters of July 12, 1991; July 16, 1991; August 10, 1991; and August 14, 1991, Paul also formulated for me a good number of problems related to the chromatic number of the plane that he did create. We look at Erdős's problems in the following sections.

Having established that the author was not Paul Erdős, I moved down the list of "candidates", and on August 8, 1991 and again on August 30, 1991, I wrote to Victor Klee, Edward Nelson, and John Isbell. I shared with them my Table 1 and asked what they knew about the creation of the problem. I also interviewed Professor Nelson on the phone on September 18, 1991.

Edward Nelson created what he named "a second four-color problem" (first being the famous four-color problem of map coloring). In his October 5, 1991, letter [Nel2], he conveyed the story of creation:

> Dear Professor Soifer:
>
> In the autumn of 1950, I was a student at the University of Chicago and among other things was interested in the four-color problem, the problem of coloring graphs topologically embedded in the plane. These graphs are visualizable as nodes connected by wires. I asked myself whether a sufficiently rich class of such graphs might possibly be subgraphs of one big graph whose coloring could be established once and for all, for example, the graph of all points in the plane with the relation of being unit distance apart (so that the wires become rigid, straight, of the same length, but may cross). The idea did not hold up, but the other problem was interesting in its own right and I mentioned it to several people.

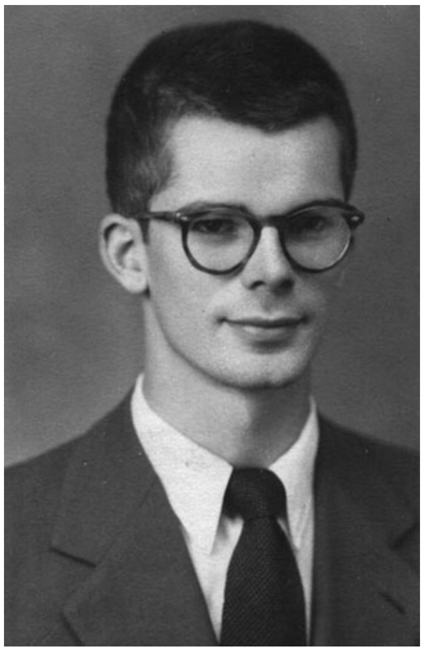
One of the people Ed Nelson mentioned the problem to was John Isbell. Half a century later, Isbell still remembered the story very vividly when on August 26, 1991 he shared it with me [Isb1]:

> ... Ed Nelson told me the problem and $\chi \geq 4$ in November 1950, unless it was October—we met in October. I said what upper bound have you, he said none, and I worked out 7. I was a senior at the time (B.S., 1951). I think Ed had just entered U. Chicago as a nominal sophomore and taken placement exams which placed him a bit ahead of me, say a beginning graduate student with a gap or two in his background. I certainly mentioned the problem to other people between 1950 and 1957; Hugh Spencer Everett III, the author of the many-worlds interpretation of quantum mechanics, would certainly be one, and Elmer Julian Brody who did a doctorate under Fox and has long been at the Chinese University of Hong Kong and is said to be into classical Chinese literature would be another. I mentioned it to Vic Klee in 1958 $\pm 1$....

Victor Klee too remembered (our phone conversation, September, 1991) hearing the problem from John Isbell in 1957–1958. In fact, it took place before September 1958 when Professor Klee left for Europe. There he passed the problem to Hugo Hadwiger who was collecting problems for the book *Open Problems in Intuitive Geometry* to be written jointly by Erdős, Fejes-Toth, Hadwiger, and Klee (this great book-to-be has never materialized).

Gustavus J. Simmons [Sim], in giving credit to "Erdős, Harary and Tutte," no doubt had in mind their joint 1965 paper [EHT] in which the three authors defined the dimension of a graph. The year 1965 was too late for our problem's creation, and besides, the three authors have not made or claimed such a discovery.

What were the roles of Paul Erdős, Martin Gardner, and Leo Moser in the story of creation? I am prepared to answer these questions, all except one: I am leaving to others to research Leo Moser's archive (maintained by his brother Willie Moser at McGill University in Montreal) and find out how and when Leo Moser came by the problem. What is important to me is that he did not create it independently from Edward Nelson, as Paul Erdős informed me in his July 16, 1991, letter [E91/7/16]: "I do not remember whether Moser in 1958 [possibly on June 16, 1958, the date from which we are lucky to have a photo record] told me how he heard the problem on the chromatic number of the plane, I only remember that it was not his problem."



Eddie Nelson, c. 1950. Courtesy of Edward Nelson

Yet, Leo Moser made a valuable contribution to the survival of the problem. He gave it to both Paul Erdős and Martin Gardner. Gardner, due to his fine taste, recognized the value of this problem and included it in his October 1960 "Mathematical Games" column in *Scientific American* [Gar2], with the acknowledgment that he received it from Leo Moser of the University of Alberta. Thus, the credit for the first publication of the problem goes to Martin Gardner. It is beyond me why so many authors of articles and books, as far back as 1973 ([Woo1], for example), gave credit for the creation of the problem to Martin Gardner, something he himself has never claimed. In our 1991 phone conversation Martin told me for a fact that the problem was not his, and he promptly listed Leo Moser as his source, both in print and in his archive.

Paul Erdős (left) and Leon Moser, June 16, 1958. Courtesy of the late Paul Erdős

Moreover, some authors ([KW], for example) who knew of Edward Nelson, still credited Martin Gardner and Hugo Hadwiger because, it seems, only written, preferably published word was acceptable to them. Following this logic, the creation of the celebrated four-color map coloring problem must be attributed to Augustus De Morgan, who first wrote about it in his October 23, 1852 letter to William Rowan Hamilton, or better yet to Arthur Cayley, whose 1878 abstract included the first nonanonymous publication of the problem.[4] Yet we all seem to agree that the 20-year-old Francis Guthrie created this problem, even though he did not publish or even write a word about it!

Of course, a lone self-serving statement would be too weak a foundation for a historical claim. On the other hand, independent disinterested testimonies corroborating each other comprise as solid a foundation for the attribution of the credit as any publication. And this is precisely what my inquiry has produced. Here is just one example of Nelson and Isbell's selflessness. Edward Nelson tells me on August 23, 1991 [Nel1]: "I proved nothing at all about the problem. . . ."

John Isbell corrects Nelson in his September 3, 1991, letter [Isb2]:

> Ed Nelson's statement which you quote, "I proved nothing at all about the problem," can come only from a failure of memory. He proved to me that the number we are talking about is $\geq 4$, by precisely the argument in Hadwiger 1961. Hadwiger's attribution (on Klee's authority) of that inequality to me can only be Hadwiger's or Klee's mistake.

This brings us to the issue of the authorship of the bounds for $\chi$,

$$4 \leq \chi \leq 7.$$

Once again, the entire literature is off the mark by giving credit for the first proofs to Hadwiger and the Mosers. Yes, in 1961 the famous Swiss geometer Hugo Hadwiger

[4] First publication could be attributed to De Morgan, who mentioned the problem in his 1860 book review in *Athenaeum* [DeM4], albeit anonymously.

published [Had4] the chromatic number of the plane problem together with proofs of both bounds. But he wrote (and nobody read!): "We thank Mr. V. L. Klee (Seattle, USA) for the following information. The problem is due to E. Nelson; the inequalities are due to J. Isbell." Hadwiger did go on to say: "Some years ago the author [i.e., Hadwiger] discussed with P. Erdős questions of this kind."

Did he imply that he created the problem independently from Nelson? We will never know for sure, but I have my doubts about Hadwiger's (co)authorship. Hadwiger jointly with H. Debrunner published an excellent, long problem paper in 1955 [HD1] that was extended to their wonderful famous book *Combinatorial Geometry of the Plane* in 1959 [HD2]; see also the 1964 English translation [HDK] with Victor Klee, and the 1965 Russian translation [HD3] edited by Isaak M. Yaglom. All these books (and Hadwiger's other papers) included a number of "questions of this kind," but did not once include the chromatic number of the plane problem. Moreover, it seems to me that the problem in question is somewhat out of Hadwiger's "character": in all problems "of this kind" he preferred to consider closed rather than arbitrary sets, in order to take advantage of topological tools.

I shared with Paul Erdős these twofold doubts about Hadwiger independently creating the problem. It was especially important because Hadwiger in the text quoted above mentioned Erdős as his witness of sorts. Paul replied in the July 16, 1991, letter [E91/7/16] as follows: "I met Hadwiger only after 1950, thus I think Nelson has priority (Hadwiger died a few years ago, thus I cannot ask him, but I think the evidence is convincing)." During his talk at the 25th Southeastern International Conference on Combinatorics, Computing and Graph Theory in Boca Raton, Florida, 9:30–10:30 A.M. on Thursday, March 10, 1994, Paul Erdős summarized the results of my historical research in the characteristically Erdősian style [E94.60]:[5]

> There is a mathematician called Nelson who in 1950 when he was an epsilon, that is he was 18, discovered the following question. Suppose you join two points in the plane whose distance is 1. It is an infinite graph. What is chromatic number of this graph?
>
> Now, de Bruijn and I showed that if an infinite graph which is chromatic number $k$, it always has a finite subgraph, which is chromatic number $k$. So this problem is really [a] finite problem, not an infinite problem. And it was not difficult to prove that the chromatic number of the plane is between 4 and 7. I would bet it is bigger than 4, but I am not sure. And the problem is still open.
>
> If it would be my problem, I would certainly offer money for it. You know, I can't offer money for every nice problem because I would go broke immediately. I was asked once what would happen if all your problems would be solved, could you pay? Perhaps not, but it doesn't matter. What would happen to the strongest bank if all the people who have money there would ask for money back? Or what would happen to the strongest country if they suddenly ask for money? Even Japan or Switzerland would go broke. You

[5] Thanks to Prof. Fred Hoffman, the tireless organizer of this annual conference, I have a videotape of Paul Erdős's memorable talk.

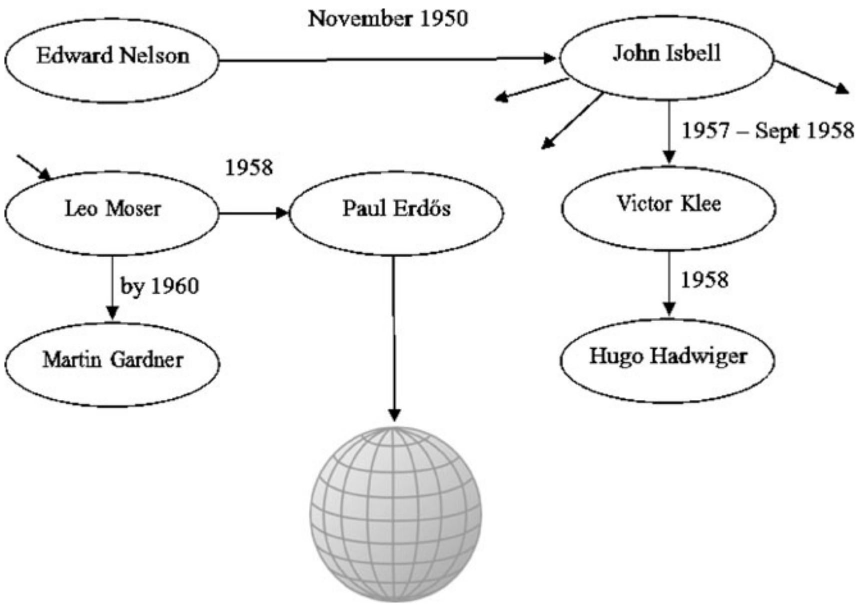



**Fig. 6** Passing the baton of the chromatic number of the plane problem

> see, Hungary would collapse instantly. Even the United States would go broke immediately...
>
> Actually it was often attributed to me, this problem. It is certain that I had nothing to do with the problem. I first learned the problem, the chromatic number of the plane, in 1958, in the winter, when I was visiting [Leo] Moser. He did not tell me from where this nor the other problems came from. It was also attributed to Hadwiger but Soifer's careful research showed that the problem is really due to Nelson.

The results of my historical research are summarized in Fig. 6, where arrows show passing the problem from one mathematician to another. In the end, Paul Erdős shared the problem with the world in numerous talks and articles.

Paul Erdős's acceptance of my findings has had a significant effect: most researchers and expositors now give credit to Edward Nelson for the chromatic number of the plane problem. There are, however, unfortunate exceptions. László Lovász and K. Vesztergombi, for example, state [LV] that "in 1944 Hadwiger and Nelson raised the question of finding the chromatic number of the plane." Of course, the problem did not exist in 1944, in Hadwiger's cited paper or anywhere else. Moreover, Eddie Nelson was just an 11–12-year-old boy at the time! In the same book, dedicated to the memory of Paul Erdős, one of the leading researchers of the problem and my friend Laszló Székely (who already in 1992 attended my talk on the history of the problem at Boca Raton), goes even further than Lovász and Vesztergombi [Sze3]: "E. Nelson and J. R. Isbell, and independently Erdős and H. Hadwiger, posed the following problem..."

The fine Russian researcher of this problem A. M. Raigorodskii repeats from Székely in his 2003 book [Raig6, p. 3], in spite of citing (thus presumably knowing) my historical investigation in his survey [Raig3]: "There were several authors. First of all, already in the early 1940s the problem was posed by remarkable mathematicians Hugo Hadwiger and Paul Erdős; secondly, E. Nelson and J. P. Isbell worked on the problem independently from Erdős and Hadwiger."[6] Raigorodskii then "discovers" a previously nonexistent connection between world affairs and the popularity of the problem:[7] "In the 1940s there was W.W.II, and this circumstance is responsible for the fact that at first chromatic numbers [sic] did not raise too thunderous an interest." Now we can finally give due credit to Edward Nelson for being first in 1950 to prove the lower bound $4 \leq \chi$. Because of this bound, John Isbell recalls in his letter [Isb1] that Nelson "liked calling it a second Four-Color Problem!"

In phone interviews with Edward Nelson on September 18 and 30, 1991, I learned some information about the problem creator. Joseph Edward Nelson was born on May 4, 1932 (an easy number to remember: 5/4/32), in Decatur, Georgia, near Atlanta. The son of the Secretary of the Italian YMCA,[8] Ed Nelson had studied at a *liceo* (Italian prep school) in Rome. In 1949 Eddie returned to the United States and entered the University of Chicago. The visionary Chancellor of the University, Robert Hutchins,[9] allowed students to avoid "doing time" at the University by passing lengthy placement exams instead. Ed Nelson had done so well on so many exams that he was allowed to go right on to graduate school without working on his bachelor's degree.

*Time* magazine reported young Nelson's fine achievements in 14 exams on December 26, 1949 [Time], next to the report on the completion of the last war-crimes trials of the World War II (Field Marshal Fritz Erich von Manstein was sentenced to 18 years in prison), assurances by General Dwight D. Eisenhower that he would not be a candidate in the 1952 presidential election (he certainly was – and won it), and promise to announce *Time*'s "A Man of the Half-Century" in the next issue (*Time*'s choice was Winston Churchill).

Upon obtaining his doctorate from the University of Chicago in 1955, Edward Nelson became the National Science Foundation's Postdoctoral Fellow at Princeton's Institute for Advanced Study in 1956. Three years later he became – and still is – a professor at Princeton University. His main areas of interest are analysis and logic. In 1975 Edward Nelson was elected to the American Academy of Arts and Sciences, and in 1997 to the National Academy of Sciences. During my 2002–2004 stay at Princeton, I had the pleasure to interact with Professor Nelson almost daily. My talk on the chromatic number of the plane problem at Princeton's Discrete Mathematics Seminar was dedicated "To Edward Nelson, who created this celebrated problem for us all."

[6] My translation from the Russian.

[7] Ibid.

[8] The Young Men's Christian Association (YMCA) is one of the oldest and largest not-for-profit community service organizations in the world.

[9] Robert Maynard Hutchins (1899–1977) was President (1929–1945) and Chancellor (1945–1951) of the University of Chicago.

John Isbell was first in 1950 to prove the upper bound $\chi \leq 7$. He used the same hexagonal 7-coloring of the plane that Hadwiger published in 1961 [Had4]. Please note that Hadwiger first used this coloring of the plane in 1945 [Had3], but for a different problem: his goal was to show that there are seven congruent closed sets that cover the plane (he also proved there that no five congruent closed sets cover the plane). Professor John Rolfe Isbell, PhD Princeton University, 1954 under Albert Tucker, has been for decades on the faculty of mathematics at the State University of New York at Buffalo, where he is now Professor Emeritus.

Paul Erdős's contribution to the history of this problem is twofold. First of all, as Augustus De Morgan did for the four-color problem, Erdős kept the flaming torch of the problem lit. He made the chromatic number of the plane problem well known by posing it in his countless problem talks and many publications. For example, we see it in [E61.21], [E63.21], [E75.24], [E75.25], [E76.49], [E78.50], [E79.04], [ESi], [E80.38], [E80.41], [E81.23], [E81.26], [E85.01], [E91.60], [E92.19], [E92.60], and [E94.60].

Secondly, Paul Erdős created a good number of fabulous related problems. We discuss one of them in the next section.

In February 1992 at the 23rd Southeastern International Conference on Combinatorics, Computing and Graph Theory in Boca Raton, Floride during his traditional Thursday morning talk, I asked Paul Erdős how much he would offer for the first solution of the chromatic number of the plane problem. Paul replied: "I can't offer money for nice problems of other people because then I will really go broke." I then transformed my question into the realm of mathematics and asked Paul "*Assume* this is *your* problem; how much would you then offer for its first solution?" Paul answered: "It is a very nice problem. If it were mine, I would offer $250 for it."

A few years ago the price went up for the first solution of just the lower bound part of the chromatic number of the plane problem. On Saturday, May 4, 2002, which by the way was precisely Edward Nelson's 70th birthday, Ramsey theory's leading mathematician and the Treasurer of the National Academy of Sciences Ronald L. Graham gave a talk on Ramsey theory at the Massachusetts Institute of Technology for about 200 participants of the USA Mathematical Olympiad. During the talk he offered $1,000 for the first proof or disproof of what he called, after Nelson, "Another 4-Color Conjecture." The talk commenced at 10:30 AM (I attended the talk and took notes).

**Another 4-Color $1,000 Conjecture 3 (Graham, May 4, 2002).** Is it possible to 4-color the plane to forbid a monochromatic distance 1?

