Introduction to Ramsey Theory

Lecture notes for undergraduate course

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To my sons, my best teachers. - Veselin Jungic

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Veselin Jungic

Preface

The purpose of these lecture notes is to serve as a gentle introduction to Ramsey theory for those undergraduate students interested in becoming familiar with this dynamic segment of contemporary mathematics that combines, among others, ideas from number theory and combinatorics.

Since this booklet contains the class lecture notes, the reader will occasionally need the help of a *more knowledgeable other*: an instructor, a peer, a book, or Google. In addition to the bibliography, links with the relevant freely available online resources are provided at the end of each section.

The only real prerequisites to fully grasp the material presented in these lecture notes, to paraphrase Professor Fikret Vajzović (1928 — 2017), is knowing how to read and write and possessing a certain level of mathematical maturity.

Any undergraduate student who has successfully completed the standard calculus sequence of courses and a standard first (or second) year linear algebra course and has a genuine interest in learning mathematics should be able to master the main ideas presented here.

My wish is to give to the reader both challenging and enjoyable experiences in learning some of the basic facts about Ramsey theory, a relatively new mathematical field.

But what is Ramsey theory?

Probably the best-known description of Ramsey theory is provided by Theodore S. Motzkin:

> *Complete disorder is impossible.*

Here are a few more:

- *Ramsey theory studies the mathematics of colouring.* — Alexander Soifer
- *Ramsey theory is the study of the preservation of properties under set partitions.* — Bruce Landman and Aaron Robertson
- *The fundamental kind of question Ramsey theory asks is: can one always find order in chaos? If so, how much? Just how large a slice of chaos do we need to be sure to find a particular amount of order in it?* — Imre Leader
- *If mathematics is a science of patterns, then Ramsey theory is a science of the stubbornness of patterns.* – V. Jungic

No project such as this can be free from errors and incompleteness. I would be grateful to anyone who points out any typos, errors, or provides any other suggestion on how to improve this manuscript.

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In Burnaby, B.C., November 2020

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Chapter 1

Introduction: Pioneers and Trailblazers

1.1 Complete Chaos is Impossible

> *Complete disorder is impossible.* — Theodore S. Motzkin, Israeli-American mathematician, 1908 — 1970.

What is Ramsey theory?

- Ramsey theory is the *mathematics of colouring*. - Soifer
- Ramsey theory is the study of the preservation of properties under set partitions. - Landman-Robertson
- The fundamental kind of question Ramsey theory asks is: can one always find order in chaos? If so, how much? Just how large a slice of chaos do we need to be sure to find a particular amount of order in it? - Leader
- If mathematics is a science of patterns, then Ramsey theory is a science of the stubbornness of patterns. - Jungic

Example 1.1.1 *A Ramsey theory problem:* If the natural numbers are finitely coloured, i.e. the set of natural numbers is partitioned into a finite number of cells, must there exist x, y (with x and y not both equal to 2) with $x + y$ and xy monochromatic, i.e., $x + y$ and xy belong to the same partition cell? (See Figure 1.1.2.) □

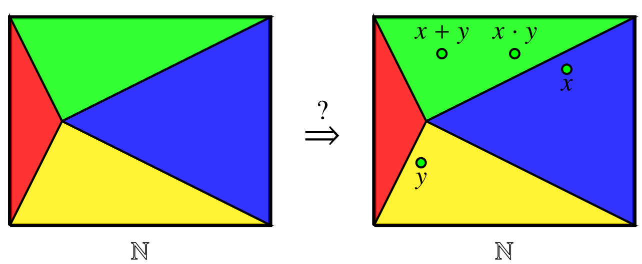


Figure 1.1.2 Monochromatic pattern?

The problem was posed by Neil Hindman in the late 1970s and resolved by Joel Moreira in 2017: Monochromatic sums and products in $\mathbb{N}$, Annals of Mathematics (2) 185 (2017), no. 3, 1069–1090. [arXiv]

What makes this problem to be *a typical Ramsey theory problem* is the following:

- The topic: the problem is to determine the relationship between the set of all finite partitions of the natural numbers and a certain pattern.
- The fact that any numerically literate person can understand the problem.
- It is a difficult problem.

Example 1.1.3

Schur's Theorem: For any partition of the positive integers into a finite number of parts, one of the parts contains three integers x, y, z with $x + y = z$. (See Figure 1.1.5.)

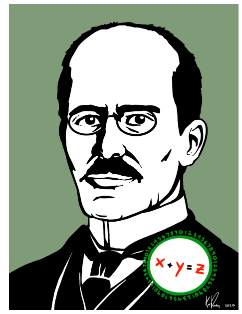


Figure 1.1.4 Issai Schur (1875 — 1941)

□

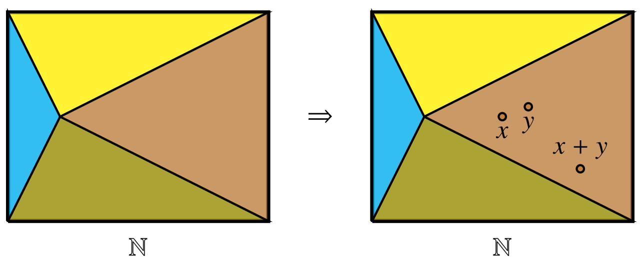


Figure 1.1.5 True, by Schur's theorem

Example 1.1.6

van der Waerden's Theorem - Special Case. For any partition of the positive integers into a finite number of parts, one of the parts contains three integers x, y, z with $x + y = 2z$. (See Figure 1.1.8.)

Figure 1.1.7 Bartel Leendert van der Waerden (1903 — 1996)

□

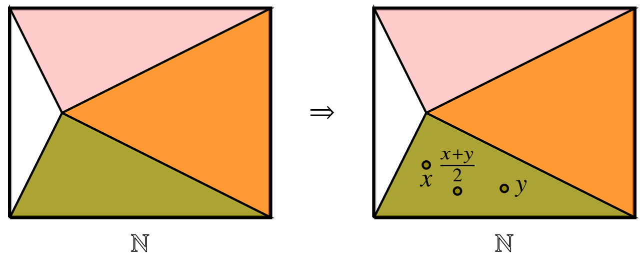


Figure 1.1.8 True, by van der Waerden's theorem

Example 1.1.9

Rado's Theorem - Special Case. For any partition of the positive integers into a finite number of parts, one of the parts contains three integers x, y, z with $ax + by + cz = 0$, $a \neq 0$, $b \neq 0$, $c \neq 0$, if and only if one of the following conditions holds $a + b + c = 0$ or $a + b = 0$ or $a + c = 0$ or $b + c = 0$. (See Figure 1.1.11.)

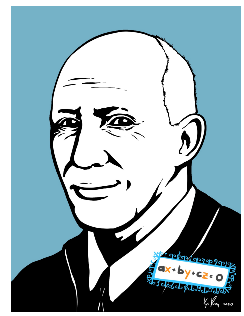


Figure 1.1.10 Richard Rado (1906 — 1989)

□

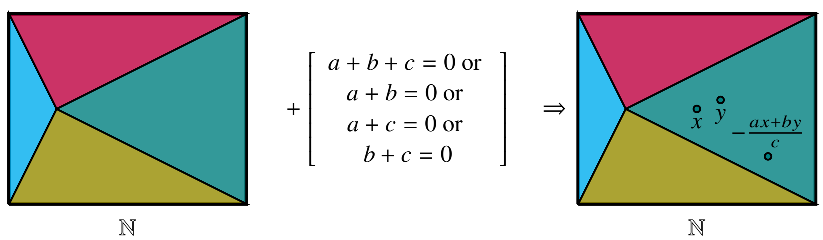


Figure 1.1.11 True, by Rado's theorem

Example 1.1.12

Ramsey's Theorem - Special case. If there are at least six people at dinner then either there are three mutual acquaintances or there are three mutual strangers. (See Figure 1.1.14.)

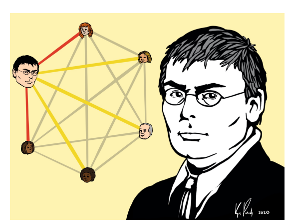
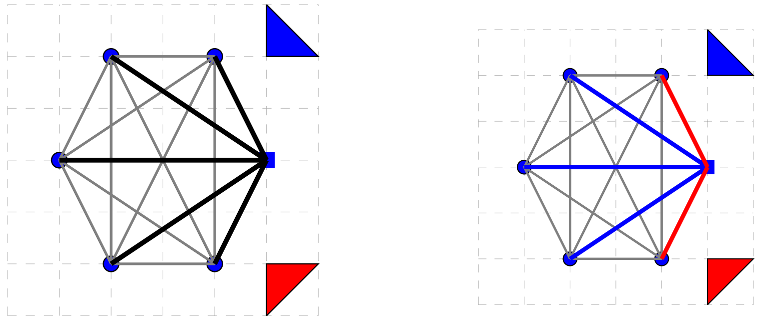


Figure 1.1.13 Frank Plumpton Ramsey (1903 — 1930)

□



Blue = acquaintances; Red = strangers. What's next?

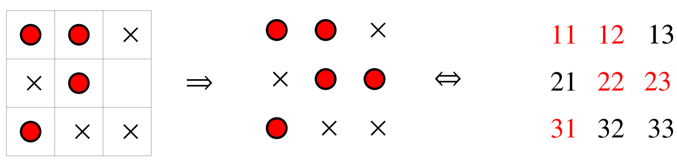
Figure 1.1.14 Proof of a special case of Ramsey's theorem.

Example 1.1.15

Hales-Jewett Theorem - Informal. In large enough dimensions, the game of Tic-Tac-Toe cannot end in a draw.

Figure 1.1.16 Alfred Hales and Robert Jewett

□



Tic-Tac-Toe It's a draw! Same but different

Figure 1.1.17 Tic-Tac-Toe

Tic-Tac-Toe - It is a win!

11	12	13		11	12	13		11	12	13
21	22	23	or	21	22	23	or	21	22	23
31	32	33		31	32	33		31	32	33

Resources.

1. See [2], [3], and [7].
2. Ramsey Theory - Wikipedia
3. Ramsey Theory by R. Graham and B. Rothschild
4. Ramsey Theory by J. Fox

1.2 Paul Erdős

> *If numbers aren't beautiful, I don't know what is.* — Paul Erdős, 1913 — 1996.

Who was Paul Erdős? Paul Erdős was a legendary mathematician who was so devoted to his subject that he lived as a mathematical pilgrim with no home and no job.

Paul Erdős made contributions to:

- *combinatorics including Ramsey theory*; a branch of mathematics concerning the study of finite or countable discrete structures
- *graph theory*; the study of graphs, which are mathematical structures used to model pairwise relations between objects
- *number theory*; a branch of pure mathematics devoted primarily to the study of the integers
- *classical analysis*; a branch of mathematics that includes the theories of differentiation, integration, measure, limits, infinite series, and analytic functions
- *approximation theory*; study of how given quantities can be approximated by other (usually simpler) ones under appropriate conditions.
- *set theory*; branch of mathematics that deals with the properties of well-defined collections of objects, which may or may not be of a mathematical nature
- *probability theory*; a branch of mathematics concerned with the analysis of random phenomena

Birth and Death. Paul Erdős was born in Budapest, Hungary, on March 26, 1913, and died at the age of 83 on September 20, 1996, in Warsaw, Poland.

Figure 1.2.1 Anna and Paul Erdős (Source: Notices of the AMS, 45(1))

World in 1913	*World in 1996*
The first wireless transmission between the USA and Germany	*The first version of the Java programming language was released*
The concept of the "isotope" introduced	*IBM's Deep Blue wins a game of chess against Gary Kasparov*
Port Coquitlam, BC, established	*The first surface photos of Pluto*
All-purpose zipper patented	*The 34th known Mersenne prime* $2^{1257787} - 1$ *discovered*
The first four engine aircraft built	*Nintendo 64 goes on sale*
The Pacific Highway between Surrey, BC, and Blaine, WA, opened as a gravel road	*Microsoft releases Internet Explorer 3.0*
The brand name "Oreo" was registered	*Netscape Browser 3.0 is released*
Mohandas Gandhi arrested for leading Indian miners march in South Africa	*South Africa adopts post-apartheid constitution*
Rabindranath Tagore presented with the Nobel Prize	*The last federally run Indian residential school closed in Saskatchewan*

Paul's Family. Paul Erdős came from a Jewish family. The original family name being Engländer. Paul's father Lajos and his mother Anna had two daughters, aged three and five, who died of scarlet fever just days before Paul was born. This had the effect of making Paul's parents protective of their son. Both of Paul's parents were teachers of mathematics.

Stanisław Ulam About Paul Erdős in 1976:

> He had been a true child prodigy, publishing his first results at the age of eighteen in number theory and in combinatorial analysis. Being Jewish he had to leave Hungary, and as it turned out, this saved his life. In 1941 he was twenty-seven years old, homesick, unhappy, and constantly worried about the fate of his mother who remained in Hungary. (...) Erdős is somewhat below medium height, an extremely nervous and agitated person. (...) His eyes indicated he was always thinking about mathematics, a process interrupted only by his rather pessimistic statements on world affairs, politics, or human affairs in general, which he viewed darkly. (...) His peculiarities are so numerous it is impossible to describe them all. (...) Now over sixty, he has more than seven hundred papers to his credit. (Source: MacTutor.)

Erdős' Work - Two Examples.

Example 1.2.2 *Happy ending problem:* □

During the winter of 1932–1933 a group of students was meeting regularly in the park Városliget, Budapest, Hungary. Among them were Pál "Paul" Erdős, Eszter "Esther" Klein, György "George" Szekeres, and Endre "Andre" Makai.

Figure 1.2.3 Statue of Anonymous, Városliget, Budapest, Hungary (Source Unknown)

One day, Esther made the following observation:

> Among any five points in general position in the Euclidean plane, it is always possible to select four points that form the vertices of a convex quadrilateral.

Vocabulary:

- "five points in general position in the Euclidean plane" = no three points are on the same line (See Figure 1.2.4.)

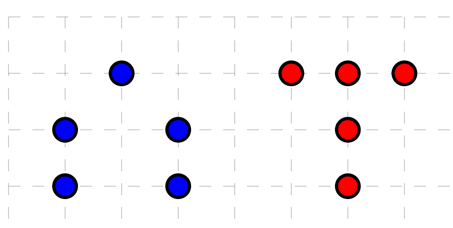


Figure 1.2.4 Points in general position and points not in general position

- "a convex quadrilateral" = a quadrilateral with the property that if two points A and B are inside of the quadrilateral then the whole segment $\overline{AB}$ is inside the quadrilateral. See Figure 1.2.5.

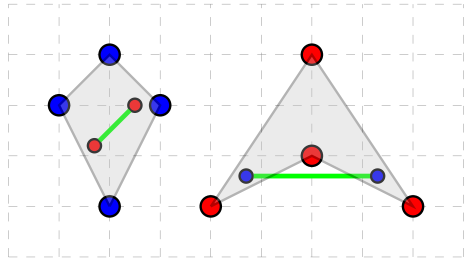


Figure 1.2.5 Convex quadrilateral and non-convex quadrilateral

Proof of Esther's observation: The *convex hull* of a set of points S in the Euclidean plane is the smallest convex set that contains all points from S. See Figure 1.2.6.

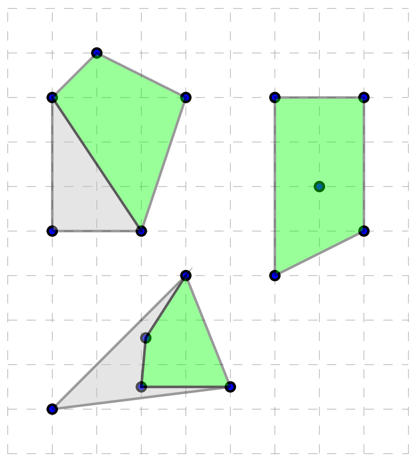


Figure 1.2.6 Five points and three cases: $(5, 0)$, $(4, 1)$, and $(3, 2)$.

Makai soon proved that among any nine points in general position, it is always possible to select five points that form the vertices of a convex pentagon. See Figure 1.2.7

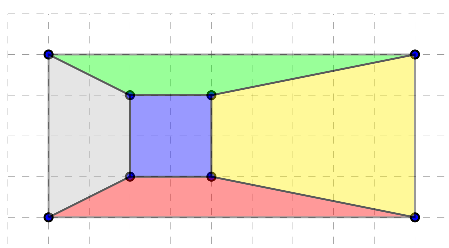


Figure 1.2.7 Eight points without a convex pentagon.

Klein suggested the following more general problem:

> Given any positive integer n, there exists a number $K(n)$ such that among any $K(n)$ points in general position, it is possible to select n points that form the vertices of a convex n-gon.

This is what Szekers wrote about what happened next:

> *I have no clear recollection how the generalization actually came about; in the paper we attributed it to Esther, but she assures me that Paul had much more to do with it. We soon realized that a simple-minded argument would not do and there was a feeling of excitement that a new type of geometric problem emerged from our circle which we were only too eager to solve. For me that it came from Epszi (Paul's name for Esther, short for "epsilon") added a strong incentive to be the first with a solution and after a few weeks I was able to confront Paul with a triumphant 'E.P. open your wise mind'. What I had really found was Ramsey's Theorem from which [the theorem] easily followed. Of course, at that time none of us knew about Ramsey.* (Source "Roots of Ramsey theory" by R. Graham.)

Esther and George married in 1937. On August 28, 2005, they died within an hour of each other.

Example 1.2.8 *Conjecture:* If A is a set of positive integers such that

$$\sum_{n \in A} \frac{1}{n} = \infty.$$

then A contains arithmetic progressions of any given length. □

Paul Erdős offered a prize of US \$3000 for a proof of this conjecture. For more details, see Wikipedia.

Two Saints in St. Gregory of Nyssa Episcopal Church in San Francisco, CA

Resources.

1. Paul Erdős - Wikipedia
2. Paul Erdős - Biography
3. Reminiscences of Paul Erdős
4. Paul Erdős (1913-1996)
5. The Erdős Number Project
6. The Man Who Loved Only Numbers
7. N is a number - Film
8. Fun Chang - Some Erdős Stories
9. Imaginary Erdős Number by Ron Graham

1.3 Frank Plumpton Ramsey

> *Philosophy must be of some use and we must take it seriously; it must clear our thoughts and so our actions.* — Frank Plumpton Ramsey, 1903 — 1930.

Who was Frank Ramsey? British mathematician, economist, and philosopher.
Ramsey made contributions to:

- *epistemology*; the branch of philosophy concerned with the nature and scope of knowledge
- *semantics*; the study of relationships between signs and symbols and what they represent
- *logic*; the study of the principles of reasoning
- *philosophy of science*; the field of rigorous academic study that deals specifically with what science is, how it works, and the logic through which we build scientific knowledge
- *decision theory*; decision theory in economics, psychology, philosophy, mathematics, and statistics is concerned with identifying the values, uncertainties and other issues relevant in a given decision, its rationality, and the resulting optimal decision.
- *metaphysics*; the part of philosophy that is concerned with the basic causes and nature of things.
- *mathematics*
- *statistics*
- *probability*
- *economics*

Birth and Death. Frank Plumpton Ramsey was born on February 22, 1903, and died at the age of 26 on January 19, 1930. Ramsey suffered from a chronic and increasingly serious liver complaint, contracted jaundice after an operation and died at Guy's Hospital in London.

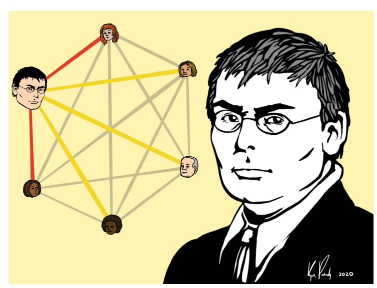


Figure 1.3.1

World in 1903	*World in 1930*
The first west–east transatlantic radio broadcast was made from the US to England	*Mao Tse-tung writes A Single Spark Can Star a Prairie Fire*
Bertrand Russell published "The Principles of Mathematics"	*The first diesel engine automobile trip completed*
Wright Brothers make the first flight	*The first radar detection of planes*
Pepsi Cola company forms	*The world's first radiosonde is launched*

Paavo Nurmi runs world record 6 mile 29:36.4	*The Mersenne number* $2^{67} - 1$ *discovered*
The Russian Social Democratic Labor Party splits into two groups; the Bolsheviks and Mensheviks	*Child labor laws established in Belgium*
The automobile electric starter patented	*The first non-stop airplane flight from Europe to US*
Nobel for physics awarded to Pierre and Marie Curie	*Nazis gain 107 seats in German election*

Frank's Family. Ramsey came from a distinguished Cambridge family. His father was a mathematician, and the President of Magdalene College. His brother, Michael Ramsey, became the Archbishop of Canterbury. His sister Bridget was a medical doctor, and his other sister Margaret was a Fellow of Lady Margaret Hall, Oxford.

In 1924, at the age of twenty-one, Ramsey himself got a Fellowship at King's College Cambridge, having graduated the year before as Cambridge's top mathematics student.

Michael Ramsey about his brother Frank:

> *He was interested in almost everything. He was immensely widely read in English literature; he was enjoying classics though he was on the verge of plunging into being a mathematical specialist; he was very interested in politics, and well-informed; he had got a political concern and a sort of left-wing caring-for-the-underdog kind of outlook about politics. I was aware that he was far cleverer than I was and knew much more, yet there was such a total lack of uppishness about him that we just conversed in a friendly way and he never made me feel inferior though I was so vastly below par intellectually, and that was the wonderful joy of it.* (Source D.H. Mellor).

Ramsey's Work - Two Examples.

Example 1.3.2 *Conditionals:* □

In "General Propositions and Causality" (1929) Frank Ramsey wrote:

> When we deliberate about a possible action, we ask ourselves what will happen if we do this or that. If we give a definite answer of the form 'If I do p, q will result', this can be properly regarded as a material implication or disjunction 'Either not p, or q'. But it differs, of course from any ordinary disjunction in that one of its members is not something of which we are trying to discover the truth, but something within our power to make true or false. Besides definite answers 'If p, q will result', we often get ones `If p, q might result or 'q would probably result'. Here the degree of probability is not a degree of belief in 'Not-p or q', but a degree of belief in q given p, which it is evidently possible to have without a definite degree of belief in p, p not being an intellectual problem. And our conduct is largely determined by these degrees of hypothetical belief. (Source Stanford Encyclopedia of Philosophy).

For example, take:

> p = Dr. J explains to his students that cheating on exams is bad for their academic growth.

and

> q = Dr. J's students do not cheat on exams.

Observe that

> *Not-* p = Dr. J does not explain to his students that cheating on exams is bad for their academic growth.

Recall that in the formal logic:

$$(p \Rightarrow q) \Leftrightarrow (\text{ Not - } p \text{ or } q).$$

Suppose that Dr. J thinks: "If I do p then q will result."
If this was an "ordinary" implication it would be equivalent to

> "Dr. J does not explain to his students that cheating on exams is bad for their academic growth" OR "Dr. J's students do not cheat on exams."

To paraphrase Ramsey's words: What Dr. J really thinks is not 'If p, q will result', but rather 'If p, q might result' or 'q would probably result'. Here the degree of probability is not a degree of Dr. J's belief in 'Not-p or q' but a degree of Dr. J's belief in q given p.

Example 1.3.3 *Ramsey's Theorem.* □

In *On a Problem of Formal Logic*, Proceedings of the London Mathematical Society, 1930 [6]:

> Given any r, n, and μ we can find an m_0 such that, if $m \geq m_0$ and the r- combinations of any Γ_m are divided in any manner into μ mutually exclusive classes C_i $(i = 1, 2, \dots, \mu)$, then Γ_m must contain a sub-class Δ_n such that all the r-combinations of members of Δ_n belong to the same C_i.

Ramsey Theory. Today Ramsey's Theorem is one of the cornerstones of Ramsey theory. Some other results that form the very base of Ramsey theory are Hilbert's Theorem (1892), Schur's Theorem (1916), and van der Waerden's Theorem (1927).

This clearly contradicts the statement by David Hugh Mellor [5] that

> Frank P. Ramsey will be known to readers of the Journal of Graph Theory as the eponymous discoverer of Ramsey numbers and founder of Ramsey theory (. . .)

So, when did Ramsey theory become *Ramsey* theory?

Alexander Soifer [7] offers a detailed account of his own investigation about this question and concludes:

> *It seems that* The Ramsey Theory *has been shaping throughout the 1970s, and the central engine of this process was new results and two surveys by Graham and Rothschild. In 1980 the long life of the name was assured when it appeared as the title of the book* Ramsey Theory *by Graham, Rothschild, and Spencer.*

Joel Spencer [8] gives us the birthplace of Ramsey theory:

> *In my opinion, Ramsey theory was born, after a long and healthy embryonic stage, at the Combinatorial Conference at Balatonfüred, Hungary, 1973.*

Who was the father? In the preface to the first edition of the book [2] the authors, Graham, Rothschild, and Spencer, attribute Paul Erdős as one

> *who can rightfully be considered the father of modern Ramsey theory.*

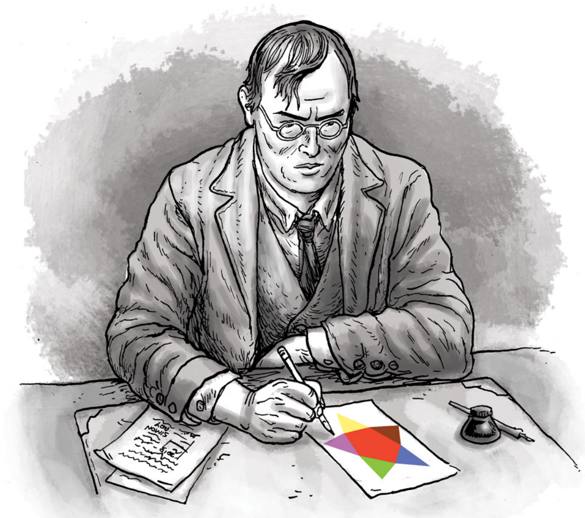


Figure 1.3.4 Frank Ramsey: by Simon Roy and Veselin Jungic

Resources.

1. Frank P. Ramsey - Wikipedia
2. Frank P. Ramsey - Biography 1
3. Frank P. Ramsey - Biography 2
4. Frank P. Ramsey - Philosopher
5. Ramsey's Model - Wikipedia

Chapter 2

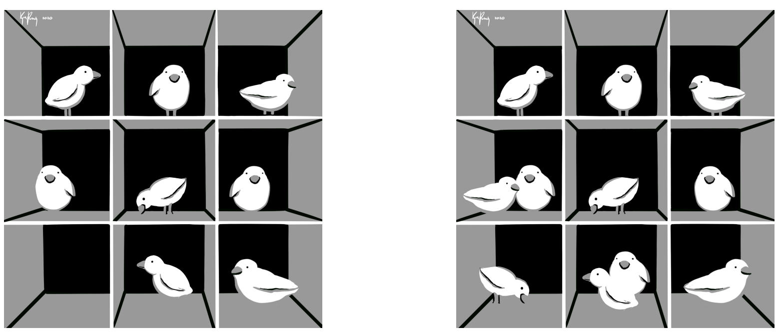
Ramsey’s Theorem

2.1 The Pigeonhole Principle

> *There are three kinds of mathematicians: Those who know how to count and those who don’t.* — Anonymous

Theorem 2.1.1 Pigeonhole Principle: *Suppose you have k pigeonholes and n pigeons to be placed in them. If $n > k$ then at least one pigeonhole contains at least two pigeons. (See Figure 2.1.2.)*

The pigeonhole principle has been attributed to German mathematician Johann Peter Gustav Lejeune Dirichlet, 1805 — 1859.



a) 8 pigeons in 9 pigeonholes

b) 11 pigeons in 9 pigeonholes

Figure 2.1.2 Pigeons and Pigeonholes

Proof. Assume that we have k pigeonholes and n pigeons, with $n > k$. We view the notion of placing of n pigeons in k holes as a function

$$f : \{1, \ldots, n\} \to \{1, \ldots, k\}.$$

The question is if it is possible for f to be an injective function, i.e., a function with the property that, for all $x, y \in \{1, \ldots, n\}$,

$$x \neq y \Rightarrow f(x) \neq f(y)?$$

If it is, then there is a one-to-one correspondence between the set $\{1, \ldots, n\}$ and the set

$$f(\{1, \ldots, n\}) = \{z \in \{1, \ldots, k\} : \exists x \in \{1, \ldots, n\}, f(x) = z\}.$$

This would mean that:

1. the set $f(\{1, \ldots, n\})$ is a subset of $\{1, \ldots, k\}$ AND

2. the set $f(\{1,\ldots,n\})$ has n elements.

This contradicts the assumption that $n > k$. Therefore f cannot be an injective function, i.e. there are $x, y \in \{1,\ldots,n\}$,

$$x \neq y \text{ and } f(x) = f(y).$$

■

Example 2.1.3 Show that among any 5 numbers one can find 2 numbers so that their difference is divisible by 4.

Solution. Say that there are four pigeonholes: 0, 1, 2, and 3. We put the number a in the pigeonhole i, $i \in \{0, 1, 2, 3\}$, if i is the remainder when a is divided by 4.

Since there are 5 numbers, at least two of them must be in the same pigeonhole:

$$a = 4k + i \text{ and } b = 4n + i.$$

It follows that

$$a - b = (4k + i) - (4n + i) = 4(k - n)$$

is divisible by 4. □

Example 2.1.4 Consider a chess board with two of the diagonally opposite corners removed. See Figure 2.1.5. Is it possible to cover the board with pieces of domino whose size is exactly two board squares?

Solution. Observe that there are sixty-two 1×1 squares on this chess board. Hence, thirty-one 2×1 dominos would be needed to cover the board. Also observe that the new board contains 32 white squares and that each domino covers one white and one black square.

Consider 31 dominos as the pigeonholes. Since there are 32 white squares, at least one domino would have to have two white squares which is impossible. □

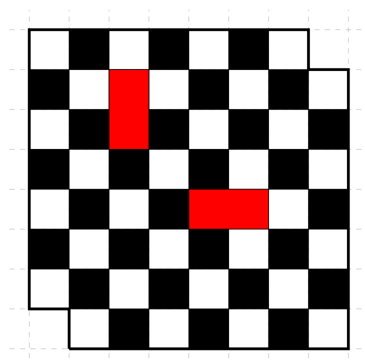


Figure 2.1.5 Two dominos on a chess board with two of the diagonally opposite corners removed

Example 2.1.6 There are 5 points in a square of side length 2. Prove that at least two of them are with the distance at most $\sqrt{2}$.

Solution. Divide the given square into for 1×1 squares. At least two points must belong to the same 1×1 square. The distance between those two points is not greater than the length of the diagonal $\sqrt{2}$. □

Example 2.1.7 A grid of 27 points in the plane is given. See Figure 2.1.8. Each point is coloured red or black. Prove that there exists a monochromatic rectangle, i.e., a rectangle with all four vertices of the same colour.

Solution. Observe that there are eight different ways to colour three points with two colours. Also observe that each coloured column contains two points of the same colour.

Let the nine columns be the pigeonholes and let the grid be coloured red and black in any of $2^{27} = 13,4217,728$ ways. Since there are only eight different ways to colour a column and since there are nine columns, there must be at least two of the columns coloured in the same way. □

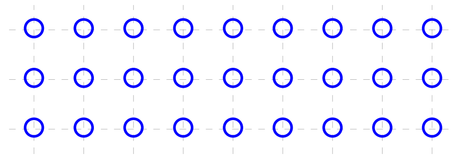


Figure 2.1.8 27 points in the plane; 3 rows and 9 columns

Theorem 2.1.9 Generalized Pigeonhole Principle: *If n pigeons are sitting in k pigeonholes, where $n > k$, then there is at least one pigeonhole with at least $\lceil \frac{n}{k} \rceil$ pigeons and at least one pigeonhole containing not more than $\lfloor \frac{n}{k} \rfloor$ pigeons.*

Proof. By definition $\lceil \frac{n}{k} \rceil$ is the integer with the property

$$\frac{n}{k} \le \left\lceil \frac{n}{k} \right\rceil < \frac{n}{k} + 1.$$

Hence, if none of the k pigeonholes contains $\lceil \frac{n}{k} \rceil$ pigeons, i.e., if the maximum number of the pigeons per pigeonhole is less than or equal to $\lceil \frac{n}{k} \rceil - 1$ then

$$\text{the number of pigeons } \le k \cdot \left(\left\lceil \frac{n}{k} \right\rceil - 1 \right) < k \cdot \left(\left(\frac{n}{k} + 1 \right) - 1 \right) = k \cdot \frac{n}{k} = n$$

which contradicts the assumption that there were n pigeons.