In August 2003, in his talk "What is Ramsey Theory?" at the Mathematical Sciences Research Institute in Berkeley, California [Gra1], Graham asked for more work for $1,000:

**$1,000 Open Problem 4 (Graham, August, 2003).** Determine the value of the chromatic number $\chi$ of the plane.

It seems that presently Graham believes that the chromatic number of the plane takes on an intermediate value, between its known boundaries, for in his two latest surveys [Gra2], [Gra3], he offers the following open problems:

**\$100 Open Problem 5 (Graham [Gra2], [Gra3]).** Show that $\chi \geq 5$.[10]

**\$250 Open Problem 6 (Graham [Gra2], [Gra3]).** Show that $\chi \leq 6$.

This prompted me to look at all published Erdős's predictions on the chromatic number of the plane. Let me summarize them here for you. First Erdős believes – and communicates it in 1961 [E61.22] and 1975 [E75.24] – that the problem creator Nelson conjectured that the chromatic number was 4; Paul enters no prediction of his own. In 1976 [E76.49] Erdős asks: "Is this graph 4-chromatic?" In 1979 [E79.04] Erdős becomes more assertive:

> It seems likely that the chromatic number is greater than 4. By a theorem of de Bruijn and myself this would imply that there are n points $x_1, \ldots, x_n$ in the plane so that if we join any two of them whose distance is 1, then the resulting graph $G(x_1, \ldots, x_n)$ has chromatic number $> 4$. I believe such an n exists but its value may be very large.[11]

A certainty comes in 1980 [E80.38] and [E80.41]: "I am sure that [the chromatic number of the plane] $\alpha_2 > 4$ but cannot prove it."

In 1981 [E81.23] and [E81.26] we read, respectively: "It has been conjectured [by E. Nelson] that $\alpha_2 = 4$, but now it is generally believed that $\alpha_2 > 4$." It seems likely that $\chi(E^2) > 4$.

In 1985 [E85.01] Paul Erdős writes: "I am almost sure that $h(2) > 4$."

Once – just once – Erdős expresses midvalue expectations, just as Ron Graham has in his problems 5 and 6. It happened on Thursday, March 10, 1994 at the 25th Southeastern International Conference on Combinatorics, Computing and Graph Theory in Boca Raton. Following Erdős's plenary talk (9:30–10:30 AM), I was giving my talk at 10:50 AM, when suddenly Paul Erdős said (and I jotted it down): "Excuse me for interrupting; I am almost sure that the chromatic number of the plane is greater than 4. It is not a proof, but any measurable set without distance 1 in a very large circle has measure less than 1/4. I also do not think that it is 7."

It is time for me to speak on the record and predict the chromatic number of the plane. I am leaning toward predicting 7 or else 4, somewhat disjointly from Graham and Erdős's apparent expectation.

**Chromatic Number of the Plane, Soft Conjecture 7.**

$$\chi = 5.5 \pm 1.5.$$

(Here $\pm$ stands for "+ or −".)

Limiting myself to just one value, I have to conjecture:

**Chromatic Number of the Plane Conjecture 7*.**

$$\chi = 7.$$

[10] Graham cites Paul O'Donnell's Theorem 48.4 (see it later in this book) as "perhaps, the evidence that $\chi$ is at least 5."

[11] If the chromatic number of the plane is 7, then for $G(x_1, \ldots, x_n) = 7$ such an $n$ must be greater than 6197 [Pri].

If you, in fact, prove the chromatic number is 7 or 4, I do not think you would lose Graham's prizes. I am sure Ron will pay his prizes for disproofs as well as for proofs. On January 26, 2007 in a personal e-mail, Graham clarified the terms of awarding his prizes: "I always assume that we are working in ZFC (for the chromatic number of the plane!). My monetary awards can vary depending on which audience I am talking to. I always give the maximum of whatever I have announced (and not the sum!)."

On May 28, 2009, during the DIMACS workshop, in the middle of my talk given 11:15–12:15, I asked the distinguished audience to determine the chromatic number of the plane by democratic means of a vote. Except one young lady voting for 6, and I voting for 7, the rest of the workshop participants equally split between 4 (including Peter D. Johnson Jr. and Mitya Karabash), and 5 (including Ron Graham). I was therefore able to determine *the democratic value of the chromatic number of the plane*: 4.5.

## 3 Polychromatic Number of the Plane & Results Near the Lower Bound

When a great problem withstands all assaults, mathematicians create many related problems. It gives them something to solve, plus sometimes there is an extra gain in this process, when an insight into a related problem brings new ways to see and conquer the original one. Numerous problems have been posed around the chromatic number of the plane. I would like to share with you my favorite among them.

It is convenient to say that a colored set $S$ *realizes distance* $d$ if $S$ contains a monochromatic pair of points distance $d$ apart; otherwise we say that $S$ *forbids distance* $d$. Our knowledge about this problem starts with the celebrated 1959 book by Hugo Hadwiger and Hans Debrunner ([HD2], and subsequently its enhanced translations into Russian by Isaak M. Yaglom [HD3] and into English by Victor Klee [HDK]). Hadwiger reported in the book the contents of the September 9, 1958 letter he received from the Hungarian mathematician A. Heppes: "Following an initiative by P. Erdős he [i.e., Heppes] considers decompositions of the space into disjoint sets rather than closed sets. For example, we can ask whether proposition 59 remains true in the case where the plane is decomposed into three disjoint subsets. As we know, this is still unresolved." In other words, Paul Erdős asked whether it was true that if the plane is partitioned (colored) into three disjoint subsets, one of the subsets must realize all distances. Soon the problem took on its current "appearance."

This important invariant had to have a name, and so in 1992 [Soi5] I named it *the polychromatic number of the plane* and denoted it by $\chi_p$.

**Erdős's Open Problem 8 (1958).** What is the smallest number $\chi_p$ of colors needed for coloring the plane in such a way that no color realizes all distances?[12]

[12] The authors of the fine problem book [BMP] incorrectly credit Hadwiger as "first" to study this problem (p. 235). Hadwiger, quite typically for him, limited his study to partitions into *closed* sets.

Since I viewed this to be a very important open problem, I asked Paul Erdős to verify his authorship, suggested, as we have seen, by Hadwiger. As always, Paul was very modest in his July 16, 1991 letter to me [E91/7/16]: "I am not even quite sure that I created the problem: Find the smallest number of colors for the plane, so that no color realizes all distances, but if there is no evidence contradicting it we can assume it for the moment."

My notes show that during his unusually long 2-week visit in December 1991–January 1992 (we were working together on the book of Paul's open problems, soon to be completed and entitled *Problems of pgom Erdős*), Paul confirmed his authorship of this problem. In the chromatic number problem, we were looking for colorings of the plane such that each color forbids distance 1. In the polychromatic number problem, we are coloring the plane in such a way that each color $i$ forbids a distance $d_i$. For distinct colors $i$ and $j$, the corresponding forbidden distances $d_i$ and $d_j$ may (but do not have to) be distinct. Of course,

$$\chi_p \leq \chi.$$

Therefore,

$$\chi_p \leq 7.$$

Nothing else had been discovered during the first 12 years of this problem's life. Then in 1970 Dmitry E. Raiskii, a student at the Moscow High School for Working Youth[13] 105, published [Rai] the lower and upper bounds for $\chi_p$. Let us look at the lower bound here, and save the upper bound for Sect.5.

**Raiskii's Lower Bound Theorem 9 (D. E. Raiskii, 1970, [Rai]).** $4 \leq \chi_p$.

Three years after Raiskii's publication, in 1973 the British mathematician Douglas R. Woodall from the University of Robin Hood (I mean Nottingham:-), published a paper [Woo1] on problems related to the chromatic number of the plane. Among other things, he gave his own proof of the lower bound for $\chi_p$. As I showed in [Soi17], Woodall's proof stemmed from a triple application of two simple ideas of Hugo Hadwiger ([HDK], Problems 54 and 59).

In 2003, the Russian-turned-Israeli mathematician Alexei Kanel-Belov communicated to me an incredibly beautiful short proof of this lower bound by the new generation of young Russian mathematicians, all his students. The proof was found by Alexei Merkov, a tenth grader from the Moscow High School 91, and communicated by Alexei Roginsky and Daniil Dimenstein in 1997 at a Moscow Pioneer Palace [Poisk]. Following is the author's proof with my gentle modifications.

*Proof of the Lower Bound Theorem 9 (A. Merkov).* Assume the plane is colored in three colors, red, white, and blue, and each color forbids a distance $r$, $w$, and $b$, respectively. Equip the 3-colored plane with the Cartesian coordinates with the origin $O$, and construct in the plane three 7-point sets $S_r$, $S_w$, and $S_b$ each being the Mosers Spindle (see Fig. 1), such that all spindles share $O$ as one of their seven vertices, and have edges all equal to $r$, $w$, and $b$, respectively. This construction defines 6 "red" vectors $v_1, \ldots, v_6$ from the origin $O$ to each remaining point of

[13] Students in such high schools hold regular jobs during the day, and attend classes at night.

$S_r$; 6 "white" vectors $v_7, \dots, v_{12}$ from $O$ to the points of $S_w$; and 6 "blue" vectors $v_{13}, \dots, v_{18}$ from $O$ to the points of $S_b$, 18 vectors in all.

Introduce now the 18-dimensional Euclidean space $R^{18}$ and a function $M$ from $R^{18}$ to the plane $R^2$ naturally defined as follows: $(a_1, \dots, a_{18}) \mapsto a_1 v_1 + \cdots + a_{18} v_{18}$. This function induces a 3-coloring of $R^{18}$ by assigning a point of $R^{18}$ the color of the corresponding point of the plane. We call the first six axes of $R^{18}$ "red," the next 6 axes "white," and the last 6 axes "blue."

Define by $W$ the subset in $R^{18}$ of all points whose coordinates include at most one coordinate equal to 1 for each of the three colors of the axes, and the rest (15 or more) coordinates 0. It is easy to verify (do) that $W$ consists of $7^3$ points. For any fixed array of allowable in $W$ coordinates on white and blue axes, we get the 7-element set $A$ of points in $W$ having these fixed coordinates on white and blue axes. The image $M(A)$ of the set $A$ under the map $M$ forms in the plane a translation of the original 7-point set $S_r$. If we fix another array of white and blue coordinates, we get another 7-element set in $R^{18}$, whose image under $M$ would form in the plane another translation of $S_r$. Thus, the set $W$ gets partitioned into $7^2$ subsets, each of which maps into a translate of $S_r$.

Now recall the observation we made after Lower Bound 1. It implies here that any translate of the Moser spindle $S_r$ contains at most two red points out of its seven points. Since the set $W$ has been *partitioned* into the translates of $S_r$, at most 2/7 of the points of $W$ are red. We can start all over again, and in a similar way show that at most 2/7 of the points of $W$ are white, and similarly to show that at most 2/7 of the points of $W$ are blue. But $2/7 + 2/7 + 2/7$ does not add up to 1! This contradiction implies that at least one of the colors realizes all distances, as required. ■

Paul Erdős proposed yet another related problem (e.g., see [E85.01]). For a given finite set $S$ of $r$ positive numbers, a set of forbidden distances if you will, we define the graph $G_S(E^2)$, whose vertices are the points of the plane, and a pair of points is adjacent if and only if the distance between them belongs to $S$. Denote

$$\chi_r = \max_S \chi\left(G_S(E^2)\right).$$

"It is easy to see that $\lim_{r\to\infty} \chi_r / r = \infty$," Erdős writes, and poses a question:

**Erdős's Open Problem 10.** Does $\chi_r$ grow polynomially?

It is natural to call the chromatic number $\chi_S(E^2)$ of the graph $G_S(E^2)$ the *S-chromatic number of the plane*. One can pose a more general and hard problem, and in fact, it is an old problem of Paul Erdős ("I asked long ago," Paul says in [E94.60]):

**Erdős's Open Problem 11.** Given a set $S$ of positive numbers, find the $S$-chromatic number $\chi_S(E^2)$ of the plane.

How difficult this problem is judge for yourselves: for the 1-element set $S$ this is the chromatic number of the plane problem!

## 4 De Bruijn–Erdős Reduction to Finite Sets and Results Near the Lower Bound

We can expand the notion of the chromatic number to any subset $S$ of the plane. The chromatic number $\chi(S)$ of $S$ is the smallest number of colors sufficient for coloring the points of $S$ in such a way that forbids monochromatic segments of length 1.

In 1951 Nicolaas Govert de Bruijn and Paul Erdős published a very powerful tool [BE2] that helps us with this and other problems. We formulate and prove it in Sect. 5. In our setting here, it implies the following.

**De Bruijn–Erdős Compactness Theorem 12.**[14] *The chromatic number of the plane is equal to the maximum chromatic number of its finite subsets.*

*Thus, as Paul Erdős used to say, the problem of finding the chromatic number of the plane is a problem about finite sets in the plane.*[15]

Victor Klee and Stan Wagon posed the following open problem in [KW]:

**Open Problem 13.** When $k$ is 5, 6, or 7, what is the smallest number $\delta_k$ of points in a plane set whose chromatic number is equal to $k$?

Of course, Problem 13 makes sense only if $\chi > 4$. In the latter case this problem suggests a way to attack the chromatic number of the plane problem by constructing new "spindles." Can we manage without unit side equilateral triangles? For four colors this for a while was an open problem, first posed by Paul Erdős in July 1975, (and published in 1976), who, as was usual with him, offered to "buy" the first solution for \$25.

**Erdős's \$25 Problem 14 [E76.49].** Let $S$ be a subset of the plane which contains no equilateral triangles of size 1. Join two points of $S$ if their distance is 1. Does this graph have chromatic number at most 3?

If the answer is no, assume that the graph defined by $S$ contains no $C_l$ [cycles of length $l$] for $3 \leq l \leq t$ and ask the same question.

It appears that Paul Erdős was not sure of the outcome, which was rare for him. Moreover, from the next publication of the problem in 1979 [E79.04], it is clear that Paul expected that triangle-free unit distance graphs had chromatic number 3, or else chromatic number 3 can be forced by prohibiting all small cycles up to $C_k$ for sufficiently large $k$:

**Erdős's \$25 Problem 14′ [E79.04].** Let our $n$ points [in the plane] be such that they do not contain an equilateral triangle of side 1. Then their chromatic number is probably at most 3, but I do not see how to prove this. If the conjecture would unexpectedly [sic] turn out to be false, the situation can perhaps be saved by the following new conjecture:

---

[14] The axiom of choice is assumed in this result.

[15] Or so we all thought until recently. Because of that, I chose to leave this section as it was written in the early 1990s. BUT: see Section X of this survey for the latest developments.

There is a $k$ so that if the girth of $G(x_1, \ldots, x_n)$ is greater than $k$, then its chromatic number is at most three – in fact, it will probably suffice to assume that $G(x_1, \ldots, x_n)$ has no odd circuit of length $\leq k$.[16]

Erdős's first surprise arrived in 1979 from Australia: Nicholas Wormald, then of the University of Newcastle, Australia, disproved the first, easier, triangle-free conjecture. Erdős paid $25 reward for the surprise, and promptly reported it in his next 1978 talk (published 3 years later [81.23]): "Wormald in a recent paper (which is not yet published) disproved my original conjecture – he found a [set] $S$ for which [the unit distance graph] $G_1(S)$ has girth 5 and chromatic number 4. Wormald's construction uses elaborate computations and is fairly complicated."

Wormald [Wor] proved the existence of a set $S$ of 6,448 (!) points without triangles and quadrilaterals with all sides 1, whose chromatic number was 4. He was aided by a computer.

The size of Wormald's example, of course, did not appear to be anywhere near optimal. Surely, it must have been possible to do the job with less than 6,448 points! In my March, 1992 talk at the Southeastern International Conference on Combinatorics, Graph Theory and Computing at Florida Atlantic University, I shared Paul Erdős's old question, but I put it in a form of competition:

**Open Problem 15.** Find the smallest number $\sigma_4$ of points in a plane set without equilateral triangles of side 1 whose chromatic number is 4. Construct (classify) all such sets $S$ of $\sigma_4$ points.

The result exceeded my wildest dreams. A number of young mathematicians, including graduate students, were inspired by this talk to enter the race. Coincidentally, during that academic year, with the participation of the celebrated geometer Branko Grünbaum, and of Paul Erdős, whose problem papers set the style, I started the new journal *Geombinatorics* dedicated to problem-posing essays on discrete and combinatorial geometry and related areas (it is still alive and well now, 19 years later). The aspirations of the journal were clear from my 1991 editor's page in issue 3 of volume I:

> In a regular journal, papers appear 1–2 years after research is completed. By then even the author may not be excited any more about his results. In *Geombinatorics* we can exchange open problems, conjectures, aspirations, work-in-progress that is still exciting to the author, and therefore exciting to the reader.

A true World Series played out on the pages of *Geombinatorics* around problem 15. The graphs obtained by the record setters were as mathematically significant as they were beautiful. See the complete report on these World Series in [Soi]. I have to show you here the two record-holding graphs; they were created by Robert Hochberg and Paul O'Donnell [HO].

**Hochberg–O'Donnell's Girth 4, 4-Chromatic Unit Distance Graph of Order 23** [HO] (Fig. 7)

[16] The symbol $G(x_1, \ldots, x_n)$ denotes the graph on the listed inside parentheses $n$ vertices, with two vertices adjacent if and only if they are unit distance apart.

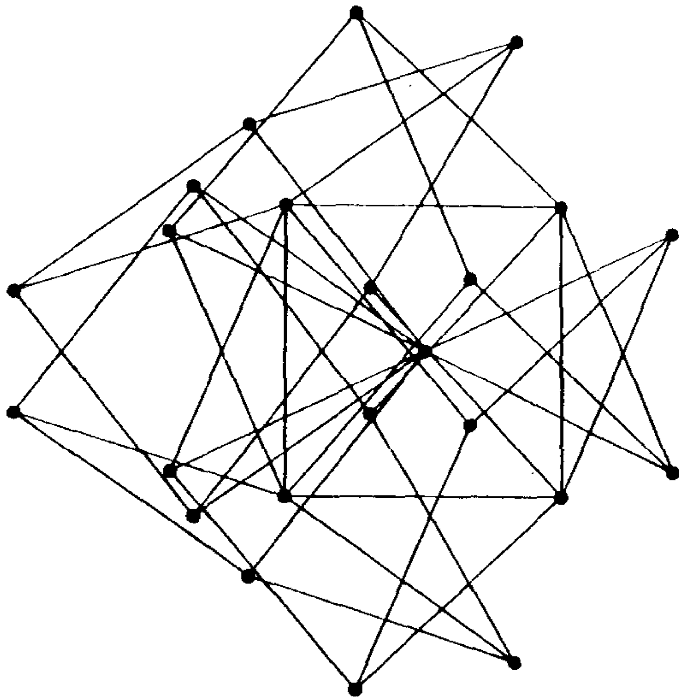


**Fig. 7** The Hochberg–O'Donnell Fish Graph

## Hochberg–O'Donnell's Girth 5, 4-Chromatic Unit Distance Graph of Order 45 [HO] (Fig. 8)

Many attempts to increase the lower bound of the chromatic number of the plane had not achieved the goal. Rutgers University's PhD student Rob Hochberg believed (and still does) that the chromatic number of the plane was 4, while his roommate and fellow PhD student Paul O'Donnell was of the opposite opinion. They managed to get along in spite of this disagreement of the mathematical kind. On January 7, 1994, Rob sent me an e-mail to that effect:

> Alex, hello. Rob Hochberg here. (The one who's gonna prove $\chi(R^2) = 4$.) ... It seems that Paul O'Donnell is determined to do his Ph. D. thesis by constructing a 5-chromatic unit distance graph in the plane. He's got several interesting 4-chromatic graphs, and great plans. We still get along.

Two months later, Paul O'Donnell's abstract in the *Abstracts* book of the Southeastern International Conference on Combinatorics, Graph Theory and Computing in Boca Raton, Florida included the following words, "The chromatic number of the plane is between four and seven. A five-chromatic subgraph would raise the lower bound. If I discover such a subgraph, I will present it."

We all came to his talk of course (it was easy for me, as I spoke immediately before Paul in the same room). At the start of his talk, however, Paul simply said,

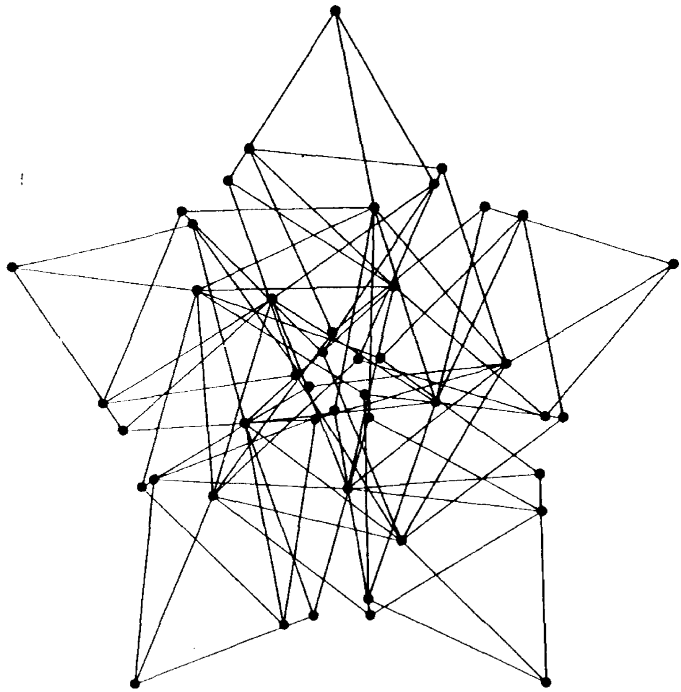


**Fig. 8** The Hochberg–O'Donnell Star Graph

"Not yet," and went on to show his impressive 4-chromatic graph of girth 4. Five years later, on May 25, 1999, Paul O'Donnell defended his doctorate at Rutgers University. I served as the outside member of his PhD defense committee. In fact, it appears that my furniture had something to do with Paul O'Donnell's remarkable dissertation, for in the dissertation's *Acknowledgements* he wrote: "Thanks to Alex. It all came to me as I drifted off to sleep on your couch."

The problem of finding a 5-chromatic unit distance graph – or proving that one does not exist – still remains open. However, much was learned about 4-chromatic unit distance graphs. The best of these results, in my opinion, was contained in this doctoral dissertation of Paul O'Donnell. He completely solved Paul Erdős's problem 14, and delivered to Paul Erdős an ultimate surprise by negatively answering his general conjecture:

**O'Donnell's Theorem 16. (1999, [Odo3,4,5]).** *There exist 4-chromatic unit distance graphs of arbitrary finite girth.*

## 5 Polychromatic Number of the Plane & Results Near the Upper Bound

Dmitry E. Raiskii's paper [Rai] also contained the upper bound:

$$\chi_p \leq 6.$$

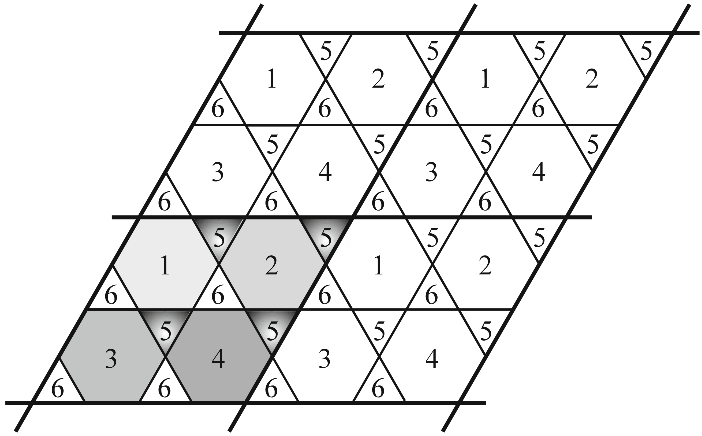



**Fig. 9** Stechkin's six-coloring of the plane

The example proving this upper bound was found by S. B. Stechkin and published with his permission by D. E. Raiskii in [Rai].

**Problem 17 (S. B. Stechkin, [Rai]).** $\chi_{p\leq}$ 6.

*Solution by S. B. Stechkin [Rai].* The "unit of the construction" is a parallelogram that consists of four regular hexagons and eight equilateral triangles, all of side lengths 1 (Fig. 9). We color the hexagons in colors 1, 2, 3, and 4. We partition triangles of the tiling into two types: we assign color 5 to the triangles with a vertex below their horizontal base; and color 6 to the triangles with a vertex above their horizontal base. While coloring, we consider every hexagon to include its entire boundary except its one rightmost and two lowest vertices; and every triangle does not include any of its boundary points.

Now we can tile the entire plane with translates of the "unit of the construction". ■

An easy construction solved problem 17, easy to understand after it was found. The trick was to find it, and Sergej B. Stechkin found it first. Christopher Columbus too "just ran into" America! I got hooked.

I felt that if our ultimate goal were to find the chromatic number $\chi$ of the plane or to at least improve the known bounds ($4 \leq \chi \leq 7$), it may be worthwhile to somehow measure how close a given coloring of the plane is to achieving this goal. In 1992 I introduced such a measurement, and named it *coloring type*.

**Definition 18 (A. Soifer [Soi5], [Soi6]).** Given an $n-$coloring of the plane such that the color $i$ does not realize the distance $d_i$ $(1 \leq i \leq n)$. Then we would say that this coloring is of *type* $(d_1, d_2, \ldots, d_n)$.

It would have been a great improvement in our search for the chromatic number of the plane if we were to find a 6-coloring of type (1,1,1,1,1,1), or to show that one does not exist. With the appropriate choice of a unit, we can make the 1970 Stechkin coloring to have type (1,1,1,1,1/2,1/2). Three years later, in 1973 Douglas

R. Woodall [Woo1] found the second 6-coloring of the plane with no color realizing all distances. Woodall's coloring had a special property that the author desired for his purposes: each of the six monochromatic sets was closed. His example, however, had three "missing distances": it had type

$$(1,1,1,\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{2\sqrt{3}}).$$

Apparently, Woodall unsuccessfully tried to reduce the number of distinct distances, for he wrote "I have not managed to make two of the three 'missing distances' equal in this way" ([Woo1], p. 193).

In 1991, in search of a "good" coloring I looked at a tiling with regular octagons and squares that I saw in some Russian public toilettes. But "The Russian toilette tiling" did not work! I then decided to shrink the squares until their diagonal became equal to the distance between two closest squares. Simultaneously (!) the diagonal of the now nonregular octagon became equal to the distance between the two octagons marked with 1 in Fig. 12. I was in business!

**First Almost Perfect 6-Coloring 19 (A. Soifer [Soi6]).** There is a 6-coloring of the plane of type $(1,1,1,1,1,1/\sqrt{5})$.

*Proof.* We start with two squares, one of side 2 and the other of diagonal 1 (Figs. 10 and 11). We can use them to create the tiling of the plane with squares and (nonregular) octagons (Fig. 12). Colors 1,..., 5 will consist of octagons; we color all squares in color 6. With each octagon and each square we include half of its boundary (bold lines in Fig. 11) without the endpoints of that half. It is easy to verify (please do) that $\sqrt{5}$ is not realized by any of the colors 1,..., 5; and 1 is not realized by the color 6. By shrinking all linear sizes by a factor of $\sqrt{5}$, we get the 6-coloring of type $(1,1,1,1,1, 1/\sqrt{5})$.

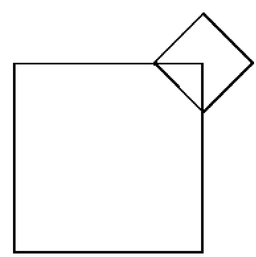


**Fig. 10**

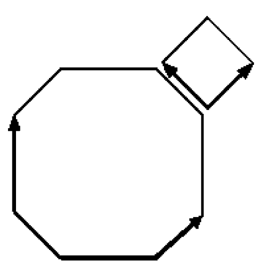


**Fig. 11**

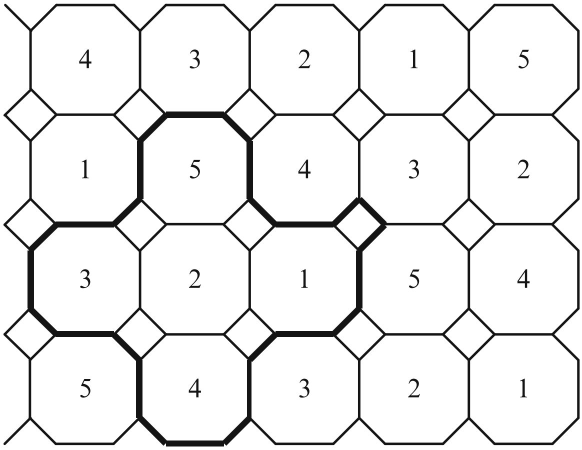



**Fig. 12** Soifer's six-coloring of the plane

To simplify a verification, observe that the unit of my construction is bounded by the bold line in Fig. 8; its translates tile the plane. ■

I had mixed feelings when I obtained the result of problem 19 in early August 1991. On the one hand, I knew the result was "close but no cigar": after all, a 6-coloring of type (1,1,1,1,1,1) was not found. On the other hand, I thought that the latter 6-coloring may not exist, and if so, my 6-coloring would be best possible. Problem 19 gave birth to a new definition and an open problem.

**Definition 20. ([HS1]).** *Almost chromatic number* $\chi_a$ of the plane is the minimal number of colors that are required for coloring the plane so that almost all (i.e., all but one) colors forbid unit distance, and the remaining color forbids a distance.

We have the following inequalities for $\chi_a$:

$$4 \leq \chi_a \left(E^2\right) \leq 6$$

The lower bound follows from Dmitry Raiskii [Rai]. I proved the upper bound in problem 19 above [Soi6]. This naturally gave birth to a new problem, which is still open:

**Open Problem 21 [HS1].** Find $\chi_a$.

## 6 Continuum of 6-Colorings of the Plane

In 1993 another 6-coloring was found jointly by Ilya Hoffman and I. Its type was $(1,1,1,1,1,\sqrt{2}-1)$. The story of this discovery is noteworthy and can be found in [Soi].

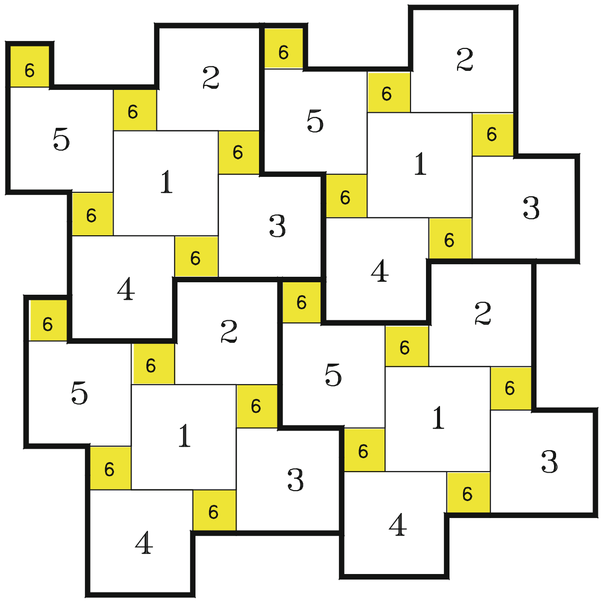
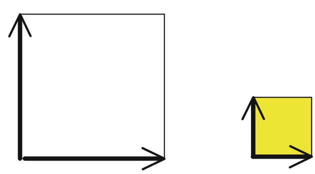



**Fig. 13** Hoffman-Soifer's six-coloring of the plane

**Fig. 14**



**Second Almost Perfect 6-Coloring 22 (I. Hoffman and A. Soifer [HS1], [HS2]).** There is a 6-coloring of the plane of type $(1{,}1{,}1{,}1{,}1{,}\sqrt{2}-1)$.

*Proof.* We tile the plane with squares of diagonals 1 and $\sqrt{2}-1$ (Fig. 13). We use colors 1,..., 5 for larger squares, and color 6 for all smaller squares. With each square we include half of its boundary, the left and lower sides, without the end-points of this half (Fig. 14).

To more easily verify that this coloring does the job, observe the unit of the construction that is bounded by the bold line in Figs. 13 and 14; its translates tile the plane. ■

The two examples, found in solutions of Problems 19 and 22 prompted me in 1993 to introduce a new terminology, and to translate the results and problems into this new language.

**Open Problem 23. (A. Soifer [Soi7], [Soi8]).** Find the *6-realizable set* $X_6$ of all positive numbers $\alpha$ such that there exists a 6-coloring of the plane of type $(1, 1, 1, 1, 1, \alpha)$.

In this new language, the results of Problems 19 and 22 can be written as follows.

$$\frac{1}{\sqrt{5}}, \sqrt{2} - 1 \in X_6$$

Now we have two examples of "working" 6-colorings. But what do they have in common? It is not obvious, is it? After a while I realized that they were two extreme examples of the general case, and in fact a much better result was possible, describing a whole continuum of "working" 6-colorings!

**Theorem 24 (A. Soifer [Soi7], [Soi8]).**

$$\left[\sqrt{2} - 1, \frac{1}{\sqrt{5}}\right] \subseteq X_6,$$

*i.e., for every* $\alpha \in \left[\sqrt{2} - 1, \frac{1}{\sqrt{5}}\right]$ *there is a 6-coloring of type* $(1, 1, 1, 1, 1, \alpha)$.[17]

*Proof Outline.* Let a unit square be partly covered by a smaller square, which cuts off the unit square vertical and horizontal segments of lengths $x$ and $y$, respectively, and forms with it an angle $\omega$ (see Fig. 15). These squares induce the tiling of the plane that consists of congruent to each other nonregular octagons and "small" squares (Fig. 16).

Now we are ready to color this tiling in six colors. Denote by $F$ the unit of our construction, bounded by a bold line (Fig. 16) and consisting of five octagons and four "small" squares. Use colors 1 through 5 for the octagons inside $F$ and color 6 for all "small" squares. Include in the colors of octagons and "small" squares the

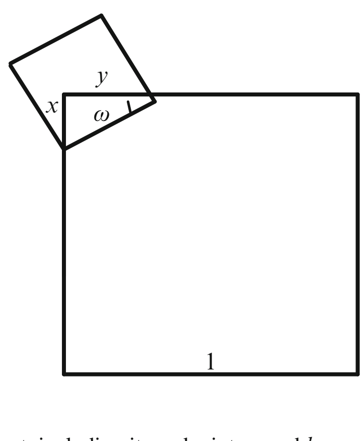



Fig. 15

[17] Symbol $[a, b]$, $a < b$, as usual, stands for the line segment, including its endpoints $a$ and $b$.

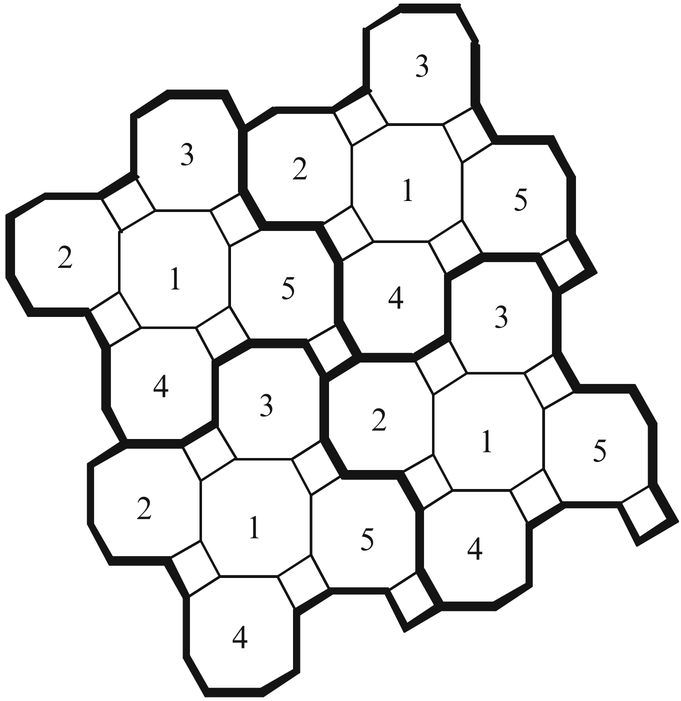
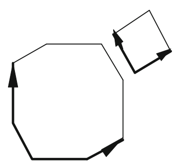



**Fig. 16** Continuum of six-colorings of the plane

**Fig. 17**



part of their boundaries that is shown in bold in Fig. 17. Translates of $F$ tile the plane and thus determine the 6-coloring of the plane. We now wish to select parameters to guarantee that each color forbids a distance.

At first, the complexity of computations appeared unassailable to me. However, a true Math Olympiad approach (i.e., good choices of variables, clever substitutions, and nice optimal properties of the chosen tilings) allowed for a successful sailing.

I have proved the required result, and much more. For every angle $\omega$ between the small and the large squares (see Fig. 15), there are (unique) sizes of the two squares (and unique squares intersection parameters $x$ and $y$), such that the constructed 6-coloring has type $(1, 1, 1, 1, 1, \alpha)$ for a uniquely determined $\alpha$.

This is a remarkable fact: the "working" solutions barely exist; they comprise something of a curve in a three-dimensional space of the angle $\omega$ and two linear variables $x$ and $y$. We thus found a continuum of permissible values for $\alpha$ and a continuum of "working" 6-colorings of the plane. ■

*Remark 6.1.* The problem of finding the 6-realizable set $X_6$ has a close relationship with the problem of finding the chromatic number $\chi$ of the plane. Its solution would shed light – if not solve – the chromatic number of the plane problem:

if $1 \notin X_6$, then $\chi = 7$;
if $1 \in X_6$, then $\chi \leq 6$.