Also, by definition $\lfloor \frac{n}{k} \rfloor$ is the integer with the property

$$\frac{n}{k} - 1 < \left\lfloor \frac{n}{k} \right\rfloor \le \frac{n}{k}.$$

This means that if each of the k pigeonholes contains more than $\lfloor \frac{n}{k} \rfloor$ pigeons then

$$\text{the number of pigeons } \ge k \cdot \left(\left\lfloor \frac{n}{k} \right\rfloor + 1 \right) > k \cdot \left(\left(\frac{n}{k} - 1 \right) + 1 \right) = k \cdot \frac{n}{k} = n$$

which again contradicts the assumption that there were n pigeons. ■

Example 2.1.10 There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?

Solution. Here, $n = 677$ classes (= pigeons) and $k = 38$ different time slots (= pigeonholes). By the generalized pigeonhole principle, there is at least one pigeonhole (time slot) with at least

$$\left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{677}{38} \right\rceil = \left\lceil 17\frac{31}{38} \right\rceil = 18$$

pigeons.

Observe that $38 \cdot 18 = 684 > 677$. Hence 18 classrooms are needed. □

Resources.

1. Pigeonhole principle - Wikipedia
2. A. Bogomolny, Pigeonhole Principle from Interactive Mathematics Miscellany and Puzzles
3. The Pigeonhole Principle by Gary MacGillivray

4. The Pigeonhole Principle by Olga Radko

2.2 Ramsey's Theorem: Friends and Strangers

> *A friend to all is a friend to none.* — Aristotle, Greek philosopher, 384 BCE - 322 BCE

Example 2.2.1 *Edge 2-Colouring.* Use TWO colours, red and blue, for example, to colour the edges of K_6, a complete graph on six vertices. See Figure 2.2.2. Each edge should be coloured by only one colour. □

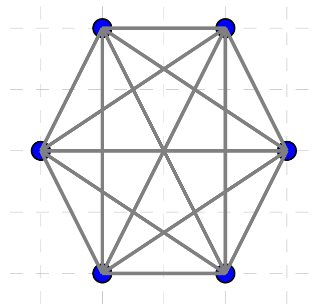


Figure 2.2.2 K_6 - a complete graph on six vertices

Two Questions:

1. How many different edge 2-colourings of K_6 are there?
2. Can you find a monochromatic triangle in your colouring, i.e., three edges coloured by the same colour that form a triangle?

BIG Question: Does any edge 2-colouring of K_6 yield a monochromatic triangle?
BIG Answer: Yes, any edge 2-colouring of K_6 yields a monochromatic triangle!

Theorem 2.2.3 Ramsey's Theorem - Special Case. *Any edge 2-colouring of K_6 yields a monochromatic K_3.*

Proof. Recall the pigeonhole principle: suppose you have k pigeonholes and n pigeons to be placed in them. If $n > k$ then at least one pigeonhole contains at least two pigeons. See Figures 2.2.4 and Figures 2.2.5.

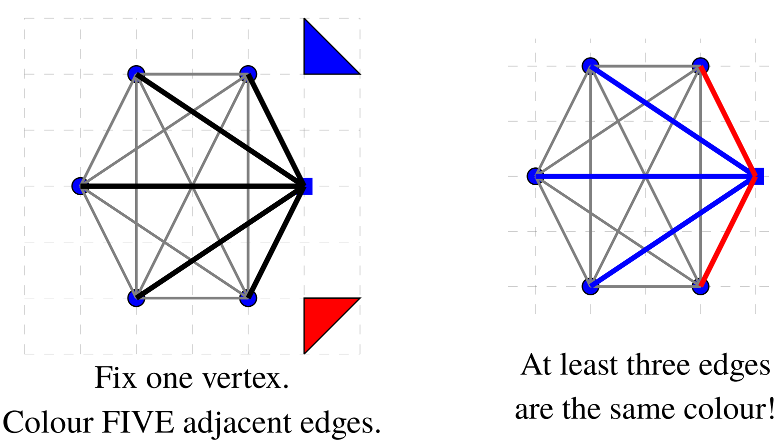


Figure 2.2.4 Proof: Step 1 and Step 2

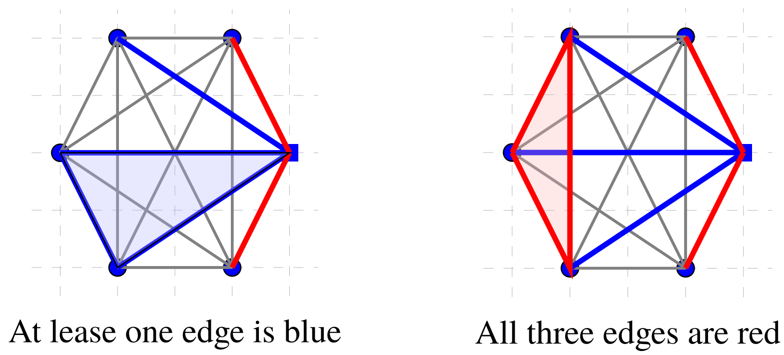


Figure 2.2.5 Proof: Step 3 — Two cases

■

Question. Is this true for any edge 2-colouring of K_5? See Figure 2.2.6.

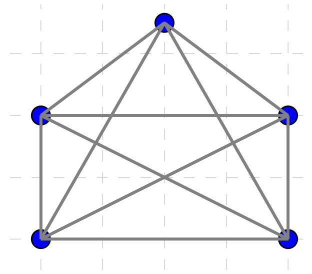


Figure 2.2.6 Find an edge 2-colouring of K_5 that avoids monochromatic triangles

A Dinner Party Problem. Suppose that six people are gathered at a dinner party. Then there is a group of three people at the party who are either all mutual acquaintances or all mutual strangers. See Figure 2.2.7.

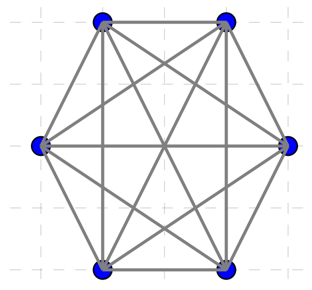


Figure 2.2.7 Do we know each other? YES or NO .

Claim 2.2.8 *Any edge 2-colouring (blue and red) of K_{10} yields a red K_4 or a blue K_3.*

Proof. See Figures 2.2.9 and Figure 2.2.10.

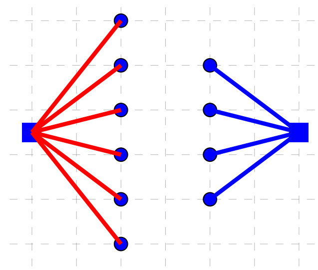
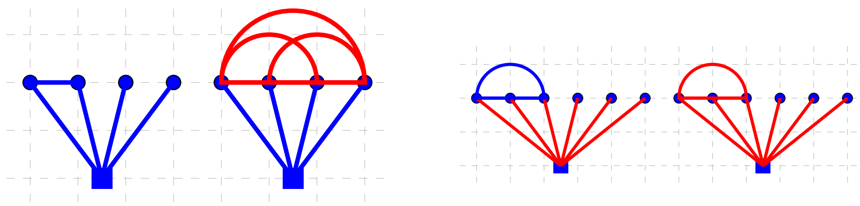


Figure 2.2.9 Fix one vertex: ■. There are at least 6 red adjacent edges OR at least 4 blue adjacent edges.



Case 2: At least six red edges

Case 1: At least four blue edges

Figure 2.2.10 Two cases

■

Claim 2.2.11 *Any edge 2-colouring (blue and red) of K_9 yields a red K_4 or a blue K_3.*

Proof. See Figures 2.2.12 and Figure 2.2.13

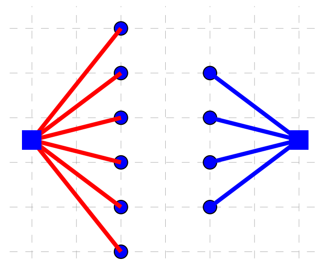


Figure 2.2.12 Step 1: If there is a vertex ■ with at least 6 red adjacent edges OR at least 4 blue adjacent edges - DONE

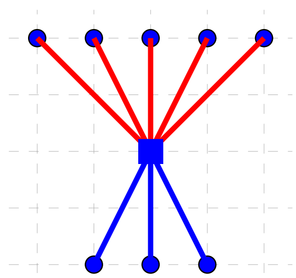


Figure 2.2.13 Step 2 — EVERY vertex ■ is adjacent with 5 red edges AND with 3 blue edges

The number of the blue edges altogether is:

$$\frac{(\#\text{ of vertices }) \cdot \#(\text{ of incident blue edges })}{2} = \frac{9 \cdot 3}{2} = 13.5$$

Something went wrong! ∎

Question. Does every blue-red edge colouring of K_8 yield a red K_4 or a blue K_3? See Figure 2.2.14.

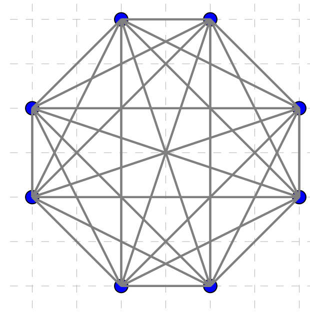


Figure 2.2.14 Find a blue-red edge colouring of K_8 with neither red K_4 nor blue K_3

Theorem 2.2.15 Ramsey's Theorem - Special Case. *Any blue-red edge colouring of K_9 yields a red K_4 or a blue K_3.*

Theorem 2.2.16 Ramsey's Theorem - Special Case. $R(4,3) = R(3,4) = 9$.

Theorem 2.2.17 Ramsey's Theorem - Special Case. $R(4,4) \leq 18$.

Proof. Consider a blue-red edge colouring of a K_{18}. See Figures 2.2.18 and Figure 2.2.19

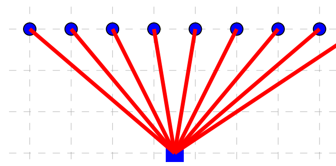
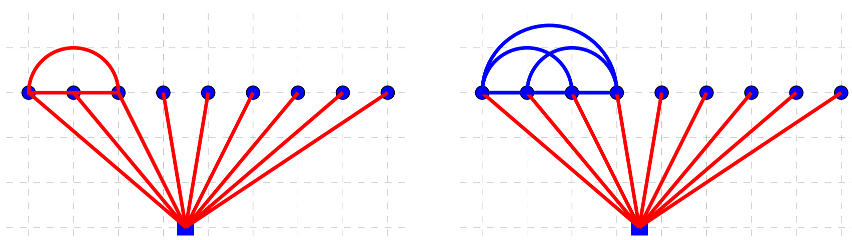


Figure 2.2.18 Step 1: Observe that each vertex ■ is adjacent to at least 9 edges of the same colour - say red



Case 1- There is a red K_3 in the induced K_9

Case 2: There is a blue K_4 in the induced K_9

Figure 2.2.19 Step 2 — Two cases

∎

Theorem 2.2.20 Actually... $R(4,4) = 18$

Resources.

1. Theorem on Friends and Strangers - Wikipedia
2. I. Leader, Friends and Strangers

2.3 Ramsey's Theorem: Two Colours

To see things in the seed, that is genius. — Laozi, Chinese philosopher, 6th century BC

Recall:

1. $R(3,3) = 6$

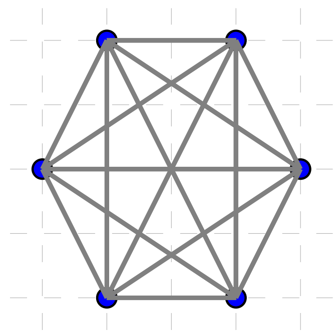


Figure 2.3.1 K_6 - a complete graph on six vertices

2. $R(4,3) = R(3,4) = 9$

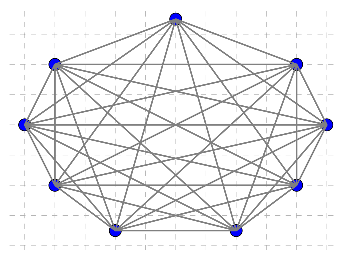


Figure 2.3.2 K_9 - a complete graph on nine vertices

3. $R(4,4) = 18$: Consider a blue-red edge colouring of a K_{18}. See Figures 2.3.3 and Figure 2.2.19.

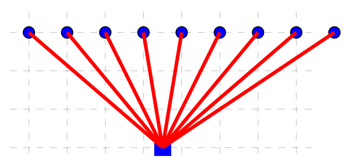
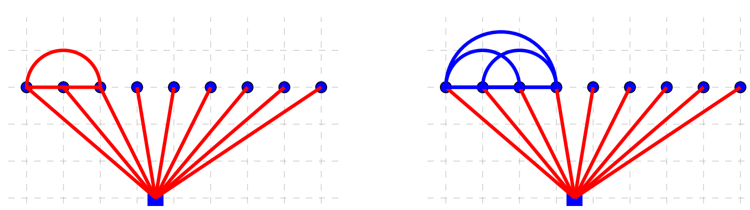


Figure 2.3.3 Step 1: Observe that each vertex ■ is adjacent to 9 edges of the same colour - say red



There is a red K_3 in the induced K_9

There is a blue K_4 in the induced K_9

Figure 2.3.4 Step 2 — Two cases

Jeopardy!

1. There are only *two* 2-colourings of K_{16} without a monochromatic K_4.

2. There is only *one* 2-colouring of K_{17} without a monochromatic K_4.

Definition 2.3.5 The Ramsey number $R(s,t)$ is the minimum number n for which any edge 2-coloring of K_n, a complete graph on n vertices, in red and blue contains a red K_s or a blue K_t. ◊

Three BIG Questions:

1. Does the Ramsey number $R(s,t)$ exist for any choice of natural numbers $s \geq 2$ and $t \geq 2$?

2. If $R(s,t)$ exists, can we find the exact value of $R(s,t)$?

3. If $R(s,t)$ exists and if we cannot find the exact value of $R(s,t)$, what are the best known bounds for $R(s,t)$?

What About...

1. $R(s,2)$*?*

2. $R(2,t)$*?*

Theorem 2.3.6 Ramsey's Theorem, Two Colours. *For any $s,t \in \mathbb{N}\backslash\{1\}$ the Ramsey number $R(s,t)$ exists and, for $s,t \geq 3$,*

$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

Observation 2.3.7 $R(2,2) = 2,\ R(3,2) = R(2,3) = 3,\ R(3,3) = 6,\ R(4,2) = R(2,4) = 4.$

Observation 2.3.8 If $s,t \in \mathbb{N}\backslash\{1\}$ are such that

$$s+t = 4 \text{ or } s+t = 5 \text{ or } s+t = 6$$

then $R(s,t)$ exists!

Observation 2.3.9 Since, for any $s \geq 2$, $R(s,2) = R(2,s) = s$, we are interested only in the question if $R(s,t)$ exists for $s,t \geq 3$.

Observation 2.3.10 $(s-1)+t = s+(t-1) = (s+t)-1$.

Observation 2.3.11 To prove that $R(s,t)$, $s,t \geq 3$, exists it is enough to prove that any 2-colouring, say red and blue, of a complete graph K_M where

$$M = R(s-1,t) + R(s,t-1)$$

yields a monochromatic (red) K_s or a monochromatic (blue) K_t. Why?

Strategy: We prove that any 2-colouring of a complete graph K_M where

$$M = R(s-1,t) + R(s,t-1)$$

yields a red K_s or a blue K_t via induction on the sum $s+t$.

Proof. (Ramsey's Theorem, Two Colours.) Let $s,t \geq 3$. We use mathematical induction on the sum $s+t$ to prove that $R(s,t)$ exists.

The base case of induction, $s+t = 6$, follows from the fact that $R(3,3) = 6$.

Suppose that $n \geq 6$ is such that for any $u,v \geq 3$ such that $u+v = n$ the Ramsey number $R(u,v)$ exists.

Let $s, t \geq 3$ by such that

$$s + t = n + 1.$$

Then, since

$$(s - 1) + t = s + (t - 1) = n,$$

by the induction hypothesis $R(s - 1, t)$ and $R(s, t - 1)$ exist. Let

$$M = R(s - 1, t) + R(s, t - 1)$$

and we consider a 2-colouring of K_M. See Figure 2.3.12.

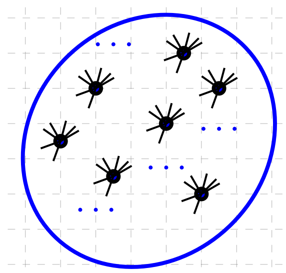
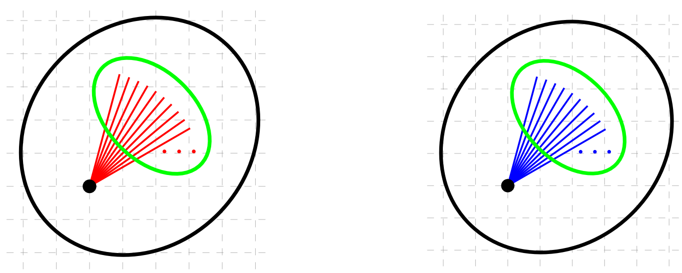


Figure 2.3.12 K_M: Each vertex is incident to $M - 1 = R(s-1, t) + R(s, t-1) - 1$ edges

Fix a vertex. There are two possibilities. See Figure 2.3.13.

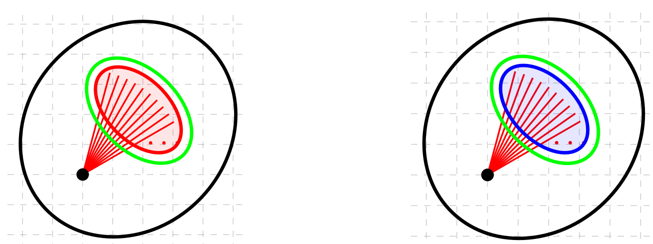


Case 1: At least $R(s - 1, t)$ red edges

Case 2: At least $R(s, t - 1)$ blue edges

Figure 2.3.13 Pigeonhole principle: Two cases

Suppose that there are at least $R(s - 1, t)$ red edges. See Figure 2.3.14.



Case 1.1: There is a red K_{s-1}

Case 1.2: There is a blue K_t

Figure 2.3.14 Recall the definition of $R(s - 1, t)$

Hence, any red/blue 2-colouring of K_M yields a red K_s or a blue K_t. Therefore $R(s, t)$ exists and

$$R(s, t) \leq R(s - 1, t) + R(s, t - 1).$$

■

Table 2.3.15 Known Ramsey Numbers

s	t	$R(s,t)$	Who and When
3	3	6	Greenwood and Gleason, 1955
3	4	9	Greenwood and Gleason, 1955
3	5	14	Greenwood and Gleason, 1955
3	6	18	Graver and Yackel, 1968
3	7	23	Kalbfleisch, 1966
3	8	28	McKay and Min, 1992
3	9	36	Grinstead and Roberts, 1982
4	4	18	Greenwood and Gleason, 1955
4	5	25	McKay and Radziszowski, 1995

Table 2.3.16 More Known Facts

s	t	$R(s,t)$	Who and When
3	10	$[40,42]$	Exoo 1989, Radziszowski and Kreher 1988
3	11	$[46,51]$	Radziszowski and Kreher 1988
4	6	$[35,41]$	Exoo, McKay and Radziszowski 1995
4	7	$[49,61]$	Exoo 1989, Mackey 1994
5	5	$[43,48]$	Exoo 1989, McKay and Radziszowski 1995
5	6	$[58,87]$	Exoo 1993, Walker 1971
6	6	$[102,165]$	Kalbfleisch 1965, Mackey 1994
6	7	$[113,298]$	Exoo and Tatarevic 2015, Xu and Xie 2002

In Erdős' Words.

> Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

How Big - Upper Bound.

$$R(s,t) \leq \binom{s+t-2}{t-1}.$$

Proof. We use mathematical induction to establish that $\binom{s+t-2}{t-1}$ is an upper bound for $R(s,t)$.

Recall that for $s \geq 2$

$$\binom{s+2-2}{2-1} = \binom{s}{1} = s = R(s,2)$$

and that

$$\binom{3+3-2}{3-1} = \binom{4}{2} = 6 = R(3,3).$$

Hence, if $s,t \geq 2$ and $s+t \leq 6$ then the inequality

$$R(s,t) \leq \binom{s+t-2}{t-1}$$

holds.

For the inductive step, suppose that $n \geq 6$ is such that the inequality holds for all u, v such that $u+v=n$ and let $s,t \in \mathbb{N}\backslash\{1,2\}$ be such that $s+t=n+1$. Observe that this implies $(s-1)+t = s+(t-1) = n$ and that by our assumption

$$R(s-1,t) \le \binom{(s-1)+t-2}{t-1} = \binom{s+t-3}{t-1}$$

and

$$R(s,t-1) \le \binom{s+(t-1)-2}{(t-1)-1} = \binom{s+t-3}{t-2}.$$

For the next step we need the following two facts:

$$(\forall s,t \in \mathbb{N}\backslash\{1\}) \;\; R(s,t) \le R(s-1,t) + R(s,t-1)$$

and

$$(\forall m,k \in \mathbb{N}) \;\; \binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}.$$

It follows that, for $s+t = n+1$,

$$R(s,t) \le \binom{s+t-3}{t-1} + \binom{s+t-3}{t-2} = \binom{s+t-2}{t-1}$$

which completes the inductive step.

By the Principle of Mathematical Induction it follows that, for all $s,t \in \mathbb{N}\backslash\{1\})$,

$$R(s,t) \le \binom{s+t-2}{t-1}.$$

■

How Big - Lower Bound. For $s \ge 3$,

$$R(s,s) > 2^{s/2}.$$

Proof. Let $s \ge 3$ and let $n = \lfloor 2^{s/2} \rfloor$. Consider a random colouring χ of K_n where each edge is coloured independently red or blue with probability $\frac{1}{2}$.

We start by choosing any s vertices of K_n and considering the corresponding complete graph K_s. Recall that K_s has $\binom{s}{2}$ edges. This implies, since each edge is coloured independently red or blue with probability $\frac{1}{2}$,

Probability all edges blue = (probability a single edge is blue)$^{\text{(number of edges)}}$ = $\frac{1}{2^{\binom{s}{2}}}$.

Similarly,

$$\text{Probability all edges red} = \frac{1}{2^{\binom{s}{2}}}$$

which implies

$$\text{Probability all edges blue or all red} = \frac{1}{2^{\binom{s}{2}}} + \frac{1}{2^{\binom{s}{2}}} = \frac{2}{2^{\binom{s}{2}}}.$$

Observe that the number of ways that we can choose s vertices of K_n equals to $\binom{n}{s}$. This implies that, for $s \ge 3$ and $n = \lfloor 2^{\frac{s}{2}} \rfloor$:

$$\text{Probability } \chi \text{ yields a monochromatic } K_s \le$$

$$\binom{n}{s} \cdot \frac{2}{2^{\binom{s}{2}}} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-s+1)}{s!} \cdot 2^{1-\frac{s^2}{2}+\frac{s}{2}} < \frac{n^s}{s!} \cdot 2^{1-\frac{s^2}{2}+\frac{s}{2}} \le$$

$$\frac{2^{\frac{s^2}{2}}}{s!} \cdot 2^{1-\frac{s^2}{2}+\frac{s}{2}} = \frac{2^{1+\frac{s}{2}}}{s!} \le 1.$$

Therefore

$$\text{Probability } \chi \text{ does not yield a monochromatic } K_s > 0$$

which implies that it is possible to 2-colour K_n and avoid a monochromatic K_s. Hence $R(s,s) > n$.

Observe that if s is even then $n = 2^{\frac{s}{2}}$ and if s is odd then $n < 2^{\frac{s}{2}} < n+1$. Since $R(s,s)$ is an integer, we conclude that, regardless if s is even or odd,

$$R(s,s) > 2^{\frac{s}{2}}.$$

■

Therefore: For $s \geq 3$

$$2^{s/2} < R(s,s) \leq \binom{2s-2}{s-1}.$$

Epilogue

Question 2.3.17 Suppose that we decide to use three colours, say blue, red, and green. Is there a something like $R(s,t,u)$, for $s,t,u \in \mathbb{N}$? In other words, is it possible to find a number n so that if the edges of K_n are coloured by one of the three colours then there will be always possible to find a blue K_s or a red K_t or a green K_u? □

Definition 2.3.18 Let $m \in \mathbb{N}\backslash\{1\}$ and $s_1, s_2, \ldots, s_m \in \mathbb{N}\backslash\{1\}$ be given. The Ramsey number $R(s_1, s_2, \ldots, s_m)$ is the minimum number n for which any edge m-colouring of K_n, a complete graph on n vertices, contains a monochromatic K_{s_i} for some $i \in [1, m]$. ◊

Three BIG Questions:

1. Does the Ramsey number $R(s_1, s_2, \ldots, s_m)$ exist for any choice of natural numbers $m, s_1, s_2, \ldots, s_m \geq 2$?

2. If $R(s_1, s_2, \ldots, s_m)$ exists, can we find the exact value of $R(s_1, s_2, \ldots, s_m)$?

3. If $R(s_1, s_2, \ldots, s_m)$ exists and if we cannot find the exact value of $R(s_1, s_2, \ldots, s_m)$, what are the best known bounds for $R(s_1, s_2, \ldots, s_m)$?

Theorem 2.3.19 Ramsey's Theorem. *For any , $s_1, s_2, \ldots, s_m \in \mathbb{N}\backslash\{1\}$ the Ramsey number $R(s_1, s_2, \ldots, s_m)$ exists.*

Resources.

1. Ramsey's theorem - Wikipedia
2. Ramsey's Theory Through Examples Part I by Veselin Jungic
3. Ramsey's Theory Through Examples Part II by Veselin Jungic
4. On Ramsey Numbers by Evelyn Lamb
5. Ramsey Theory by G.E.W. Taylor, pp 1–8
6. Ramsey Theory by Alan Frieze
7. Cut The Not - Ramsey's Theorem
8. Cut The Not - Ramsey's Number $R(5,3)$
9. Ramsey Number - Wolfram - MathWorld
10. Applications of Ramsey theory to computer science

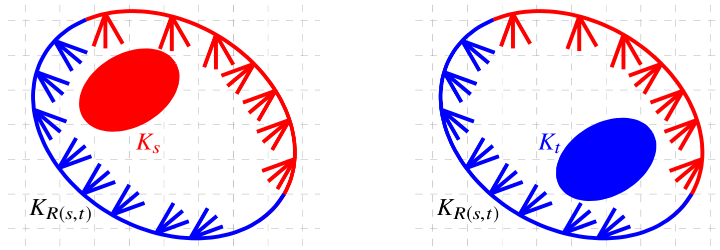
2.4 Ramsey's Theorem, Infinite Case

No finite point has meaning without an infinite reference point. — Jean-Paul Sartre, French philosopher, playwright, novelist, screenwriter, political activist, biographer, and literary critic, 1905 - 1980.

Reminder. For any $s, t \in \mathbb{N}\backslash\{1\}$ the Ramsey number $R(s,t)$ exists and, for $s, t \geq 3$,

$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

See Figure 2.4.1



There is a red K_s There is a blue K_t

Figure 2.4.1 $K_{R(s,t)}$: There is a red K_s or there is a blue K_t

Infinite Case - Notation:

- The set of natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- For $r \in \mathbb{N}$ and any set X we define $X^{(r)}$ to be the set of all subsets on X with exactly r elements:
$$X^{(r)} = \{A \subset X : |A| = r\}.$$
- For $k \in \mathbb{N}$ we define a k-colouring of $\mathbb{N}^{(r)}$ as a function from $\mathbb{N}^{(r)}$ to $\{1, 2, \ldots, k\}$:
$$c : \mathbb{N}^{(r)} \longrightarrow \{1, 2, \ldots, k\} = [1, k].$$
- If c is a k-colouring of $\mathbb{N}^{(r)}$ and $A \subset \mathbb{N}$ such that, for all $x, y \in A^{(r)}$, $c(x) = c(y)$, we say that the set A is *monochromatic*. See Figure 2.4.2.

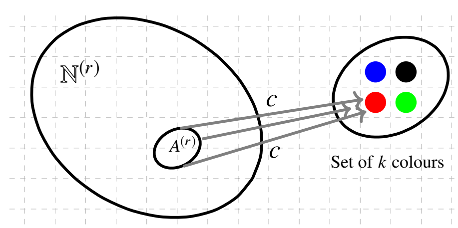


Figure 2.4.2 $A \subset \mathbb{N}$ is monochromatic!

We re-state Theorem 2.3.6 in the following form:

Theorem 2.4.3 *Whenever $\mathbb{N}^{(2)}$ is 2-coloured, there exist arbitrarily large monochromatic sets.*

Observation 2.4.4 An infinite monochromatic set is much more than having arbitrarily large monochromatic sets.

Example 2.4.5 Colour

$$\{1, 2\}, \{3, 4, 5\}, \{6, 7, 8, 9\}, \cdots$$

i.e., colour red all edges within the sets above. Colour all other edges blue.

For example, is the edge between 500500 and 500501 red or blue? What about the edge between 499499 and 500500?

What about the existence of an **infinite** red set in this colouring? In other words, can you find an infinite set $A \subset \mathbb{N}$ such that the edge between any $x, y \in A$ is red? □

Theorem 2.4.6 Ramsey Theorem - Two Colours - Infinite Case: *Whenever $\mathbb{N}^{(2)}$ is 2-coloured, there exists an infinite monochromatic set.*

Proof. We colour elements of $\mathbb{N}^{(2)}$ red and blue:

$$c : \mathbb{N}^{(2)} \to \{\bullet, \bullet\},$$

See Figures 2.4.7– Figure 2.4.10.

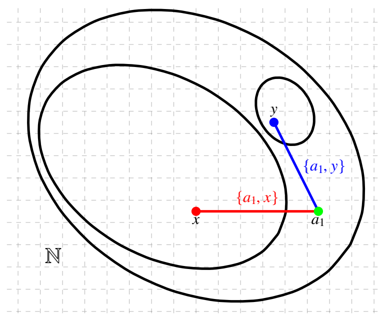


Figure 2.4.7 Step 1: Pick $a_1 \in \mathbb{N}$. Look at $\{a_1, x\}$ and $\{a_1, y\}$, $x, y \in \mathbb{N}$.

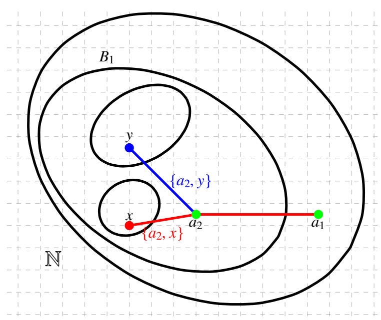


Figure 2.4.8 Step 2: Say that $B_1 = \{x \in \mathbb{N} : \{a_1, x\}\}$ is infinite. Pick $a_2 \in B_1$. Look at $\{a_2, x\}$ and $\{a_2, y\}$, $x, y \in B_1$.

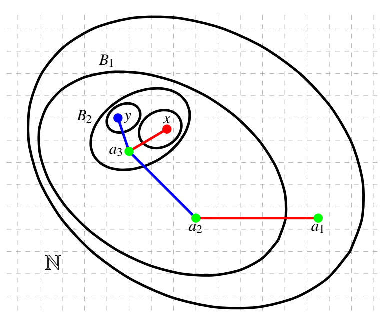


Figure 2.4.9 Step 3: Say that $B_2 = \{y \in B_1 : \{a_2, y\}\}$ is infinite. Pick $a_3 \in B_2$. Look at $\{a_3, x\}$ and $\{a_3, y\}$, $x, y \in B_2$.