**Open Problem 25 (A. Soifer [Soi5]).** Find $X_6$.

I am sure you understand that this problem, formulated in just two words, is extremely difficult.

## 7 Chromatic Number of the Plane in Special Circumstances

In 1973 Douglas R. Woodall [Woo1] formulated and attempted to prove a lower bound for the chromatic number of the plane for the special case of map-type coloring of the plane. This was the main result of [Woo1]. However, in 1979 Stephen Phillip Townsend from the University of Aberdeen found an error in Woodall's proof, and constructed a counterexample demonstrating that one essential idea of Woodall's proof was false. By that time, Townsend had already proved the same result, and his proof was much more elaborate than Woodall's unsuccessful attempt. You can find the complete definition of the map-type coloring and the proof in [Soi].

**Townsend-Woodall's Theorem 26 (Townsend, 1979).** *The chromatic number of the plane under map-type coloring is 6 or 7.*

Woodall showed that this result implies one more meritorious statement:

**Closed Chromatic Number of the Plane 27 [Woo1].** The chromatic number of the plane under coloring with closed monochromatic sets is 6 or 7.

In 1993–1994 a group of three undergraduate students Nathanial Brown, Nathan Dunfield, and Greg Perry, in a series of three essays, their first publications, proved on the pages of *Geombinatorics* [BDP1,2,3][18] that a similar result is true for coloring with open monochromatic sets. Now the youngsters are professors of mathematics, Nathan at the University of Illinois at Urbana-Champaign, and Nathanial at Pennsylvania State University.

**Open Chromatic Number of the Plane 27′ [BDP1, 2, 3].** The chromatic number of the plane under coloring with open monochromatic sets is 6 or 7.

While a graduate student in Great Britain, Kenneth J. Falconer proved the following important result [Fal].

[18] The important problem book [BMP] mistakenly cites only one of this series of three papers. It also incorrectly states that the authors proved only the lower bound 5, whereas they raised the lower bound to 6.

**Falconer's Theorem 28.** *Let $E^2 = \bigcup_{i=1}^{4} A_i$ be a covering of the Euclidean plane $E^2$ by 4 disjoint measurable sets. Then one of the sets $A_i$ realizes distance 1. In other words, the measurable chromatic number $\chi_m$ of the plane is equal to 5, 6, or 7.*

I found his 1981 publication [Fal1] to be too concise and not self-contained for the result that I viewed as very important. I asked Kenneth Falconer, currently professor and dean at the University of St. Andrews in Scotland, for a more detailed and self-contained exposition. In February 2005, I received Kenneth's wonderfully clear handwritten proof; see it in [Soi].

## 8 Colored Space

Paul Erdős generalized the problem of finding the chromatic number of the plane to $n$-dimensional Euclidean space $E^n$. On October 2, 1991 I received a letter from him, which contained an historical remark [E91/10/2]:[19] "I certainly asked for the chromatic number of $E^{(n)}$ long ago (30 years)." Paul was interested in both asymptotic behavior as $n$ increased, and in exact values of the chromatic number for small $n$, and first of all $n = 2$ and 3. In 1970 Dmitry E. Raiskii [Rai] proved the following lower bound for $n-$dimensional Euclidean spaces.

**Raiskii's Lower Bound 29 (Raiskii, 1970).** For $n > 1$,

$$n + 2 \leq \chi\left(E^n\right).$$

For $n = 3$ this, of course, gives $5 \leq \chi(E^3)$. This lower bound for the three-dimensional space had withstood 30 years, until in 2000 Oren Nechushtan of Tel Aviv University improved it (and published 2 years later [Nec]).

**Best Lower Bound for $R^3$ 30 (Nechushtan, 2000).**

$$6 \leq \chi\left(E^3\right).$$

David Coulson [Cou2] achieved a truly amazing improvement in the upper bound: he obtained the upper bound of 15 by using a face-centered cubic lattice (see Conway and Sloane [CS] for more about three-dimensional lattices). The proof of the upper bound of 15 was submitted to *Discrete Mathematics* on December 9, 1998. It took 4 years to appear in print.

**Best Upper Bound for $R^3$ 31 (Coulson, 1998).**

$$\chi(E^3) \leq 15.$$

[19] Curiously, Paul wrote an improbable date on the letter: "1977 VII 25".

Coulson [Cou2] informally conjectured that the upper bound of 15 is best possible for lattice-based coloring. I dare to conjecture much more: I think it is the exact value for 3-space every bit as likely as 7 is for the plane:

**Chromatic Number of 3-Space Conjecture 32.**

$$\chi(E^3) = 15.$$

Life in 4 and 5 dimensions was studied by Kent Cantwell in his 1996 work [Can1]. His lower bounds are still best known today.

**Best Lower Bounds for *$E^4$* and *$E^5$* 33 (Cantwell, 1996).**

$$\chi(E^4) \geq 7;$$
$$\chi\left(E^5\right) \geq 9.$$

On March 31, 2008, I received an impressive submission [Cib] to *Geombinatorics* from Josef Cibulka of Charles University in Prague. His main result offered the new lower bound for the chromatic number of $E^6$:

**Best Lower Bounds for *$E^6$* 34 (Cibulka, 2008).**

$$\chi(E^6) \geq 11.$$

A long time ago Paul Erdős conjectured, and often mentioned in his problem talks [E75.24], [E79.04], [E80.38], [E81.23], [E81.26], that the chromatic number $\chi(E^n)$ of the Euclidean $n-$space $E^n$ grows exponentially in $n$. In his words:

**Erdős's Conjecture on Asymptotic Behavior of the Chromatic Number of *$R^n$* 35.** $\chi(E^n)$ tends to infinity exponentially.

This conjecture was settled in the positive by a set of two results, the 1972 exponential upper bound, found by D. G. Larman and C. A. Rogers, and the 1981 exponential lower bound established by P. Frankl and R. M. Wilson:

**Frankl–Wilson's Asymptotic Lower Bound 36 (1981, [FW]).**

$$(1 + o(1))1.2^n \leq \chi(E^n).$$

**Larman-Rogers' Asymptotic Upper Bound 37 (1972, [LR]).**

$$\chi\left(E^n\right) \leq \left(3 + o\left(1\right)\right)^n.$$

Asymptotically Larman–Rogers' upper bound remains best possible still today. Frankl–Wilson's asymptotic lower bound has recently been improved.

**Raigorodskii's Asymptotic Lower Bound 38 (2000, [Raig2]).**

$$\left(1.239\cdots + o\left(1\right)\right)^n \leq \chi\left(E^n\right).$$

Obviously, there is a gap between the lower and upper bounds, and it would be very desirable to narrow it down.

The notion of polychromatic number $\chi_p$ of the plane naturally generalizes to the *polychromatic number* $\chi_p(E^n)$ *of Euclidean n-dimensional space* $E^n$. Dmitry E. Raiskii was first to publish a relevant result [Rai].

**Raiskii's Lower Bound 39.**

$$n + 2 \leq \chi_p (E^n) .$$

Larman and Rogers [LR] upper bound 37 implies the same asymptotic upper bound for the polychromatic number:

**Larman–Rogers Upper Bound 40.**

$$\chi_p (E^n) \leq (3 + o (1))^n .$$

Larman and Rogers also conjectured that $\chi_p(E^n)$ grows exponentially in $n$. The positive proof of this conjecture was completed by Frankl and Wilson [FW]:

**Frankl–Wilson Lower Bound 41.**

$$(1 + o (1)) \, 1.2^n \leq \chi_p (E^n) .$$

The problem of forbidding a set of distances can be generalized to $n-$dimensional Euclidean space too. For a given finite set $S$ of $r$ positive numbers, called *a set of forbidden distances*, we define the graph $G_S(E^n)$, whose vertices are points of the Euclidean $n$-space $E^n$, and a pair of points is adjacent if the distance between them belongs to $S$. We naturally call the chromatic number $\chi_S(E^n)$ of the graph $G_S(E^n)$ the *S-chromatic number of n-space $E^n$*. The following problem is as general as it is hard.

**Erdős's Open Problem 42.** Given $S$, find the $S-$chromatic number $\chi_S(E^n)$ of space $E^n$. The De Bruijn–Erdős compactness theorem reduces the problem of investigating $S$-chromatic numbers to finite subgraphs of $R^n$.[20]

## 9 Rational Coloring

The following problem naturally arises when one works on finding the chromatic number of the plane.

**Open Problem 43.** Find a countable subset $C$ of the set of real numbers $R$ such that the chromatic number $\chi(C^2)$ is equal to the chromatic number $\chi(E^2)$ of the plane.

[20] The De Bruijn–Erdős theorem assumes the axiom of choice.

The goal here is to find such a countable set $C$, which will help us attack the main problem of the chromatic number of the plane. The set $Q$ of all rational numbers would not work, as Douglas R. Woodall showed in 1973.

**Chromatic Number of $Q^2$ 44 (D. R. Woodall, [Woo1]).**

$$\chi(Q^3) = 2.$$

Then there came a "legendary unpublished manuscript," as Peter D. Johnson, Jr. referred [Joh8] to the paper by Miro Benda, then of the University of Washington, and Micha Perles, then of the Hebrew University, Jerusalem. The widely circulated and admired manuscript was called *Colorings of Metric Spaces.* Peter Johnson told its story on the pages of *Geombinatorics* [Joh8]. Johnson's story served as an introduction and homage to the conversion of the unpublished manuscript into the Benda–Perles publication [BP] in *Geombinatorics*' January 2000 issue.

**Chromatic Number of $Q^3$ 45 (Benda and Perles [BP]).**

$$\chi(Q^4) = 4.$$

**Chromatic Number of $Q^4$ 46 (Benda and Perles [BP]).**

$$\chi(Q^4) = 4.$$

Benda and Perles then pose important problems.

**Open Problem 47 (Benda and Perles [BP]).** Find $\chi(Q^5)$ and, in general, $\chi(Q^n)$.

**Open Problem 48 (Benda and Perles [BP]).** Find the chromatic number of $Q^2(\sqrt{2})$ and, in general, of any algebraic extension of $Q^2$.

This direction was developed by Peter D. Johnson, Jr. from Auburn University [Joh1], [Joh2], [Joh3], [Joh4], [Joh5] and [Joh6]; Joseph Zaks from the University of Haifa, Israel [Zak1], [Zak2], [Zak4], [Zak6], [Zak7]; Klaus Fischer from George Mason University [Fis1], [Fis2]; Kiran B. Chilakamarri [Chi1], [Chi2], [Chi4]; Douglas Jungreis, Michael Reid, and David Witte ([JRW]); and Timothy Chow (Cho]).

In fact, in 2006 Peter Johnson published in *Geombinatorics* "A Tentative History and Compendium" of this direction of research inquiry [Joh9]. I refer the reader to this survey and the works cited there for many exciting results of this algebraic direction, and to Peter's talk at DIMACS May 27–29, 2009 international workshop, which resulted in the paper [Joh10] that appears in this volume.

Recently Matthias Mann from Germany entered the scene and published his results in *Geombinatorics* [Man1].

**Lower Bound for $Q^5$ 49 (Mann [Man1]).**

$$\chi(Q^5) \geq 7.$$

This jump from $\chi(Q^4) = 4$ explains the difficulty in finding $\chi(Q^5)$, the exact value of which remains open. Matthias then found a few important lower bounds [Man2].

**Lower Bounds for $Q^6$, $Q^7$ and $Q^8$ 50(Mann [Man2]).**

$$\chi(Q^6) \geq 10;$$
$$\chi(Q^7) \geq 13;$$
$$\chi(Q^8) \geq 16.$$

On March 31, 2008, Josef Cibulka, a first-year graduate student at Charles University in Prague, submitted to, and in October 2008 published in, *Geombinatorics* new lower bounds for the chromatic numbers of rational spaces, improving some of Mann's results:

**Newest Lower Bounds for $Q^5$ and $Q^7$ 51 (Cibulka, [Cib]).**

$$\chi(Q^5) \geq 8;$$
$$\chi\left(Q^7\right) \geq 15.$$

## 10 Axioms of Set Theory and the Chromatic Number of Graphs

*A prudent question is one-half of wisdom.*
– Francis Bacon[21]

*Theories come and go; examples live forever.*
– I. M. Gelfand

I felt that a wide range for the chromatic number of the plane (CNP) – from 4 to 7 – was an embarrassment for mathematicians. The 4-color map-coloring problem, for example, from its beginning in 1852 had a conjecture: 4 colors suffice. Since 1890, thanks to Percy John Heawood [Hea], we knew that the answer was 4 or else 5. The CNP problem is an entirely different matter. After 60 years of very active work on the problem, we have not even been able to confidently conjecture the answer! Have mathematicians been so bad, or has the problem been so good? Have we been missing something in our assault on the CNP? Perhaps the presence or absence of the axiom of choice (that matters only for infinite sets) affects the (finite) chromatic number of the plane?

These were the questions that occupied me as I was flying cross-country from Colorado Springs to Rutgers University of New Jersey in October 2002 for a week of joint research with Saharon Shelah. We were able to break some new ground.

Our first task was to expand the definition of the chromatic number.[22] How important is it to select a productive definition? Socrates thought highly of this undertaking: "The beginning of wisdom is the definition of terms."[23] Without the

[21] Quoted from [Pet], p. 494.

[22] It *is* the first task, but we did not think of it then, and so this definition appears for the first time in [Soi].

[23] Quoted from [Pet], p. 494.

axiom of choice, the minimum, and thus the chromatic number of a graph, may not exist. In allowing a system of axioms for set theory not to include the axiom of choice, we need to come up with a much broader definition of the chromatic number of a graph than the usual one, if we want the chromatic number to exist. In fact, instead of the chromatic number we ought to talk about the *chromatic set*. There are several meaningful ways to define it. I have chosen the following definition.

**Definition 52.** Let $G$ be a graph and $\boldsymbol{A}$ a system of axioms for set theory. *The set of chromatic cardinalities* $\chi^A(G)$ of $G$ is the set of all cardinal numbers $\tau \leq |G|$ such that there is a proper coloring of the vertices of $G$ in $\tau$ colors and $\tau$ is minimum with respect to this property.

As you can easily see, the set of chromatic cardinalities does not have to have just one element as was the case when $A$ = **ZFC**. It can also be empty.

The advantage of this definition is its simplicity. Best of all, we can use inequalities on sets of chromatic cardinalities as follows. The *inequality* $\chi^A(G) > \beta$, where $\beta$ is a cardinal number, means that for every $\alpha \in \chi^A(G)$, $\alpha > \beta$. The inequalities $<, \leq,$ *and* $\geq$ are defined analogously. We also agree that the empty set is greater than or equal to any other set. Finally, if $\beta$ is a cardinal number, $\chi^A(G) = \beta$ means that $\chi^A(G) = \{\beta\}$.

The Zermelo–Fraenkel–Choice system of axioms is denoted as usual, **ZFC**; the countable axiom of choice by $\mathbf{AC}_{\aleph_0}$; the principle of dependent choices **DC**. We use one more axiom, **LM**: Every set of real numbers is Lebesgue measurable.

Assuming the existence of an inaccessible cardinal,[24] Robert M. Solovay, using Paul Cohen's forcing, constructed in 1964 (and published in 1970) a model that proved a remarkable theorem [Sol1]. We honor this achievement with the definitions.

*The Zermelo–Fraenkel–Solovay system of axioms* for set theory, which we denote by **ZFS**, is defined as follows,

$$\mathbf{ZFS} = \mathbf{ZF} + \mathbf{AC}_{\aleph_0} + \mathbf{LM},$$

and **ZFS+** stands for

$$\mathbf{ZFS+} = \mathbf{ZF} + \mathbf{DC} + \mathbf{LM}.$$

Now Solovay's theorem formulates very concisely.

**The Solovay Theorem 53.** **ZFS+** *is consistent.*[25]

**Example 54 (Shelah–Soifer [SS1]).** Define a graph $G$ as follows. The set $R$ of real numbers serves as the vertex set, and the set of edges is $\{(s,t) : s - t - \sqrt{2} \in Q\}$.

[24] A cardinal $\kappa$ is called *inaccessible* if $\kappa > \aleph_0$, $\kappa$ is regular, and $\kappa$ is strong limit. An infinite cardinal $\aleph_\alpha$ is *regular*, if $\mathrm{cf}\omega_\alpha = \omega_\alpha$. A cardinal $\kappa$ is a *strong limit* cardinal if for every cardinal $\lambda$, $\lambda < \kappa$ implies$2^\lambda < \kappa$.

[25] Assuming the existence of an inaccessible cardinal.

**Result 55 (Result 55 (Shelah–Soifer [SS1]).** For the distance graph $G$ on the line, $\chi^{ZFC}(G) = 2$, while $\chi^{\mathbf{ZFS}}(G) > \aleph_0$.

Similar examples for the plane – or $E^n$ – as the vertex set, were constructed in [SS2] and [Soi23]. These examples illuminate the influence of the system of axioms for set theory on combinatorial results. They also suggest that the chromatic number of $E^n$ may not exist "in the absolute" (i.e., in **ZF**), but depend upon the system of axioms we choose for set theory. The examples we have seen naturally pose the following open problem.

**Open AC Problem 56.** For which values of $n$ is the chromatic number $\chi(E^n)$ of the $n$-space $E^n$ defined "in the absolute", i.e., in **ZF** regardless of the addition of the axiom of choice or its relative?

The best example, however, came from the Australian student Michael Payne. On July 10, 2007, I received his e-mail from Melbourne, Australia:

> Dear Professor Soifer,
>
> I am a student from Monash Uni[versity] in Australia and I have done some work on the chromatic number of the plane problem. I found your various publications on the topic extremely helpful. I particularly liked your recent work with Saharon Shelah and as part of my [Honours] bachelor's thesis I found another example of a graph with 'ambiguous' chromatic number. This graph is a unit distance graph so it may be considered even further evidence that the plane chromatic number may also be ambiguous as you have suggested. It has been submitted for review but you can find a pre-print of it here http://arxiv.org/abs/0707.1177 if you are interested. As you will notice, I am greatly indebted to your work since my proof is essentially analogous to yours.
>
> Kind regards,
>
> Michael Payne

Indeed, the paper Michael has submitted to arXiv the day before his e-mail to me, contains a fabulous example. He starts with unit distance graph $G_1$ whose vertex set is the rational plane $Q^2$ and, of course, two vertices are adjacent if and only if they are distance 1 apart.

**Example 57 (Payne [Pay1]).** The desired unit distance graph $G$ on the vertex set $R^2$ is obtained by tiling of the plane by translates of the graph $G_1$; i.e., its edge set is

$$\left\{(p_1, p_2) : p_1, p_2 \in R^2; p_1 - p_2 \in Q^2; |p_1 - p_2| = 1\right\}.$$

*Claim 1.* $\chi^{ZFC}(G) = 2$.

*Claim 2.* $3 \le \chi^{ZFS}(G) \le 7$.

Michael Payne shows first that any measurable set $S$ of positive (Lebesgue) measure contains the endpoints of a path of length 3 in $G$. Of course, this would rule out 2-coloring of $S$. Payne continues: "We can then proceed in a similar fashion to Shelah and Soifer's proof in [SS1]."

In August 2009 Michael Payne [Pay2] constructed a new class of unit distance graphs on the vertex set $E^n$ whose chromatic number depends upon the system of axioms for set theory.

## 11 Predicting the Future

Saharon Shelah and I obtained the following surprising result:

**Conditional Chromatic Number of the Plane Theorem 58.**[26] **(Shelah-Soifer [SS1]).** *Assume that any finite unit distance plane graph has chromatic number not exceeding 4. Then:*

$$\chi^{\mathrm{ZFC}}(E^2) = 4;$$

yet

$$\chi^{\mathrm{ZFS+}}(E^2) \geq 5.$$

Can we get anything unconditionally, a piece of mathematical truth-forever result? Yes, we can, but not yet in **ZFC**.

**Unconditional Chromatic Number of the Plane Theorem 59 [SS1].**

$$\chi^{\mathbf{ZFS+}}(E^2) \geq 5.$$

I have been asked – and asked others – what is the most reasonable expected value of the chromatic number of the plane, and more generally of $E^n$ – in **ZFC**? I believe that the chromatic number of the plane is 4 or 7.

**Chromatic Number of the Plane Conjecture 60.**

$$\chi(E^2) = 4 \text{ or } 7.$$

It would be lovely to have 4 as the chromatic number of the plane: this is when our conditional chromatic number of the plane Theorem 58 would make the chromatic number dependent upon the system of axioms for set theory. Yet, if you were to insist on my choosing just one value, I would choose the latter:

**Chromatic Number of the Plane Conjecture 61.**

$$\chi(E^2) = 7.$$

"OK," I hear my reader reply, "but then a unit-distance 7-chromatic graph must exist in the plane!" This is true, but it could be quite large. In fact, in 1998 Dan Pritikin published the lower bound for the size of such a graph.

**Lower Bound for a Unit Distance 7-Chromatic Graph 62 [Pri].** Any unit distance 7-chromatic graph $G$ satisfies the following inequality:

[26] Due to the use of the Solovay's theorem, we assume the existence of an inaccessible cardinal.

$$|v(G)| \geq 6198,$$

where $v(G)$ is the vertex set of $G$.

In fact, the size of the smallest such graph may have to be even much larger. Now, try to construct it! For 3-space I conjecture:

**Chromatic Number of 3-Space Conjecture 63.**

$$\chi(E^3) = 15.$$

In general, I believe in the following conjecture:

**Chromatic Number of $E^n$ Conjecture 64.**

$$\chi(E^n) = 2^{n+1} - 1.$$

As Paul Erdős used to say about some of his problems and results, "This conjecture will likely withstand for centuries, but, we will see!"

I view the problems presented here to be part of Euclidean Ramsey theory, which is much broader than the scope of my paper. I therefore am taking the pleasure of recommending to your attention the fine survey "Open Problems in Euclidean Ramsey Theory" by the leader of the field Ronald L. Graham and his graduate student Eric Tressler that appears in this volume.

I thank Peter D. Johnson Jr., Stanislaw Radziszowski, and Pawel Radziszowski for valuable suggestions, and Col. Dr. Robert Ewell for computer-aided implementation of my Figs. 5 and 6 and some of the illustrations.

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# Euclidean Distance Graphs on the Rational Points

Peter Johnson, Jr.

## 1 Definitions

Throughout, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the usual rings of integers, rational numbers, and real numbers, respectively. If $X$ is a set and $n$ is a positive integer, $X^n$ denotes, as usual, the set of $n$-tuples with entries from $X$.

If $X$ is nonempty, a *distance function on* $X$ is a function $\rho : X^2 \to [0, \infty)$ satisfying: for all $x, y \in X$:

(i) $\rho(x, y) = 0$ if and only if $x = y$, and
(ii) $\rho(x, y) = \rho(y, x)$.

That is, a distance function satisfies the requirements to be a metric, except, possibly, the triangle inequality.

If $\rho$ is a distance function on $X$ and $D \subseteq (0, \infty)$, the *distance graph* associated with $X$, $\rho$, and $D$, denoted $G(X, D)$ (suppressing mention of $\rho$, which is usually fixed in the discussion; if it isn't, the distance graph is denoted $G_\rho(X, D)$), is the graph with vertex set $X$, and with $x, y \in X$ adjacent in $G(X, D)$ if and only if $\rho(x, y) \in D$. If $D = \{d\}$, a singleton, then we write $G(X, d)$ rather than $G(X, \{d\})$. If $P$ is a graph parameter, we shorten $P(G(X, D))$ to $P(X, D)$.

When $X \subseteq \mathbb{R}^n$, $\rho$ is the usual Euclidean distance: for $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$,

$$\rho(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{1/2},$$

also denoted $|x - y|$. When $X \subseteq \mathbb{R}^n$ and $X$ is closed under multiplication by positive scalars (real numbers), then clearly all of the single-distance graphs $G(X, d)$, $d \in (0, \infty)$, are isomorphic.

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A. Soifer (ed.), *Ramsey Theory: Yesterday, Today, and Tomorrow*,
Progress in Mathematics 285, DOI 10.1007/978-0-8176-8092-3_9,

As usual, $\chi$ denotes the chromatic number and $\omega$ the clique number. Therefore, $\omega(\mathbb{R}^n, d) = n + 1$ for all $n = 1, 2, \ldots$ and $d > 0$, and $\chi(\mathbb{R}^n, d) = \chi(\mathbb{R}^n, 1)$, the famous "chromatic number of $\mathbb{R}^n$," i.e., the chromatic number of the Euclidean unit distance graph on $\mathbb{R}^n$. There is more on this in the next section.

Suppose $\rho$ is a distance function on a nonempty set $X$, and $k$ is a positive integer. The $k$th *Babai number* of $(X, \rho)$, denoted $B_k(X)$, for short, is

$$B_k(X) = \sup[\chi(X, D); D \subseteq (0, \infty) \text{ and } |D| = k]$$

Very often that "sup" is a "max". For instance, if $X \subseteq \mathbb{R}^n$ and $D \subseteq (0, \infty)$, $|D| = k$, we have, as shown in Sect. 3, that $\chi(X, D) \leq \chi(\mathbb{R}^n, 1)^k$, and so the sup is a max for all such $X$ and all $n$. However, as shown in [21], it is quite easy to find $(X, \rho)$ such that $\{\chi(X, d) | d > 0\} = \{1, 2, 3, \ldots\}$, whence $B_1(X) = \sup\{1, 2, 3, \ldots\} = \aleph_0$, a sup, not a max.

Clearly the idea behind the definition of the $B_k$ can be used to define a sequence of parameters on pairs $(X, \rho)$ associated with any graph parameter whatever, not just the chromatic number. Only one of these sequences plays a role here; we define the $k$th clique number of $(X, \rho)$ to be

$$C_k(X) = \sup[\omega(X, D) : D \subseteq (0, \infty) \text{ and } |D| = k].$$

Clearly $C_k(X) \leq B_k(X)$ for all $(X, \rho)$ and all $k$.

The Babai numbers are so named because the idea for their definition was supplied by Lázlo Babai, in conversation with Aaron Abrams, some time in the late 1990s. This communication eventually resulted in [1], in which the definition of the Babai numbers first appeared in print, so far as we know. The debut was marred by "max" appearing where "sup" should have been: blame for the error belongs entirely to the second author of [1].

The Babai numbers were also discovered independently by Paul Erdös in [8]; the definition is confined to the case $X = \mathbb{R}^2$, but it is the same definition. The authors of [1] did not find out about this until quite recently. It seems reasonable to stick to terminology of [1] and [18–21] while acknowledging that credit for the underlying idea belongs also to the great Erdös.

## 2 The Search for $\chi(\mathbb{R}^n, 1)$ Leads to the Search for $\chi(\mathbb{Q}^n, 1)$

In 1950, almost 100 years after the 20-year-old Francis Guthrie started a mathematical avalanche by posing the four-color conjecture to his brother Frederick, who transmitted it to Augustus DeMorgan, Edward Nelson, then 18, posed the following problem to John Isbell, a fellow student at the University of Chicago: how many colors are needed to color the Euclidean plane so that no two points in the plane a distance one apart are the same color? In notation defined in Sect. 1, the problem was to determine $\chi(\mathbb{R}^2, 1)$. Nelson himself proved that $\chi(\mathbb{R}^2, 1) \geq 4$, Isbell proved that $\chi(\mathbb{R}^2, 1) \leq 7$, and neither estimate has been improved since.

Neither Isbell nor Nelson published any of their work on this new problem, yet, somehow, by word of mouth, evidently from Isbell by an untraced route to Leo Moser, to his brother Willy, Martin Gardner, and Paul Erdös, the problem became known by the 1960s. (As full an account as will ever be known of the emergence of this problem may be found in [35].) Publication on the matter was sparse; [12, 13], and [29] were the first green shoots, and then the topic was nursed along by Erdös, who mentioned it frequently in papers appearing in the 1960s. In 1972 Larman and Rogers proved [26] that $\chi(\mathbb{R}^n, 1) \leq (3 + o(1))^n$, the first (as far as I know) major advance in the problem of estimating $\chi(\mathbb{R}^n, 1)$, and still the best-known general upper bound on $\chi(\mathbb{R}^n, 1) = B_1(\mathbb{R}^n)$.

In 1973, suddenly, off-handedly, in a major paper about $G(\mathbb{R}^2, 1)$, Woodall proved [38] that $\chi(\mathbb{Q}^2, 1) = 2$. This seems to be the very first published result concerning a Euclidean distance graph with vertex set $\mathbb{Q}^n$ for some $n$. The subject had been broached, and there ensued a drip of results, quickening to a trickle, over the next 30 years, almost all of which concerned $\chi(\mathbb{Q}^n, 1)$, although there were some productive forays into questions about the connectedness of $G(\mathbb{Q}^n, 1)$ [2, 5] and even of $G(\mathbb{F}^n, 1)$, with $\mathbb{F}$ a real finite-dimensional algebraic extension of $\mathbb{Q}$ [10, 32]. In almost all of this there was only one distance involved, and that distance was 1. Here is a sketch, with commentary, of the developments after Woodall around the question of $\chi(\mathbb{Q}^n, 1)$. This will be an update and elaboration of some of [17], leaving out the discussion there of matters such as distance graphs $G(\mathbb{Q}(\sqrt{m})^2, D)$ for square-free integers $m > 1$, which are extraneous to our purpose here.

Benda and Perles proved in [2] that $\chi(\mathbb{Q}^3, 1) = 2$ and $\chi(\mathbb{Q}^4, 1) = 4$. Although [2] appeared 28 years after Woodall [38], it was actually circulating as a manuscript in the late 1970s and early 1980s, and its results were quite well known. (For an account of the curious history of this influential manuscript, see [16].) These results on $\mathbb{Q}^3$ and $\mathbb{Q}^4$ also follow from stronger results in [15].

The exact value of $\chi(\mathbb{Q}^n, 1)$ is not known for any $n > 4$. All progress toward finding those numbers has been in the form of progress on bounds, and all of that progress has been on lower bounds. That progress is sketched below.

Why has there been no progress on upper bounds of $\chi(\mathbb{Q}^n, 1)$? A standard approach to obtaining an upper estimate of $\chi(\mathbb{Q}^n, 1)$ would be to exhibit, or describe, a proper coloring of $G(\mathbb{Q}^n, 1)$; this was how it was shown that $\chi(\mathbb{Q}^2, 1) = \chi(\mathbb{Q}^3, 1) = 2$ and $\chi(\mathbb{Q}^4, 1) = 4$, and there the proper colorings were achieved via a mixture of very elementary algebra and number theory quite suitable to the task. After all, if you are dealing with rational numbers, you are really dealing with integers! Although these methods certainly become more difficult to apply to $\mathbb{Q}^n$ when $n > 4$, there is hope that they may be brought to bear to produce significantly better upper bounds on $\chi(\mathbb{Q}^n, 1)$, for $n = 5, 6, 7, \ldots$, than the upper bounds on $\chi(\mathbb{R}^n, 1)$, of which the result of Larman and Rogers [26],

$$\chi(\mathbb{R}^n, 1) \leq (3 + o(1))^n,$$

is still the champion, for $n \geq 5$. [In [30] it is shown that $\chi(\mathbb{R}^4, 1) \leq 54$.] In view of how big these upper bounds on $\chi(\mathbb{R}^n, 1)$ are, and even how big some of the recently found lower bounds on $\chi(\mathbb{Q}^n,1)$ are, for $n = 5,6,7$, and 8, to achieve a breakthrough proper coloring of $G(\mathbb{Q}^n,1)$ for small $n \geq 5$ you are allowed so many more colors than $4 = \chi(\mathbb{Q}^4,1)$ that the resulting freedom may well offset the algebraic and number theoretic difficulties. And, as we show in the next section, the introduction of the Babai numbers into the mix has strengthened the algebraic approach, although actual results on, say, $\chi(\mathbb{Q}^5,1)$, are still awaited.

*Lower bounds on $\chi(\mathbb{Q}^n,1)$, 1980–2008.* In 1981 Frankl and Wilson proved [11] that $\chi(\mathbb{R}^n,1) \geq (1+o(1))(1.2)^n$ and indicated how their proof might yield a similar estimate for $\chi(\mathbb{Q}^n,1)$. The best I can do from the proof is $\chi(\mathbb{Q}^n,1) \geq (1+o(1))r^n$ for any $r < 3^{3/4}/2$; $r = 1.138$ will work. Relatively recently, Raigorodskij [31] raised the value of $r$ in this inequality to 1.173.

In the late 1980s and early 1990s Kiran Chilakamarri ([5], for instance) and Joseph Zaks [39] provided the first nontrivial lower estimates of $\chi(\mathbb{Q}^n,1)$ for $5 \leq n \leq 13$.

These estimates have been eclipsed by Matthias Mann [27, 28] and Joseph Cibulka [6]. Here is where things now stand:

$\chi(\mathbb{Q}^5,1) \geq 8$ (Cibulka [6]);
$\chi(\mathbb{Q}^6,1) \geq 10$ (Mann [28]);
$\chi(\mathbb{Q}^7,1) \geq 15$ (Cibulka [6]);
$\chi(\mathbb{Q}^8,1) \geq 16$ (Mann [28]);

since 16 is bigger than 15, which was Zaks' upper estimate for $\chi(\mathbb{Q}^{13},1)$ [38], Cibulka and Mann's results are the reigning champion lower estimates of $\chi(\mathbb{Q}^n,1)$ for $5 \leq n \leq 13$.

Both Cibulka and Mann use computer programs to do some of the checking necessary to their proofs; many will find this alarming, as I did at first, but there is something to be learned from their methods. Here is an attempted elaboration, mainly of Cibulka's methods.

First of all, for any positive rational number $r$ and any positive integer n, $G(\mathbb{Q}^n,r)$ and $G(\mathbb{Q}^n,1)$ are isomorphic (scalar multiplication!), so, in searching for finite subgraphs of $G(\mathbb{Q}^n,1)$ with large chromatic number, we can search in $G(\mathbb{Q}^n,r)$ instead, and it may be convenient to do so. Cibulka obtains his results by finding subgraphs of $G(\mathbb{Q}^5,4)$ and $G(\mathbb{Q}^7,2)$.

Mann used a computer search to find the subgraphs that give his estimates [28], and Cibulka may have; he does not say. To the purist it does not matter how the subgraphs are found, as long as they are explicitly described, and they are. But surely the ideas behind the searches are of interest. Mann gives a description of his search methods in [28].

Where Mann and Cibulka risk purist displeasure is in showing that the chromatic numbers of the graphs they describe are at least as large as they claim. This is accomplished by a computer program; but in Cibulka's case, at least, the general idea is transparent, and the calculation can, perhaps, be executed with pencil and paper. Here is an elaboration of Cibulka's concise account.

Suppose that $G$ is a finite simple graph, and $w : V(G) \to (0, \infty)$ is a weighting of the vertices of $G$. For $S \subseteq V(G)$, let $w(S) = \sum_{v \in S} w(v)$, and let $\alpha_w(G) = \max[w(I); I \subseteq V(G)$ is an independent set of vertices], the weighted independence number of $G$ with respect to $w$. So $\alpha = \alpha_1$, with 1 denoting the constant function 1, is the usual vertex independence number.

By the standard proof that $|V(G)| \leq \alpha(G)\chi(G)$ one sees that $w(V(G)) \leq \alpha_w(G)\chi(G)$, and thus that $\chi(G) \geq w(V(G))/\alpha_w(G)$. Notice that because the lower estimate of $\chi(G)$ thus obtained is a ratio of sums of values of $w$, we may as well confine ourselves to $w : V(G) \to [1, \infty)$. And because $\chi(G) \in \mathbb{Z}$ we may as well further restrict to $w : V(G) \to \{1, 2, \ldots\}$.

For instance, to show that $\chi(\mathbb{Q}^7, 1) \geq 15$, Cibulka exhibits 388 points in $\mathbb{Q}^7$, considers them as vertices in $G(\mathbb{Q}^4, 2)$, gives them a weighting $w$ with total weight 280, 128, and then reports that $\alpha_w \leq 20, 009$, whence $\chi(\mathbb{Q}^7, 1) = \chi(\mathbb{Q}^7, 2) \geq 280, 128/20, 009 > 14$. So the unverified-by-hand computer-assist is in the inequality $\alpha_w \leq 20, 009$, but it seems very likely that the logic of the program can be followed to verify this estimate without computer assistance, in view of the simplicity of the description of the graph.

It would be very interesting to know how Cibulka hunts for his graphs and the weightings; does he find them together, or first one and then the other? In any case, there are lessons to be learned! Here is one of them, left as an exercise: Show that for any finite simple graph $G$, $\chi(G) = \max[\lceil w(V(G))/\alpha_w(G) \rceil; w : V(G) \to \{1, 2, \ldots\}]$.