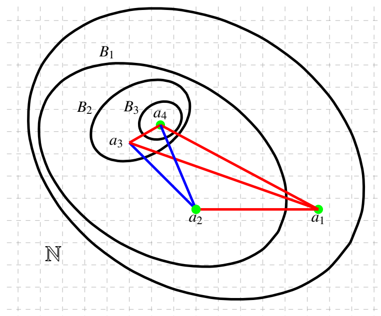


Figure 2.4.10 Step 4: Say that $B_3 = \{x \in B_2 : \{a_3, x\}\}$ is infinite. Pick $a_4 \in B_3$. Look at $\{a_4, x\}$ and $\{a_4, y\}$, $x, y \in B_3$. Note that $\{a_1, a_2\}$ $\{a_1, a_3\}$ $\{a_1, a_4\}$, $\{a_2, a_3\}$ $\{a_2, a_4\}$, and $\{a_3, a_4\}$.

Continue. . .

Summary: We obtain an infinite sequence of natural numbers

$$a_1, a_2, a_3, \ldots$$

and an infinite sequence of sets

$$\mathbb{N} \supseteq B_1 \supseteq B_2 \supseteq B_3 \supseteq \ldots$$

with the property that, for any $i \in \mathbb{N}$

1. B_i is an infinite set
2. $a_{i+1} \in B_i$
3. $c(\{a_i, a_{i+1}\}) = c(\{a_i, a_{i+2}\}) = c(\{a_i, a_{i+3}\}) = \ldots$.

See Figure 2.4.11.

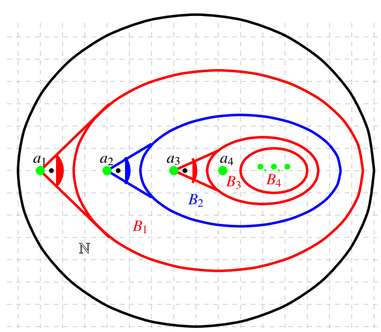


Figure 2.4.11 Conclusion: There must be an infinite number of a_i's that see only red or an infinite number of a_i's that see only blue.

■

Example 2.4.12 Let $a_1, a_2, a_3, \ldots$ be a sequence of mutually distinct real numbers. Prove that it contains a monotone subsequence. □

Challenge. Whenever $\mathbb{N}^{(2)}$ is k-coloured, there exists an infinite monochromatic set.

Example 2.4.13 Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers. Prove that it contains either a constant or strictly monotonic subsequence. □

Recall Theorem 2.4.6: Whenever $\mathbb{N}^{(r)}$ is 2-coloured, there exists an infinite monochromatic set.

Theorem 2.4.14 Ramsey Theorem. *Let $m, r \in \mathbb{N}$. Whenever $\mathbb{N}^{(r)}$ is m-coloured, there exists an infinite monochromatic set.*

Resources.

1. Ramsey's theorem - Wikipedia
2. Ramsey Theory by I. Leader
3. Ramsey Theory by G.E.W. Taylor, pp 1–10
4. A couple of questions using Ramsey Theorem
5. Applications of the Canonical Ramsey Theorem to Geometry by W. Gasarch and S. Zbarsky
6. An Application of Ramsey Theorem to Stopping Games by A. Shmaya at al.

2.5 Exercises

Exercise 2.5.1 Essay. Write a short essay (300-400 words) on the life and work of Frank Ramsey.

Exercise 2.5.2 Essay. Write a short essay (300-400 words) on the life and work of Paul Erdős.

Exercise 2.5.3 Pigeonhole principle. Prove that if there are 10 pairs of shoes on a shelf, picking 11 shoes randomly from the shelf will result in picking up at least one pair of shoes.

Solution. If there are 11 pigeons (shoes) sitting in 10 pigeonholes (one for each pair of shoes), at least one pigeonhole has 2 pigeons (so a pair of shoes) by the pigeonhole principle..

Exercise 2.5.4 Pigeonhole principle. Between 1972 and 2012, the 411 Senior Centre occupied the historic 411 Dunsmuir Street building in Vancouver, British Columbia. The Centre was an important part of life for generations of elderly Vancouverites.

With this example, we honour the memory of ten members of the Centre: Mirko, Wanda, John, Hubert, Ursula, Gadafi, two ladies remembered as the Librarian and the Volunteer, and two gentlemen remembered as the Miner and the Sailor.

The 10 friends formed the "411 Ping Pong Club."

For the purpose of this example, we assume that each day between January 1, 1997, and December 31, 1998, four members of the 411 Ping Pong Club got together and played exactly one game of ping pong doubles.

Prove that in this time period, there was some particular set of four members that had played at least four games of ping pong doubles together.

Solution. Observe that the number of different groups of four members of the 411 Ping Pong is equal to

$$r = \binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} = 210$$

and that the number of days between January 1, 1997, and December 31, 1998, is equal to $m = 2 \cdot 365 = 730$.

Let r be the number of pigeonholes and let m be the number of pigeons. We need to prove that at least one of the pigeonholes contains at least four pigeons.

If this is not true, then each of r pigeonholes contains at most three pigeons. This would imply that m, the number of pigeons, satisfies the inequality

$$730 = m \leq 3 \cdot r = 3 \cdot 210 = 630.$$

This is not true, so at least one pigeonhole contains at least four pigeons.

Hence there was some particular set of four members of the 411 Ping Pong Club that had played at least four games of ping pong doubles together between January 1, 1997, and December 31, 1998.

Exercise 2.5.5 Pigeonhole principle. Show that in a group of n people where $n > 2$, there are at least two people who have the same number of friends.

Solution. Let j be the number of people with no friends. If $j \geq 2$ then there are at least two people who have the same number (0) of friends.

If $j \in \{0, 1\}$ then there are $n - j > 0$ people who have between 1 and $n - j - 1$ friends.

By the Pigeonhole Principle there must be at least two people with the same number of friends..

Exercise 2.5.6 Pigeonhole principle. Colour each point in the xy plane having integer coefficients Red or Blue. Then some rectangle has all its vertices the same colour.

Solution. There exists a 3 by 9 grid in the xy-plane. By the result from Example 2.1.7, there exists a rectangle with monochromatic vertices in the grid and hence in the whole xy-plane.

Exercise 2.5.7 Pigeonhole principle. Color each point in the integer grid $[1, 257] \times [1, 4]$ Red, Green, or Blue.

Show that some rectangle has all its vertices the same colour. In other words, show that for any function

$$f : \{1, 2, \ldots, 257\} \times \{1, 2, 3, 4\} \to \{R, G, B\}$$

there are $a, b \in \{1, 2, \ldots, 257\}$, $a < b$, and $c, d \in \{1, 2, 3, 4\}$, $c < d$, such that

$$f(a, c) = f(a, d) = f(b, c) = f(b, d).$$

Solution. Let a colouring

$$f : \{1, 2, \ldots, 257\} \times \{1, 2, 3, 4\} \to \{R, G, B\}$$

be given.

For each $i \in \{1, 2, \ldots, 257\}$ we define g_i, a colouring of the set $\{1, 2, 3, 4\}$, by

$$g_i(1) = f(i, 1),\ g_i(2) = f(i, 2),\ g_i(3) = f(i, 3),\ g_i(4) = f(i, 4).$$

Let P be the the set of all functions $g : \{1, 2, 3, 4\} \to \{R, G, B\}$, i.e. let P be the set of all 3-colouring of the set $\{1, 2, 3, 4\}$.

Observe that

- $|P| = 4^3 = 256$,
- by the Pigeonhole Principle, for each $g \in P$ there are $c, d \in \{1, 2, 3, 4\}$, $c < d$, such that $g(c) = g(d)$.

Let the elements of the set P be pigeonholes and let the elements of the set $\{g_i : i \in \{1, 2, \ldots, 257\}\}$ be pigeons.

By the Pigeonhole Principle, at least one of 256 pigeonholes contains at least two of 257 pigeons. In other words, there is $g \in P$ and $a, b \in \{1, 2, \ldots, 257\}$, $a < b$, such that $g = g_a$ and $g = g_b$. This means that, for each $j \in \{1, 2, 3, 4\}$,

$$g_a(j) = g_b(j).$$

Also observe that there are $c, d \in \{1, 2, 3, 4\}$, $c < d$, such that

$$g_a(c) = g_a(d) = g_b(c) = g_b(d).$$

By definition this means that

$$f(a, c) = f(a, d) = f(b, c) = f(b, d).$$

Therefore the rectangle with vertices (a, c), (a, d), (b, c), and (b, d) has all its vertices in the same colour.

Exercise 2.5.8 Pigeonhole principle. There are many beads in a box. There are two colours of beads, red and blue, divided equally. If someone were to make three bracelets, using 10 beads each, prove that when the bracelets are stacked on top of each other, there will be a rectangle with each vertex the same colour.

Solution. Observe that each vertical line of the stacked bracelets has three beads, and there are two colours, so each vertical line has a dominant colour by the pigeonhole principle, which means one colour appears a least twice in the vertical line.

Next recall that there are $2^3 = 8$ possible colour different configurations of three beads stacked vertically.

By the pigeonhole principle there must be at least two (out of ten) vertical lines with the same colour configurations. In two of those identically configured vertical lines we chose four beads coloured by the dominant colour that form a rectangle.

Exercise 2.5.9 Pigeonhole principle. If $a_1, a_2, \ldots, a_{n+1} \in [1, 2n]$ are distinct, then there exist i, j, $i \neq j$, such that a_i divides a_j. (This was one of Erdős' favourite questions to ask of an ϵ.)

Solution. Choose $\{a_1, \ldots, a_{n+1}\}$ from $[1, 2n]$, so that they are distinct. We will show that there exists $i \neq j$ such that a_i divides a_j or a_j divides a_i.

For each i, we may write ai as follows:

$$a_i = 2^{b_i} q_i$$

where q_i is an odd number. Consider the numbers $\{q_1, q_2, \ldots, q_{n+1}\}$, a set of of $n + 1$ odd numbers in $[1, 2n]$. Since there are only n odd numbers in the range $[1, n]$, we conclude that, for some $i \neq j$, $q_i = q_j$. Let $q = q_i = q_j$. Then

$$a_i = 2^{b_i} q \text{ and } a_j = 2^{b_j} q.$$

Since $a_i \neq a_j$, we have that either $b_i > b_j$ or $b_j > b_i$. In the former case, we have that $a_j | a_i$ and in the latter case, $a_i | a_j$, as required.

Exercise 2.5.10 Pigeonhole principle. Place the numbers $1, 2, \ldots, 12$ around a circle, in any order. Then there are three consecutive numbers which sum to at least 19.

Solution. Given the number $1, 2, \ldots., 12$ placed around a circle, in some order, partition the circle into 4 equal sections, each containing 3 consecutive numbers. Let A_1, A_2, A_3, and A_4 be the four sets of consecutive numbers.

We wish to show that $\displaystyle\sum_{a \in A_j} a \geq 19$ for some $j \in \{1, 2, 3, 4\}$. Let $a_j = \sum_{a \in A_j} a$ for $j = 1, 2, 3, 4$. We know that

$$a_1 + a_2 + a_3 + a_4 = \sum_{b=1}^{12} b = 78.$$

By the generalized pigeon-hole principle, we have that some a_i for $i \in \{1, 2, 3, 4\}$

has the property that

$$a_i \geq \left\lfloor \frac{78}{4} \right\rfloor = \lfloor 19.5 \rfloor = 19,$$

as required.

Exercise 2.5.11 Pigeonhole principle. You ask your computer to randomly pick, one by one, 50 positive integers. This generates a sequence $a_1, a_2, \ldots, a_{50}$, where the index i means that the integer a_i was the i-th randomly picked positive integer. Observe that, since the whole process is random, the sequence $a_1, a_2, \ldots, a_{50}$ may not be ordered. In other words, for any $i \leq 49$, a_{i+1} may be greater than or equal to or less than a_i.

After generating several sequences, you notice that each time you can find at least 8 members of the sequence, say $a_{i_1}, a_{i_2}, \ldots, a_{i_8}$, that form a nondecreasing subsequence, i.e. for each $j \in \{1, \ldots, 7\}$,

$$a_{i_{j+1}} \geq a_{i_j},$$

OR that they form a nonincreasing subsequence, i.e. for each $j \in \{1, \ldots, 7\}$,

$$a_{i_{j+1}} \leq a_{i_j}.$$

You wonder if this is just a coincidence or it is true that something like this must always happen.

What would you do?

Note: Actually, it is true that any sequence of $n^2 + 1$ positive integers, there exists a nondecreasing or a nonincreasing sequence of length $n + 1$. Can you prove this statement?

Solution. Let $a_1, a_2, \ldots, a_{50}$ be a sequence of positive integers.

For each $i \in \{1, \ldots, 50\}$, let m_i be the length of the longest nondeacreasing subsequence starting at and including a_i. This means that there are integers $j_1, j_2, \ldots, j_{m_i} \in \{1, 2, \ldots, 50\}$ such that

$i = j_1 \leq j_2 \leq \cdots \leq j_{m_i}$ and $a_i = a_{j_1} \leq a_{j_2} \leq \cdots \leq a_{j_{m_i}}$

but for any $s \in \{j_{m_i} + 1, \ldots, 50\}$,

$$j_{m_i} > a_s.$$

Since if, for some i, $m_i \geq 8$, our observation has been supported, let us suppose that for all $i \in \{1, 2, \ldots, 50\}$, $m_i \leq 7$.

Next, let the elements of the set $\{1, 2, 3, 4, 5, 6, 7\}$ be the pigeonholes and let the elements of the set $\{m_i : i \in \{1, 2, \ldots, 50\}\}$ be the pigeons. We put the pigeon m_i in the pigeonhole j if and only if $m_i = j$.

Observe that one of the pigeonholes contains at least eight pigeons. Otherwise, the number of pigeons would be at most $7 \cdot 7 = 49$.

Say that $i_1, i_2, \ldots, i_8 \in \{1, 2, \ldots, 50\}$ and $m \in \{1, 2, \ldots, 7\}$ are such that

$i_1 \leq i_2 \leq \cdots \leq i_8$ and $m_{i_1} = m_{i_2} = \cdots = m_{i_8} = m$.

What can we tell about the subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_8}$?

Recall that, for each $j \in \{1, 2, \ldots, 8\}$, $m_{i_j} = m$ represents the length of the longest nondecreasing subsequence starting at a_{i_j}.

Let $j, k \in \{1, 2, \ldots, 8\}$, $j < k$. Observe that, under our assumptions, $a_{i_j} > a_{i_k}$. Otherwise we will have a nondecreasing sequence starting at a_{i_j} and of length $m + 1$. (In this scenario, a_{i_j} would be followed by an m-term nondicreasing subsequence determined by a_{i_k}.)

Therefore

$$a_{i_1} > a_{i_2} > \cdots > a_{i_8}$$

and we have found an 8-term nonincreasing subsequence.

Exercise 2.5.12 Ramsey's Theorem. Prove the following statement: Suppose that nine people are gathered at a dinner party. Then there is a group of four people at the party who are all mutual acquaintances or there is a group of three people at the party who are all mutual strangers.

Solution. Observe that the question asks for a proof that $R(4,3) \leq 9$.

See Claim 2.2.11.

Exercise 2.5.13 Ramsey's Theorem. Show that $R(3,3,3) \leq 17$. (This means: Every 3-colouring of the edges of K_{17} gives a monochromatic K_3.)

Solution. Consider any 3–edge–colouring c of K_{17} with colours c_1, c_2, c_3. (This means that for e, an edge of K_{17}, $c(e)$ denotes the colour of the edge e.) We will show that there exists a monochromatic K_3.

Let x be any vertex in K_{17}. The vertex x is incident to 16 edges, coloured with one of the three colours. By the generalized pigeonhole principle, we have that there is a colour c_i such that x is incident to 6 edges with colour c_i.

Let $Y = \{y_1, \ldots, y_6\}$ be the six neighbours of x, i.e. the six vertices which are joined to x by an edge of the colour c_i. If any edge joining two vertices in Y is also coloured with colour c_i, then they form a monochromatic K_3 with x as its vertex and we are done.

Otherwise, every edge joining vertices of Y is not coloured with colour c_i. Then, the vertices of Y induce a subgraph of K_{17} that is a copy of K_6, edge–coloured with two colours. By Ramsey's theorem every 2-colouring of K_6 contains a monochromatic K_3, and the result follows.

Exercise 2.5.14 Ramsey number. Prove that the upper bound for a diagonal Ramsey number is:

$$R(s,s) \leq \binom{2s-2}{s-1}.$$

Solution. Recall (Section 2.3) that, for any $s, t \geq 3$, $\binom{s+t-2}{t-1}$. Hence, for $s = t$

$$R(s,s) \leq \binom{2s-2}{s-1}.$$

Note: For the latest developments (as of December 2020) regarding the upper bounds for a diagonal Ramsey number see this article by Ashwin Sah, an undergraduate student from MIT: "Diagonal Ramsey via effective quasirandomness"..

Exercise 2.5.15 Ramsey's Theorem, Infinite Case. Let S be an infinite set of points in the plane. Show that there is an infinite subset A of S such that either no three points of A are on a line, or all points of A are on a line.

Solution. Let $\mathcal{A} = \{\{x,y,z\} : x,y,z \in S\} = S^{(3)}$, i.e. let $\mathcal{A}$ be the set of all triples of points from S.

Let $f : \mathcal{A} \longrightarrow \{\bullet, \blacksquare\}$ be defined in the following way:

$f(\{x,y,z\}) = \bullet \Leftrightarrow x, y, z$ are not colinear

and

$f(\{x,y,z\}) = \blacksquare \Leftrightarrow x, y, z$ are colinear.

By Ramsey's Theorem, Infinite Case, it follows that there is a monochromatic infinite subset $A \subset S$.

If

$$\forall \{x,y,z\} \in A^{(3)}, \quad f(\{x,y,z\}) = \bullet$$

then no three points of A lie on a line.

Let

$$\forall \{x,y,z\} \in A^{(3)}, \quad f(\{x,y,z\}) = \blacksquare.$$

Let $x, y \in A$ and let L be the line through x and y. Let z be any other point in A. Since x, y, z are collinear, z lies on L.

Exercise 2.5.16 Ramsey's Theorem, Infinite Case. Let

$$y_1, y_2, y_3, \dots$$

be a sequence of mutually distinct real numbers. Use Ramsey's theorem to prove that the sequence

$$(1, y_1), (2, y_2), (3, y_3), \dots$$

of points in $\mathbb{R}^2$ contains a subsequence such that the induced function is convex or concave.

Solution. A function is convex if its epigraph (the set of points on or above the graph of the function) is a convex set, i.e., any line segment that connect two points in the epigraph also belongs to the epigraph. A function f is concave if the function $-f$ is convex. Equivalently, a function f is concave if its hypograph (the set of points on or below the graph of the function) is a convex set,

Note that any linear function is both convex and concave. For the purpose of this problem we will take a linear function to be convex.

Let $i, j, k \in \mathbb{N}$ be such that $i < j < k$. Since $y_i \neq y_j$, $y_i \neq y_k$, and $y_j \neq y_k$ there are these possibilities:

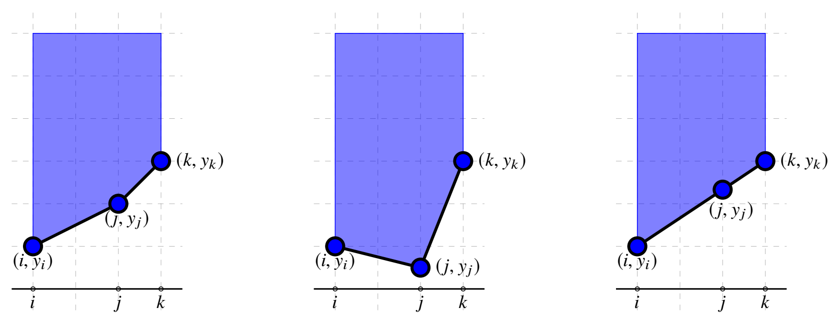


Figure 2.5.17 The induced function is convex.

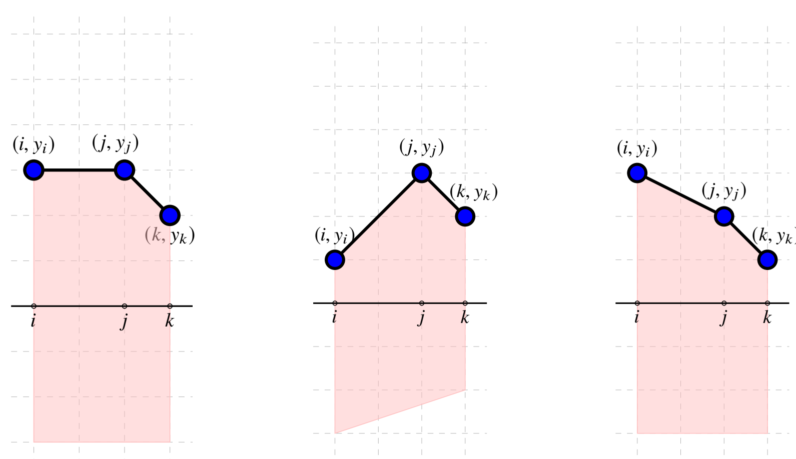


Figure 2.5.18 The induced function is concave.

We 2-colour $\mathbb{N}^{(3)}$ in the following way:

- Colour $\{i, j, k\}, i < j < k$, blue if the points $(i, y_i), (j, y_j), (k, y_k)$ induce a convex function.

- Colour $\{i, j, k\}$, $i < j < k$, red if the points $(i, y_i), (j, y_j), (k, y_k)$ induce a concave function.

By Ramsey's theorem there is an infinite monochromatic set

$$i_1 < i_2 < i_3 < \dots.$$

This means that that, for any $p, q, r \in \mathbb{N}$, the set $\{i_p, i_q, i_r\}$ is always of the same colour.

Say that colour is blue. In particular this means that for any j the function induced by

$$(i_j, y_{i_j}), (i_{j+1}, y_{i_{j+1}}), (i_{j+2}, y_{i_{j+2}})$$

is convex.

Consider two points (x, y) and (x', y'), $x < x'$, in the epigraph of the function f induced by the sequence of points

$$(i_1, y_{i_1}), (i_2, y_{i_2}), (i_3, y_{i_3}), \dots.$$

If there is $j \in \mathbb{N}$ such that

$$i_j \leq x < x' \leq i_{j+1}$$

then the line segment with the end points (x, y) and (x', y') is on or above the line segment with the end points $(i_j, y_{i_j}), (i_{j+1}, y_{i_{j+1}})$ and thus in the epigraph of the function f.

Otherwise, there are j and $k \geq 3$ such that

$$i_j < x \leq i_{j+1} < i_{j+k-1} \leq x' < i_{j+k}.$$

Let

$$z_{i_{j+1}}, z_{i_{j+2}}, \dots, z_{i_{j+k-1}}$$

be such that, for any $l \in [1, k-1]$, the point $(i_{j+l}, z_{i_{j+l}})$ belongs to the line segment with the end points (x, y) and (x', y'). See Figure 2.5.19.

As above, line segments with the end points $(i_{j+l}, z_{i_{j+l}})$ and $(i_{j+l+1}, z_{i_{j+l+1}})$, $l \in [1, k-2]$, belong to the epigraph of the function f. It follows that their union, the line segment with the end points (i_j, z_{i_j}) and $(i_{j+k-1}, z_{i_{j+k-1}})$ belongs to the epigraph of the function f. Finally, if $x \neq i_j$ and/or $x' \neq i_{j+k-1}$ the segments with end points (x, y) and (i_j, z_{i_j}) and with the endpoints $(i_{j+k-1}, z_{i_{j+k-1}})$ and (x', y') belong to the epigraph of the function f. This implies that the line segment with the end points (x, y) and (x', y') belongs to the epigraph of the function f.

Therefore, the function f is convex.

Now we consider the case that for any p, q, r, the set $\{i_p, i_q, i_r\}$ is always coloured red. In particular this means that for any p, q, r the function induced by

$$(i_p, y_{i_p}), (i_q, y_{i_q}), (i_r, y_{i_r})$$

is concave. By definition, for any p, q, r the function induced by

$$(i_p, -y_{i_p}), (i_q, -y_{i_q}), (i_r, -y_{i_r})$$

is convex, and as we have already seen, the function g induced by by the sequence of points

$$(i_1, -y_{i_1}), (i_2, -y_{i_2}), (i_3, -y_{i_3}), \dots$$

is convex. It follows that the function $-g$ induced by by the sequence of points

$$(i_1, y_{i_1}), (i_2, y_{i_2}), (i_3, y_{i_3}), \dots$$

is concave.

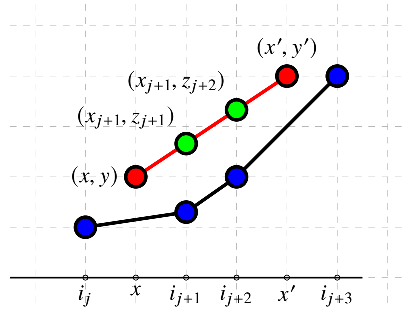


Figure 2.5.19 It is convex!

Exercise 2.5.20 Ramsey's Theorem, Infinite Case. Let

$$y_1, y_2, y_3, \ldots$$

be a sequence of real numbers. Use Ramsey's theorem to prove that that the sequence

$$(1, y_1), (2, y_2), (3, y_3), \ldots$$

of points in $\mathbb{R}^2$ contains a subsequence such that the induced function is constant, convex or concave.

Solution. Let a 3-colouring of $\mathbb{N}^{(3)}$ be defined in the following way:

- Colour $\{i, j, k\}$, $i < j < k$, green if $y_i = y_j = y_k$.
- Colour $\{i, j, k\}$, $i < j < k$, blue if not all of y_i, y_j, y_k are not mutually equal and if the points $(i, y_i), (j, y_j), (k, y_k)$ induce a convex function.
- Colour $\{i, j, k\}$, $i < j < k$, red if not all of y_i, y_j, y_k are not mutually equal and if the points $(i, y_i), (j, y_j), (k, y_k)$ induce a concave function.

By Ramsey's theorem there is an infinite monochromatic set:

$$i_1 < i_2 < i_3 < \ldots.$$

This means that, for any p, q, r, the set $\{i_p, i_q, i_r\}$ is always of the same colour.

If $\{\{i_p, i_q, i_r\}\}$ for any p, q, r then $y_{i_p} = y_{i_q} = y_{i_r}$ and the induced function is a constant.

If $\{\{i_p, i_q, i_r\}\}$ for any p, q, r then y_p, y_q, y_r are not mutually equal and the points $(i_p, y_{i_p}), (i_q, y_{i_q}), (i_r, y_{i_r})$ induce a convex function. By the previous problem the function induce by $i_1 < i_2 < i_3 < \ldots$ is convex.

If $\{\{i_p, i_q, i_r\}\}$ for any p, q, r then y_p, y_q, y_r are not mutually equal and the points $(i_p, y_{i_p}), (i_q, y_{i_q}), (i_r, y_{i_r})$ induce a concave function. By the previous problem the function induced by $i_1 < i_2 < i_3 < \ldots$ is concave.

Chapter 3

van der Waerden's Theorem

3.1 Bartel van der Waerden

> *None of the three of my children had any interest in mathematics.* —
> Bartel Leendert van der Waerden, 1903 — 1996.

Who was Bartel Leendert van der Waerden? Bartel Leendert van der Waerden was a Dutch mathematician and historian of mathematics and science.

Table 3.1.1 Bartel Leendert van der Waerden made contributions to:

abstract algebra	algebraic geometry	analysis
combinatorics	geometry	group theory
history of ancient science	history of astronomy	history of mathematics
history of modern physics	mathematical statistics	number theory
probability theory	quantum mechanics	topology

Birth and Death. Bartel Leendert van der Waerden was born in Amsterdam, Netherlands, on February 2, 1903 and died at the age of 92 on January 12, 1996, in Zürich, Switzerland.

Figure 3.1.2

Timeline.

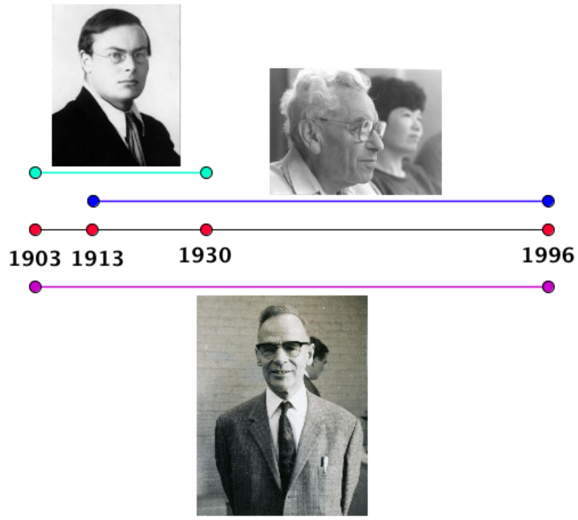


Figure 3.1.3 Ramsey, Erdős, van der Waerden

Bartel's Family. Bartel van der Waerden's parents were Theodorus van der Waerden and Dorothea Adriana Endt. Theo was born in Eindhoven on 21 August 1876 and studied civil engineering at the Delft Technical University. Then, after teaching mathematics and mechanics in Leeuwarden and Dordrecht, he moved to Amsterdam in 1902 where again he taught mathematics and mechanics. At university he had become interested in politics and played a role in politics throughout his life as a left wing Socialist. He married Dorothea on 28 August 1901. Bartel was the eldest of their three children, the other two boys being Coenraad (born 29 December 1904) and Benno (born 2 October 1909). (Source MacTutor).

Van Dalen:

> Van der Waerden was an extremely bright student, and he was well aware of this fact. He made his presence in class known through bright and sometimes irreverent remarks. Being quick and sharp (much more so than most of his professors) he could make life miserable for the poor teachers in front of the blackboard. During the, rather mediocre, lectures of Van der Waals Jr. he could suddenly, with his characteristic stutter, call out: "Professor, what kind of nonsense are you writing down now?" He did not pull such tricks during Brouwer's lectures, but he was one of the few who dared to ask questions. [9]

Time of War.

Between 1931-1945 van der Waerden was a professor of mathematics at the University of Leipzig, Germany. During the rise of the Third Reich and through World War II, van der Waerden remained at Leipzig, and passed up opportunities to leave Nazi Germany for Princeton and Utrecht. (van der Waerden's photo from Alchetron.)

Figure 3.1.4

An extensive description of this part of van der Waerden's life is published by Soifer [7].

Facts:

- At the peak of their activity, between the outbreak of World War I in 1914 and the Nazis' rise to power in 1933, one-third of all math professors in Germany were Jewish - although Jews constituted less than 1 percent of the total population. These mathematicians served on the editorial boards of leading academic journals and were involved in the founding of the mathematical society.

- Of the 90 Jewish mathematicians chronicled in a recent historic study, three committed suicide after the Nazis rose to power and two were killed in the Holocaust. The rest managed to emigrate.

- The situation was particularly dire at Göttingen: Three out of four of the heads of the university's mathematics and physics institutes had been Jews. Not long after the mass expulsion, a reception was held at the university, at which Nazi education minister Bernhard Rust met the former director of the mathematics institute. Rust asked him if it had been harmed by the expulsion of the Jews. "It has not been harmed, sir," replied the former director. "It has simply ceased to exist."

(All from *Setting the record straight about Jewish mathematicians in Nazi Germany*, by Ofer Aderet, Haaretz, November 25, 2011)

Fact: van der Waerden was not a Nazi.

Question 3.1.5 Why did he stay in Germany during the Nazi era and witnessed the terrible destruction of private and professional lives of his Jewish friends and colleagues? □

van der Waerden's Work - Two Examples.

Example 3.1.6 *Burnside Group:*

The Burnside group $B(m, 3)$ has exactly 3^c elements where

$$c = m + \frac{m(m-1)}{2} + \frac{m(m-1)(m-2)}{6}.$$

(van der Waerden, 1933) □

Vocabulary:

A *group* is an ordered pair $(G, *)$ where G is a set and $* : G \times G \to G$ is a function, called a *group operation*, with the following properties (we write $a * b$ for $*(a, b)$):

1. *Associativity:* For all $a, b, c \in G$, $(a * b) * c = a * (b * c)$.

2. *Identity element:* There exists an element $e \in G$, such that for every element $a \in G$, $e * a = a * e = a$. Such an element is unique and it is called the *identity element*.

3. *Inverse element:* For each $a \in G$, there exists an element $b \in G$ such that $a * b = b * a = e$ where e is the identity element. We say that a and b are *inverse* to each other and write $b = a^{-1}$ and $a = b^{-1}$.

A *generating set of a group* is a subset such that every element of the group can be expressed as the combination (under the group operation) of finitely many elements of the subset and their inverses.

The *free Burnside group* of rank m and exponent n, denoted $B(m, n)$, is a group with m distinguished generators $x_1, \ldots, x_m$ in which the identity $x^n = 1$ holds for all elements x, and which is the "largest" group satisfying these requirements.

Example 3.1.7 *History of Mathematics:*

Yvonne Dold-Samplonius:

> His [van der Waerden's] last book, *A History of Algebra from al-Khwārizmī to Emmy Noether*, appeared in 1985 and provided a personal account of the development of algebra. Starting with al-Khwārizmī, from whose

treatise the word "algebra" is derived, van der Waerden traced the development through the ages to modern times. In his view, modern algebra began with Galois, who first investigated the structure of fields and groups and showed that these two structures are closely connected. After Galois, the efforts of the leading algebraists were mainly directed toward the investigation of the structure of rings, fields, algebras and the like. van der Waerden tracked such investigations up through roughly the middle of the 20th century in a discussion that comprises some two-thirds of the book and that includes some of his own contributions. Only van der Waerden could have given us this fascinating account. [1]

Figure 3.1.8 Abū Abdallāh Muhammad ibn Mūsā al-Khwārizmī, c. 780–c. 850, Évariste Galois, 1811–1832, and Emmy Noether, 1882–1935. (Source Images of Mathematicians on Postage Stamps).

□

On a lighter note... From J. H. van Lint:

> This note allows me to save for posterity a humorous experience of the late sixties. Van der Waerden, by then retired, had decided to attend a meeting on combinatorics, a field that he never seriously worked in. There was a talk by a young mathematician, who was desperately trying to explain his complete thesis in 20 minutes. I was sitting in the front row, next to van der Waerden, when the famous conjecture [related to the minimal permanent among all doubly stochastic matrices] was mentioned by a speaker and the alleged author inquired what the famous conjecture stated! The exasperated speaker spent a few seconds of his precious time to explain and at the end of his talk wandered over to us to read that badge of the person who had asked this inexcusable question. I knew it was going to happen and still remember happily how he recoiled. Do not worry; he had recovered and is now a famous combinatorialist. The lesson for the reader is the following. If you did not know of the 'conjecture' then it is comforting to realize that it was 40 years old before van der Waerden heard that it had this name. [11]

Van der Waerden's Theorem: For any given positive integers r and k, there is some number N such that if the set of integers $\{1, 2, \ldots, N\}$ is r-coloured then there there is a k-term monochromatic arithmetic progression.

Vocabulary:

- For a given $r \in \mathbb{N}$, an r-colouring of a set A is any function $c : A \longrightarrow \{1, 2, \ldots, r\}$.
- Let $c : A \longrightarrow \{1, 2, \ldots, r\}$ be an r-colouring of the set A and let $B \subseteq A$. We say that the set B is monochromatic if, for any $x, y \in B$

$$c(x) = c(y).$$

- We say that a set A is k-term monochromatic arithmetic progression if there are $a, d \in \mathbb{R}$, $d \neq 0$, such that

$$A = \{a + jd : j \in \{0, 1, \ldots, k-1\}\}$$

is a monochromatic set.

Resources.

1. Bartel Leendert van der Waerden - Wikipedia
2. Bartel Leendert van der Waerden - Biography
3. A short biography of B.L. van der Waerden
4. Interview with Bartel Leendert van der Waerden
5. Setting the record straight about Jewish mathematicians in Nazi Germany

3.2 van der Waerden's Theorem: 3–term APs

> *Say what you know, do what you must, come what may.* — Sofia Vasilyevna Kovalevskaya, Russian mathematician, 1850 — 1891

Three Reminders.

1. An l–term arithmetic progression is any set of the form

$$a, a + d, a + 2d, \ldots, a + (l-1)d$$

where $a, d \in \mathbb{R}$, $d \neq 0$.

2. A k-colouring of a set A is any function

$$c : A \longrightarrow \{1, 2, \ldots, k\} = [1, k].$$

3. If c is a k-colouring of the set A and if $B \subseteq A$ is such that for any $x, y \in B$

$$c(x) = c(y)$$

then we say that the set B is monochromatic.

Challenge. Colour with 2–colours avoiding monochromatic 3–term arithmetic progressions:

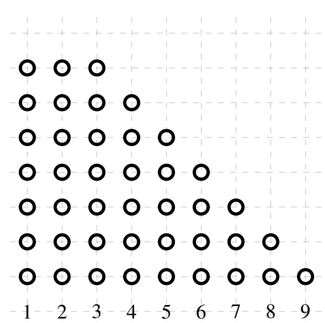


Figure 3.2.1 Colour each set with 2–colours avoiding monochromatic 3–term arithmetic progressions

Check and Extend: Check if the following 3–colouring of the the set $\{1, 2, \ldots, 17\}$ avoids monochromatic 4–term arithmetic progressions:

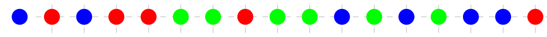


Figure 3.2.2 Can you find a monochromatic 4–term arithmetic progression?

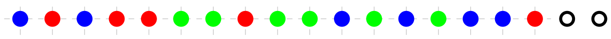


Figure 3.2.3 Can you colour numbers 18 and 19 to avoid a monochromatic 4–term arithmetic progression?

Question: Do you think that it is possible to extend the colouring above (and keep it with no monochromatic 4–term arithmetic progression) to the interval $[1, 25]$? $[1, 50]$? $[1, 100]$? Forever?

Conjecture 3.2.4 Pierre Joseph Henry Baudet, 1891 — 1921: *If the sequence of integers $1, 2, 3, \ldots$ is divided into two classes, at least one of the classes contains an arithmetic progression of l terms, no matter how large the length l is.*

Theorem 3.2.5 van der Waerden's Theorem: *If the sequence of integers $1, 2, 3, \ldots$ is divided into two classes, at least one of the classes contains an arithmetic progression of l terms, no matter how large the length l is. [10]*

"Beweis einer Baudetschen Vermutung" = "Proof of a Baudet Conjecture"
In van der Waerden's Words:

> Once in 1926, while lunching with Emil Artin and Otto Schreier, I told them about the conjecture of the Dutch mathematician Baudet: If a sequence of integers of 1, 2, 3, etc. is divided into two classes, at least one of the classes contains an arithmetic progression of l terms - $a, a + b, a + 2b, \ldots, a + (l - 1)b$ - no matter how large the length l is. After lunch we went into Artin's office . . . and tried to find a proof.
>
> . . . One of the main difficulties in the psychology of invention is that most mathematicians publish their results with condensed proofs, but do not tell us how they found them. In many cases they do not even remember their original ideas. Moreover, it is difficult to explain our vague ideas and tentative attempts in such a way that others can understand them.
>
> . . . All ideas we formed in our minds were at once put into words and explained by little drawings on the blackboard. We represented the integers 1, 2, 3, etc. in two classes by means of vertical strokes on two parallel lines. Whatever one makes explicit and draws is much easier to remember and to reproduce than mere thoughts.

(For the whole essay "How the proof of Baudet's conjecture was found" see [7].)

Theorem 3.2.6 Van der Waerden's theorem - any number of colours, length 3: *Let $k \in \mathbb{N}$. Any k-colouring of positive integers contains a monochromatic 3-term arithmetic progression. Moreover, there is a natural number N such that any k colouring of the segment of positive integers $[1, N]$ contains a monochromatic 3-term arithmetic progression.*

Note: The smallest N guaranteed by the theorem is annotated by $W(3, k)$. We have seen that $W(3, 2) = 9$.
Proof.

1. *Colour-focused arithmetic progressions and spikes:* Let c be a finite colouring of an interval of positive integers $[1, m]$ and $l, r \in \mathbb{N}$. We say that the set of l-term arithmetic progressions $A_1, A_2, \ldots, A_r$, i.e., for all $i \in [1, r]$ we have, for

some $a_i, d_i \in \mathbb{N}$,

$$A_i = \{a_i + jd_i : j \in [0, l-1]\}$$

is *colour-focused* at $f \in \mathbb{N}$ if

(a) $A_i \subseteq [1, m]$ for each $i \in [1, r]$.

(b) Each A_i is monochromatic.

(c) If $i \neq j$ the A_i and A_j are not of the same colour.

(d) $a_1 + ld_1 = a_2 + ld_2 = \cdots = a_r + ld_r = f$.

We call elements of a colour-focused set *spikes*.

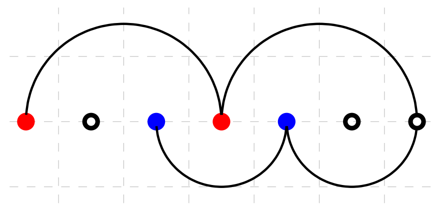


Figure 3.2.7 $\{1, 4\}$ and $\{3, 5\}$ are *colour-focused* at 7.

2. *Warm up - $k = 2$:*

(a) Consider a two colouring of $[1, 3]$.

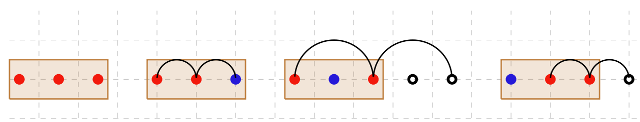


Figure 3.2.8 Any 2-colouring of the set $\{1, 2, 3\} = [1, 3]$ produces or a monochromatic 3-term arithmetic progression or *one* coloured-focused 2-term arithmetic progression.

(b) Consider the interval of positive integers $[1, (2 \cdot 3) \cdot (2^6 + 1)] = [1, 390]$. Divide this interval into 65 consecutive blocks of length 6. See Figure 3.2.9.

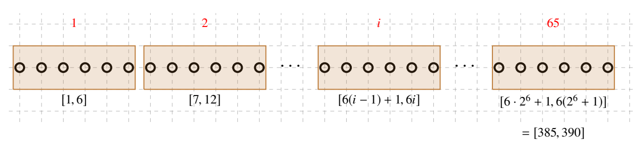


Figure 3.2.9 $(2 \cdot 3) \cdot (2^{2 \cdot 3} + 1) = 65$ consecutive blocks of length $2 \cdot 3 = 6$.

(c) In how many ways can we 2-colour six consecutive integers?

(d) Let c be a 2-colouring of the interval $[1, 2 \cdot 390]$.

(e) Every 2-colouring of $[1, 780]$ contains a monochromatic 3-term arithmetic progression! See Figure 3.2.10.

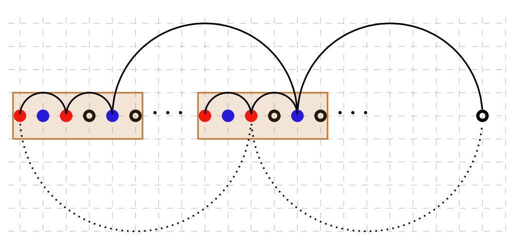


Figure 3.2.10 Every 2-colouring of $[1, 780]$ contains a monochromatic 3-term arithmetic progression!

(f) Thus $W(3, 2) \leq 780$.

3. Next we consider a k-colouring, for any $k \geq 2$.

   (a) *Strategy:* We use induction on r to prove the following statement:

   > For all $r \leq k$, there exists a natural number n such that whenever $[1, n]$ is k-coloured, *either* there exists a monochromatic 3-term arithmetic progression *or* there exist r coloured-focused arithmetic progressions of length 2.

   (b) *The base case:* Take $r = 1$ and $n = k + 1$. See Figure 3.2.11.

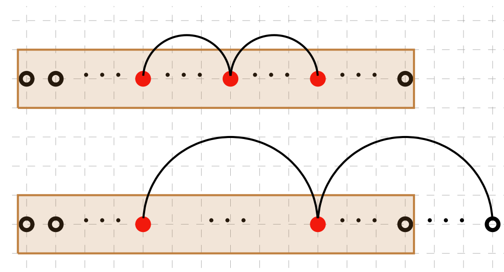


Figure 3.2.11 The base step: If the interval $[1, k + 1]$ is k-coloured then there is a monochromatic 3-term arithmetic progression or *one* coloured-focused 2-term arithmetic progression.

4. *The inductive step:* Suppose that for $r \in [2, k]$ there is an n such that any k-colouring of $[1, n]$ contains a monochromatic 3-term arithmetic progression or $r - 1$ 'spikes', i.e. $r - 1$ colour focused 2-term arithmetic progressions. See Figure 3.2.12.

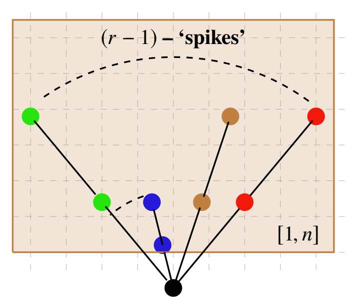


Figure 3.2.12 The inductive step: For $1 < r \leq k$ there is an n such that any k-colouring of $[1, n]$ contains a monochromatic 3-term arithmetic progression or $r - 1$ 'spikes', i.e., $r - 1$ colour focused 2-term arithmetic progressions.

   (a) How many different k-colourings of the interval $[1, 2n]$ are there?

   (b) Consider the interval of positive integers $[1, (2 \cdot n) \cdot (k^{2n} + 1)]$. Divide this interval into $k^{2n} + 1$ consecutive blocks of length $2n$. Call those blocks B_i, $1 \leq i \leq k^{2n} + 1$. See Figure 3.2.13.

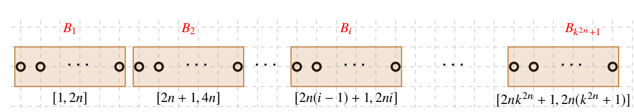


Figure 3.2.13 The interval $[1, 2n(k^{2n} + 1)]$ is divided in $k^{2n} + 1$ blocks of length $2n$ and k-coloured.

(c) Let c be a k-colouring of the interval $[1, (2 \cdot n) \cdot (k^{2n} + 1)]$. Suppose that c does not contain a monochromatic 3-term arithmetic progression.

(d) Note that by the inductive hypothesis each block B_i contains $r - 1$ spikes together with their focus. See Figure 3.2.14.

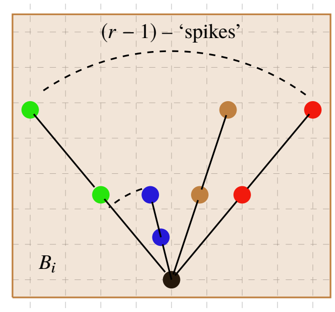


Figure 3.2.14 Each block B_i contains $r - 1$ spikes together with their focus.

(e) There must be two blocks coloured in the same way. See Figures 3.2.15 and Figure 3.2.16.

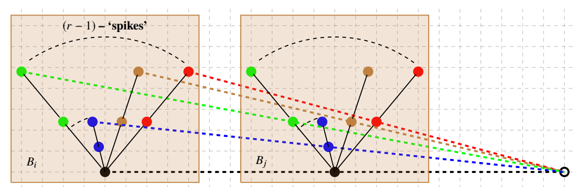


Figure 3.2.15 There must be two two blocks coloured in the same way. Each of them contains $r - 1$ spikes together with their focus. r spikes with the same focus emerge.

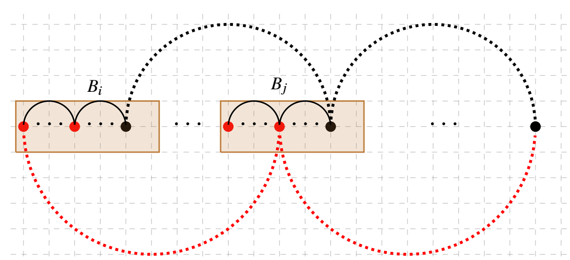


Figure 3.2.16 A closer look: Two pairs of spikes in B_i and B_j produce a new pair of spikes.

(f) This completes the induction step:

(g) What happens when $r = k$? See Figure 3.2.17.

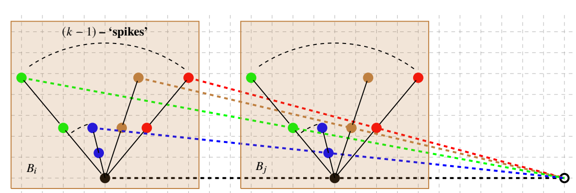


Figure 3.2.17 What happens when $r = k$? Do you see how a monochromatic 3-term arithmetic progression emerges?

■

BIG Question. How big is $W(3, k)$?
Resources.

1. van der Waerden's theorem - Wikipedia
2. Ramsey Theory by I. Leader
3. The ergodic and combinatorial approaches to Szeméredi's theorem by Terrence Tao, pp 4–6
4. Van der Waerden's theorem on arithmetic progressions by R. Swan
5. Commentary by N. G. de Bruijn

3.3 Proof of van der Waerden's Theorem

> *Mathematics is really there for you to discover.* — Ron Graham, American mathematician, 1935 — 2020

Recall Theorem 3.2.6:

Van der Waerden's Theorem - any number of colours, length 3: Let $k \in \mathbb{N}$. Any k-colouring of positive integers contains a monochromatic 3-term arithmetic progression. Moreover, there is a natural number N such that any k colouring of the segment of positive integers $[1, N]$ contains a monochromatic 3-term arithmetic progression.

1. *Note:* The smallest N guaranteed by the theorem is annotated by $W(3, k)$.
2. *Proof - the main tool: Colour-focused arithmetic progressions and spikes:* Let c be a finite colouring of an interval of positive integers $[1, m]$ and $l, r \in \mathbb{N}$. We say that the set of l-term arithmetic progressions $A_1, A_2, \ldots, A_r$, i.e., for all $i \in [1, r]$ we have, for some $a_i, d_i \in \mathbb{N}$,

$$A_i = \{a_i + jd_i : j \in [0, l-1]\}$$

is *colour-focused* at $f \in \mathbb{N}$ if

(a) $A_i \subseteq [1, m]$ for each $i \in [1, r]$.
(b) Each A_i is monochromatic.
(c) If $i \neq j$ the A_i and A_j are not of the same colour.
(d) $a_1 + ld_1 = a_2 + ld_2 = \cdots = a_r + id_r = f$.

We call elements of a colour-focused set *spikes*.

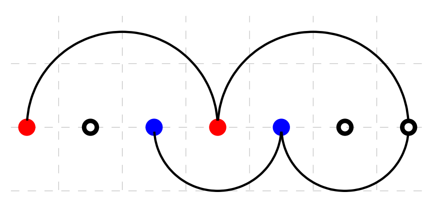


Figure 3.3.1 $\{1, 4\}$ and $\{3, 5\}$ are *colour-focused* at 7.

3. *Proof - a detail:* What happens when $r = k$?

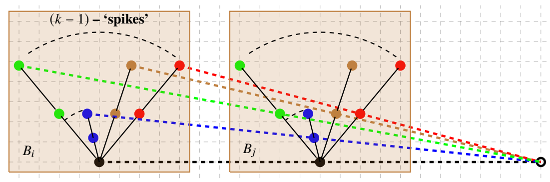


Figure 3.3.2 What happens when $r = k$? Do you see how a monochromatic 3-term arithmetic progression emerges?

Recall that Conjecture 3.2.4 was about two colours and a monochromatic arithmetic progression of any (finite) length:

Baudet's Conjecture: If the sequence of integers $1, 2, 3, \ldots$ is divided into two classes, at least one of the classes contains an arithmetic progression of l terms, no matter how large the length l is.

So what about any finite number of colours and a monochromatic arithmetic progression of any (finite) length?

Theorem 3.3.3 Van der Waerden's Theorem - any number of colours, any length: *Let $l, k \in \mathbb{N}$. Any k-colouring of positive integers contains a monochromatic l-term arithmetic progression. Moreover, there is a natural number N such that any k-colouring of the segment of positive integers $[1, N]$ contains a monochromatic l-term arithmetic progression.*

Definition 3.3.4 The smallest N guaranteed by Theorem 3.3.3 is annotated by $W(l, k)$. ◊

We have seen that $W(3, 2) = 9$ and that $W(3, k)$ exists for any $k \in \mathbb{N}$.

Proof.

1. *Strategy:* We use induction on l.

2. *The base case:* We already know that $W(l, k)$ exists if $l \leq 3$ and $k \in \mathbb{N}$, i.e., that the claim of the theorem is true for $l = 1, 2, 3$.

3. *The inductive step:* Let $l \geq 4$ be such that $W(l-1, k)$ exists for all k.

   (a) *Claim:* For all $r \leq k$, there exists a natural number M such that whenever $[1, M]$ is k-coloured, *either* there exists a monochromatic l-term arithmetic progression *or* there exist r coloured-focused $(l-1)$-term arithmetic progressions.

      i. *The base case:* Let $r = 1$ and let $M = 2W(l-1, k)$. Any k-colouring of $[1, M]$ contains a monochromatic l-term arithmetic progression or at least *one* coloured-focused $(l-1)$-term arithmetic progression focused at some $f \in [1, M]$.

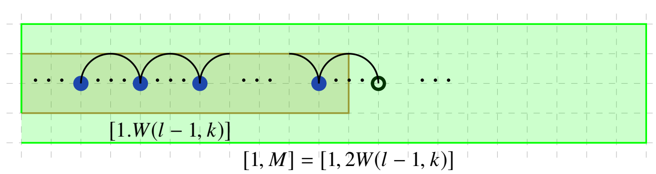


Figure 3.3.5 Any k-colouring of the set $[1, M]$ produces or a monochromatic l-term arithmetic progression or *one* coloured-focused $(l-1)$-term arithmetic progression.

      ii. *The inductive step:* Suppose that $r \in [2, k]$ is such that there is an M such that any k-colouring of $[1, M]$ contains a monochromatic l-term arithmetic progression or $r-1$ 'spikes', i.e., $r-1$ colour focused $(l-1)$-term arithmetic progressions.

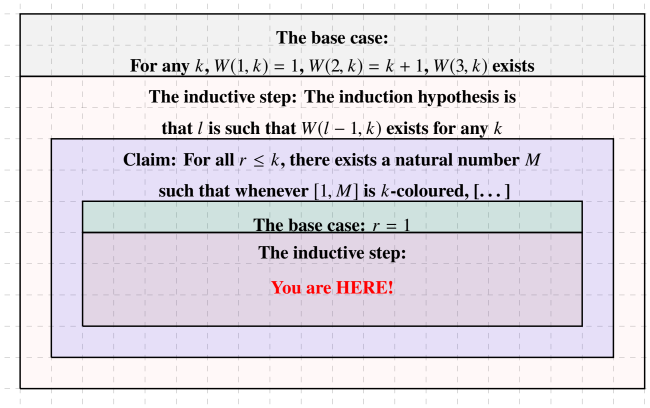


Figure 3.3.6 Where are you?

iii. Observe that any k-colouring of $[1,2M]$ contains a monochromatic l-term arithmetic progression or at least $r-1$ coloured-focused $(l-1)$-term arithmetic progression focused at some $f \in [1,2M]$. See Figure 3.3.7.

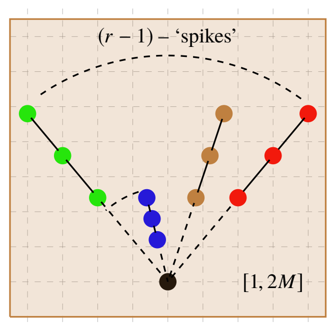


Figure 3.3.7 There are $r-1$ spikes.

iv. Consider the interval of positive integers $[1, 2M \cdot W(l-1, k^{2M})]$. (How do we know that $W(l-1,k^{2M})$ exists?) Divide this interval into $W(l-1,k^{2M})$ consecutive blocks B_i, $1 \le i \le W(l-1,k^{2M})$, of length $2M$. See Figure 3.3.8.

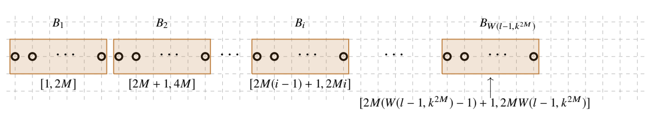


Figure 3.3.8 The interval $[1, 2M \cdot W(l-1,k^{2M})]$ is divided into $W(l-1,k^{2M})$ consecutive blocks B_i, $1 \le i \le W(l-1,k^{2M})$, of length $2M$.

v. Why $W(l-1,k^{2M})$?

vi. Suppose that c is a k-colouring of $[1, 2M \cdot W(l-1,k^{2M})]$ that does not contain a monochromatic l-term arithmetic progression. Each block B_i is k-coloured in one of the possible k^{2M} ways. See Figure 3.3.9.

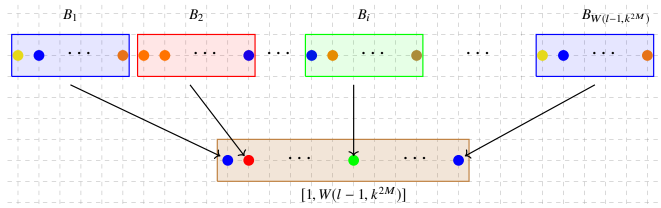


Figure 3.3.9 The k-colouring c of $[1, 2M \cdot W(l-1, k^{2M})]$ induces a k^{2M}-colouring of $[1, W(l-1, k^{2M})]$.

vii. Any k^{2M}-colouring of $[1, W(l-1, k^{2M})]$ contains a monochromatic $(l-1)$-term arithmetic progression. See Figure 3.3.10.

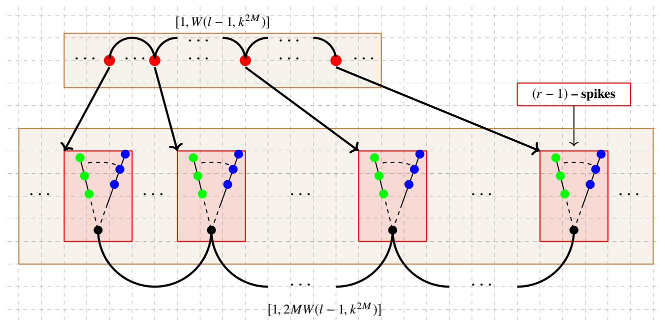


Figure 3.3.10 The k^{2M}-colouring of $[1, W(l-1, k^{2M})]$ induced by the colouring c contains a monochromatic $(l-1)$-term arithmetic progression. This means that there are $l-1$ blocks B_{i_j}, $1 \leq j \leq l-1$, that are coloured by c in the same way and they are equally spaced between each other.

viii. Every B_{i_j}, $1 \leq j \leq l-1$:
   - is k-coloured the same way
   - contains $r-1$ spikes (monochromatic $(l-1)$-term arithmetic progressions) together with their focus. Note that there are no two spikes of the same colour (by definition!) and that the focus is of a different colour. (Why?)

ix. The key step! The r^{th} spike appears! See Figures 3.3.11 and Figure 3.3.12 .

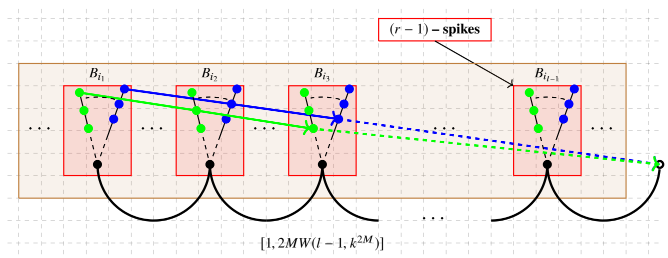


Figure 3.3.11 The k^{2M}-colouring of $[1, W(l-1, k^{2M})]$ induced by the colouring c contains a monochromatic $(l-1)$-term arithmetic progression. This means that there are $l-1$ blocks B_{i_j}, $1 \leq j \leq l-1$, that are coloured by c in the same way and they are equally spaced between each other.

Take a closer look:

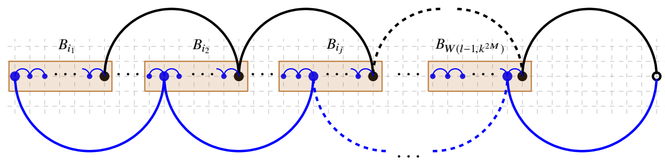


Figure 3.3.12 Do you see how $r-1$ initial spikes generate r new spikes?

(b) Where are we?

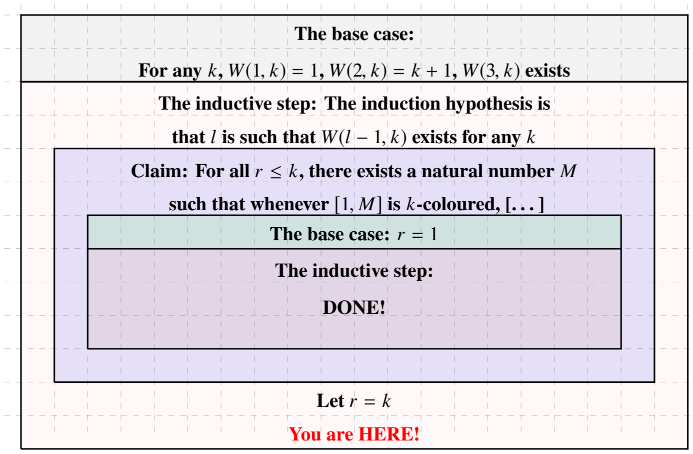


Figure 3.3.13 Almost there!

(c) Let $r=k$:

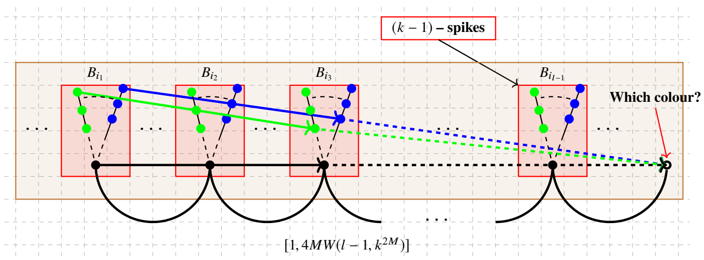


Figure 3.3.14 Done!

■

Resources.

1. van der Waerden's theorem - Wikipedia
2. Ramsey Theory by I. Leader pp 4–6
3. The ergodic and combinatorial approaches to Szeméredi's theorem by Terrence Tao, pp 4–6
4. Proof of van der Waerden's Theorem in Nine Figures by A. Blondal and V. Jungic
5. Van der Waerden's theorem on arithmetic progressions by R. Swan
6. Commentary by N. G. de Bruijn

3.4 van der Waerden's Theorem: How Far and Where?

Do not, however, confuse elementary with simple. — Aleksandr Yakovlevich Khinchin, Soviet mathematician, 1894 —1959

Figure 3.4.1 Aleksandr Yakovlevich Khinchin

Recall Theorem 3.3.3:

Van der Waerden's Theorem: Let $l, k \in \mathbb{N}$. Any k-colouring of positive integers contains a monochromatic l-term arithmetic progression. Moreover, there is a natural number N such that any k-colouring of the segment of positive integers $[1, N]$ contains a monochromatic l-term arithmetic progression.

Reminder: The smallest N guaranteed by the theorem is annotated by $W(l, k)$. We have seen that $W(3, 2) = 9$ and that $W(3, k)$ exists for any $k \in \mathbb{N}$.

Two Questions.

1. How big is $W(l, k)$?
2. If $\mathbb{N}$ is k-coloured can we be sure that a certain colour contains an l-term arithmetic progression?

van der Waerden Numbers. In 1951, Paul Erdős and Richard Rado introduced the van der Waerden's function:

$$W : (l, k) \to W(l, k).$$

The values of van der Waerden's function are called *van der Waerden numbers*.