## 3 Distances Other Than 1?

Could it have just plain escaped notice that $\chi(\mathbb{Q}^n, d)$ might not be the same as $\chi(\mathbb{Q}^n, 1)$ for all distances $d > 0$ actually realized between points of $\mathbb{Q}^n$? While I am quite sure that the thought crossed my mind during the preparation of [15], I must sheepishly admit that it was over 15 years later that it occurred to me that, while $\chi(\mathbb{Q}^3, 1) = 2$, the points $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$ induce a $K_4$ in $G(\mathbb{Q}^3, \sqrt{2})$, and so $\chi(\mathbb{Q}^3, \sqrt{2}) \geq \omega(\mathbb{Q}^3, \sqrt{2}) = 4$. (The last equality holds because clearly $\omega(\mathbb{Q}^3, \sqrt{2}) \geq 4$, and, meanwhile, $\omega(\mathbb{Q}^n, d) \leq \omega(\mathbb{R}^n, d) = n + 1$ for any $n$ and any $d > 0$; $\omega(\mathbb{R}^n, d) = n + 1$ for well-known geometric/dimensional reasons.)

But, I'm slow – thorough, I hope – but very slow. My guess is that the fact that $\chi(\mathbb{Q}^3, \sqrt{2}) \geq 4$ was well known, but ignored; the herd was grazing on $G(\mathbb{Q}^n, 1)$. Why? Historical accident and intellectual inertia, perhaps.

This is where the Babai numbers, another historical accident but with a halo of inevitability that the numbers $\chi(\mathbb{Q}^n, 1)$ just do not possess, come in. If $X$ is a nonempty set with distance function $\rho$, what has a better claim to the title, chromatic number of $(X, \rho)$, than $B_1(X)$? So, it ought to be $B_1(\mathbb{Q}^n)$ we're after. But if you are emotionally attached to $\chi(\mathbb{Q}^n, 1)$, there is no cause for alarm: $\chi(\mathbb{Q}^n, 1) \leq B_1(\mathbb{Q}^n) \leq B_1(\mathbb{R}^n) = \chi(\mathbb{R}^n, 1)$, so $B_1(\mathbb{Q}^n)$ is in the same neighborhood as the big stars of Euclidean coloring.

*A result on $B_k(X)$, for an arbitrary $(X, \rho)$, and one on $B_k(\mathbb{Q}^n)$, with short proofs.*

B1. $B_{k+m}(X) \leq B_k(X)B_m(X)$ [1].

*Proof.* Suppose $D \subseteq (0, \infty)$ and $|D| = k + m$. Let $D = D_0 \cup D_1$ with $|D_0| = k$ and $|D_1| = m$. Let $\varphi_1$ be a proper coloring of $G(X, D_i)$ with $\chi(X, D_i)$ colors, $i = 0, 1$. Then coloring each $x \in X$ with the ordered pair $(\varphi_0(x), \varphi_1(x))$ gives a proper coloring of $G(X, D)$ with $\chi(X, D_0)\chi(X, D_1) \leq B_k(X)B_m(X)$ colors. Therefore $B_{k+m}(X) = \sup_{|D|=k+m} \chi(X, D) \leq B_k(X)B_m(X)$. □

**Corollary 3.1.** $B_k(X) \leq B_1(X)^k, k = 1, 2, \ldots$

B2. *For each* $n, k = 1, 2, \ldots, B_k(\mathbb{Q}^n) = B_k(\mathbb{Z}^n)$ [17].

*Proof.* Clearly $B_k(\mathbb{Q}^n) \geq B_k(\mathbb{Z}^n)$. Suppose $D \subseteq (0, \infty)$, $|D| = k$, with $\chi(\mathbb{Q}^n, D) = B_k(\mathbb{Q}^n)$. [Since $B_k(\mathbb{Q}^n) \leq B_1(\mathbb{Q}^n)^k \leq B_1(\mathbb{R}^n)^k = \chi(\mathbb{R}^n, 1)^k \leq (3 + o(1))^{nk}$ is finite, $B_k(\mathbb{Q}^n) = \max_{|D|=k} \chi(\mathbb{Q}^n, D)$.] By the famed result of de Bruijn and Erdös [7] on the chromatic numbers of infinite graphs, there is a finite set $F \subseteq \mathbb{Q}^n$ such that $\chi(F, D) = \chi(\mathbb{Q}^n, D) = B_k(\mathbb{Q}^n)$. Let $m$ be a positive common multiple of the denominators of the coordinates of the points in $F$. Then $B_k(\mathbb{Q}^n) = \chi(F, D) = \chi(mF, mD) \leq \chi(\mathbb{Z}^n, mD) \leq B_k(\mathbb{Z}^n)$. □

Therefore $\chi(\mathbb{Q}^n, 1) \leq B_1(\mathbb{Q}^n) = B_1(\mathbb{Z}^n)$. $\mathbb{Q}^n$ is a mess, but $\mathbb{Z}^n$ is just the set of all $n$-tuples of whole numbers! It is the fact that $\chi(\mathbb{Q}^n, 1) \leq B_1(\mathbb{Z}^n)$ that makes us feel that upper bounds for $\chi(\mathbb{Q}^n, 1)$ that are not just upper bounds of $\chi(\mathbb{R}^n, 1)$ are within reach.

The proof of the de Bruijn–Erdös result depends greatly on the axiom of choice. There is a question here for the mavens of mathematical logic: is there a model of Zermelo–Fraenkel set theory in which $B_1(\mathbb{Z}^n) < B_1(\mathbb{Q}^n)$, for some $n > 1$?

In the remainder of this section we complete our survey of known results on chromatic and clique numbers of Euclidean distance graphs $G(\mathbb{Q}^n, D)$, $D \subseteq (0, \infty)$; almost all of the preceding has been about the case $D = \{1\}$. Most developments in this area have been recent, emerging after the debut of the Babai numbers [1], but there were some twentieth century stirrings to report; the account is roughly organized by historical chronology. Some notation:

$$\text{If } D \subseteq (0, \infty), \sqrt{D} = \{\sqrt{d} | d \in D\}; \mathbb{Q}_0 = \{p/q \mid p, q$$

are odd positive integers}.

**DGQ1:** *Evaluation of* $C_1(\mathbb{Q}^n)$.

If $d > 0$, $\omega(\mathbb{Q}^n, d) \leq \omega(\mathbb{R}^n, d) = n + 1$, so $C_1(\mathbb{Q}^n) \leq n + 1$. On the other hand, for $n \geq 2$, $(0, \ldots, 0), (1, 1, 0, \ldots, 0)$, $(1, 0, 1, 0, \ldots, 0), \ldots, (1, 0, \ldots, 0, 1)$ are $n$ vectors in $\mathbb{Z}^n \subseteq \mathbb{Q}^n$ which induce a clique in $G(\mathbb{Q}^n, \sqrt{2})$. Consequently, noting that $C_1(\mathbb{Q}) = 2$, we have that $C_1(\mathbb{Q}^n) \in \{n, n + 1\}$ for each $n$. The question is, for which $n$ is $C_1(\mathbb{Q}^n) = n$?

The answer was given in 1937, about 70 years before the question arose. By the same kind of argument as that given in the proof of $B2$, above, except that there is no appeal to the de Bruijn–Erdös theorem necessary, it is easy to see that $C_k(\mathbb{Q}^n) = C_k(\mathbb{Z}^n)$ for all positive integers $n$ and $k$. Clearly $C_1(\mathbb{Z}^n)$ is the greatest order (number of extreme points) in a regular simplex with extreme points in $\mathbb{Z}^n$. This number was given by I. J. Schoenberg in [33], and here it is: $C_1(\mathbb{Z}^n) = n + 1$ if and only if either:

(i) $n \equiv 3 \bmod 4$, or
(ii) $n$ is one less than an odd square, or
(iii) $n$ is one less than the sum of two odd squares.

Otherwise, $C_1(\mathbb{Z}^n) = C_1(\mathbb{Q}^n) = n$.

**DGQ2:** A 1980s *result in which distances other than one are considered*.

Let $A_n$ be the set of points in $\mathbb{Q}^n$ with "odd denominator," i.e., points with a representation $(a_1/b, \ldots, a_n/b)$ in which the $a_i$ and $b$ are integers, and $b$ is odd. It is shown in [15] that $\chi(A_n, \sqrt{\mathbb{Q}_0}) = 2$ for all $n = 1, 2, \ldots$, and this result is used to prove that $\chi(\mathbb{Q}^3, \sqrt{\mathbb{Q}_0}) = 2$ and that $\chi(\mathbb{Q}^4, \sqrt{\mathbb{Q}_0}) = 4$, a considerable strengthening of the corresponding results about $G(\mathbb{Q}^n, 1)$, $n \in \{2, 3, 4\}$.

Jungreis, Reid, and Witte [24] proved a sort of strong converse of each of the results above, except one:

Suppose $B \subseteq \{\mathbb{Q}, \mathbb{Q}^2, \mathbb{Q}^3\} \cup \{A_n | n = 1, 2, \ldots\}$, and $d > 0$ is a distance actually occurring between points of $B$. If $\chi(B, \{1, d\}) = 2$, then $d \in \sqrt{\mathbb{Q}_0}$.

The remaining open questions of interest in this vein are: If $n \geq 4$, for which $d > 0$ is it the case that $\chi(\mathbb{Q}^n, \{1, d\}) = \chi(\mathbb{Q}^n, 1)$? We may as well worry only about those $d$ that actually occur as distances between points of $\mathbb{Q}^n$! The Jungreis–Reid–Witte answer for $n = 1, 2, 3$ is: $d \in \sqrt{\mathbb{Q}_0}$ (if $d$ is a realized distance). I think that the same answer holds for $n = 4$, but this has not been shown. It may be implied by the methods, in [2] and [15], of coloring that show that $\chi(\mathbb{Q}^4, 1) = 4$, and by the work of Joseph Zaks in [40], where he proves that for every proper 4-coloring of $G(\mathbb{Q}^4, 1)$ each of the 4 color classes is dense in $\mathbb{Q}^4$.

We leave the resolution of this problem – for which $d > 0$ realized as a distance between points of $\mathbb{Q}^4$ is it the case that $\chi(\mathbb{Q}^4, \{1, d\}) = 4$? – as a puzzle for the pleasure of those interested. The more general question – for which $n$ and $d > 0$ is it the case that $\chi(\mathbb{Q}^n, \{1, d\}) = \chi(\mathbb{Q}^n, 1)$? – is a bit of a side issue associated with the problem of finding $B_2(\mathbb{Q}^n)$. But it is a side issue of more interest than most because of the historic interest in $\chi(\mathbb{Q}^n, 1)$.

**DGQ3:** $B_1(\mathbb{Q}^n)$, $n = 1, 2, 3, 4$.

It is shown in [1] that $B_1(\mathbb{Q}^2) = 2$. (It is clear that $B_1(\mathbb{Q}) = 2$.) As a corollary we obtain: not only is there no regular polygon with an odd number of sides with vertices in $\mathbb{Q}^2$, there is no closed walk in $\mathbb{Q}^2$ with an odd number of steps of equal lengths.

The observation above that $B_1(\mathbb{Q}^3) \geq \omega(\mathbb{Q}^3, \sqrt{2}) = 4$ appears in [19]. In [20] it was shown that $B_1(\mathbb{Q}^3) = 4$, and then this result was eclipsed in [18]: $B_1(\mathbb{Q}^4) = 4$. As with $\chi(\mathbb{Q}^n, 1)$, $B_1(\mathbb{Q}^n)$ is unknown for $n > 4$. (Of course, all lower bounds of

$\chi(\mathbb{Q}^n, 1)$ are lower bounds, also, of $B_1(\mathbb{Q}^n)$.) We can make one simplifying observation toward the computation of $B_1(\mathbb{Q}^n)$, besides the fact that $B_1(\mathbb{Q}^n) = B_1(\mathbb{Z}^n)$, noted in $B2$, above. For $D \subseteq (0, \infty)$ and $r \in \mathbb{Q}$, $G(\mathbb{Q}^n, D)$ and $G(\mathbb{Q}^n, rD)$ are isomorphic graphs; consequently, to determine $B_1(\mathbb{Q}^n) = \max_{d>0} \chi(\mathbb{Q}^n, d)$, it suffices to determine $\chi(\mathbb{Q}^n, \sqrt{m})$ and $\chi(\mathbb{Q}^n, \sqrt{2m})$ for all square-free odd positive integers $m$. By the proof of $B2$, it suffices, alternatively, to determine $\chi(\mathbb{Z}^n, d)$, $d$ an integer multiple of $\sqrt{m}$ or $\sqrt{2m}$.

There is a logical infelicity in the proofs that $B_1(\mathbb{Q}^4) = B_1(\mathbb{Q}^3) = 4$ and that $B_1(\mathbb{Q}^n) = 2$: the axiom of choice is applied to achieve the results in the following way. First, suppose that $A$ is an additive subgroup of $\mathbb{R}^n$ and $D \subseteq (0, \infty)$. Let $A_D = \{x \in A | \, |x| \in D\}$. Then a coloring $\varphi$ of $A$ is a proper coloring of $G(A, D)$ if and only if $\varphi$ "forbids (translation by) $A_D$," meaning, if $y \in A$ and $x \in A_D$ then $\varphi(y) \neq \varphi(y + x)$. Suppose $B$ is the subgroup of $A$ generated by $A_D$; suppose $B \neq A$, and suppose $\varphi$ properly colors $G(B, D)$. Then $\varphi$ can be extended to $A$ by "copying" $\varphi$ on each coset of $B$ in $A$. This involves choosing a representative $y$ of each coset $y + B$ in $A$ and then defining $\varphi(y + x) = \varphi(x)$ for each $x \in B$. The resulting extension will forbid $A_D$ if $\varphi$ does, and it uses the same colors as $\varphi$, and thus is a proper coloring of $G(A, D)$, if $\varphi$ is a proper coloring of $G(B, D)$, with the same number of colors.

The axiom of choice – or, if $B$ has countable index in $A$, the axiom of countable choice – enters at the choosing of the coset representatives. In the proofs in [1], [18], and [20], concerning $B_1(\mathbb{Q}^n)$, $n = 2, 3, 4$, or in [21], concerning $\chi(\mathbb{Q}^3, D)$ for some choices of $D \subseteq (0, \infty)$, in which $A$ is always a subgroup of $\mathbb{Q}^n$ (usually $\mathbb{Q}^n$ itself) the appeal to the ACC can plausibly be excised by explicitly, recursively describing a system of representatives of the cosets of $B$ in $A$. No attempt to do this has been made; perhaps it would be a worthy exercise to try.

Recently Jeffrey Burkert gave another proof that $B_1(\mathbb{Q}^4) = 4$ by applying $B2$; in [3] he gives, for each positive integer $m$, an explicit proper coloring of $G(\mathbb{Z}^4, \sqrt{m})$ with four or fewer colors. Because the proof of $B2$ uses the axiom of choice, or, at least, the ACC, this proof does not give an ACC-free proof that $B_1(\mathbb{Q}^4) = 4$. However, if you are not choice-squeamish, the methods of the proof show promise for getting upper bounds on $B_1(\mathbb{Z}^n) = B_1(\mathbb{Q}^n) \geq \chi(\mathbb{Q}^n, 1)$ for $n \geq 5$.

**DGQ4:** $B_k(\mathbb{Q}) = C_k(\mathbb{Q}) = k + 1$ for all $k$; $B_2(\mathbb{Q}^2) = C_2(\mathbb{Q}^2) = 4$; *further results on distance graphs on* $\mathbb{Q}^3$.

There are but $k$ distances among $k + 1$ consecutive integers, so, for all $k = 1, 2, \ldots B_k(\mathbb{R}) \geq B_k(\mathbb{Q}) = B_k(\mathbb{Z}) \geq C_k(\mathbb{Z}) \geq k + 1$. As noted in [19], one of the main results in [25], restated, is that $B_k(\mathbb{R}) = k + 1$ for all $k$, so equality holds throughout.

The proof in [25] uses the de Bruijn–Erdös theorem, and therefore the axiom of choice, as follows. Suppose that $D \subseteq (0, \infty)$, $|D| = k$, and $H$ is a finite subgraph of $G(\mathbb{R}, D)$. We want to show that $\chi(H) \leq k + 1$. Let the vertices of $H$ be $u_1 < u_2 < \cdots < u_m$. Each $u_j$ has exactly $k$ neighbors, in $G(\mathbb{R}, D)$, in the interval $(-\infty, u_j)$, and thus at most $k$ neighbors $u_i$, $i < j$, in $H$. Therefore $H$ can be properly colored from a stock of $k + 1$ colors by coloring $u_1, \ldots, u_n$, in that order, coloring each $u_j$ with a color not appearing on any neighbor of $u_j$ among the $u_i$, $i < j$.

In [34] Shelah and Soifer show that if $D = \{|r + \sqrt{2}| \mid r \in \mathbb{Q}\}$ then $\chi(\mathbb{R}, D) = 2$ if the axiom of choice is assumed true, but that if the axiom of choice is replaced by the axiom of countable choice together with the assumption that all sets of real numbers are Lebesgue measurable, axioms known to be consistent with the Zermelo–Fraenkel axioms of set theory, then $\chi(\mathbb{R}, D) > \aleph_0$. This is quite shocking, and raises all sorts of questions. For instance, could it be that $B_k(\mathbb{R}) > k + 1$ if the axiom of choice is negated? It seems unlikely; the Shelah–Soifer set of distances is infinite, while $B_k(\mathbb{R})$ is the maximum of $\chi(\mathbb{R}, D)$ for sets of $k$ distances only. And yet, there is that application of the de Bruijn–Erdös theorem in Kemnitz and Murangio's argument! Can it be dispensed with?

In the case of $\mathbb{Q}$ and $\mathbb{Z}$ it appears that it can. Given $D \subseteq \mathbb{Z} \cap (0, \infty)$, with $|D| = k$, we can easily properly color $G(\mathbb{Z}, D)$ with $k + 1$ colors by employing the Kemnitz–Murangio ploy on $0, 1, 2, \ldots$, and then on $-1, -2, \ldots$, taking care just to color each integer with a color not on any already colored neighbor of it. In the case of $\mathbb{Q}$, supposing $D \subseteq \mathbb{Q} \cap (0, \infty)$, $|D| = k$, since $G(\mathbb{Q}, D)$ is isomorphic to $G(\mathbb{Q}, rD)$ for every positive rational $r$, we can suppose that $D \subseteq \mathbb{Z}$, and then carry out orderly greedy colorings, with $k + 1$ colors, of $\mathbb{Z}$, then $\frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$, and so on, coloring, at the $n$th stage, the elements of $(1/n)\mathbb{Z}$ that have not yet been colored at earlier stages. So $B_k(\mathbb{Q}) = B_k(\mathbb{Z}) = k + 1$, even if one does not allow the axiom of choice.