Best known lower bounds to van der Waerden numbers.

Table 3.4.2 Best known lower bounds to van der Waerden numbers.

	l = length of AP						
k = # of colours	3	4	5	6	7	8	9
2	9	35	178	1132	> 3703	> 7584	> 27113
3	27	> 292	> 1209	> 8886	> 43855	> 238400	
4	76	> 1948	> 10437	> 90306	> 387967		
5	> 125	> 2254	> 24045	> 246956			
6	> 207	> 9778	> 56693	> 600486			

Big Question:

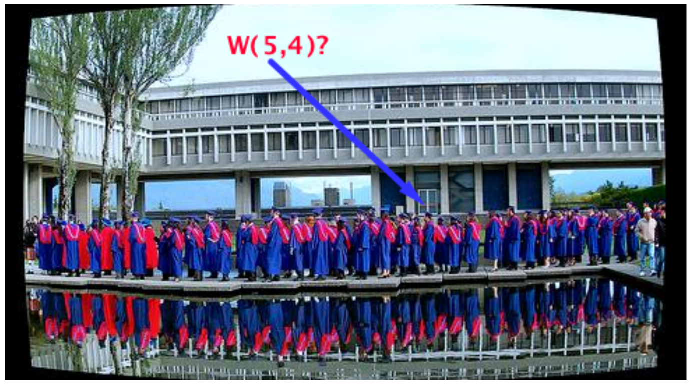


Figure 3.4.3 $W(l, k)$: Can you find me?

Two Lower Bounds: It is a convention to write $W(l)$ instead of $W(l, 2)$. Hence,

- $W(3) = 9$ and $W(4) = 35$ (Chvatal, 1970)
- $W(5) = 178$ (Stevens and Shantaram, 1978)
- $W(6) = 1132$ (Kouril and Paul, 2008)

If l is a prime then $W(l + 1) > l \cdot 2^l$. (Berlekamp, 1969)

Figure 3.4.4 Elwyn Ralph Berlekamp (1940 — 2019)

For any $\varepsilon > 0$,

$$W(l) \geq \frac{2^l}{l^\varepsilon}$$

for large enough l. (Szabó, 1990)

Figure 3.4.5 Zoltán Szabó (1965–)

Upper Bounds.

Prelude:

- $f_1(x) = \text{DOUBLE}(x) = 2x$
- $f_2(x) = \text{EXPONENT}(x) = 2^x$ Note that

  $f_1^{(2)}(1) = f_1(f_1(1))) = f_1(2\cdot 1) = 2\cdot 2 = 2^2 = f_2(2)$, $f_1^{(3)}(1) = f_1(2^2) = 2\cdot 2^2 = f_2(3)$

  and in general

$$f_2(x) = f_1^{(x)}(1).$$

- $f_3(x) = \text{ TOWER }(x) = \left.2^{2^{2^{\cdot^{\cdot^{2}}}}}\right\}x = f_2^{(x)}(1)$
- $f_4(x) = \text{ WOW }(x) = f_3^{(x)}(1)$
- $f_{i+1}(x) = f_i^{(x)}(1)$
- $f_\omega(x) = \text{ ACKERMANN }(x) = f_x(x)$

		1	2	3	4	5	6
DOUBLE	f_1	2	4	6	8	10	12
EXPONENT	f_2	2	4	8	16	32	64
TOWER	f_3	2	4	16	65536	2^{65536}	$\vdots$
WOW	f_4	2	4	65536	WOW!	$\vdots$	$\vdots$
	f_5	2	4	WOW!	$\vdots$	$\vdots$	$\vdots$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ACKERMANN	f_ω	2	4	16	WOW!	$\vdots$	$\vdots$

Figure 3.4.6 Wilhelm Friedrich Ackermann (1896 -1962)

van der Waerden's proof implies, for $k \geq 10$

$$W(l) \leq \text{ ACKERMANN }(l).$$

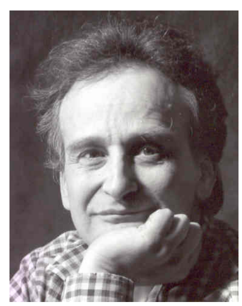


Figure 3.4.7 Saharon Shelah (1945–)

$W(l) < \text{ WOW }(l+2)$. (Shelah, 1988)

Figure 3.4.8 Timothy Gowers (1963–)

$W(l) \leq 2^{2^{2^{2^{2^{l+9}}}}}$. (Gowers, 1998)

Ron Graham offered \$ 1000 for a proof or disproof of the bound that $W(l) \leq 2^{l^2}$.

Figure 3.4.9 Ron Graham (1935–2020)

Celebrating Erdős:

Figure 3.4.10 Budapest 1999: Ron Graham and Timothy Gowers (Photo by Tom Brown)

Closer to Home: Given any positive integer r and positive integers $k_1, k_2, \ldots, k_r$, there is an integer m such that given any partition $\{1, 2, \ldots, m\} = P_1 \cup P_2 \cup \ldots \cup P_r$, there is always a class P_j containing an arithmetic progression of length k_j. Let us denote the least m with this property by $w(r; k_1, k_2, \ldots, k_r)$.

Tom Brown, an SFU professor, in 1974 found the following:

$w(3; 2, 3, 3) = 14$	$w(3; 2, 4, 4) = 40$	$w(4; 2, 2, 3, 3) = 17$	$w(4; 2, 3, 3, 3) = 40$
$w(3; 2, 3, 4) = 21$	$w(3; 2, 4, 5) = 71$	$w(4; 2, 2, 3, 4) = 25$	
$w(3; 2, 3, 5) = 32$		$w(4; 2, 2, 3, 5) = 43$	
$w(3; 2, 3, 6) = 40$		$w(4; 2, 2, 4, 4) = 53$	

Where to look for monochromatic arithmetic progressions?

Prelude: Let A be a subset of the set of natural numbers $\mathbb{N}$. For any $n \in \mathbb{N}$ let

$$A(n) = \{1, 2, \ldots, n\} \cap A \text{ and } a(n) = |A(n)|.$$

We define the upper density $\overline{d}(A)$ of the set A by

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{a(n)}{n}.$$

Similarly, $\underline{d}(A)$, the lower density of A, is defined by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{a(n)}{n}.$$

We say that A has density $d(A)$ if

$$\underline{d}(A) = \overline{d}(A).$$

Thus

$$d(A) = \lim_{n\to\infty} \frac{a(n)}{n}.$$

Two examples:

Example 3.4.11 What is the density of the set of all natural numbers divisible by 3? □

Example 3.4.12 What is the density of the set of all powers of 2? □

Note: For more examples see: Natural density - Wikipedia

Paul Erdős and Paul Turán conjectured in 1936 that any set of integers with positive density contains a 3-term arithmetic progression.

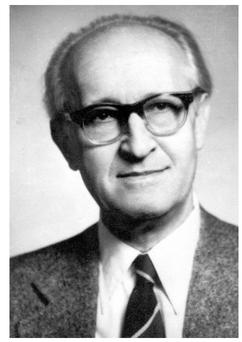


Figure 3.4.13 Paul Turán (1910-1976)

Any set of integers with positive density contains a 3-term arithmetic progression. (Roth, 1953)

Figure 3.4.14 Klaus Friedrich Roth (1925– 2015)

Any set of integers with positive density contains an arithmetic progression of any length. (Szemerédi, 1975)

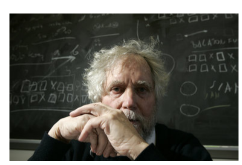


Figure 3.4.15 Endre Szemerédi (1940 —)

Any set of integers with positive density contains an arithmetic progression of any length. (Furstenberg, 1977)

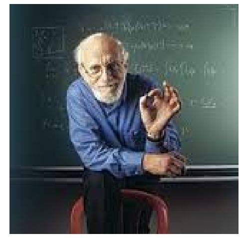


Figure 3.4.16 Hillel (Harry) Furstenberg (1935 —)

For any $k \in \mathbb{N}$, there is a k-term progression consisting of primes. (Green-Tao Theorem, 2004)

Figure 3.4.17 Ben Green (1977–)

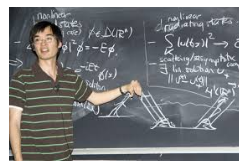


Figure 3.4.18 Terence Chi-Shen Tao (1975–)

Resources.

1. For more details see [2], [3], and [7].
2. van der Waerden's number - Wikipedia
3. van der Waerden number - Wolfram Math World
4. Van der Waerden's theorem on arithmetic progressions by R. Swan
5. Van der Wearden's Theorem: Variants and "Applications" by W. Gasarch, C. Kruskal, and A. Parrish, pp 40–45
6. On the history of van der Waerden's theorem on arithmetic progressions by Tom Brown and Peter Jau-Shyong Shiue
7. Mathematicians Catch a Pattern by Figuring Out How to Avoid Its by Kevin Hartnett

3.5 van der Waerden's Theorem: A Few Related Questions

Never measure the height of a mountain until you have reached the top. Then you will see how low it was. — Dag Hjalmar Agne Carl Hammarskjöld, Swedish diplomat, economist, and author, 1905 –1961

Recall Theorem 3.3.3:

Van der Waerden's Theorem: Let $l, k \in \mathbb{N}$. Any k-colouring of positive integers contains a monochromatic l-term arithmetic progression. Moreover, there is a natural number N such that any k-colouring of the segment of positive integers $[1, N]$ contains a monochromatic l-term arithmetic progression.

Question 3.5.1 Is it true that any 2-colouring of positive integers contains an infinite monochromatic arithmetic progression? □

Question 3.5.2 Is it true that any infinite colouring of positive integers contains a monochromatic l-term arithmetic progression, for $l \in \mathbb{N}$? □

Theorem 3.5.3 Canonical form of van der Waerden's theorem: *If f is an arbitrary function from the positive integers to the positive integers, then there are arbitrarily large arithmetic progressions P such that the restriction of f to P is either constant or one-to-one. (See Figures 3.5.4 and Figure 3.5.5.)*

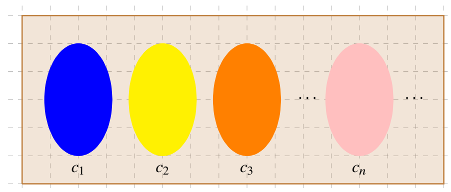


Figure 3.5.4 Divide positive integers in as many parts as you wish. Possibly infinite...

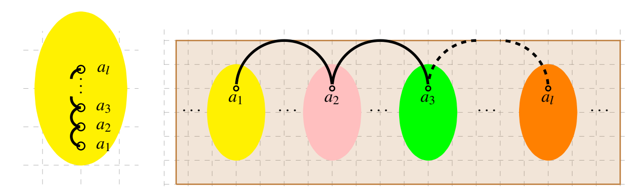


Figure 3.5.5 Canonical form: monochromatic or rainbow

Question 3.5.6 Is it true that any finite colouring of positive integers contains a monochromatic l-term arithmetic progression with an **odd common difference**? □

Question 3.5.7 Is it true that any finite colouring of positive integers contains a monochromatic l-term arithmetic progression with an **even common difference**? □

Theorem 3.5.8
Polynomial van der Waerden Theorem: *Let $l, r \in \mathbb{N}$ and let p be a polynomial with integer coefficients such that $p(0) = 0$, . Then for any r-colouring of $\mathbb{Z}$ there are $a, d \in \mathbb{Z}$ such that the l-term arithmetic progression*

$$a, a + p(d), a + 2p(d), \ldots, a + (l - 1)p(d)$$

is monochromatic.

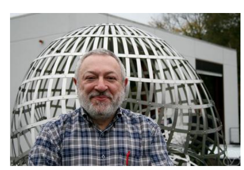


***Figure 3.5.9** Vitaly Bergelson (1950–)*

Note: *This is a very special case of the Polynomial van der Waerden Theorem proved by Bergelson and Leibman (Polynomial extensions of van der Waerden's and Szemeredi's theorems, Journal of the American Math Society, Vol. 9, 1996, 725-753.)*

Figure 3.5.10 *Alexander Leibman*

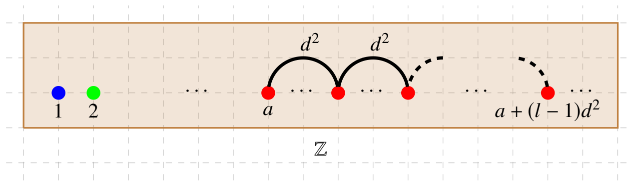


Figure 3.5.11 A monochromatic l-term arithmetic progression with step d^2.

2-*Large and Large Sets:*

We say that a set $L \subseteq \mathbb{N}$ is 2-large if any 2-colouring of $\mathbb{N}$ contains long monochromatic arithmetic progressions with common difference in L.

We say that a set $L \subseteq \mathbb{N}$ is large if any finite colouring of $\mathbb{N}$ contains long monochromatic arithmetic progressions with common difference in L.

Example 3.5.12

1. Is the set of all natural numbers large?
2. Is the set of all odd numbers large?
3. Is the set of all numbers divisible by 3 large?
4. Is the set of all perfect squares large?
5. Is the set $\{2, 6, 12, , 20, 30, 42, \ldots\}$ large?

□

Conjecture 3.5.13

Every 2-large set is large. *(T.C. Brown, R.L. Graham, and B.M. Landman, On the set of common differences in van der Waerden's theorem on arithmetic progressions, Canad. Math. Bull. 42 (1999), 25-36.)*

Figure 3.5.14 *Bruce Landman*

Example 3.5.15 Colour the 12 points below with three colours so that you use each colour four times. Can you avoid 3-term rainbow arithmetic progressions? □

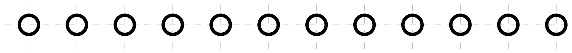


Figure 3.5.16 Colour with three colours; use each colour four times; look for a rainbow 3-term arithmetic progression.

Example 3.5.17 *What About...* Colour the 15 points below with three colours so that you use each colour five times. Can you avoid 3-term rainbow arithmetic progressions? □

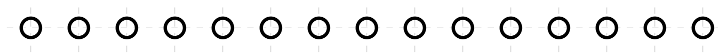


Figure 3.5.18 Can you avoid rainbow 3-term arithmetic progressions?

Theorem 3.5.19

Every equinumerous 3-colouring of $[1, 3n]$ contains a rainbow 3-term arithmetic progression. *(Jungić, V., Radoičić, R., Rainbow Arithmetic Progressions, Integers, Electron. J. Combin. Number Theory 3 (2003) A18)*

Figure 3.5.20 *Radoš Radoičić*

Resources.

1. For more details see [2], [3], and [7].
2. Large Sets - Wikipedia
3. On the history of van der Waerden's theorem on arithmetic progressions by Tom Brown and Peter Jau-Shyong Shiue
4. Rainbow Ramsey Theory by V. Jungic, J Nesetril, and R. Radoicic

3.6 Exercises

The following exercises are based on the material covered in Chapter 3.

Exercise 3.6.1 Arithmetic Progressions. In this question we interested in counting arithmetic progressions of the given length k in the given interval $[1, n]$.

1. Let $m \in \mathbb{N}$ be given and let $n = 3m + 1$.

   Let d be such that for some $a \in [1, n]$, the 4-term arithmetic progression $a, a + d, a + 2d, a + 3d$ is contained in $[1, n]$.

   (a) Show that the number of 4-term arithmetic progressions with the step d contained in $[1, n]$ is equal to $s(d) = n - 3d$.

   (b) Show that the maximum value of d equals to m.

   (c) Denote by $A_n(4)$ the number of 4-term arithmetic progressions contained in the interval $[1, n]$. Show that $A_n(4) \leq \frac{n^2}{2 \cdot 3}$.

2. Let $k \in \mathbb{N}$ and let d be such that for some $a \in [1, n]$, the k-term arithmetic progression $a, a+d, a+2d, \ldots, a+(k-1)d$ is contained in $[1, n]$.

   (a) Show that the number of k-term arithmetic progressions with the step d contained in $[1, n]$ is equal to $s(d) = n - (k-1)d$.

   (b) Show that the maximum value of d equals to $\left\lfloor \dfrac{n-1}{m-1} \right\rfloor$.

   (c) Denote by $A_n(k)$ the number of k-term arithmetic progressions contained in the interval $[1, n]$. Show that $A_n(k) \leq \frac{n^2}{2\cdot(k-1)}$.

Solution.

1. (a) Observe that if $a, a+d, a+2d, a+3d$ is contained in $[1, n]$, then for any $b \in [1, a)$ the arithmetic progression $b, b+d, b+2d, b+3d$ is also contained in $[1, n]$. Hence the question is to find the largest $a \in [1, n]$ with the the property that $a, a+d, a+2d, a+3d$ is contained in $[1, n]$. Clearly, if a is the largest than $a+3d = n$ what is the same as $a = n - 3d$.
   It follows that $s(d) = n - 3d$.

   (b) Say that $d = m = \dfrac{n-1}{3}$. From

$$1 + 3\cdot d = 1 + 3 \cdot \frac{n-1}{3} = 1 + (n-1) = n$$

   we conclude that the arithmetic progression $1, 1+d, 1+2d, 1+3d$ is contained in $[1, n]$.
   What about the arithmetic progression $1, 1+(d+1), 1+2(d+1), 1+3(d+1)$? From

$$1 + 3\cdot(d+1) = 1 + 3\cdot\left(\frac{n-1}{3} + 1\right) = 1 + (n-1) + 3 = n + 3 > n$$

   we conclude that this arithmetic progression is not contained in $[1, n]$.
   Hence the maximum value of d equals to $m = \dfrac{n-1}{3}$.

   (c) It follows from (a) and (b) that

$$\begin{aligned}
A_n(4) &= \sum_{d=1}^{\frac{n-1}{3}} s(d) = \sum_{d=1}^{\frac{n-1}{3}} (n-3d) = \sum_{d=1}^{\frac{n-1}{3}} n - 3\cdot \sum_{d=1}^{\frac{n-1}{3}} d \\
&= n\cdot\frac{n-1}{3} - 3\cdot\frac{1}{2}\cdot\frac{n-1}{3}\cdot\left(\frac{n-1}{3}+1\right) \\
&= \frac{1}{2}\cdot\frac{n-1}{3}\cdot\left(2n - 3\cdot\left(\frac{n-1}{3}+1\right)\right) \\
&= \frac{1}{2}\cdot\frac{n-1}{3}\cdot(n-2) < \frac{n^2}{2\cdot 3}.
\end{aligned}$$

2. (a) Observe that if $a, a+d, \ldots, a+(k-1)d$ is contained in $[1, n]$, then for any $b \in [1, a)$ the arithmetic progression $b, b+d, \ldots, b+(k-1)d$ is also contained in $[1, n]$. Hence the question is to find the largest $a \in [1, n]$ with the property that $a, a+d, \ldots, a+(k-1)d$ is contained in $[1, n]$. Clearly, if a is the largest then $a+(k-1)d = n$ what is the same as $a = n-(k-1)d$.

(b) Say that $d = \left\lfloor \dfrac{n-1}{k-1} \right\rfloor$.

From

$1 + (k-1) \cdot d = 1 + (k-1) \cdot \left\lfloor \frac{n-1}{k-1} \right\rfloor \leq 1 + (k-1) \cdot \frac{n-1}{k-1} = 1 + (n-1) = n$

we conclude that the arithmetic progression $1, 1+d, \ldots, 1+(k-1)d$ is contained in $[1, n]$.

What about the arithmetic progression $1, 1+(d+1), \ldots, 1+(k-1)(d+1)$?

From

$1+(k-1)\cdot(d+1) = 1+(k-1)\cdot\left(\left\lfloor \frac{n-1}{k-1} \right\rfloor + 1\right) > 1+(k-1)\cdot\frac{n-1}{k-1} = 1+(n-1) = n$

we conclude that this arithmetic progression is not contained in $[1, n]$.

Hence the maximum value of d equals to $\left\lfloor \dfrac{n-1}{k-1} \right\rfloor$.

3. It follows from (a) and (b) that

$$
\begin{aligned}
A_n(k) = \sum_{d=1}^{\lfloor \frac{n-1}{k-1} \rfloor} s(d) &= \sum_{d=1}^{\lfloor \frac{n-1}{k-1} \rfloor} (n-(k-1)d) = \sum_{d=1}^{\lfloor \frac{n-1}{k-1} \rfloor} n - (k-1) \cdot \sum_{d=1}^{\lfloor \frac{n-1}{k-1} \rfloor} d \\
&= n \cdot \left\lfloor \frac{n-1}{k-1} \right\rfloor - (k-1) \cdot \frac{1}{2} \cdot \left\lfloor \frac{n-1}{k-1} \right\rfloor \cdot \left(\left\lfloor \frac{n-1}{k-1} \right\rfloor + 1 \right) \\
&= \frac{1}{2} \cdot \left\lfloor \frac{n-1}{k-1} \right\rfloor \cdot \left(2n - (k-1) \cdot \left(\left\lfloor \frac{n-1}{k-1} \right\rfloor + 1 \right) \right) \\
&\leq \frac{1}{2} \cdot \frac{n-1}{k-1} \cdot \left(2n - (k-1) \cdot \frac{n-1}{k-1} \right) = \frac{1}{2} \cdot \frac{n-1}{k-1} \cdot (n+1) \\
&= \frac{n^2-1}{2 \cdot (k-1)} < \frac{n^2}{2 \cdot (k-1)}.
\end{aligned}
$$

Exercise 3.6.2 Arithmetic progressions.

1. Check if the following 4-colouring of the set $\{1, 2, \ldots, 13\}$ avoids monochromatic 3-term arithmetic progressions:

$$R\ B\ B\ G\ Y\ R\ Y\ G\ B\ Y\ Y\ B\ G$$

2. Add 3 colours to the end to produce a monochromatic 3-term arithmetic progression.

Solution.

1.
   - Since there are only two red (R) elements, 1 and 6, there is no a red 3-term arithmetic progression.
   - There are four blue (B) elements, $2, 3, 9$ and 12. Clearly this set does not contain a 3-term arithmetic progression.
   - There are three green (G) elements, $4, 8$ and 13. These three elements do not form a 3-term arithmetic progression.
   - There are four yellow (Y) elements, $5, 7, 10$ and 11. This set does not contain a 3-term arithmetic progression.

2. For example,

$$R\ \underline{B}\ B\ G\ Y\ R\ Y\ G\ \underline{B}\ Y\ Y\ B\ G\ Y\ B\ \underline{B}.$$

Exercise 3.6.3 van der Waerden's theorem. Let $\chi : \mathbb{N} \to \{0,1\}$ be a 2-colouring of positive integers and let $k \in \mathbb{N}$. Use van der Waerden's theorem to prove that there is a χ-monochromatic k-term arithmetic progression with a common difference that is divisible by 3.

Solution. Let $\chi : \mathbb{N} \to \{0,1\}$ be a 2-colouring of positive integers and let $k \in \mathbb{N}$. We define a new colouring $\chi' : \mathbb{N} \to \{0,1\}$ by

$$\chi'(i) = \chi(3i),\ i \in \mathbb{N}.$$

By van der Waerden's theorem there are $a, d \in \mathbb{N}$ such that

$$\chi'(a) = \chi'(a+d) = \chi'(a+2d) = \ldots = \chi'(a+(k-1)d).$$

By definition of the 2-colouring χ' this implies that

$$\chi(3a) = \chi(3a+3d) = \chi(3a+2\cdot 3d) = \ldots = \chi(3a+(k-1)\cdot 3d).$$

Therefore the k-term monochromatic progression

$$3a, 3a+3d, 3a+2\cdot 3d, \ldots, 3a+(k-1)\cdot 2d$$

is χ-monochromatic. Note that the common difference of this arithmetic progression is $3d$, a number divisible by 3.

Exercise 3.6.4 Arithmetic progressions. Prove that for any $k, l \in \mathbb{N}$ there is $S(k,l) \in \mathbb{N}$ such that any k-colouring of the set of positive integers $[1, S(k,l)]$ contains a monochromatic arithmetic progression of length l together with its difference.

Solution. We prove the claim by mathematical induction on k, the number of colours.
For $k = 1$ and any $l \in \mathbb{N}$ we take $S(1,l) = l$.
Assume that the claim is true for a fixed $k \geq 1$ and any $l \in \mathbb{N}$. We fix $l \in \mathbb{N}$ and denote by $S(k,l)$ the corresponding number.
Next we define

$$S(k+1,l) = w(k+1,(l-1)S(k,l)+1),$$

where $w(*,*)$ denotes a van der Waerden number.

Let the set of integers $[1, S(k+1,l)]$ be $(k+1)$-coloured. Then, by van der Waerden's Theorem, there is a $((l-1)S(k,l)+1)$-term monochromatic arithmetic progression

$$a, a+d, \ldots, a+(l-1)S(k,1)d.$$

For every $x = 1, 2, \ldots, S(k,l)d$ this monochromatic arithmetic progression contains the following l-term arithmetic progression:

$$a, a+xd, \ldots, a+(l-1)xd.$$

If for one of the numbers x, the difference xd is coloured by the same colour as the original progression, we have concluded the proof of our induction step.

Otherwise, the $S(k,l)$-term arithmetic progression

$$d, 2d, \ldots S(k,l)d$$

is coloured in only k colours. Now we apply the inductive hypothesis to conclude that the k-coloured set $\{d, 2d, \ldots S(k,l)d\} \subset [1, S(k+1,l)]$ contains a monochromatic arithmetic progression of length l together with its difference.

Exercise 3.6.5 Syndetic sets. This question is about so–called syndetic sets.

A syndetic set is any set that can be represented as an increasing sequence of positive integers with bounded gaps, i.e. as a sequence $a_1, a_2, a_3, \ldots$ such that $a_1 < a_2 < \cdots$ and such that there is $M > 0$ for which $a_{i+1} - a_i \le M$, for all $i \in \mathbb{N}$.

1. Which of the following two sets is syndetic:
   (a) The set of all (positive) powers of 2.
   (b) The set of all positive integers that are divisible by 3 or by 5.
2. Prove that if $\mathbb{N} = A \cup B$, then either A contains arbitrarily long intervals or B is syndetic.
3. Which of the following two sets contains long arithmetic progressions:
   (a) The set of all (positive) powers of 2.
   (b) The set of all positive integers that are divisible by 3 or by 5.
4. Prove that every syndetic set contains long arithmetic progressions
5. If the set of positive integers is partitioned into two classes, then at least one of the following holds:
   (a) One class contains arbitrarily long strings of consecutive integers.
   (b) Both classes contain arithmetic progressions of arbitrary length.

Solution.

1. (a) The set of all positive powers of 2 may be represented as a sequence $2, 2^2, \ldots, 2^i, 2^{i+1}, \ldots$.
   Let $M > 0$ and let $i \in \mathbb{N}$ be such that $i > \log_2 M$.
   Then
   $$2^{i+1} - 2^i = 2^i \cdot (2-1) = 2^i > 2^{\log_2 M} = M.$$
   Hence there are two consecutive powers of 2 with the gap between them greater than M. Since M was arbitrarily, it follows that the set of the powers of 2 is not syndetic.

   (b) The first several elements of the sequence representing this set are $3, 5, 6, 9, 10, 12, 15, \ldots$
   We observe that any interval of the form $(5k, 5(k+1))$ contains at least one number divisible by 3. Otherwise, there would an $l \in \mathbb{N}$ such that $3 = 3(l+1) - 3l \ge 4$.
   Consider two consecutive elements of the sequence above, a_i and a_{i+1}. If both of them are divisible by 3 then $a_{i+1} - a_i = 3$. If, say, a_i is not divisible by 3 then, by the observation above, a_{i+1} must be divisible by 3. Since $a_{i+1} - 3 < a_i < a_{i+1}$ we conclude that $1 \le a_{i+1} - a_i \le 2$.
   Therefore, for any $i \in \mathbb{N}$,
   $$1 \le a_{i+1} - a_i \le 3$$
   and this set is syndetic.

2. Suppose that $B = \{b_1 < b_2 < \cdots\}$ is not syndetic. Then for any $M > 0$ there are two consecutive elements of B, say b_i and b_{i+1}, such that $b_{i+1} - b_i > M$.

   It follows that the set A contains the interval $[b_i + 1, b_{i+1} - 1]$, an interval of the length $b_{i+1} - 1 - (b_i + 1) + 1 = b_{i+1} - b_i - 1 \ge M$.

   Therefore if B is not syndetic then A contains arbitrarily long intervals.

3. (a) Suppose that $2^i, 2^j$ and 2^k, with $i < j < k$, form a 3-term arithmetic progression.

   This would imply that

$$2^j = \frac{2^i + 2^k}{2} \Leftrightarrow 2^{j+1} = 2^i + 2^k \Leftrightarrow 2^{j-i+1} = 1 + 2^{k-i},$$

   i.e. that a power of 2 is an odd number.

   Therefore, the set of the powers of 2 does not contain a 3-term arithmetic progression.

   (b) Let $A = \{a_1, a_2, a_3, \dots\}$ be the set of all positive integers that are divisible by 3 or by 5. Recall that, for all $i \in \mathbb{N}$, $a_{i+1} - a_i \leq 3$.

   We partition (colour) the set of all positive integers in the following way:

$$C_0 = A = \{a_1, a_2, a_3, \dots\}$$

$$C_1 = \{a_1 + 1, a_2 + 1, a_3 + 1, \dots\} \setminus C_0$$
$$C_2 = \{a_1 + 2, a_2 + 2, a_3 + 2, \dots\} \setminus (C_0 \cup C_1)$$

   Observe that if a is a positive integer that does not belong to the set A then its remainder after dividing by 3 must 1 or 2. Hence, a is in C_1 or C_2, which proves that this partition is a three colouring of $\mathbb{N}$. By van der Waerden's theorem, for any $k \in \mathbb{N}$, there is monochromatic k–term arithmetic progression, $a, a + d, \dots, a + (k - 1)d$.

   Three cases:

   i. $\{a, a + d, \dots, a + (k - 1)d\} \subset C_0 = A$

   ii. If $\{a, a+d, \dots, a+(k-1)d\} \subset C_1$ then $(a-1), (a-1)+d, \dots, (a-1)+(k-1)d$ is an arithmetic progression and $\{(a-1), (a-1)+d, \dots, (a-1) + (k-1)d\} \subset A$.

   iii. If $\{a, a+d, \dots, a+(k-1)d\} \subset C_2$ then $(a-2), (a-2)+d, \dots, (a-2)+(k-1)d$ is an arithmetic progression and $\{(a-2), (a-2)+d, \dots, (a-2) + (k-1)d\} \subset A$.

   Therefore for any $k \in \mathbb{N}$, the set A contains a k–term monochromatic progression.

   Solution 2: Notice that the set A contains the k–term arithmetic progression $3, 6, 9, \dots, 3k$.

4. **Solution 1:**

   Let $A = \{a_1 < a_2 < a_3 < \cdots\}$ be a syndetic set and let $M \in \mathbb{N}$ be such that, for all $i \in \mathbb{N}$, $a_{i+1} - a_i \leq M$.

   We partition (colour) the set of all positive integers in the following way:

$$C_0 = A = \{a_1, a_2, a_3, \dots\}$$

$$C_1 = \{a_1 + 1, a_2 + 1, a_3 + 1, \dots\} \setminus C_0$$
$$\dots$$
$$C_i = \{a_1 + i, a_2 + i, a_3 + i, \dots\} \setminus (C_0 \cup C_1 \cup \dots \cup C_{i-1})$$
$$\dots$$
$$C_{M-1} = \{a_1 + M - 1, a_2 + M - 1, a_3 + M - 1, \dots\} \setminus (C_0 \cup C_1 \cup \dots \cup C_{M-2})$$

   Observe that if a is a positive integer that does not belong to the set A then its remainder after dividing by M belongs to the set $\{1, 2, \dots, M - 1\}$. Hence, a

belongs to C_i for some $i \in \{1, \dots, M-1\}$, which proves that this partition is an M colouring of $\mathbb{N}$. By van der Waerden's theorem, for any $k \in \mathbb{N}$, there is monochromatic k–term arithmetic progression $a, a+d, \dots, a+(k-1)d$.