As mentioned in $DGQ3$, it was shown in [1] that $B_1(\mathbb{Q}^2) = 2$. From this it follows, by $B1$, that $B_2(\mathbb{Q}^2) \leq B_1(\mathbb{Q}^2)^2 = 4$. It was not noticed until [21] that $(0, 0), (0, 1), (1, 0), (1, 1)$ form a 4-clique in $G(\mathbb{Q}^2, \{1, \sqrt{2}\})$, whence $B_2(\mathbb{Q}^2) \geq C_2(\mathbb{Q}^2) \geq 4$, so $B_2(\mathbb{Q}^2) = C_2(\mathbb{Q}^2) = 4$. Amusingly or embarassingly, depending on your point of view, the fact that $C_2(\mathbb{Q}^2) = 4$ was used in [21] to draw the corollary that no regular pentagon in the plane has all five vertices rational; the reasoning was that since only two distances occur between the vertices of a regular pentagon the existence of such a pentagon with rational vertices would imply a set $D$ of two distances such that $5 \leq \omega(\mathbb{Q}^2, D) \leq C_2(\mathbb{Q}^2) = 4$, a contradiction.

But, as mentioned early in $DGQ3$, the mere fact that $B_1(\mathbb{Q}^2) = 2$ implies that no odd-sided regular polygon in the plane has all vertices rational, and even more, no closed walk in $\mathbb{Q}^2$ with steps of equal length can have an odd number of steps.

However, not everything in [21] is either an oversight or the correction of one. It is shown that $C_2(\mathbb{Q}^3) \geq 5$; we expect equality here, but the problem is open. And while $B_2(\mathbb{Q}^3) \geq C_2(\mathbb{Q}^3) \geq 5$, it seems that $\chi(\mathbb{Q}^3, D) \leq 4$ for many sets $D$ of two distances. In particular, the first inspiration for [21] was the discovery that $\chi(\mathbb{Q}^3, \{1, \sqrt{2}\}) = 4$.

Let $\mathbb{Q}_1 = \{p/q \mid p$ and $q$ are odd positive integers and $p \equiv q \bmod 4\}$, and $\mathbb{Q}_2 = \{p/q \mid p, q$ are odd positive integers and $p \not\equiv q \bmod 4\}$. The main result of [21] is that $\chi(\mathbb{Q}^3, \sqrt{2\mathbb{Q}_0} \cup \sqrt{\mathbb{Q}_i}) = 4$ for each $i = 1, 2$. Earlier results were that $\chi(\mathbb{Q}^3, \sqrt{\mathbb{Q}_0}) = 2$ [15], and that $\chi(\mathbb{Q}^3, \sqrt{2\mathbb{Q}_0}) = 4$ [20]. Pretty clearly the road to determining $B_2(\mathbb{Q}^3)$ might lie through the attempt to determine

$$\chi(\mathbb{Q}^3, \sqrt{2^r\mathbb{Q}_0} \cup \sqrt{\mathbb{Q}_i}), \quad i = 1, 2, \quad r = 2, 3, \ldots$$

An alternative approach would be to evaluate $\chi(\mathbb{Z}^3, \{s, t\})$, for positive integers $s, t$.

## 4 Problems

As noted in [21], $n = 3$ is the only value for which it is known that $\chi(\mathbb{Q}^n, 1) < B_1(\mathbb{Q}^n)$. On the grounds that, as $n$ increases, the importance of the choice of distance $d \in \sqrt{\mathbb{Q} \cap (0, \infty)}$ might diminish in determining $\chi(\mathbb{Q}^n, d)$, might it be the case that $n = 3$ is indeed the only value such that $\chi(\mathbb{Q}^n, 1) < B_1(\mathbb{Q}^n)$? A related deeper question: for each positive integer $n$, how many different isomorphism classes are represented by the graphs $G(\mathbb{Q}^n, d)$, $d \in \sqrt{\mathbb{Q} \cap (0, \infty)}$, and what is the corresponding partition of $\sqrt{\mathbb{Q} \cap (0, \infty)}$ into equivalence classes of the relation $\sim$ defined by $d \sim d'$ if and only if $G(\mathbb{Q}^n, d) \simeq G(\mathbb{Q}^n, d')$? In the case $n = 1$ there are two isomorphism classes, one represented by $G(\mathbb{Q}, 1)$ and the other the empty graph on $\mathbb{Q}$. The equivalence classes on $\sqrt{\mathbb{Q} \cap (0, \infty)}$ are $\mathbb{Q} \cap (0, \infty)$ and its complement, the set of square roots of positive rationals which are not perfect squares. Not very interesting! But the story gets better as $n$ goes up, as I leave it to interested readers to discover.

Here is a much more modest question list some of which would be easy to answer if the isomorphism classes of the graphs $G(\mathbb{Q}^n, d)$, $n = 3, 4$, $d \in \sqrt{\mathbb{Q} \cap (0, \infty)}$, were known. Observe that if $n \in \{2, 3, 4\}$ and $d \in \sqrt{\mathbb{Q} \cap (0, \infty)}$, then $B_1(\mathbb{Q}^4) = 4$ implies that $\chi(\mathbb{Q}^n, d) \in \{1, 2, 3, 4\}$. When $n = 2, 3$, values of $d$ are known that are not realized as distances between points of $\mathbb{Q}^n$ ($d = \sqrt{7}$, for instance); for such $d$, $\chi(\mathbb{Q}^n, d) = 1$. There are no such $d$ for $n \geq 4$. (Why not?) For all $d \in \sqrt{\mathbb{Q} \cap (0, \infty)}$ realized as a distance between points of $\mathbb{Q}^2$, $\chi(\mathbb{Q}^2, d) = 2$. Values of $d$ are known for which $\chi(\mathbb{Q}^3, d) = 2$ ($d = 1$, for instance), for which $\chi(\mathbb{Q}^3, d) = 4$ ($d = \sqrt{2}$, for instance), and for which $\chi(\mathbb{Q}^4, d) = 4$ ($d = 1$ or $\sqrt{2}$, for instance). Which brings us to our questions: Is there any $d > 0$ such that $\chi(\mathbb{Q}^3, d) = 3$? Do there exist $d > 0$ such that $\chi(\mathbb{Q}^4, d) = 2$ or 3?

Is it the case that for every $n$, or for every $n$ sufficiently large, the numbers $\chi(\mathbb{Q}^n, d)$, $d \in \sqrt{\mathbb{Q} \cap (0, \infty)}$, form a block of consecutive integers?

Does there exist $D \subset (0, \infty)$, $|D| = 2$, such that $\chi(\mathbb{Q}^2, D) = 3$?

There are Euclidean Ramsey problems that are not just about the chromatic numbers of distance graphs on $\mathbb{R}^n$, and many of these are quite interesting when $\mathbb{R}^n$ is replaced by $\mathbb{Q}^n$. Since the paper is about distance graphs on $\mathbb{Q}^n$, $n = 1, 2, \ldots$, we confine the last remarks here to two classes of these problems that bear on or involve those distance graphs.

If $n$ is a positive integer and $D \subseteq (0, \infty)$, a *rather red* coloring of $X \subseteq \mathbb{R}^n$ with reference to $D$ is a coloring of $X$ with two colors, red and blue, such that if $u$ and $v$ are both blue, then $|u - v| \notin D$. Such colorings came to fame because of a result and a problem posed in [9]. The result: if $T$ is a set of three points in $\mathbb{R}^2$, the plane, then for every rather red coloring of $\mathbb{R}^2$ with reference to $\{1\}$ (or any other single distance), the red set contains a translate of $T$. The problem: is there a rather red coloring of $\mathbb{R}^2$ with reference to $\{1\}$ such that the red set does not contain the vertices of a unit square? This question was soon answered in the negative by R. Juhász [23] who proved that for any rather red coloring of $\mathbb{R}^2$ with reference to $\{1\}$, or any other single distance, the red set will contain congruent copies of every 4-point planar set. (Two subsets of $\mathbb{R}^n$ are congruent if one is the image of the other under a composition of a translation and a rotation.)

If $D \subseteq (0, \infty)$ and $X \subseteq \mathbb{R}^n$ we define $m_c(X, D)$, respectively $m_t(X, D)$, to be the smallest size $|Y|$ of a subset $Y$ of $X$ such that there exists a rather red coloring of $X$ with reference to $D$ such that no congruent copy, respectively, translate, of $Y$ in $X$ is all red. Clearly $m_t(X, D) \le m_c(X, D)$. It is a consequence of a generalization [14] of a theorem first appearing in [36] that if $X$ is closed under vector addition then

$$\chi(X, D) \le m_t(X, D);$$

see [4] for a fuller account of this matter.

So, finding or estimating $m_t(\mathbb{Q}^n, d)$ and $m_c(\mathbb{Q}^n, d)$ for various $d \in \sqrt{\mathbb{Q} \cap (0, \infty)}$ is of interest, not only because these problems are worthy in themselves but also because upper bounds on these numbers will bound $\chi(\mathbb{Q}^n, d)$, and thus may lead to estimates of $B_1(\mathbb{Q}^n)$.

On the other hand, $B_1(\mathbb{Q}^2) = 2$ implies that $m_t(\mathbb{Q}^2, d) = m_c(\mathbb{Q}^2, d) = 2$ for every distance $d$ actually realized between points of $\mathbb{Q}^2$, and $\chi(\mathbb{Q}^3, 1) = 2$ implies that $m_t(\mathbb{Q}^3, 1) = m_c(\mathbb{Q}^3, 1) = 2$. So it appears that the place to start, the very first basic open question in this vein, is on the problems of determining $m_t(\mathbb{Q}^3, \sqrt{2})$ and $m_c(\mathbb{Q}^3, \sqrt{2})$. I volunteer the conjecture that

$$m_c(\mathbb{Q}^3, \sqrt{2}) = m_t(\mathbb{Q}^3, \sqrt{2}) = 4.$$

Finally, there are myriad mixed hypergraph coloring problems in which one of the hypergraphs is a distance graph on $\mathbb{Q}^n$. For the full Monty on mixed hypergraph colorings, see [37], but for some particular geombinatorial cases, see [22]. The general form of these geombinatorial problems involving distance graphs is this: for $X \subseteq \mathbb{R}^n$ and $S \subseteq (0, \infty)$, and a collection $\mathcal{Y}$ of subsets of $X$, for which $k \ge \chi(X, D)$, if any, is there a proper coloring of $G(X, D)$ with $k$ colors such that every $Y \in \mathcal{Y}$ is monochromatic? Or, turning the question around, given $X, D$, and $k \ge \chi(X, D)$, for which $Y \subseteq X$ is there a proper coloring of $G(X, D)$ with $k$ colors such that $Y$ is monochromatic?

In the most promising subclass of these problems, $X$ is an additive subgroup of $\mathbb{R}^n$ – possibly $\mathbb{Z}^n, \mathbb{Q}^n$, or $\mathbb{R}^n$ itself – and the collection $\mathcal{Y}$ is the set of all translates in $X$ of some $Y \subseteq X$. Suppose that $X$ is a subgroup of $\mathbb{R}^n$ and $f : X \to C$ is a coloring of $X$ ($C$ is some set of "colors"). Let $P(f, X) = \{u \in X \mid$ for all $v \in X, f(u + v) = f(v)\}$, the set of "periods" of the coloring $f$ (allowing the zero vector honorary status as a period). It is shown in [22], and is quite easy to see, that, in these circumstances:

(a) $P(f, X)$ is a subgroup of $X$.
(b) Every translate of $Y \subseteq X$ is monochromatic (i.e., $f(x + Y)$ is a singleton for every $x \in X$) if and only if $Y$ is a translate of a subset of $P(f, X)$.

So the obvious initial path to follow in exploring this territory is indicated by the question: given $X$, a subgroup of $\mathbb{R}^n$, and $D \subseteq (0, \infty)$, which subgroups $P(f, X)$ of $X$ do we get by proper colorings $f$ of $G(X, D)$? Or, what properties must $P(f, X)$

have? For instance, it is shown in [22] that if $n \in \{2, 3\}$, and $f$ is a proper coloring of $G(\mathbb{Q}^n, 1)$ with two colors, then $P(f, \mathbb{Q}^n)$ is dense in $\mathbb{Q}^n$. Do there exist $k \geq 3$ and a proper coloring $f$ or $G(\mathbb{Q}^n, 1)$ ($n \in \{2, 3\}$) with $k$ colors such that $P(f, \mathbb{Q}^n)$ is not dense in $\mathbb{Q}^n$? Can we have $P(f, \mathbb{Q}^n) = \{\underline{0}\}$ for such an $f$? In [22] similar questions are asked about proper 4-colorings of $G(\mathbb{Q}^4, 1)$ and $G(\mathbb{Q}^3, \sqrt{2})$. Obviously there is a rich trove of such questions; the interested reader will not need much instruction on how to unwrap this package. Perhaps it should be mentioned that the key to answering the question about $G(\mathbb{Q}^4, 1)$ might be found in [40].

It must be reported that in [22] it is asserted that if $d$ is a distance realized between points of $\mathbb{Q}^2$, and $f$ is a proper coloring of $G(\mathbb{Q}^2, d)$ with two colors, then $P(f, \mathbb{Q}^2)$ is dense in $\mathbb{Q}^2$, "by an argument similar to the proof of" the corresponding assertion for $d = 1$. I wish to declare this question reopened! If the rational points are dense on the circle $\{u \in \mathbb{Q}^2 \mid |u| = d\}$, then the conclusion does hold, with the same proof as in the case $d = 1$, but, doubtless due to lacunae in my education, I do not know whether this is true for all distances $d$ realized between points of $\mathbb{Q}^2$.

What of proper colorings of $G(X, D)$ for which all subsets of $X$ which are congruent to a given $Y = X$ are monochromatic? It is shown in [22] that if $Y \subseteq \mathbb{R}^n$ has at least two points, then there is no such proper coloring of $G(\mathbb{R}^n, 1)$, for $n > 1$. The facts that $\chi(\mathbb{Q}^2, 1) = \chi(\mathbb{Q}^3, 1) = 2$ show that the analogous statement with $\mathbb{R}^n$ replaced by $\mathbb{Q}^n$, $n = 2$ or 3, does not hold; just take $Y = \{\underline{0}, u\}$ where $u \in \mathbb{Q}^n$ has length $|u| = 2$, and color $G(\mathbb{Q}^n, 1)$ properly with two colors. So, the problems of properly coloring $G(\mathbb{Q}^n, D)$ so that congruent copies in $\mathbb{Q}^n$ of nonsingletons $Y \subseteq \mathbb{Q}^n$ are forced to be monochromatic are open for consideration! I wonder if any such coloring is possible when $\chi(\mathbb{Q}^n, D) > 2$. When $\chi(\mathbb{Q}^n, D) = 2$, $n = 2$ or 3, then one can force monochromatic congruent copies of arbitrarily large, even infinite, $Y \subseteq \mathbb{Q}^n$, with a proper 2-coloring of $G(\mathbb{Q}^n, D)$, by taking $Y$ to be a set of collinear points in $\mathbb{Q}^n$, spaced at even integer multiples of distances in $D$ along the line. Are these the only such instances of such phenomena with $|Y| > 2$?

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# Open Problems Session

The workshop is over: Peter D, Johnson, Jr. (left) and Alexander Soifer.

During the workshop *Ramsey Theory Yesterday, Today and Tomorrow* at Rutgers University on May 27-29, 2009, I offered a Problem Posing Session. All 30 participants of the workshop attended the session, and almost everyone came to the board and posed favorite open problems. The session was scheduled for an hour and lasted twice as long. I asked for problem submissions in writing for this volume. Below you will find all submitted problems (which is far from all the problems orally presented at the workshop).

A. Soifer (ed.), *Ramsey Theory: Yesterday, Today, and Tomorrow*,
Progress in Mathematics 285, DOI 10.1007/978-0-8176-8092-3_10,

In addition, see many more open problems in the surveys of this volume. The survey by Ronald L. Graham and Eric Tressler, for one, consists entirely of open problems.

The goal of posing problems is, of course, have them solved. I am happy to report that one solution has already been submitted to *Geombinatorics*. You will find the story of this affair in Peter D. Johnson Jr's problem offering below.

Alexander Soifer October 28, 2009
Workshop organizer

# 1 Problems Submitted by William Gasarch[1]

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Consider the following known theorem.

**Theorem 1.1.** *For all 2-colorings of the lattice points of the plane there exists $d \in \mathsf{N}$, $d \geq 2$, and there exist four points that are the same color and form a $d \times d^2$ rectangle whose sides are parallel to the $x$-axis and $y$-axis.*

The only proof of Theorem 1.1 that we know uses the polynomial Hales–Jewett theorem, which we do not state here. It was first proven (using ergodic theory) in [2]. A purely combinatorial proof is in [3]. Even though it has a purely combinatorial proof, it is rather complicated.

**Open Problem:** Provide a proof of Theorem 1.1 that does not use the polynomial Hales–Jewett theorem. It may use the Hales–Jewett theorem (which can be found in any text on Ramsey theory) and/or the polynomial van der Waerden theorem. (See [1] for the original ergodic theory proof or [3] for a purely combinatorial proof that appeared later.)

**Note 1.2.** There is a much more general version of Theorem 1.1 that can also be proven by the polynomial Hales–Jewett theorem. It is rather complicated to state; however, we state a corollary of it. For all $n \in \mathsf{N}$, for all $p_1, \ldots, p_{2^n} \in \mathsf{Z}[x]$ such that $p_i(0) = 0$, for all finite colorings of $\mathsf{Z}^n$ there exist $d \in \mathsf{N}$, $d > 0$, and an $n$-dimensional rectangle (all sides parallel to some axis) with all corners the same color with sides of length $p_1(d), \ldots, p_{2^n}(d)$. A proof of this theorem, with the same restrictions as that of the Open Problem, would also be interesting.

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## 2 Problems Submitted by Peter Johnson, Jr.[2]

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Throughout, let $\mathbb{Z}$ denote the set of integers, $\mathbb{Q}$ the set of rational numbers, and $\mathbb{R}$ the set of real numbers.

1. Suppose $A$, $B \subseteq \mathbb{Z}$, and $|A| = |B| = 3$. Is it necessarily possible to color $\mathbb{Z}$ with two colors so that no translate of either $A$ or $B$ is monochromatic?

   *Background:* It is possible to find $A$, $B \subseteq \mathbb{Z}$, $|A| = 3$, $|B| = 2$ so that for every 2-coloring of $\mathbb{Z}$, some (actually, infinitely many) translate(s) of one of $A$, $B$ is (are) monochromatic. For instance, take $A = \{0, 1, 3\}$, $B = \{0, 4\}$. Also, it is possible to find $A, B, C \subseteq \mathbb{Z}$, $|A| = |B| = |C| = 3$, such that for every 2-coloring of $\mathbb{Z}$, infinitely many translates of one of $A, B, C$ are monochromatic. For instance, take $A = \{0, 1, 2\}$, $B = \{0, 1, 3\}$ and $C = \{0, 4, 8\}$.

   As explained in "An easier analogue of a difficult old Euclidean coloring problem," by A. W. Bohannon, P. D. Johnson, and E. G. Thomas, *Geombinatorics* 12 (January, 2003), 94–101, if the answer to the question in this problem is "yes" then for every pair of 3-point sets in the Euclidean plane it is possible to 2-color the plane so that no translate of either set is monochromatic.

   *Foreground:* Since this problem was posed at the Workshop, it has been solved *in the negative*, to its poser's great surprise, by a Hungarian student, Balázs Gosztonyi, who heard it from András Gyárfás. His example: $A = \{0, 2, 6\}$ and $B = \{0, 1, 8\}$. The proof that for every 2-coloring of $\mathbb{Z}$ some translate of one of these is monochromatic is nontrivial.; it will appear soon in *Geombinatorics*.

   It would be of interest to characterize those pairs of triples $A$ and $B$ of integers such that monochromatic translates of both of them cannot be forbidden by a 2-coloring of $\mathbb{Z}$. It is known, and is easy to see, that for such a pair, every ratio (difference within $A$)/(difference within $B$) must be, in lowest terms, even/odd or odd/even; what else?

2. Consider the integers $\mathbb{Z}$ to be an additive subgroup of the plane $\mathbb{R}^2$ in a natural way: $\mathbb{Z} \cong \{(0, 0), (\pm 1, 0), (\pm 2, 0), \ldots\}$. Consider the $2^c$ 4-colorings of the plane obtainable by coloring each coset of $\mathbb{Z}$ in $\mathbb{R}^2$ with red and blue so that the distance 1 is forbidden (so, the coloring of the coset is one of the two colorings in which red and blue alternate as you count through the integers), then coloring the collection $\mathbb{R}^2/\mathbb{Z}$ of all cosets somehow with two colors, say green and yellow, and finally coloring each $u \in \mathbb{R}^2$ with the ordered pair (color of $u$ in $u + \mathbb{Z}$, color of $u + \mathbb{Z}$).

   Do any of these 4-colorings of $\mathbb{R}^2$ forbid the distance 1?

   *Background:* If the chromatic number of the unit distance graph in the plane is greater than 4, then the answer to the question is no, of course. But perhaps the answer is yes!

   In "Coloring Abelian groups," *Discrete Math.* 40 (1982), 219–223, P. D. Johnson suggests a plan like this for obtaining a 4-coloring of the plane that forbids the distance 1, but with the role of $\mathbb{Z}$ to be played by a large subgroup of

$\mathbb{R}^2$ which can be 2-colored so as to forbid the distance one within the subgroup. There is still hope for the success of that program, if only Zorn can provide a large 2-colorable subgroup with the right properties! But curiosity whispers: why cannot the program succeed using the smallest nontrivial subgroup of them all, $\mathbb{Z}$?

3. Suppose that $X$ and $Y$ are sets, $\varphi : X \to \{1, \ldots, r\}$ is an $r$-coloring of $X$, and $\psi : Y \to \{1, \ldots, k\}$ is a $k$-coloring of $Y$. We call the $rk$-coloring of $X \times Y$ defined by $(x, y) \to (\varphi(x), \psi(y))$ an $r \times k$ *simple product coloring*. The following is a conjecture of W. Kuperberg from the early 1980s: There is no positive integer $k$ such that there is a $2 \times k$ simple product coloring of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ which forbids the Euclidean distance 1. We may add to the challenge of this conjecture: should this conjecture be true, what about $2 \times \aleph_0$ simple product colorings of $\mathbb{R} \times \mathbb{R}$? Can the distance 1 be forbidden by such a coloring?

   *Background:* It is easy to see that there is a $3 \times 3$ simple product coloring of $\mathbb{R} \times \mathbb{R}$ which forbids the distance 1: just color each $(x, y) \in \mathbb{R}^2$ with the ordered pair $(\lfloor x\sqrt{2} \rfloor \mod 3, \lfloor y\sqrt{2} \rfloor \mod 3)$. It is shown in “Simple product colorings,” by P. D. Johnson, *Discrete Math.* 48 (1984), 83–85, that there is a $2 \times 2$ simple product coloring of $\mathbb{Q} \times \mathbb{Q}$ which forbids the distance 1, but no $2 \times 2$ simple product coloring of $\mathbb{Q} \times \mathbb{Q}(\sqrt{15})$ which does so. Kuperberg’s conjecture was inspired by a prepublication discussion of that paper with the author, and the conjecture has already appeared in that paper and in a list of problems and conjectures associated with the 1985 British Combinatorial Conference.

# 3 Problems on Topological Stability of Chromatic Numbers Submitted by Dmytro Karabash[3]

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Setting: Let $M$ be a (smooth) $n$-dimensional Riemannian manifold. For distance $t > 0$, let $G_t$ be the $t$-distance graph defined on vertex set $M$; i.e., two vertices of $G_t$ are adjacent if and only if they are at ($M$-geodesic) distance $t$ apart. Let $(G_t)$ be the chromatic number of this graph $G_t$. For simplicity let us restrict ourselves to $ZFC$.

**Question 1.** How does $\chi(G_t)$ behave as $t$ goes to 0? Is it necessarily a constant near 0 or can it oscillate between two or more numbers?

Let us call set $S(M) = \{t \in R_+ : \chi(G_t) = \chi(\mathbb{R}^n)\}$ of distances $t$ for which $G_t$ has the same chromatic number as $\mathbb{R}^n$. We say that elements of $S(M)$ are $M$-stable.

For example, consider $n = 1$ and let $M$ be a circle of circumference 1. Then $\chi(G_t) = 3$ if and only if $G_t$ has an odd cycle and otherwise $\chi(G_t) = 2$. Note that $G_t$ has an odd cycle when $t < \frac{1}{2}$ and $t = p/q$ , where $p$ and $q$ are integers and $q$ is odd. Hence $S(M)$ in this case all of positive reals except for countably many points. This leads us to the following conjecture:

*Conjecture 1.* Set $R_+ \setminus S(M)$ is countable; i.e., for any smooth Riemannian manifold M all but countably many positive reals are M-stable.

For dimensions higher than 1, we have the following conjecture:

*Conjecture 2.* If $M$ is of dimension $n > 2$, then for some $s > 0$, $(0, s) \subset S(M)$.

**Related Questions and Generalization:**

Given a set $D$ of positive reals one can also consider the same questions for graph $G_t$ where two points are adjacent if and only if they are at ($M$-geodesic) distance $ts$, where $s$ is some element of $D$.

Corresponding questions can be asked with various restrictions on the chromatic sets:

1. We can restrict all chromatic sets in the coloring to be (Lebesgue) measurable; this is similar (and most likely equivalent) to considering chromatic number in Solovay system.
2. We can consider only map-type colorings (where the set corresponding to each color is the union of regions on $M$); this might be the most feasible direction as we understand map-type colorings better due to absence of need of axiom of choice.

## 4 Problem on the Gallai–Ramsey Structure, Submitted by Colton Magnant[4]

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We say that a copy of a graph $H$ is rainbow if each edge is colored with a distinct color. We consider colorings of the edges of a complete graph $K_n$ which contain no rainbow copies of a fixed graph $H$.

**Definition 1.** Let $\mathscr{H}$ be the set of (connected) graphs $H$ such that in any rainbow $H$-free coloring of $K_n$, there exists a (bi)partition of the vertices of $K_n$ with only $k$ colors on the edges between the parts where $k$ is a constant depending only on $H$ (not $n$).

This definition leads to the following problem.

**Problem 1.** Classify the graphs in $\mathscr{H}$.

It is known [1, 2] that $K_3 \in \mathscr{H}$ and from this, one may easily show that any tree or tree with an extra edge (forming a triangle) is also in $\mathscr{H}$. Unfortunately, that's all we know for graphs that are in $\mathscr{H}$.

On the other end of the spectrum, suppose $H$ is a graph containing two cycles. Let $k_0$ be any fixed constant and consider a graph $G$ with girth $g > |H|$ and edge connectivity $k > k_0$. We color each edge of $G$ with a distinct color (rainbow) and, in order to finish a coloring of a complete graph, we color the complement $\bar{G}$ with a single (different from all others) color. Suppose this final color is red.

Since the girth $g(G) > |H|$ and $H$ contains two cycles, any copy of $H$ in this coloring must use at least two red edges. Hence, this coloring contains no rainbow copy of $H$. Also this construction has no (bi)partition with at most $k$ colors on edges between the parts. Since $k > k_0$ was chosen arbitrarily, $H \notin \mathscr{H}$. Hence, if $H$ is not unicyclic, then $H \notin \mathscr{H}$.

It seems as though the set of unicyclic graphs is precisely the set of graphs in $\mathscr{H}$. The following conjecture appears to be, by far, the most interesting case of this problem.

*Conjecture 1.* $C_4 \in \mathscr{H}$.

It is known that the number of colors $k$ for the graph $C_4$ would be at least 4 but we have, so far, been unable to show that the number is bounded.

A classification of the graphs in $\mathscr{H}$ and bounds on the corresponding values of $k$ would be of great interest in the field of Gallai–Ramsey theory and other rainbow-coloring-related problems.

This is a joint work with Shinya Fujita. The authors would like to thank Jacob Fox, András Gyárfás, and Daniel M. Martin for fruitful discussions.

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## 5 Problems Involving Triangles, Submitted by Stanisław P. Radziszowski[5]

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**Definition 1.** For graphs $G$ and $H$, $R(G, H) = n$ if and only if $n$ is the least positive integer such that in any 2-coloring of the edges of $K_n$ there is a monochromatic $G$ in the first color or a monochromatic $H$ in the second color. ♢

We write simply $R(k, l) = R(K_k, K_l)$ if the avoided graphs are complete. Two-colorings of the edges of $K_n$ are often seen as graphs consisting of the edges in the first color, while their complements correspond to the edges in the second color. The $(k, l; n)$-graphs are $n$-vertex graphs lower-bounding $R(k, l)$, i.e., 2-colorings of the edges of $K_n$ proving $n < R(k, l)$. The $(k, l)$-graphs stand for $(k, l; n)$-graphs for some $n$. If $n = R(k, l) - 1$ then $(k, l; n)$-graphs are called critical. These concepts naturally generalize to $r$ colors, $r$ graphs, and the multicolor Ramsey numbers $R(G_1, \dots, G_r)$.

### Computational Problems for Ramsey Numbers, Two Colors

For detailed references to the subproblems listed in this and the next section please see the dynamic survey "Small Ramsey Numbers" in the *Electronic Journal of Combinatorics* [2]. Many historical comments and background information can be found in *The Mathematical Coloring Book* by Alexander Soifer [4].

The first open case of a Ramsey number of the form $R(3, k)$ is $40 \le R(3, 10) \le 43$. It seems that in order to determine the largest $(3, 10)$-graph we need to know more about $(3, 9; n \le 35)$-graphs, which in turn requires the knowledge of $(3, 8; n \le 27)$, which in turn requires the knowledge of $(3, 7; n \le 22)$. All $(3, 6)$-graphs and all critical $(3, 7; 22)$-graphs are known, and there are 761,692 and 191 of them, respectively. Thus the sequence of smaller, but still difficult, tasks towards solving $R(3, 10)$ could be as follows.

(a) Enumerate more $(3, 7)$-graphs.
Enumerating all graphs in $(3, 7; 21)$ should be easy, more difficult for $(3, 7; 20)$ and perhaps feasible for $(3, 7; 19)$. More complete enumeration of $(3, 7)$ can make it easier to progress on the further steps below.

(b) Enumerate all critical $(3, 8; 27)$-graphs.
More than 430 K such graphs are already known, but there may be more of them. Full enumeration of $(3, 8; 26)$ seems to be very difficult, but it could likely be done for some well defined part, like graphs with at most 78 edges.

(c) Enumerate all critical $(3, 9; 35)$-graphs.
There is only one $(3, 9; 35)$-graph known, but there might be more of them. Finding all $(3, 9; 34)$-graphs also could be feasible.

(d) Finish off $37 \leq R(3, K_{10} - e) \leq 38$.
This number is between $R(3, 9) = 36$ and $R(3, 10)$, and the type of computations needed to decide the existence of $(3, K_{10} - e; 37)$-graphs is similar to what is needed in (c) and (e). (d) may possibly be easier, hence attacking it first is a good choice.

(e) Attack $R(3, 10)$.
We know that $40 \leq R(3, 10) \leq 43$. The author feels that 40 is likely the correct value. First, try to prove computationally that $R(3, 10) \leq 42$. The results from (a) through (d) should help.

## Computational Problems for Ramsey Numbers, Multiple Colors

The computational tasks related to the smallest and most-studied open cases for multicolor Ramsey numbers are as follows.

(f) Improve on $45 \leq R(3, 3, 5) \leq 57$.
The task of just improving the inequality should not be too hard. We are not aware of any published dedicated attack on this number. The exact evaluation of $R(3, 3, 5)$ is a different matter, apparently well beyond what we can currently do.

(g) Finish off $30 \leq R(3, 3, 4) \leq 31$.
This is perhaps the only open case of a classical multicolor Ramsey number, for which we can anticipate exact evaluation in the not too distant future. A complete solution is likely feasible with a large-scale computational effort similar to that in [PR1, PR2] as referenced in [2].

(h) Improve on $51 \leq R_4(3) \leq 62$.
This is the most studied and intriguing open multicolor case. We believe the lower bound to be close, if not equal, to the actual value. Improving the upper bound, while difficult, should be feasible with large-scale computational effort, for example, by extending work [FKR] referenced in [2]. We are not aware of any heuristic approaches which would come even close to the lower bound 51. This could be used as an interesting novel test of strength of general heuristic search techniques. As of now, we do not seem to understand well why known heuristics are inadequate for this task.

## Computational Folkman Problems

The Folkman problems we are concerned with in this part can be expressed using the usual Ramsey arrowing operator restricted to graphs not containing $K_m$ (or not containing some other graph). For detailed references to the background, history, and problems similar to those listed below see *The Mathematical Coloring Book* by Alexander Soifer [4]. Many technical comments and further references can be found in [1] and [3].

**Definition 2.**

- $F \to (s_1, \ldots, s_r)^e$ if and only if for every $r$-coloring of the edges, the graph $F$ contains a monochromatic copy of $K_{s_i}$ in some color $i$, $1 \le i \le r$.
- $F \to (G, H)^e$ if and only if for every red/blue edge-coloring of $F$, the graph $F$ contains a blue copy of $G$ or a red copy of $H$.
- $\mathcal{F}_e(s,t;k) = \{G \to (s,t)^e : K_k \not\subseteq G\}$ is called the set of *edge Folkman graphs*.
- $F_e(s,t;k)$ is defined as the smallest integer $n$ such that there exists an $n$-vertex graph $G$ in $\mathcal{F}_e(s,t;k)$. These are called the *edge Folkman numbers*. $\diamondsuit$

**Theorem (Folkman 1970).**
*For all $k > max(s,t)$ edge Folkman numbers $F_e(s,t;k)$ exist.*

The most wanted edge Folkman number $F_e(3,3;4)$ involves the smallest parameters for which the problem is nontrivial, and quite surprisingly it is already extremely difficult to compute. Equivalently, $F_e(3,3;4)$ is equal to the order of the smallest $K_4$-free graph which is not a union of two triangle-free graphs. We know that $19 \le F_e(3,3;4) \le 941$, where the lower bound was established in [3] and the upper bound in [1]. Much of the history of work on such cases is reported in [3] and [4]. In particular, it seems that even the question if $50 \le F_e(3,3;4) \le 100$ could be very hard to answer.

## Computational Folkman Problems to Work on

(i) Improve on $F_e(3,3;4) \le 941$.
This bound was established by Dudek and Rödl in 2008 [1], after a few decades of colorful history reported in [4] and [3].
One of the possible options to proceed forward is as follows. In 1982, Hill and Irving defined the graph $G_{127} = (Z_{127}, E)$, $E = \{(x,y) | x - y = \alpha^3 \pmod{127}\}$ in the context of Ramsey numbers. It is a $(4,12;127)$-graph, and also the monochromatic subgraph in each of three colors of a $(4,4,4;127)$ witness to the lower bound $128 \le R(4,4,4)$. Exoo suggested studying if $G_{127} \to (3,3)^e$. If true it would prove that $F_e(3,3;4) \le 127$.

(j) Improve on $19 \le F_e(3,3;4)$.
No reasonable, even large-scale, computation seems to be sufficient to improve on the lower bound of 19, which was established with significant computational effort in [3].

(k) Study $F_e(K_4 - e, K_4 - e; K_4)$.
We know that $19 \le F_e(3,3;4) \le F_e(K_4 - e, K_4 - e; K_4) \le 30193$. The lower bound follows from monotonicity of $F_e()$; the upper bound, probably not a very strong one, was observed by Lu in his work on $F_e(3,3;4)$.

(l) Study $F_e(3,3;G)$ for $G \in \{K_5 - e, W_5 = C_4 + x\}$.
Similar to (k), but this time vary the forbidden graph while still considering arrowing triangles. We are not aware of any work related to these cases.

(m) Don't study $F_e(3,3;K_4 - e)$.
Because after a moment of thought the reader can certainly discover that this number doesn't exist.

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## 6 Problems on Chromatic Number of the Plane and Its Relatives, Submitted by Alexander Soifer[6]

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**I. General Problem.**

$$\text{Find } \chi^{ZFC}(E^2) \text{ and } \chi^{ZFS+}(E^2).$$

$$\text{Find } \chi^{ZFC}(E^n) \text{ and } \chi^{ZFS+}(E^n) \text{ for } n>2.$$

**II. Chromatic Number of the Plane Soft Conjecture.**

$$\chi\left(E^2\right) = 5.5 \pm 1.5.$$

**III. Chromatic Number of the Plane Conjecture.**

$$\chi\left(E^2\right) = 7.$$

**IV. Chromatic Number of 3-Space Conjecture.**

$$\chi\left(E^3\right) = 15.$$

**V. Chromatic Number of $E^n$ Conjecture.**

$$\chi\left(E^n\right) = 2^{n+1} - 1.$$

**VI. Minimum Girth 4, 4-Chromatic Unit Distance Graph.**

Find the minimum $n$ such that there is a 4-chromatic graph G of order $n$ of girth 4. Construct such a graph $G$.

We know that $11 \leq n \leq 23$.

(Lower bound is due to Mycielski–Grötsch graph. R. Hochberg and P. O'Donnell constructed the Fish graph thus proving that $n \leq 23$.)

**VII. AC Problem.**

For which values of $n$ is the chromatic number $\chi\left(R^n\right)$ of the $n$-space $R^n$ defined "in the absolute", i.e., in **ZF** regardless of the addition of the axiom of choice or its relative?

### Reference

1. Soifer, A., *The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of Its Creators*, Springer, New York, 2009.