We distinguish two cases:

(a) $\{a, a+d, \dots, a+(k-1)d\} \subset C_0 = A$

(b) If for some $i \in \{1, 2, \dots, M-1\}$, $\{a, a+d, \dots, a+(k-1)d\} \subset C_i$ then $(a-i), (a-i)+d, \dots, (a-i)+(k-1)d$ is an arithmetic progression and $\{(a-i), (a-i)+d, \dots, (a-i)+(k-1)d\} \subset A$.

Solution 2: (By James Andrews.)

Let $S = \{a_1 < a_2 < a_3 < \dots\}$ be a syndetic set and let $M = \max_{i \in \mathbb{N}} |a_{i+1} - a_i|$.

We consider the set of intervals $I_1 = [a_1, a_1 + M]$, $I_2 = [a_1 + 1 + M, a_1 + 2M], \dots$ and observe that, for each $i \in \mathbb{N}$, the interval I_i contains at least one element from S.

Next, we M-colour the natural numbers in the following way: $c(i) = k$ if k is the place in which the first element in $S \cap I_i$ appears. (So, $c(1) = 1$.)

Now apply van der Waerden's theorem to obtain the required property of syndetic sets.

Therefore for any $k \in \mathbb{N}$, the set A contains a k–term monochromatic progression.

5. Suppose that neither of the classes contains arbitrarily long strings of consecutive integers. This implies that both classes are syndetic sets and by the previous exercise, contain arithmetic progressions of arbitrary length.

Exercise 3.6.6 Monochromatic arithmetic progressions. Consider the following 2-colouring of positive integers:

$$\underbrace{1}_{1}\ \underbrace{00}_{2}\ \underbrace{1111}_{4}\ \underbrace{0\cdots0}_{8}\underbrace{1\cdots1}_{16}\underbrace{0\cdots0}_{32}11\cdots$$

1. Check if the claim of van der Waerden's theorem holds: For any $k \in \mathbb{N}$ find a monochromatic k-term arithmetic progression.

2. Show that there is no monochromatic arithmetic progression of infinite length.

Solution.

1. Let $k \in \mathbb{N}$ and let $n \in \mathbb{N}$ be such that $k < 2^n$. By definition, the above colouring contains 2^n consecutive integers coloured by 1. Therefore, there is positive integer a such that all terms of the k-term arithmetic progression

$$a, a+1, a+2, \dots, a+(k-1)$$

are coloured by the colour 1.

2. Let

$$a, a+d, a+2d, \dots, a+kd, \dots$$

be an infinite arithmetic progression with the common difference d and let $n \in \mathbb{N}$ be such that

$$d < 2^n.$$

Since $d < 2^n$, any interval of consecutive integers with at least 2^n elements must contain a term from the infinite arithmetic progression with the common difference d.

On the other hand, by definition of the colouring we have

$$\underbrace{11\cdots1}_{2^n}\underbrace{00\cdots0}_{2^{n+1}} \text{ or } \underbrace{00\cdots0}_{2^n}\underbrace{11\cdots1}_{2^{n+1}}.$$

Hence

$$a, a+d, a+2d, \dots, a+kd, \dots$$

is not monochromatic.

Exercise 3.6.7 Monochromatic arithmetic progressions. Consider the following infinite word on the alphabet $\{A, B\}$:

$$\mathcal{W} = \underbrace{A}_{1}\ \underbrace{BB}_{2}\ \underbrace{AAAA}_{4}\underbrace{B\cdots B}_{8}\underbrace{A\cdots A}_{16}\underbrace{B\cdots B}_{32}AA\cdots$$

1. Show that for any $k \in \mathbb{N}$ there is a subword of the form $\underbrace{XX\cdots X}_{k}$, where $X = A$ or $X = B$.
2. Show that, in the given word, for any $d \in \mathbb{N}$, any sequence of the form

$$X\underbrace{\ \cdots\ }_{d}X\underbrace{\ \cdots\ }_{d}X\underbrace{\ \cdots\ }_{d}X\underbrace{\ \cdots\ }_{d}X\cdots$$

must terminate at some point. In other words if, for example $X = A$, after certain numbers of jumps of the length d, you will have to land at the letter B.

Solution.

1. Let $k \in \mathbb{N}$ and let $n \in \mathbb{N}$ be such that $k < 2^n$.

   By definition of the word $\mathcal{W}$ we have

   $\underbrace{AA\cdots A}_{2^n}\underbrace{BB\cdots B}_{2^{n+1}}$ or $\underbrace{BB\cdots B}_{2^n}\underbrace{AA\cdots A}_{2^{n+1}}$.

   Therefore, there is the subword $\underbrace{AA\cdots A}_{k}$ and a subword $\underbrace{BB\cdots B}_{k}$.

2. Let $d \in \mathbb{N}$ and let

$$\mathcal{S} = A\underbrace{\ \cdots\ }_{d}A\underbrace{\ \cdots\ }_{d}A\underbrace{\ \cdots\ }_{d}A\underbrace{\ \cdots\ }_{d}A\cdots$$

   be a sequence of equally spaced letters A in the given infinite word $\mathcal{W}$.

   Let $n \in \mathbb{N}$ be such that $d < 2^n$. If $\mathcal{S}$ is an infinite sequence, then any subword of $\mathcal{W}$ of length at least 2^n would contain a letter A that belongs to $\mathcal{S}$.

   On the other hand, by (1) we know that there is a subword $\underbrace{BB\cdots B}_{2^n}$, i.e. a subword of length 2^n that does not contain the letter A.

   Therefore $\mathcal{S}$ must be finite.

Exercise 3.6.8 van der Waerden numbers. Prove that $w(3; 2, 3, 3) \le 18$.

Solution. Let c be a 3-colouring of the interval $[1, 18]$. Say that the first colour is red, the second colour is blue, and the third colour is green. If there are at least two elements coloured red, then there is a red 2-term arithmetic progression.

Suppose that there is only one element $i \in [1, 18]$ coloured red. If $i \in [1, 9]$ then $[i + 1, 18]$ contains at least nine consecutive integers coloured blue or green. Since $w(3, 3) = 9$ there is monochromatic 3-term arithmetic progression contained in $[i + 1, 18]$. If $i \in [10, 18]$ then $[1, i - 1]$ contains at least nine consecutive integers coloured blue or green. Since $w(3, 3) = 9$ there is monochromatic 3-term arithmetic progression contained in $[1, i - 1]$.

If there is no element coloured red, then the interval $[1, 18]$ is 2-coloured. Since $w(3, 3) = 9$ there is monochromatic 3-term arithmetic progression contained in $[1, 18]$.

Therefore any 3-colouring of $[1, 18]$ contains a 2-term arithmetic progression in the first colour or a 3-term arithmetic progression in the second or third colour.

In 1974 Tom Brown proved that $w(3; 2, 3, 3) = 14$.

Exercise 3.6.9 van der Waerden numbers. Show that if $k \equiv \pm 1 (\mathrm{mod}\ 6)$ then $w(k, 2, 2; 3) = 3k$.

Solution. Suppose that $k \equiv \pm 1 (\mathrm{mod}\ 6)$. The claim is obviously true for $k = 1$. Hence suppose that $k \geq 5$. Let $l \geq 1$ be such that $k = 6l + 1$ or $k = 6l - 1$. Observe that k is an odd number.

To show that $w(k, 2, 2; 3) = 3k$, it is sufficient to prove that $w(k, 2, 2; 3) \leq 3k$ and $w(k, 2, 2; 3) \geq 3k$.

To prove that $w(k, 2, 2; 3) \leq 3k$ we need to show that any 3-colouring (say, red, blue, and green) of the interval $[1, 3k]$ will yield a red k-term arithmetic progression or a blue 2-term arithmetic progression or a green 2-term arithmetic progression.

Clearly, any colouring of $[1, 3k]$ that contains two blue or two green elements will yield a blue 2-term arithmetic progression or a green 2-term arithmetic progression.

Hence we consider a three colouring $c : [1, 3k] \to \{\text{red, blue, green}\}$ that colours at most one element of $[1, 3k]$ blue and at most one element green.

Observe that if one of the colours blue or green is not used then there is a sequence of at least k consecutive red elements. Clearly, in this case there is a red k-term arithmetic progression.

Hence assume that $c : [1, 3k] \to \{\text{red, blue, green}\}$ is a colouring such that there is a unique $x \in [1, 3k]$ such that $c(x) = \text{blue}$ and a unique $y \in [1, 3k]$ such that $c(y) = \text{green}$.

Suppose that $y < x$.

If $y > k$ or $x < 2k$ then there is a sequence of at least k consecutive red elements between $[1, y-1]$ or between $[x+1, 3k]$, i.e. there is a red k-term arithmetic progression.

Suppose that $1 \leq y \leq k < 2k \leq x \leq 3k$. Observe that the number of elements between x and y is given by $d = (x - 1) - (y + 1) + 1 = x - y - 1$.

Note that if $y < k$ or $x > 2k$ then $d \geq k$, which means that there is a sequence of at least k consecutive red elements between y and x.

Hence the only remaining case is if $y = k$ and $x = 2k$. Observe that, since $k \equiv \pm 1 (\mathrm{mod}\ 6)$, neither k nor $2k$ is divisible by 3.This implies that all elements of the k-term arithmetic progression

$$3, 3 + 3, \ldots, 3 + 3i, \ldots, 3 + 3(k - 1) = 3k$$

are coloured red.

Therefore $w(k, 2, 2; 3) \leq 3k$.

To prove $w(k, 2, 2; 3) \geq 3k$, we must show that there is a 3-colouring of $[1, 3k - 1]$ which does not produce a red k-term arithmetic progression or a blue 2-term arithmetic progression or a green 2-term arithmetic progression.

Let the colouring $c : [1, 3k - 1] \to \{\text{red, blue, green}\}$ be defined in the following way:

$$c(k) = \text{green},\ c(2k) = \text{blue},\ \text{and } c(x) = \text{red, otherwise}.$$

Let

$$a, a + d, \ldots, a + id, \ldots, a + (k - 1)d$$

be a k-term arithmetic progression contained in $[1, 3k-1]$.

Observe that $d \in \{1, 2, 3\}$. Otherwise $a + (k-1)d \geq 1 + 4(k-1) = 4k-3$ which is greater than $3k-1$ for $k \geq 5$.

Since any set of k consecutive integers in $[1, 3k-1]$ must contain the integer k or the integer $2k$ we conclude that an arithmetic progression contained in $[1, 3k]$ with the step $d = 1$ cannot be c-monochromatic.

If the arithmetic progression $a, a+2, \ldots, a+2i, \ldots, a+2(k-1)$ is contained in $[1, 3k-1]$ then $a + 2(k-1) \leq 3k-1$ implies that $a \leq k+1$.

If a is an odd number then, for some $i \in \{0, 1, \ldots, k-1\}$, $a + 2i = k$ and the corresponding k-term arithmetic progression is not c-monochromatic.

If a is an even number then, for some $i \in \{0, 1, \ldots, k-1\}$, $a + 2i = 2k$ and the corresponding k-term arithmetic progression is not c-monochromatic.

If the arithmetic progression $a, a+3, \ldots, a+2i, \ldots, a+3(k-1)$ is contained in $[1, 3k-1]$ then $a + 3(k-1) \leq 3k-1$ implies that $a \leq 2$.

If $a = 1$ and $k = 6l+1$ then $1 + 3 \cdot (2l) = k$ and the corresponding k-term arithmetic progression is not c-monochromatic.

If $a = 1$ and $k = 6l-1$ then $1 + 3 \cdot (4l-1) = 12l - 2 = 2(6l-1) = 2k$ and the corresponding k-term arithmetic progression is not c-monochromatic.

If $a = 2$ and $k = 6l+1$ then $2 + 3 \cdot (4l) = 12l + 2 = 2 \cdot (6l+1) = 2k$ and the corresponding k-term arithmetic progression is not c-monochromatic.

If $a = 2$ and $k = 6l-1$ then $1 + 3 \cdot (2l) = 6l + 1 = k$ and the corresponding k-term arithmetic progression is not c-monochromatic.

Hence, the colouring c of $[1, 3k-1]$ does not produce a red k-term arithmetic progression or a blue 2-term arithmetic progression or a green 2-term arithmetic progression which proves that $w(k, 2, 2; 3) \geq 3k$.

Therefore, $k = \pm 1 (\text{mod } 6)$ implies $w(k, 2, 2; 3) = 3k$.

Exercise 3.6.10 Arithmetic progressions. Prove that for any $k, l \in \mathbb{N}$ there is $S(k, l) \in \mathbb{N}$ such that any k-colouring of the set of positive integers $[1, S(k, l)]$ contains a monochromatic arithmetic progression of length l together with its difference.

Solution. We prove the claim by mathematical induction on k, the number of colours.

For $k = 1$ and any $l \in \mathbb{N}$ we take $S(1, l) = l$.

Assume that the claim is true for a fixed $k \geq 1$ and any $l \in \mathbb{N}$. We fix $l \in \mathbb{N}$ and denote by $S(k, l)$ the corresponding number.

Next we define

$$S(k+1, l) = w(k+1, (l-1)S(k, l) + 1),$$

where $w(*, *)$ denotes a van der Waerden number.

Let the set of integers $[1, S(k+1, l)]$ be $(k+1)$-coloured. Then, by van der Waerden's Theorem, there is a $((l-1)S(k, l) + 1)$-term monochromatic arithmetic progression

$$a, a+d, ..., a + (l-1)S(k, 1)d.$$

For every $x = 1, 2, \ldots, S(k, l)d$ this monochromatic arithmetic progression contains the following l-term arithmetic progression:

$$a, a + xd, \ldots, a + (l-1)xd.$$

If for one of the numbers x, the difference xd is coloured by the same colour as the original progression, we have concluded the proof of our induction step.

Otherwise, the $S(k, l)$-term arithmetic progression

$$d, 2d, \ldots S(k, l)d$$

is coloured in only k colours. Now we apply the inductive hypothesis to conclude that the k-coloured set $\{d, 2d, \ldots S(k, l)d\} \subset [1, S(k+1, l)]$ contains a monochromatic arithmetic progression of length l together with its difference.

Exercise 3.6.11 Density. What is the density of the set of all powers of 3?

Solution. Consider the set of all powers of 3, i.e. consider the set $A = \{1, 3, 9, 27, \ldots\}$.
Let $n \in \mathbb{N}$ and let $k \in \mathbb{N} \cup \{0\}$ be such that $3^k \leq n < 3^{k+1}$. This is the same as

$$k \leq \log_3(n) < k + 1.$$

It follows that

$$a(n) = |A \cap [1, n]| \leq 1 + \log_3(n).$$

By definition,

$$\text{Density of } A = \lim_{n\to\infty} \frac{a(n)}{n} \leq \lim_{n\to\infty} \frac{1 + \log_3(n)}{n} = 0.$$

.

Exercise 3.6.12 Szemerédi's theorem. Szemerédi's theorem claims that any set of integers with positive *upper* density contains an arithmetic progression of any length.
Let $A = \{a_i : i \in \mathbb{N}\}$ be a set such that

$$0 < a_{i+1} - a_i \leq 2, \text{ for all } i \in \mathbb{N}.$$

1. Show that for any even $n \in \mathbb{N}$
$$\frac{|A \cap [1, n]|}{n} \geq \frac{1}{2}.$$
2. Conclude that the upper density of A is at least $1/2$.
3. Use Szemerédi's theorem to conclude that A contains arithmetic progressions of any finite length.

Solution.

1. Note that
$$0 < a_{i+1} - a_i \leq 2, \text{ for all } i \in \mathbb{N}$$
implies that for a_i and a_{i+1} one of the following must be true:
   - $a_{i+1} - a_i = 1$ which means that a_i and a_{i+1} are consecutive integers
   - $a_{i+1} - a_i = 2$ which means that a_i and a_{i+1} are consecutive even integers or consecutive odd integers.

   Therefore, for any $k \in \mathbb{N}$
$$A \cap \{2k - 1, 2k\} \neq \emptyset.$$
   Hence for a chosen even integer n, the set A intersects each of $n/2$ sets
$$\{1, 2\}, \{3, 4\}, \ldots, \{2n - 1, n\}$$
   which implies that
$$|A \cap [1, n]| \geq \frac{n}{2}.$$
   Therefore for any even $n \in \mathbb{N}$
$$\frac{|A \cap [1, n]|}{n} \geq \frac{1}{2}.$$

2. From (a) it follows that there is an infinite sequence $\{x_i = \frac{|A\cap[1,2i]|}{2i}\}_{i\in\mathbb{N}}$ such that
$$\frac{1}{2} \leq x_i \leq 1 \text{ for all } i \in \mathbb{N}.$$
The limit of any convergent subsequence of this sequence is greater than $\frac{1}{2}$ and hence
$$\overline{d}(A) = \limsup \frac{|A \cap [1,n]|}{n} \geq \frac{1}{2}.$$
Therefore the upper density of A is at least $1/2$.

3. Since the upper density of the set A is positive, by Szemerédi's theorem the set A contains arithmetic progressions of any finite length.

Exercise 3.6.13 Powers of 2. Show that the set
$$A = \{2^n : n \in \mathbb{N}\}$$
does not contain any 3-term arithmetic progressions.

Solution. Suppose that $i, j, k \in \mathbb{N}$, $i < j < k$, are such that $2^i, 2^j, 2^k$ form a 3-term arithmetic progression. This means that
$$2^j - 2^i = 2^k - 2^j.$$
It follows that
$$2^i(2^{j-i} - 1) = 2^j(2^{k-j} - 1)$$
and
$$2^{j-i} - 1 = 2^{j-i}(2^{k-j} - 1).$$
But this is impossible since $2^{j-i} - 1$ is an odd integer and $2^{j-1}(2^{k-j} - 1)$ is an even integer.

Therefore he set
$$A = \{2^n : n \in \mathbb{N}\}$$
does not contain any 3-term arithmetic progressions.

Exercise 3.6.14 Two examples. Give an example of:

1. An infinite colouring of positive integers that does not contain a monochromatic 2-term arithmetic progression.

2. A 3-colouring that avoids 2-term arithmetic progressions with a common difference that is equal 1 (mod 3).

Solution.

1. Colour every integer differently.

2. Consider the colouring
$$c(i) = \begin{cases} \text{red} & \text{if} \quad i \equiv 0 \pmod 3 \\ \text{blue} & \text{if} \quad i \equiv 1 \pmod 3 \\ \text{green} & \text{if} \quad i \equiv 2 \pmod 3. \end{cases}$$
It follows that if a and b are of the same colour then $a \equiv b \pmod 3$. Therefore the colouring c avoids 2-term arithmetic progressions with a common difference that is equal 1 (mod 3).

Chapter 4

Schur's Theorem and Rado's Theorem

4.1 Issai Schur

> *To live without hope is to cease to live.* — Fyodor Mikhailovich Dostoyevsky, Russian novelist, 1821 — 1881.

Who was Issai Schur? A mathematician who was born in the Russian Empire, worked in Germany for most of his life, and died in Palestine on his 66th birthday.

Table 4.1.1 List of mathematical objects and techniques named after Issai Schur:

Frobenius-Schur indicator	Herz-Schur multiplier	Jordan-Schur theorem
Lehmer-Schur algorithm	Schur algebras	Schur complement
Schur complement method	Schur decomposition	Schur function
Schur index	Schur indicator	Schur multiplier
Schur orthogonality relations	Schur polynomial	Schur product
Schur's inequality	Schur-convex function	Schur-Horn theorem
Schur-Weyl duality	Schur-Zassenhaus theorem	Schur test
Schur's lemma	Schur's lower bound	Schur's property
Schur's theorems in Ramsey theory, differential geometry, linear algebra, analysis		

Birth and Death. Issai Schur was born on January 10, 1875, in Mogilev, Russian Empire (now Belarus), and died at the age of 66 on January 10, 1941 in Tel Aviv, Palestine (now Israel).

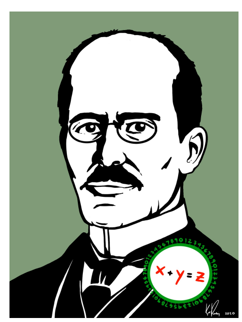


Figure 4.1.2 Issai Schur

World in 1875

World in 1941

- Electric dental drill is patented
- The first recorded hockey game (Montréal)
- Georges Bizet's opera "Carmen" premieres
- The Supreme Court of Canada is created
- Alexander Graham Bell makes the first voice transmission
- Tanaka Seisakusho (now Toshiba) company established
- The first woman licensed to practise medicine in Canada
- The Metre Convention Treaty signed
- Louis Riel is granted amnesty

Figure 4.1.3 War!

Issai's Family. Issai Schur was born in the family of a merchant Moses Schur and Golde Landau. In Berlin, on September 2, 1906, he married Regina Malka Frumkin, a medical doctor. They had two children, George, born in 1907, and Hilde, born in 1911.

Walter Ledermann about Issai Schur:

> I attended many courses. But Schur's lectures were for me the most impressive and inspiring ones. It seemed to me that they were perfect both in content and in form. When I was in Berlin the elementary cycle consisted of Determinants (for a whole semester!), Algebra, Number Theory, Theory of Invariants. The more advanced cycle consisted of Galois Theory, Analytic Number Theory I and II, Ideals. Sometimes additional courses were offered by Schur; for example, the Theory of Matrices, Group Representations, Elliptic Functions. Schur was a superb lecturer. He spoke slowly and clearly and his writing on the blackboard was very legible. All his courses were carefully structured into chapters and sections, each bearing a number and an appropriate heading. His lectures were meticulously prepared. It is known that he had very full lecture notes, written on loose sheets which he carried in the breast pocket of his jacket. But I can remember only one occasion when he consulted his notes: during one of the lectures on invariants he wrote down a list of invariants of a certain quintic polynomial. He furtively pulled out a sheet of paper from his pocket in order to check whether he had remembered the rather complicated formulae correctly (he had!). He never got stuck in his lecture or failed to remember what he had said in the previous lecture. [4]

On 7 April 1933 the Nazis passed a law which ordered the retirement of civil servants who were not of Aryan descent. Schur was 'retired'.

> When Schur's lectures were cancelled there was an outcry among the students and professors, for Schur was respected and very well liked. [Schiffer]

Many years later, Menahem Max Schiffer recalled:

> Schur told me [in Palestine] that the only person at the Mathematical Institute in Berlin who was kind to him was Grunsky, then a young lecturer. Long after the war, I talked to Grunsky about that remark and

> he literally started to cry: "You know what I did? I sent him a postcard to congratulate him on his sixtieth birthday. I admired him so much and was very respectful in that card. How lonely he must have been to remember such a small thing." (Source MacTutor.)

Schur left Germany for Palestine in 1939, broken in mind and body.

"This volume is dedicated to the memory of Issai Schur. It opens with some biographical reminiscences of the famous school he established in Berlin, his brutal dismissal by the Nazi regime and his tragic end in Palestine. This is followed by an extensive review of the extraordinary impact of his lesser known analytic work. Finally, leading mathematicians in the representation theory of the symmetric groups, of semisimple and affine Lie algebras and of Chevalley groups have contributed original and outstanding articles. These concern many areas inspired by Schur's work as well as more recent developments involving crystal and canonical bases, Hecke algebras, and the geometric approach linking orbits to representations."

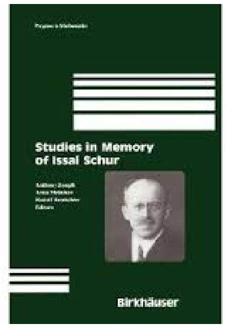


Figure 4.1.4 Studies in Memory of Issai Schur

Schur's Work - Three Examples.

Example 4.1.5 *Schur complement:* In linear algebra and the theory of matrices, the Schur complement of a matrix block (i.e., a submatrix within a larger matrix) is defined as follows. Suppose A, B, C, D are respectively $p \times p$, $p \times q$, $q \times p$ and $q \times q$ matrices, and D is invertible. Let

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

so that M is a $(p+q) \times (p+q)$ matrix. Then the Schur complement of the block D of the matrix M is the $p \times p$ matrix

$$A - BD^{-1}C.$$

It is named after Issai Schur who used it to prove Schur's lemma.

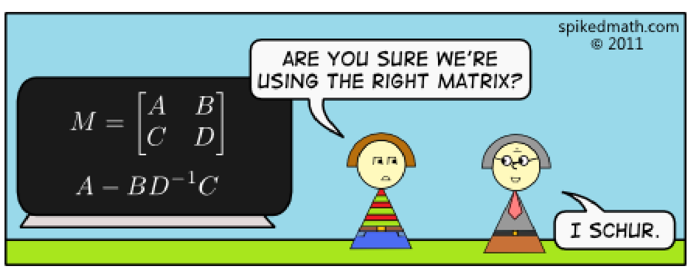


Figure 4.1.6 Source Spiked Math Comics.

Applications: in solving linear equations, probability theory, and statistics. □

Example 4.1.7 *Schur's inequality.* For all $x, y, z \geq 0$ and $t > 0$,

$$x^t(x-y)(x-z) + y^t(y-z)(y-x) + z^t(z-x)(z-y) \geq 0$$

with equality if and only if $x = y = z$ or two of them are equal and the other is zero. When t is an even positive integer, the inequality holds for all real numbers x, y, and z. □

Proof. Since the inequality is symmetric in x, y, z we take that $x \geq y \geq z \geq 0$.
Use the fact that

$$x^t(x-y)(x-z) + y^t(y-z)(y-x) + z^t(z-x)(z-y) =$$

$$= (x-y)[x^t(x-z) - y^t(y-z)] + z^t(x-z)(y-z)$$

to finish the proof. ∎

Example 4.1.8 *Schur product.* The Schur product (also known as the Hadamard product) of two matrices of the same dimensions, $A = (a_{ij})$ and $B = (b_{ij})$ is the matrix $C = (c_{ij})$ such that

$$c_{ij} = a_{ij} \cdot b_{ij}.$$

The Schur product $\circ$ is commutative, associative and distributive over addition. That is,

$$\begin{aligned} A \circ B &= B \circ A, \\ A \circ (B \circ C) &= (A \circ B) \circ C, \\ A \circ (B + C) &= A \circ B + A \circ C. \end{aligned}$$

Question: What is the identity matrix under the Schur product? *Applications:* The Schur product appears in algorithms such as JPEG. For more applications see Schur product - Wikipedia □

Schur and Ramsey Theory.

Schur's Theorem: If the set of positive integers $\mathbb{N}$ is finitely coloured then there exist x, y, z having the same colour such that

$$x + y = z.$$

Schur and van der Waerden's Theorem: A. Soifer in his "The mathematical coloring book" (pages 331-332) proved that Schur conjectured the statement that we know as van der Waerden's theorem. Schur called it "a helpful lemma."

Schur and Rado's Theorem: Schur's Ph.D. student Richard Rado in his doctoral thesis "Studien zur Kombinatorik" (1933) completely solved the following problem:

Let $Ax = 0$ be a system of linear equations, where A is a matrix with integer entries. Under which conditions for every r-coloring of the set of positive integers $\mathbb{N}$ the system has a monochromatic solution?

Resources.

1. For mored details see [7], pp. 321-334.
2. Issai Schur - Wikipedia
3. Issai Schur - Biography by J J O'Connor and E F Robertson
4. Interview with Walter Ledermann
5. Issai Schur and his algebraic school in Berlin by Reinhard Siegmund-Schultze
6. Issai Schur - Mathematics Genealogy Project

4.2 Schur's Theorem

> *The hardest thing to see is what is in front of your eyes.* — Johann Wolfgang von Goethe, German writer and politician, 1749 — 1832.

Reminder: Schur's Theorem. If the set of positive integers $\mathbb{N}$ is finitely coloured then there exist x, y, z having the same colour such that

$$x + y = z.$$

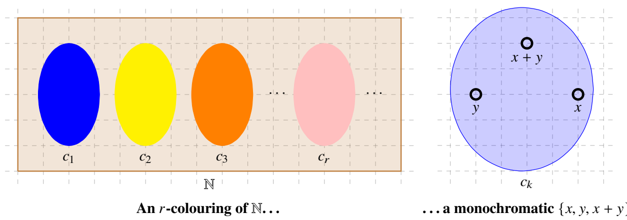


Figure 4.2.1 Schur's Theorem: If the set of positive integers $\mathbb{N}$ is finitely coloured then there exist x, y, z having the same colour such that $x + y = z$.

Definition 4.2.2 A triple x, y, z that satisfies $x + y = z$ is called a Schur triple. ◊

Reminder: The Ramsey number $R(s, t)$ is the minimum number n for which any edge 2-coloring of K_n, a complete graph on n vertices, in red and blue contains a red K_s or a blue K_t.

Recall Definition 2.3.18:

The Ramsey number $R(s_1, s_2, \ldots, s_r)$ is the minimum number n for which any edge r-colouring of K_n, a complete graph on n vertices, contains an i-monochromatic K_{s_i}, for some $i \in [1, r]$.

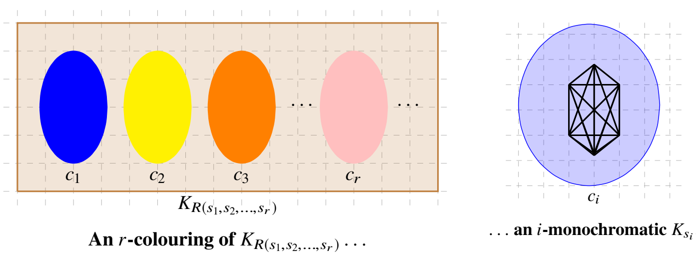


Figure 4.2.3 Ramsey Theorem: If the the complete graph $K_{R(s_1,s_2,\ldots,s_r)}$ is r-coloured then, for some $i \in [1, r]$, there exists a complete graph K_{s_i} that is i-monochromatic.

Example 4.2.4 $R(3, 3, 3) \leq 17$. □

Proof. See Figures 4.2.5 and Figure 4.2.6 .

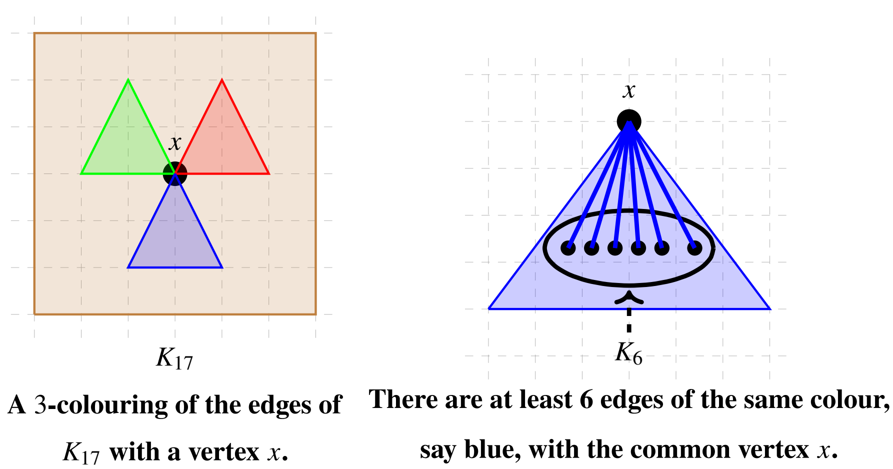
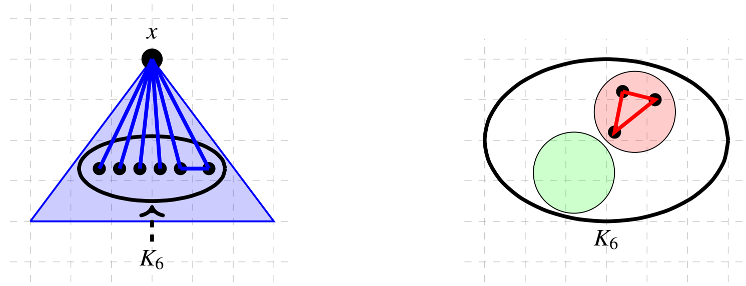


Figure 4.2.5 Use the the pigeonhole principle to conclude that if the edges of K_{17} are 3-coloured then each vertex is incident to at least six edges that are of the same colour.



Case 1: K_6 **contains at least one blue edge.** **Case 2:** K_6 **does not contain any blue edges.**

Figure 4.2.6 Two cases . . . Done!

■

Question 4.2.7 What is the meaning of $R(3, 4, 5, 6)$? $R(3, 3, 3, 3, 3)$? □

Theorem 4.2.8 Schur's Theorem. *If the set of positive integers* $\mathbb{N}$ *is finitely coloured then there exist* x, y, z *having the same colour such that*

$$x + y = z\,,$$

i.e. there is a monochromatic Schur triple.

Proof. Let $c : \mathbb{N} \to [1, 2, \ldots, r]$ and let $M = R(\underbrace{3, 3, \ldots, 3}_{r})$. See Figures 4.2.9 and Figure 4.2.10 .

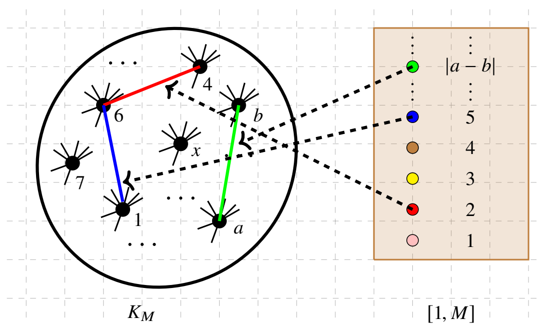


Figure 4.2.9 Denote vertices of K_M by $1, 2, \ldots, M$. For any $a, b \in [1, M]$, colour the edge $\{a, b\}$ by $c(|a-b|)$. Observe that all we need is the restriction of the r-colouring c on the interval $[1, M]$.

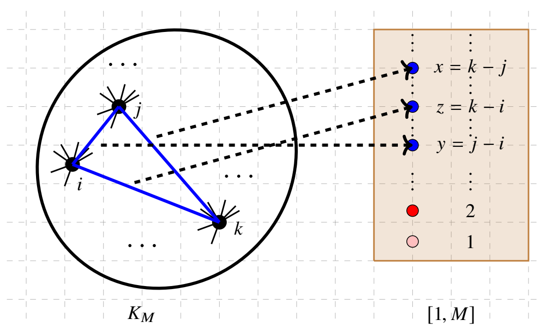


Figure 4.2.10 There is a monochromatic triangle with vertices $i < j < k$. (Why?) Take $x = k - j$, $y = j - i$, and $z = k - i$. Done! (Do you see why?)

■

Theorem 4.2.11 Actually . . . Schur's Theorem. *For any $r \in \mathbb{N}$ there is a natural number M such that any r- colouring of* $[1, M]$ *contains x, y, z having the same colour such that*

$$x + y = z.$$

The least M with such property is called a Schur number *and it is detonated by* $s(r)$.

Example 4.2.12 What is $s(2)$? □

1. Can you 2-colour, say in red and blue, the interval of positive integers $[1, 4]$ and avoid monochromatic Schur triples? Note that $1, 1, 2$ and $2, 2, 4$ are Schur triples. See Figure 4.2.13.

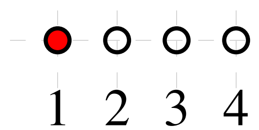


Figure 4.2.13 $s(2) > 4$

2. Can you 2-colour, say in red and blue, the interval of positive integers $[1, 5]$ and avoid monochromatic Schur triples? Note that $1, 1, 2$ and $2, 2, 4$ are Schur triples. See Figure 4.2.14.

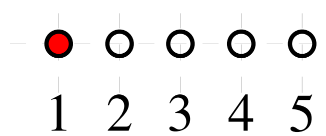


Figure 4.2.14 $s(2) = 5$

Known Schur Numbers.

$$s(1) = 2, s(2) = 5, s(3) = 14, s(4) = 45.$$

Time Machine.

In 1637 Fermat scribbled into the margins of his copy of *Arithmetica* by Diophantus, that

> It is impossible for a cube to be the sum of two cubes, a fourth power to be the sum of two fourth powers, or in general for any number that is a power greater than the second to be the sum of two like powers. I have discovered a truly marvellous demonstration of this proposition that this margin is too narrow to contain.

The margin note became known as Fermat's Last Theorem. It was proved by Andrew Wiles in 1995.

In 1916 Schur proved the following:

> Let $n > 1$. Then, for all primes $p > s(n)$, the congruence
>
> $$x^n + y^n \equiv z^n (\text{mod } p)$$
>
> has a solution in the integers, such that p does not divide xyz.

Fact:

> For any odd prime p, the multiplicative group
>
> $$\mathbb{Z}_p^* = \mathbb{Z}/p\mathbb{Z} = \{1, 2, \ldots, p - 1\}$$
>
> is *cyclic*.

Example 4.2.15 Take $p = 5$. Then $\mathbb{Z}_5^* = \{1, 2, 3, 4\}$ and the multiplication is given by

$\cdot$	1	2	3	4
1				
2				
3				
4				

Also, $\mathbb{Z}_5^* = \{2, 2^2, 2^3, 2^4\} = \{2, 4, 3, 1\}$ and $\mathbb{Z}_7^* = \{3, 3^2, 3^3, 3^4, 3^5, 3^6\} = \{3, 2, 6, 4, 5, 1\}$. □

In general, for any odd prime p there is $q \in \{1, \ldots, p-1\}$ such that $\mathbb{Z}_p^* = \{q, q^2, \ldots, q^{p-1}\}$.

Theorem 4.2.16 (Schur, 1916): *Let $n > 1$. Then, for all primes $p > s(n)$, the congruence*

$$x^n + y^n \equiv z^n (\mathrm{mod}\ p)$$

has a solution in the integers, such that p does not divide xyz.

Proof. See Figures 4.2.17 — Figure 4.2.19.

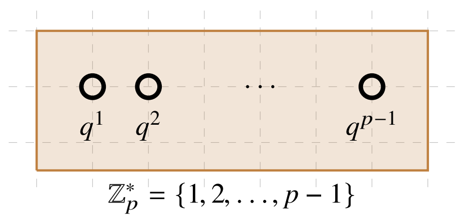


Figure 4.2.17 $\mathbb{Z}_p^* = \{1, 2, \ldots, p-1\} = \{q, q^2, \ldots, q^{p-1}\}$

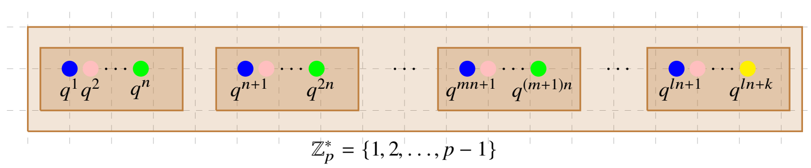


Figure 4.2.18 An n-colouring of $\{1, 2, \ldots, p-1\}$, $p > s(n)$

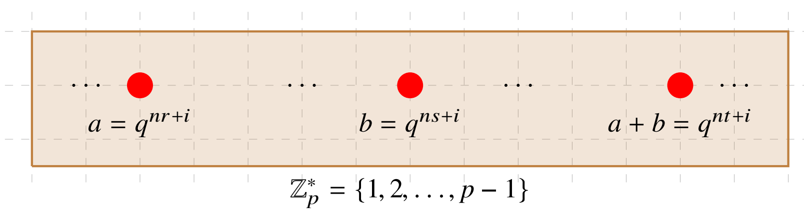


Figure 4.2.19 There is a monochromatic Schur triple!

From

$$q^{nr+i} + q^{ns+i} = q^{nt+i} \Leftrightarrow q^i(q^{nr} + q^{ns} - q^{nt}) \equiv 0 \pmod{p}$$

we conclude that

$$p | q^i(q^{nr} + q^{ns} - q^{nt}).$$

Since $0 \leq q^i < n < s(n) \leq p - 1$ it follows that $p \nmid q^i$. Therefore

$$p | (q^{nr} + q^{ns} - q^{nt})$$

or, what is the same

$$q^{nr} + q^{ns} - q^{nt} \equiv 0 \pmod{p}.$$

By taking $x = q^r$, $y = q^s$, and $z = q^t$ we obtain

$$x^n + y^n \equiv z^n \pmod{p}.$$

■

Example 4.2.20 Let P be the set of points in the plane $x + y - z = 0$ whose coordinates are positive integers. Let an r-colouring of the set of positive integers be given.

For each $(a, b, c) \in P$, do the following. If a, b, c are of the same colour, then colour (a, b, c) with that colour. Otherwise, mark (a, b, c) with an X.

Three Questions:

1. Can all of the points be marked with an X?
2. Can we tell if, under any given finite colouring, the plane must contain an infinite number of coloured points?
3. Same for the plane $x + y - 2z = 0$.

□

Resources.

1. For more details see [2], pp. 69-70, [3], and [7].
2. Schur's Theorem - Wikipedia
3. Schur's theorem and related topics in Ramsey theory - by Summer Lynne Kisner, pp 19-53
4. Ramsey Theory - by Jacob Fox (p 3)

4.3 Richard Rado

> *There are almost as many types of mathematicians as there are types of human being. Among them are technicians, there are artists, there are poets, there are dreamers, men of affairs, and many more.* — Richard Rado

Who was Richard Rado? A mathematician who earned two Ph.D.s: in 1933 from the University of Berlin under Issai Schur, and in 1935 from the University of Cambridge under Godfrey Harold Hardy.

Table 4.3.1 Mathematical work of Richard Rado:

Convergence of sequences and series	Inequalities	Ramsey theory
Geometry and measure theory	Number theory	Graph theory

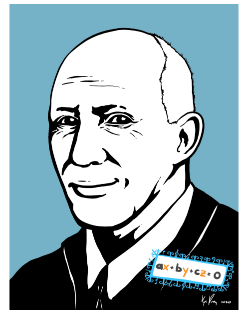


Figure 4.3.2 Richard Rado

Birth and Death. Richard Rado was born on 28 April 1906 in Berlin, Germany, and died on 23 December 1989 in Henley-on-Thames, Oxfordshire, England.

World in 1906

- The first radio set advertised
- The first woman elected to American Society of Civil Engineers
- The first animated cartoon copyrighted
- Albert Einstein introduces his Theory of Relativity
- Mahatma Gandhi coins the term "Satyagraha"
- Alberta adopts Mountain Standard Time
- SOS adopted as warning signal
- Lee de Forest patents a 3-diode amplification valve

World in 1989

Figure 4.3.3 Berlin Wall came down!

Richard's Family. Richard was born in Berlin. He was the second son of Leopold Rado, who was a Hungarian from Budapest. As a young man he had to choose between being a concert pianist or a mathematician. He chose to become a mathematician in the belief that he could continue with music as a hobby, but that he could never treat mathematics in that way.

In 1933 he married Luise Zadek, whom he had met when he needed a partner to play piano duets. They had one son, Peter Rado, born in 1943.

> Rado and his wife had a double partnership: she went with him to mathematical conferences and meetings and kept contact with his mathematical friends, he was an accomplished pianist and she was a singer of professional standard. They gave many recitals both public and private, often having musical evenings in their home in Reading. Rado was the kindest and gentlest of men. (Source MacTutor.)

Paul Erdős about Richard Rado:

> I first became aware of Richard Rado's existence in 1933 when his important paper *Studien zur Kombinatorik* appeared. I thought a great deal about the many fascinating and deep unsolved problems stated in this paper but I never succeeded to obtain any significant results here (...) Our joint work extends to more than 50 years; we wrote 18 joint papers (...) Our most important work in undoubtedly in set theory and, in particular, the creation of the partition calculus. The term partition calculus is, of course, due to Rado. Without him, I often would have been content in stating only special cases. (Source My joint work with Richard Rado).

Canadian Connection.

- Canadian Commonwealth Fellow, University of Waterloo, 1971-1972
- Visiting Professor, University of Calgary, 1973-1974
- Hon. D. Mathematics, University of Waterloo, 1986

Rado's Work - Two Examples.

Example 4.3.4 *Partition Calculus:* □

In Erdős and Rado's words:

> The investigation centres round what we call partition relations connecting given cardinal numbers or order types and in each given case the problem arises of deciding whether a particular partition relation is true or false. It appears that a large number of seemingly unrelated arguments in set theory are, in fact, concerned with just such a problem. It might therefore be of interest to study such relations for their own sake and to build up a partition calculus which might serve as a new and unifying principle in set theory. (Source Project Euclid.)

In the early 1950s, Rado introduced the partition calculus notation. For example

Ramsey's Theorem: For any $n, m < \omega$, one has $\omega \to (\omega)^n_m$.

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THEOREM 43. *If* $r<s\leqq\beta_0$; $\alpha\not\to(\beta_0)^r_k$; $\beta_1\to(s)^r_k$, *then*

$$\alpha \to (\beta_0, \beta_1)^s.$$

This proposition remains valid if the types α, β_0, β_1 *are replaced by cardinals.*

PROOF. Let $r<s\leqq\beta_0$; $\alpha\to(\beta_0,\beta_1)^s$; $\beta_1\to(s)^r_k$. We have to deduce that

(102) $$\alpha \to (\beta_0)^r_k.$$

Let $\bar{S}=\alpha$; $[S]^r=\sum'\,[\nu<k]K_\nu$. Then $[S]^s=K_0'+'K_1'$, where

$$K_0' = \sum[\nu<k]\{A : A\in[S]^s;\quad [A]^r\subset K_\nu\}.$$

Then there are $B\subset S$; $\lambda<2$ such that $[B]^s\subset K_\lambda'$; $\bar{B}=\beta_\lambda$. If $\lambda=1$, then $\bar{B}\to(s)^r_k$, and therefore there are $A\in[B]^s$; $\nu<k$ such that $[A]^r\subset K_\nu$. Then $A\in K_0'$; $A\notin K_1'$, which is false. Hence $\lambda=0$. Let $\{X, Y\}_{\neq}\subset[B]^r$. Then we can write $X=\{x_0,\cdots,x_{r-1}\}$; $Y=\{x_m,\cdots,x_{m+r-1}\}$, where $1\leqq m\leqq r$; $\{x_0,\cdots,x_{m+r-1}\}_{\neq}$. Put

$$X_\mu = \{x_\mu,\cdots,x_{\mu+r-1}\} \qquad (\mu\leqq m).$$

Figure 4.3.5 How complicated?

Example 4.3.6 *Rado Graph.* □

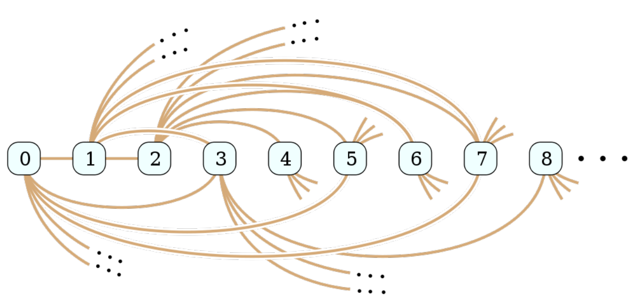


Figure 4.3.7 Rado graph (Source David Eppstein/Public domain.)

In 1964 Rado constructed the Rado graph by identifying the vertices of the graph with the natural numbers $0, 1, 2, \ldots$. An edge connects vertices x and y in the graph (with $x < y$) whenever the xth bit of the binary representation of y is nonzero. Thus, for instance, the neighbours of vertex 0 consist of all odd-numbered vertices, while the neighbours of vertex 1 consist of vertex 0 (the only vertex whose bit in the binary representation of 1 is nonzero) and all vertices with numbers congruent to 2 or 3 modulo 4.

Resources.

1. For mored details see [7], pp. 304-308.
2. Richard Rado - Wikipedia
3. Richard Rado - Biography by J J O'Connor and E F Robertson
4. My Joint Work WIth Richard Rado - by Paul Erdős
5. A Partition Calculus in Set Theory
6. Rado Graph - Wikipedia
7. Mathematicians Begin to Tame Wild "Sunflower" Problem by Kevin Hartnett

4.4 Rado's Theorem

> *One must still have chaos in oneself to be able to give birth to a dancing star.* — Friedrich Wilhelm Nietzsche, German philologist, philosopher, cultural critic, poet and composer, 1844 — 1900

Reminder: Schur's Theorem. If the set of positive integers $\mathbb{N}$ is finitely coloured then there exist x, y, z having the same colour such that

$$x_1 + x_2 - x_3 = 0.$$

Reminder: van der Waerden's Theorem. If the set of positive integers $\mathbb{N}$ is finitely coloured then there exist x, y, z having the same colour such that

$$x_1 + x_2 - 2x_3 = 0$$

Question 4.4.1 Does every 2-colouring of natural numbers contain a monochromatic solution of the equation

$$x_1 - 2x_2 = 0?$$

See Figure 4.4.2. □

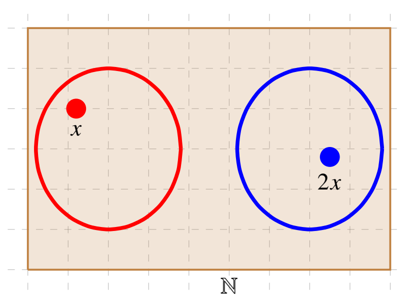


Figure 4.4.2 If x then $2x$.

Question 4.4.3 Does every finite colouring of positive integers contain a monochromatic solution of the equation

$$x_1 - 2x_2 + 3x_3 = 0?$$

□

Proof. We define a colouring $c : \mathbb{N} \to \{1, 2, \ldots, 6\}$ in the following way:

$$\text{If } n = 7^k \cdot (7 \cdot l + i),\ i \in \{1, 2, \ldots, 6\}, k, l \geq 0, \text{ then } c(n) = i.$$

See Figure 4.4.4. For example, in this colouring

$$c(5) = 5,\ c(14) = 2,\ c(25) = 4, \text{ and } c(49) = 1.$$

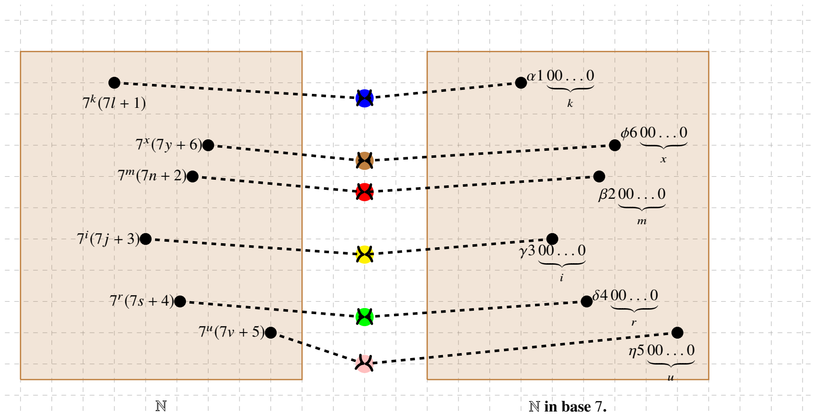


Figure 4.4.4 A 6 colouring of $\mathbb{N}$.

Suppose that there is a c- monochromatic solution of the given equation. Hence suppose that there are

$$x_1 = 7^k(7l + i),\ x_2 = 7^s(7t + i),\ x_3 = 7^p(7q + i),$$

with $i \in \{1, 2, \cdots, 6\}$ and $k, l, s, t, p, q \geq 0$, such that

$$x_1 - 2x_2 + 3x_3 = 0 \Leftrightarrow 7^k(7l + i) - 2 \cdot 7^s(7t + i) + 3 \cdot 7^p(7q + i) = 0$$

or, which is the same,

$$(7^k - 2 \cdot 7^s + 3 \cdot 7^p) \cdot i = 2 \cdot 7^{s+1} \cdot t - 7^{k+1} \cdot l - 3 \cdot 7^{p+1} \cdot q.$$

Observe that there is one "extra" factor of 7 on the right hand side of the expression above. What happens if we divide the expression by 7^r, where $r = \min\{k, s, p\}$? Will the right-hand side of the expression be divisible by 7? What about the left-hand side?

■

Question 4.4.5 Under what conditions does a homogeneous linear equation

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0, c_1, c_2, \ldots, c_n \in \mathbb{Z}$$

have a monochromatic solution whenever $\mathbb{N}$ is finitely coloured? □

Definition 4.4.6 We say that a homogeneous linear equation

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0, c_1, c_2, \ldots, c_n \in \mathbb{Z}$$

is **partition regular over** $\mathbb{N}$ if it has a monochromatic solution whenever $\mathbb{N}$ is finitely coloured. ◊

Question 4.4.7 Under what conditions is a homogeneous linear equation

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0, c_1, c_2, \ldots, c_n \in \mathbb{Z}$$

partition regular over $\mathbb{N}$? □

Observation 4.4.8

1. Equations $x_1 + x_2 - x_3 = 0$ and $x_1 + x_2 - 2x_3 = 0$ are partition regular.
2. Equations $x_1 - 2x_2 = 0$ and $x_1 - 2x_2 + 3x_3 = 0$ are not partition regular.

Definition 4.4.9 ***p*-Primer.** Let p be a prime. The p-primer is a $(p-1)$-colouring of natural numbers obtained in the way demonstrated at the Figure 4.4.10. ◊

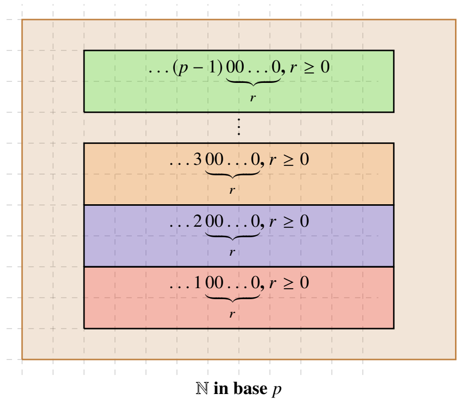


Figure 4.4.10 The p-primer, a $(p-1)$-colouring of $\mathbb{N}$.

Proposition 4.4.11 *Let integers $c_1, c_2, \ldots, c_n$ be such that for any subset $J \subseteq [n]$*

$$\sum_{i \in J} c_i \neq 0.$$

Then the equation $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$ is NOT *partition regular over $\mathbb{N}$.*

Proof. Let p be a prime such that

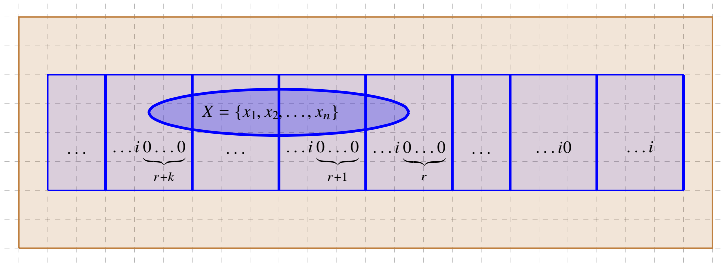
$$p > \sum_{i=1}^{n} |c_i|$$

and let $\chi : \mathbb{N} \to [p-1]$ be the p-primer.

Suppose that $X = \{x_1, x_2, \ldots, x_n\}$ is a p-primer monochromatic solution of the given equation. Then there are $i \in \{1, \ldots, p-1\}$ and $r_1, r_2, \ldots, r_n \geq 0$ such that

$$\begin{aligned}
x_1 &= \ldots i\underbrace{0\cdots0}_{r_1} \\
x_2 &= \ldots i\underbrace{0\cdots0}_{r_2} \\
&\vdots \\
x_n &= \ldots i\underbrace{0\cdots0}_{r_n}.
\end{aligned}$$

See Figure 4.4.12.



$\mathbb{N}$ in base p and the p-primer colour i

Figure 4.4.12 ... there is a monochromatic solution $X = \{x_1, x_2, \ldots, x_n\}$.

Let $r = \min\{r_1, r_2, \ldots, r_n\}$ and $J = \{j \in [1, n] : r_j = r\}$:

$$
\begin{array}{rccccccccc}
 & & & & & r+1 & & & \\
 & & & & & \downarrow & & & \\
x_1 = & \ldots & & & & i & 0 & \ldots & 0 \\
x_2 = & \ldots & & & i & 0 & 0 & \ldots & 0 \\
x_3 = & \ldots & i & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & & & & & & & & \\
x_j = & \ldots & & & & i & 0 & \ldots & 0 \\
\vdots & & & & & & & & \\
x_n = & \ldots & & i & \ldots & 0 & 0 & \ldots & 0
\end{array}
$$

Divide

$$0 = \sum_{j \in J} c_j x_j + \sum_{j \in [n] \setminus J} c_j x_j = \sum_{j \in J} c_j \cdot p^r (pk_j + i) + \sum_{j \in [n] \setminus J} c_j \cdot p^{r_j} (pk_j + i)$$

by p^r to obtain

$$i \sum_{j \in J} c_j + p \cdot A = 0, \ A \in \mathbb{Z}.$$

Contradiction. ■

Therefore:

Proposition 4.4.13 *Let the equation $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$ be partition regular over $\mathbb{N}$. Then there is $J \subseteq [n]$ such that*

$$\sum_{i \in J} c_i = 0.$$

Lemma 4.4.14 *Let $q \in \mathbb{Q}$ and $k \in \mathbb{N}$. Every k-colouring of natural numbers contains a monochromatic solution of the equation $x + qy = z$.*

Proof.

1. $q = 0$:

2. $q < 0$:

3. $q > 0$: Let $r, s \in \mathbb{N}$ be such that $q = \dfrac{r}{s}$.

   (a) We prove by induction on k that for any $k \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that any k-colouring of $[n]$ contains a monochromatic solution of the equation $x + qy = z$.

   (b) If $k = 1$, take $n = \max\{s, r + 1\}$ and $x = 1$, $y = s$, and $z = r + 1$.

(c) Suppose that $k \geq 1$ and $n \in \mathbb{N}$ are such that any k-colouring of $[n]$ contains a monochromatic solution of the equation $x + qy = z$. See Figure 4.4.15.

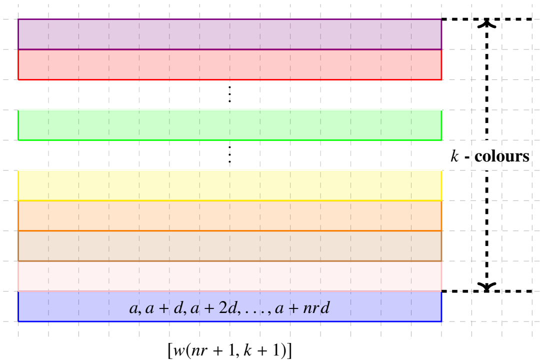


Figure 4.4.15 Let $w(nr+1, k+1)$ be a van der Waerden number, i.e., the least positive integer such that any $(k+1)$-colouring of $[1, w(nr+1, k+1)]$ contains a monochromatic $nr + 1$ arithmetic progression.

(d) *Case 1.* See Figure 4.4.16.

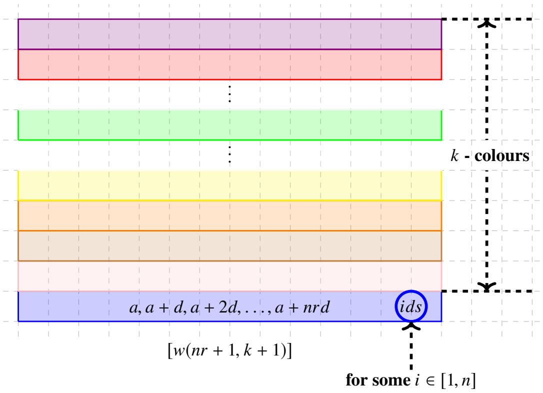


Figure 4.4.16 Take $x = a$, $y = ids$, and $z = a + ird$.

(e) *Case 2.* See Figure 4.4.17.

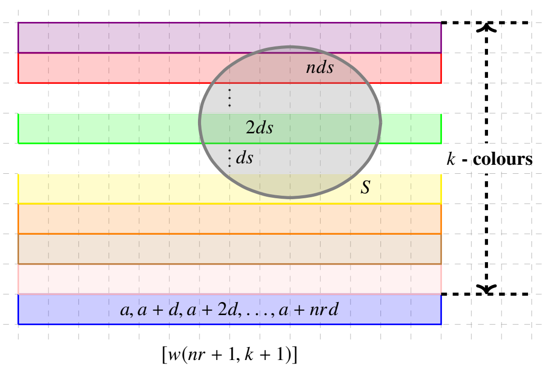


Figure 4.4.17 The set $S = \{ds, 2ds, \ldots, nds\}$ is k-coloured.

■

Proposition 4.4.18 *If the set of non-zero integers* $\{c_1, c_2, \ldots, c_n\}$ *is such that*

$$\sum_{i \in J} c_i = 0$$

for some $J \subseteq [n]$*, then the equation* $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$ *is partition regular over* $\mathbb{N}$*.*

Proof. Let χ be a finite colouring of $\mathbb{N}$. Let $i_0 \in J$ and let $\{x, y, z\}$ be a χ-monochromatic solution of the equation

$$x + \frac{\sum_{i \notin J} c_i}{c_{i_0}} y = z$$

Let

$$x_i = \begin{cases} x & \text{if } i = i_0 \\ y & \text{if } i \notin J \\ z & \text{if } i \in J \backslash \{i_0\}. \end{cases}$$

Then

$$\begin{aligned} c_1x_1 + c_2x_2 + \cdots + c_nx_n &= \sum_{i \in J} c_i x_j + \sum_{i \notin J} c_i x_j \\ &= c_{i_0} x_{i_0} + \sum_{i \in J \backslash \{i_0\}} c_i x_j + \sum_{i \notin J} c_i x_j \\ &= \end{aligned}$$

■

Theorem 4.4.19 Rado's Theorem. *The equation* $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$ *is partition regular over* $\mathbb{N}$ *if and only if there is* $J \subseteq [n]$ *such that*

$$\sum_{i \in J} c_i = 0.$$

Resources.

1. For more details see [7].
2. Rado's Theorem - Wikipedia
3. Ramsey Theory - by I. Leader - pp 12-13
4. Rado's single equation theorem - by Neil Lyall
5. Shur's Theorem and Related Topics in Ramsey Theory - by Summer Lynne Kisner - pp 71-76
6. Ramsey Theory - by G. Taylor - pp 25-28

4.5 Exercises

These exercises are based on the material covered in Chapter 4.

Exercise 4.5.1 **Essay.** Write a short essay (300-400 words) on the life and work of Issai Schur.

Exercise 4.5.2 **Essay.** Write a short essay (300-400 words) on the life and work of Richard Rado.

Exercise 4.5.3 Schur number. Show that

$$s(r) \leq R(\underbrace{3, 3, \ldots, 3}_{r}) - 1,$$

where $s(r)$ is the Schur number for r colours.

Solution. Let c be an r-colouring of the interval $[1, R(\underbrace{3, 3, \ldots, 3}_{r}) - 1]$. Consider the complete graph on $R(\underbrace{3, 3, \ldots, 3}_{r})$ vertices denoted by $1, 2, \ldots, R(\underbrace{3, 3, \ldots, 3}_{r})$. Define an r-colouring c' of the edges of this complete graph by

$$c'(\{a, b\}) = c(|a - b|)$$

where $\{a, b\}$ is the edge between the vertices a and b. We observe that, for any $a, b \in [1, R(\underbrace{3, 3, \ldots, 3}_{r})]$,

$$|a - b| \leq R(\underbrace{3, 3, \ldots, 3}_{r}) - 1$$

and hence the colouring c' is well-defined. By definition of $R(\underbrace{3, 3, \ldots, 3}_{r})$ the complete graph contains a monochromatic triangle, i.e., there are i, j, k, $i < j < k$, such that

$$c'(\{i, j\}) = c'(\{i, k\}) = c'(\{j, k\})$$

which is the same as

$$c(j - i) = c(k - i) = c(k - j).$$

Setting $x = j - i$, $y = k - j$, and $z = k - i$ we observe that

$$x, y, z \in [1, R(\underbrace{3, 3, \ldots, 3}_{r}) - 1], \ x + y = z, \text{ and } c(x) = x(y) = c(z).$$

Therefore x, y, z is a c-monochromatic Schur triple.

This proofs that any r-colouring of $[1, R(\underbrace{3, 3, \ldots, 3}_{r}) - 1]$ contains a monochromatic Schur triple. Therefore

$$s(r) \leq R(\underbrace{3, 3, \ldots, 3}_{r}) - 1.$$

Exercise 4.5.4 Schur triples.

1. Find a red/blue colouring of $\{1, 2, 3, 4, 5, 6, 7\}$ that contains neither a red Schur triple nor a blue 3-term arithmetic progression.

2. Show that any red/blue colouring of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ contains a red Schur triple or a blue 3-term arithmetic progression.

Solution.

1. Consider the positive numbers from 1 to 7.

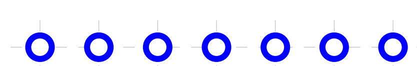


Figure 4.5.5 Positive numbers from 1 to 7.

Colour the number 1 red:

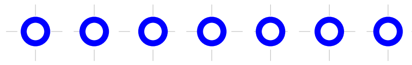


Figure 4.5.6 Colour 1 red.

We try to colour $[1,7]$ to avoid red Schur triples and blue 3-term arithmetic progressions. Note that 2 must be blue. (Figure 4.5.7.)

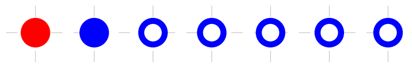


Figure 4.5.7 Colour 1 red and 2 blue.

Suppose that 3 is red. Then 6 must blue. This forces 4 to be red, but then there is a red Schur triple $1,3,4$. (Figure 4.5.8.)

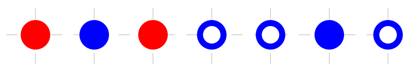


Figure 4.5.8 Colour 3 red.

Suppose that 3 is blue. Then 4 must red, 5 must be blue, 7 must be red, and 6 must be blue. (Figure 4.5.9.)

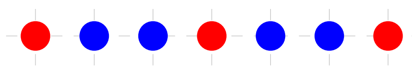


Figure 4.5.9 Colouring R-B-B-R-B-B-R.

Colour 1 blue and 2 red. Then 4 must be blue, 7 must be red, 5 must be blue. None of 3 and 6 can be blue, but if both of them are red, then there is a red Schur triple $3,3,6$. (Figure 4.5.10.)

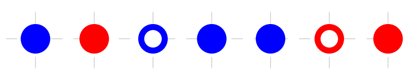


Figure 4.5.10 Colour 1 blue and 2 red.

Colour 1 and 2 blue. Then 3 must be red, 6 must be blue, 4 must be red, 7 must be blue, and 5 must be red. (Figure 4.5.11.)

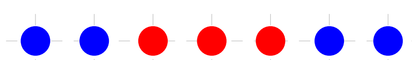


Figure 4.5.11 Colouring B-B-R-R-R-B-B.

Hence there are only two blue-red colourings of $[1,7]$ that avoid red Schur triples and blue 3-term arithmetic progressions: R-B-B-R-B-B-R and B-B-R-R-R-B-B.

2. Consider the R-B-B-R-B-B-R colouring of $[1,7]$. If we colour 8 red then there is a red Schur triple $1,7,8$. If we colour 8 blue then there is able 3-term arithmetic progression $3,5,8$.

   Consider the B-B-R-R-R-B-B colouring of $[1,7]$. If we colour 8 red then there is a red Schur triple $3,5,8$. If we colour 8 blue then there is a blue 3-term arithmetic progression $6,7,8$.

   Therefore any blue/red-colouring of $[1,8]$ contains a red Schur triple or a blue 3-term arithmetic progression.

Exercise 4.5.12 Schur triples and arithmetic progressions. Show that the minimum integer n such that any red/blue colouring of $[1, n]$ must admit either a red strict Schur triple, or a blue 3-term arithmetic progression is $n = 10$.

Solution. The question is to show that any red/blue colouring of $[1, 10]$ must contain a red solution to $x + y = z$ (of distinct integers) or a blue solution to $x + y = 2z$.

We also need to show that there is a red/blue colouring of $[1, 9]$ with no red solution to $x + y = z$ (of distinct integers) or a blue solution to $x + y = 2z$. The following colouring of $[1, 9]$ achieves this:

Next we try to build a red/blue colouring of $[1, 10]$ that avoids a red (R) solution to $x + y = z$ (of distinct integers) or a blue (B) solution to $x + y = 2z$.

We start by observing that such a colouring has to avoid monochromatic triples $\{1, 2, 3\}$, $\{2, 4, 6\}$, and $\{3, 6, 9\}$ because these triples are both 3-term arithmetic progressions and Schur's triples.

In our attempt to avoid a red solution to $x + y = z$ (of distinct integers) or a blue solution to $x + y = 2z$, we consider all possible colourings of the triple $\{2, 4, 6\}$ with two colours, R and B:

- Case 1: If $2 = R, 4 = B, 6 = B$ then $5 = R$ (because of the arithmetic progression $4, 5, 6$). This implies $7 = B$ (because of $2 + 5 = 7$) and $8 = R$ (because of the arithmetic progression $6, 7, 8$) and $10 = R$ (because of the arithmetic progression $4, 7, 10$), But now, $2, 8, 10$ is a red Schur's triple.

- Case 2: If $2 = B, 4 = R, 6 = B$ then $10 = R$ (because of the arithmetic progression $2, 6, 10$).

  Now, if $3 = B$ then $1 = R$ (because of the arithmetic progression $1, 2, 3$) and $9 = B$ (because of $1 + 9 = 10$.) But then $3, 6, 9$ is a blue arithmetic progression.

  If $3 = R$ then $7 = B$ (because of $3 + 4 = 7$) and $5 = R$ (because of the arithmetic progression $5, 6, 7$) and $8 = R$ (because of the arithmetic progression $6, 7, 8$). But then $3 + 5 = 8$ is a red Schur's triple.

- Case 3: If $2 = B, 4 = B, 6 = R$ then $3 = R$ (because of the arithmetic progression $2, 3, 4$). It follows that $9 = B$ (because of $3 + 6 = 9$).

  Now, if $1 = R$ then $5 = B$ (because of $1 + 5 = 6$) and $7 = B$ (because of $1 + 6 = 7$.) But then $5, 7, 9$ is a blue arithmetic progression.

  If $1 = B$ then $5 = R$ (because of the arithmetic progression $1, 5, 9$) and $7 = B$ (because of $1 + 6 = 7$.) Also, $8 = B$ (because of $3 + 5 = 8$). This implies $7 = R$ (because of the arithmetic progression $7, 8, 9$) and then $10 = R$ (because of the arithmetic progression $8, 9, 10$). Now, $3 + 7 = 10$ is a red Schur's triple.

- Case 4: Let $2 = R, 4 = R, 6 = B$.

  Now, if $3 = R$ then $5 = B$ (because of $2 + 3 = 5$) and $7 = B$ (because of $3 + 4 = 7$.) But then $5, 6, 7$ is a blue arithmetic progression.

  If $3 = B$ then $9 = R$ (because of the arithmetic progression $3, 6, 9$) and $5 = B$ (because of $4 + 5 = 9$) and $7 = B$ (because of $2 + 7 = 9$). Now, $5, 6, 7$ is a blue arithmetic progression.

- Case 5: If $2 = R, 4 = B, 6 = R$ then $8 = B$ (because of $2 + 6 = 8$).

  Now, if $5 = R$ then $1 = B$ (because of $1+5 = 6$), $3 = B$ (because of $2+3 = 5$. and $7 = B$ (because of $2 + 5 = 7$.) But then $1, 4, 7$ is a blue arithmetic progression.

  If $5 = B$ then $3 = R$ (because of the arithmetic progression $3, 4, 5$) and $1 = B$ (because of $1 + 2 = 3$) and $7 = R$ (because of the arithmetic progression $1, 4, 7$). Now, if $9 = B$ (because of $2+7 = 9$), then $1, 5, 9$ is a blue arithmetic progression.

- Case 5: If $2 = B, 4 = R, 6 = R$ then $10 = B$ (because of $4 + 6 = 10$).

  Now, if $3 = R$ then $1 = B$ (because of $1 + 3 = 4$), $7 = B$ (because of $3 + 4 = 7$. and $9 = B$ (because of $3+6 = 9$.) It follows that $8 = R$ (because of the arithmetic progression $8, 9, 10$) and $5 = B$ (because of $3 + 5 = 8$). But then $1, 5, 9$ is a blue arithmetic progression.

  If $3 = B$ then $1 = R$ (because of the arithmetic progression $1, 2, 3$) and $5 = B$ (because of $1 + 4 = 5$) and $7 = B$ (because of $1 + 6 = 7$) . But then $3, 5, 7$ is a blue arithmetic progression.

Therefore, it is impossible to colour $[1, 10]$ red and blue and to avoid a red Schur's triple or a blue 3-term arithmetic progression.

Exercise 4.5.13 Schur's theorem. Prove that any finite colouring of positive integers admits a monochromatic solution to $xy = z$.

Solution. Let f be a finite colouring of positive integers. Define a finite colouring g by

$$g(i) = f(2^i),\ i \in \mathbb{N}.$$

By Schur's theorem there is a g-monochromatic triple (x_1, x_2, x_3) such that $x_1+x_2 = x_3$.
Let

$$x = 2^{x_1},\ y = 2^{x_2},\ \text{and } z = 2^{x_3}.$$

Then

$$g(x_1) = g(x_2) = g(x_3) \Rightarrow f(x) = f(y) = f(z)$$

and

$$xy = 2^{x_1} \cdot 2^{x_2} = 2^{x_1+x_2} = 2^{x_3} = z.$$

Exercise 4.5.14 Schur's theorem. Prove that any finite colouring of positive integers admits a monochromatic solution to $xy + x + y = z$.

Solution. Let f be a finite colouring of positive integers. Define a finite colouring g by

$$g(i) = f(2^i - 1),\ i \in \mathbb{N}.$$

By Schur's theorem there is a g-monochromatic triple (x_1, x_2, x_3) such that $x_1+x_2 = x_3$.
Let

$$x = 2^{x_1} - 1,\ y = 2^{x_2} - 1,\ \text{and } z = 2^{x_3} - 1.$$

Then

$$g(x_1) = g(x_2) = g(x_3) \Rightarrow f(x) = f(y) = f(z)$$

and

$$xy + x + y = (2^{x_1} - 1) \cdot (2^{x_2} - 1) + 2^{x_1} - 1 + 2^{x_2} - 1 = 2^{x_1+x_2} - 1 = 2^{x_3} - 1 = z.$$

Exercise 4.5.15 Generalized Schur number. The generalized Schur number $S(4, 5)$ is defined as the smallest positive integer n such that any blue/red colouring of the set $\{1, 2, \ldots, n\}$ contains a blue solution to the equation $\mathcal{L}(4) : x_1 + x_2 + x_3 = x_4$ or a red solution to the equation $\mathcal{L}(5) : x_1 + x_2 + x_3 + x_4 = x_5$.

1. Consider the following 2-colouring of the interval $[1, 13] = \{1, 2, \ldots, 13\}$:

   $$B = \{1, 2, 12, 13\} \text{ and } R = [3, 11].$$

   Check that this colouring does not contain a blue solution to $\mathcal{L}(4)$ or a red solution to $\mathcal{L}(4)$.

2. To show that any blue/red colouring of $[1, 14]$ contains a blue solution to $\mathcal{L}(4)$ or a red solution to $\mathcal{L}(5)$ do the following:

(a) Suppose that $1 \in B$ and build a blue/red colouring trying to avoid a blue solution to $\mathcal{L}(4)$ AND a red solution to $\mathcal{L}(5)$.

(b) Suppose that $1 \in R$ and build a blue/red colouring trying to avoid a blue solution to $\mathcal{L}(4)$ AND a red solution to $\mathcal{L}(5)$.

3. Carefully justify your conclusion that $S(4,5) = 14$.

Solution.

1. Consider $x_1 + x_2 + x_3$, with $x_1, x_2, x_3 \in B$. For this sum to be less than or equal to 13, we have to have $\{x_1, x_2\} \subseteq \{1, 2\}$. But in that case

$$3 \leq x_1 + x_2 + x_3 \leq 6$$

which implies that $\mathcal{L}(4)$ has no a blue solution. If $\{x_1, x_2, x_3, x_4\} \subseteq R$ then

$$x_1 + x_2 + x_3 + x_4 \geq 3 + 3 + 3 + 3 = 12$$

which implies that there is no a red solution to $\mathcal{L}(5)$.

2. (a) Let $B \cup R = [1, 14]$, $B \cap R = \emptyset$. Suppose that $1 \in B$.

   - To avoid a blue solution to $\mathcal{L}(4)$, $3 \in R$. Hence, so far, $\{1\} \subseteq B$ and $\{3\} \subseteq R$.
   - To avoid a red solution to $\mathcal{L}(5)$, $12 \in B$. Hence, so far, $\{1, 12\} \subseteq B$ and $\{3\} \subseteq R$.
   - To avoid a blue solution to $\mathcal{L}(4)$, $14 \in R$. Hence, so far, $\{1, 12\} \subseteq B$ and $\{3, 14\} \subseteq R$.
   - To avoid a red solution $3 + 3 + 3 + 5 = 14$, $5 \in B$. Hence, so far, $\{1, 5, 12\} \subseteq B$ and $\{3, 14\} \subseteq R$.
   - To avoid a blue solution $1 + 2 + 2 = 5$, $2 \in R$. Hence, so far, $\{1, 5, 12\} \subseteq B$ and $\{2, 3, 14\} \subseteq R$.
   - To avoid a red solution $2 + 2 + 3 + 7 = 14$, $7 \in B$. Hence, so far, $\{1, 5, 7, 12\} \subseteq B$ and $\{2, 3, 14\} \subseteq R$.

   But now, we have a blue solution $1 + 1 + 5 = 7$.

   (b) Let $B \cup R = [1, 14]$, $B \cap R = \emptyset$. Suppose that $1 \in R$.

   - To avoid a red solution to $\mathcal{L}(5)$, $4 \in B$. Hence, so far, $\{4\} \subseteq B$ and $\{1\} \subseteq R$.
   - To avoid a blue solution to $\mathcal{L}(4)$, $12 \in R$. Hence, so far, $\{4\} \subseteq B$ and $\{1, 12\} \subseteq R$.
   - To avoid a red solution $1 + 1 + 1 + 9 = 12$, $9 \in B$. Hence, so far, $\{4, 9\} \subseteq B$ and $\{1, 12\} \subseteq R$.
   - To avoid a blue solution $3+3+3 = 9$, $3 \in R$. Hence, so far, $\{4, 9\} \subseteq B$ and $\{1, 3, 12\} \subseteq R$.

   But now, we have a red solution $3 + 3 + 3 + 3 = 12$.

3. In (1) we proved that $S(4, 5) \geq 14$. In (1) we proved that $S(4, 5) \leq 14$. Hence, $S(4, 5) = 14$.

Exercise 4.5.16 Rado's theorem. Is the following equation partition regular over $\mathbb{N}$:

$$\frac{1}{3}x_1 - \frac{1}{4}x_2 + 2x_3 - \frac{1}{12}x_4 = 0?$$

Justify your answer.

Solution. Notice that the given equation is equivalent to the equation

$$4x_1 - 3x_2 + 24x_3 - x_4 = 0.$$

Since $a_1 + a_2 + a_4 = 4 - 3 - 1 = 0$, by Rado's theorem this equation is partition regular over $\mathbb{N}$, i.e., any finite colouring of positive integers contains a monochromatic solution of this equation. But that implies that any finite colouring of positive integers contains a monochromatic solution of the original equation. Therefore, the given equation is partition regular over $\mathbb{N}$.

Exercise 4.5.17 Rado's theorem. Let the equation

$$x_1 + x_2 - 4x_3 = 0$$

be given. Is it possible to find a finite colouring that does not contain monochromatic solutions of the given equation? If it is, find such a colouring. If it is not, justify your answer.

Solution. Here $a_1 = a - 2 = 1$ and $a_3 = -4$. From $a_1 + a_2 = 2$, $a_1 + a_3 = -3$, $a_2 + a_3 = -3$, and $a_1 + a_2 + a_3 = -2$ we conclude, via Rado's theorem, that the given equation is not partition regular over $\mathbb{N}$. Therefore, there is a finite colouring of positive integers that does not contain a monochromatic solution of the given equation. By the proof of Rado's theorem that was demonstrated in the class, the following colouring would do.

Note that $|a_1| + |a_2| + |a_3| = 1 + 1 + 4 = 6$. We take $p = 7$, a prime number greater than $|a_1| + |a_2| + |a_3|$, and defined a 6-colouring $c : \mathbb{N} \to [1, 6]$ in the following way.

For $n \in \mathbb{N}$ we find $k, l \in \{0, 1, 2 \ldots\}$ and $i \in \{1, 2, 3, 4, 5, 6\}$ such that $n = 7^k(7 \cdot l + i)$. Then, by definition,

$$c(n) = c(7^k(7 \cdot l + i)) = i.$$

Now if there is a c-monochromatic solution of the given equation then

$$x_1 = 7^k(7 \cdot l + i), x_2 = 7^m(7 \cdot n + i), x_3 = 7^r(7 \cdot s + i),$$

for some $k, l, m, n, r, s \in \{0, 1, 2 \ldots\}$ and $i \in \{1, 2, 3, 4, 5, 6\}$. Hence

$$0 = \quad 7^k(7 \cdot l + i) + 7^m(7 \cdot n + i) - 4 \cdot 7^r(7 \cdot s + i)$$

If $k = m = r$ then it follows that

$$\begin{aligned} 0 &= \quad (7 \cdot l + i) + (7 \cdot n + i) - 4 \cdot (7 \cdot s + i) \\ &= \quad 7 \cdot (l + n + s) - 2i. \end{aligned}$$

This would imply that $2i$ is divisible by 7, what is impossible because $i \in \{1, 2, 3, 4, 5, 6\}$.

Other cases, lead to a contradiction in a similar way.

Exercise 4.5.18 Rado's theorem. Check if the equation $x - y + 5z - 3w = 0$ is partition regular over $\mathbb{N}$.

Solution. Yes, the given equation is partition regular over $\mathbb{N}$. If we look at all the coefficients: $c_1 = 1, c_2 = -1, c_3 = 5, c_4 = -3$, the sum of the coefficients c_1 and c_2 turns out to be equal to 0. Hence, by Rado's theorem this equation is partition regular, i.e it has a monochromatic solution for any finite colouring of the set of natural numbers.

Exercise 4.5.19 Rado graph. On the Rado graph, will vertex 3 be connected to vertex 9?

Solution. On the Rado graph, vertex 3 will be connected to vertices that have a non-zero 3rd bit of the binary representation. Vertex 9 in binary representation is 1001. The 3rd bit of 1001 is 0, therefore vertex 3 will not be connected to vertex 9.

Chapter 5

The Hales-Jewett Theorem

5.1 Combinatorial Lines

> *Last year I went fishing with Salvador Dali. He was using a dotted line. He caught every other fish.* — Steven Alexander Wright, American comedian, actor and writer, 1955–

Alphabet. For $m \in \mathbb{N}$, any set A such that $|A| = m$ is called an alphabet on m symbols.

Example 5.1.1 Let $A = \{a, 1, \triangle\}$. Then A is an alphabet on $|A| = 3$ symbols. □

Words. Let A be an alphabet on m symbols. For $n \in \mathbb{N}$, any function $w : [1, n] \to A$ is called a word of length n on the alphabet A. If $w(i) = a_i$, $i \in [1, n]$ then we write

$$w = a_1 a_2 \cdots a_n.$$

The set of all words of length n on the alphabet A is denoted by A^n. We say that A^n is *the n-dimensional cube on alphabet A.*

Example 5.1.2 Let $A = \{a, 1, \triangle\}$ be an alphabet on three symbols. Then $w = a1a1a1$ is a word of length 6 on the alphabet A. Here $w : [1, 6] \to A$ is defined as $w(1) = w(3) = w(5) = a$ and $w(2) = w(4) = w(6) = 1$.

Also, $A^2 = \{w : w : [1, 2] \to A\} = \{aa, a1, a\triangle, 1a, 11, 1\triangle, \triangle a, \triangle 1, \triangle\triangle\}$. □

Roots. Let A be an alphabet (on m symbols) and let $*$ be a symbol such that $* \notin A$. We consider the alphabet $A_* = A \cup \{*\}$. Any word on the alphabet A_*, i.e, any element of $(A_*)^n = A_*^n$, for some $n \in \mathbb{N}$, that contains the symbol $*$ is called a root.

Example 5.1.3 Let $A = \{a, 1, \triangle\}$ be an alphabet on three symbols. Then $A_* = A \cup \{*\} = \{a, 1, \triangle\, *\}$. By definition, $1 * \triangle$ and $a * a*$ are two roots. □

Words From Roots. For a root $\tau \in A_*^n$ and a symbol $a \in A$ we define the word $\tau_a \in A^n$ in the following way. For $i \in [1, n]$

$$\tau_a(i) = \begin{cases} \tau(i) & \text{if} \quad \tau(i) \neq *, \\ a & \text{if} \quad \tau(i) = *. \end{cases}$$

Example 5.1.4 Let $A = \{a, b, c\}$ and let $\tau = *\, b\; c\; b \in A_*^4$ be a root. Then

$$\begin{aligned} \tau_a &= && \square\, b\; c\; b \\ \tau_b &= && \square\, b\; c\; b \\ \tau_c &= && \square\, b\; c\; b. \end{aligned}$$

□

Example 5.1.5 *Example:* Let $A = [1,4]$ and let $\tau = *\ 1\ 3\ *\ 4 * \in A^6_*$ be a root. Then

$$\tau_2 = \square\ 1\ 3\ \square\ 4\ \square.$$

$\square$

Combinatorial Line: Let A be an alphabet, let $n \in \mathbb{N}$, and let $\tau \in A^n_*$ be a root. A combinatorial line in A^n rooted in τ is the set of words

$$L_\tau = \{\tau_a : a \in A\}.$$

Observation 5.1.6 $L\tau \subseteq A^n$.

Example 5.1.7 Let $A = \{1,2,3\}$ and $n = 2$. Find all combinatorial lines in A^2.

1. All roots in A^2_*:

$$\begin{aligned} \tau &= & *\,1 \\ \sigma &= & *\,2 \\ \theta &= & *\,3 \\ \rho &= & 1\,* \\ \chi &= & 2\,* \\ \phi &= & 3\,* \\ \mu &= & *\ *. \end{aligned}$$

2. All combinatorial lines:

$$\begin{aligned} L_\tau &= & \{1\ 1, 2\ 1, 3\ 1\} \\ L_\sigma &= & \{1\ 2, 2\ 2, 3\ 2\} \\ L_\theta &= & \{1\ 3, 2\ 3, 3\ 3\} \\ L_\rho &= & \{1\ 1, 1\ 2, 1\ 3\} \\ L_\chi &= & \{2\ 1, 2\ 2, 2\ 3\} \\ L_\phi &= & \{3\ 1, 3\ 2, 3\ 3\} \\ L_\mu &= & \{1\ 1, 2\ 2, 3\ 3\}. \end{aligned}$$

$\square$

All combinatorial lines - Another view:

L_τ	L_σ	L_θ	L_ρ	L_χ	L_ϕ	L_μ
1 1	1 2	1 3	1 1	2 1	3 1	1 1
2 1	2 2	2 3	1 2	2 2	3 2	2 2
3 1	3 2	3 3	1 3	2 3	3 3	3 3

All combinatorial lines - Another view (see Figure 5.1.8):

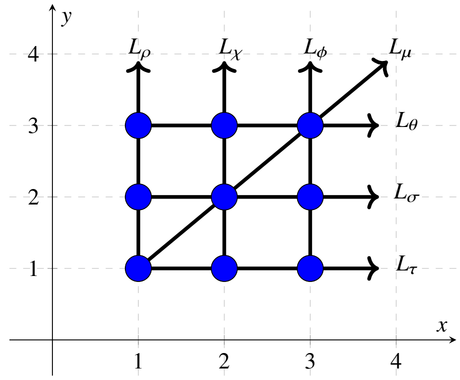


Figure 5.1.8 All combinatorial lines in $[1, 3]^2$.

It looks like... (See Figure 5.1.9.)

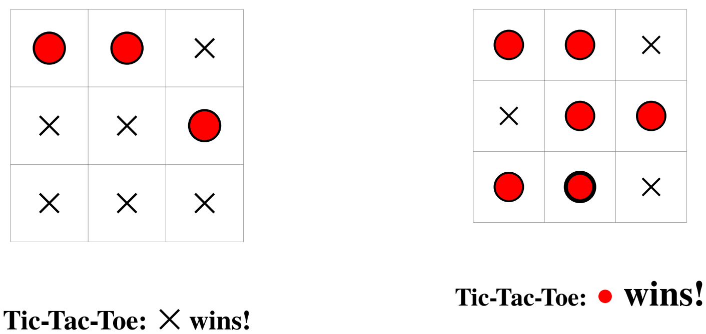


Figure 5.1.9 Tic-Tac-Toe: it's a win!

Example 5.1.10 *What About Combinatorial Lines in* $[1, 3]^3$*?* □

Consider roots:

$$\tau = *\,2\,3, \quad \sigma = *\ *\ 3, \ \theta = *\ *\ *.$$

Then (also see Figure 5.1.11):

L_τ	L_σ	L_θ
123	113	111
223	223	222
323	333	333

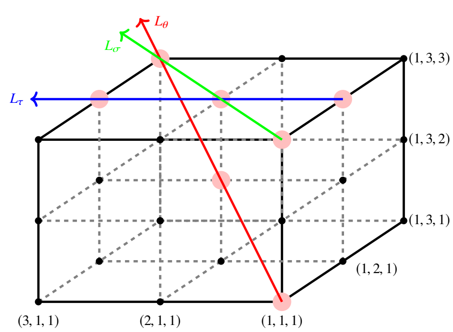


Figure 5.1.11 Three combinatorial lines in $[1, 3]^3$.

$4 \times 4 \times 4$ *Tic-Tac-Toe*

This is just a 2-player Tic-Tac-Toe game on a 4 x 4 x 4 cube. The player wins who first gets four in a row of his own pieces - either horizontal, vertical, or diagonal. See Figure 5.1.12. (Source BoardGameGeek.)

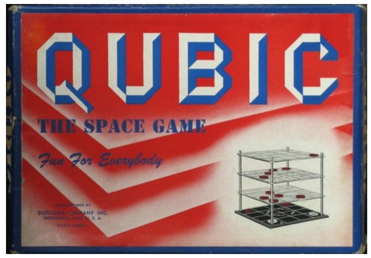


Figure 5.1.12 The first edition of Qubic by any company was produced by Duplicon in 1946 or 1947.

Rules of the Game. Create a monochromatic combinatorial line in $[1,4]^3$.

Question 5.1.13 Let A be an alphabet on m symbols and let A^n be the n-dimensional cube on alphabet A:

$$A^n = \{a_1 a_2 \cdots a_n : a_i \in A, i \in [1,n]\}.$$

If A^n is k-coloured, can we be sure that A^n contains a monochromatic combinatorial line? □

More precisely . . . Let $m, k \in \mathbb{N}$ and let A be an alphabet on m symbols. Does there exist an $n \in \mathbb{N}$ such that whenever A^n is k-coloured there exists a monochromatic line? See Figure 5.2.4.

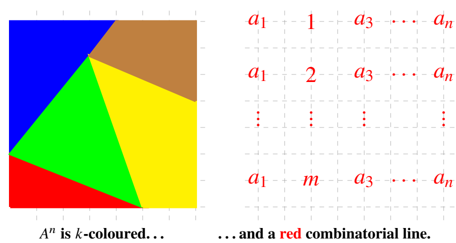


Figure 5.1.14 Is it true that whenever A^n is k-coloured there exists a monochromatic line?

5.2 The Hales-Jewett Theorem

> *The truth is outside of all fixed patterns.* — Bruce Lee, a Hong Kong American martial artist and actor, 1940 — 1973.

Theorem 5.2.1 The Hales-Jewett Theorem. *Let $m, k \in \mathbb{N}$ and let A be an alphabet on m symbols. There exists an $n \in \mathbb{N}$ such that whenever A^n is k-coloured there exists a monochromatic line.*

Definition 5.2.2 The smallest such n is denoted by $HJ(m,k)$. ◊

From "The Mathematical Coloring Book" - page 518 [7]:

> This result — as is often case in mathematics — was obtained by the young mathematicians: Alfred W. Hales was 23, and Robert I. Jewett 24. Alfred email to me [A. Soifer], on January 3, 2007, and recalled how it all come about: "Bob and I were undergraduates at Caltech together - he was a year ahead of me. We had common interest in both math and volleyball. We also both worked in Sol Golomb's coding theory group at the Jet Propulsion Lab, and we continued doing this when we were in graduate school - he at the University of Oregon and I at Caltech."

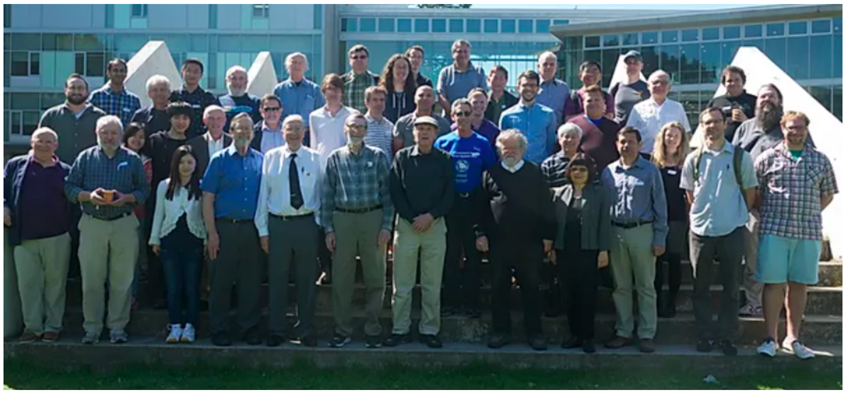


Figure 5.2.3 50 Years of the Hales-Jewett Theorem Conference, May 6-8, 2016, WWU

Proof. (The Hales-Jewett theorem)

1. *Settings:* Let $m, k \in \mathbb{N}$. As an alphabet on m symbols we take $A = [1, m]$. *Reminder:* A root $\tau \in [1, m]^n_*$ is an n-word on $m + 1$ symbols, $1, 2, \ldots, m$ and $*$, that contains the symbol $*$. A combinatorial line in $[1, m]^n$ rooted in τ is the set of words

$$L_\tau = \{\tau_a : a \in [1, m]\}.$$

Here, for $a \in [1, m]$ and $i \in [1, n]$,

$$\tau_a(i) = \begin{cases} \tau(i) & \text{if} \quad \tau(i) \neq *, \\ a & \text{if} \quad \tau(i) = *. \end{cases}$$

Focussed and Colour-Focussed Lines:

- Let $r \in \mathbb{N}$ and let and $\tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(r)} \in [1, m]^n_*$ be r roots. We say that the corresponding combinatorial lines are focussed at $f \in [1, m]^n$ if

$$\tau^{(1)}_m = \tau^{(2)}_m = \cdots = \tau^{(r)}_m = f.$$

Example:

Consider $\tau^{(1)}, \tau^{(2)}, \tau^{(3)} \in [1, 4]^4_*$ given by

$$\tau^{(1)} = * \; * \; 3 \, *, \; \tau^{(2)} = * \, 4 \, 3 \, *, \; \tau^{(3)} = * \, 4 \, 3 \, 4.$$

Then

$$\tau^{(1)}_4 = \square \, \square \, 3 \, \square, \; \tau^{(2)}_4 = \square \, 4 \, 3 \, \square, \; \tau^{(3)}_4 = \square \, 4 \, 3 \, 4.$$

See Figures 5.2.4 and Figure 5.2.5.

Hence the corresponding combinatorial lines are focussed at $f = 4\,4\,3\,4$:

$L_{\tau^{(1)}}$	$L_{\tau^{(2)}}$	$L_{\tau^{(3)}}$
1 1 3 1	1 4 3 1	1 4 3 4
2 2 3 2	2 4 3 2	2 4 3 4
3 3 3 3	3 4 3 3	3 4 3 4
4 4 3 4	4 4 3 4	4 4 3 4

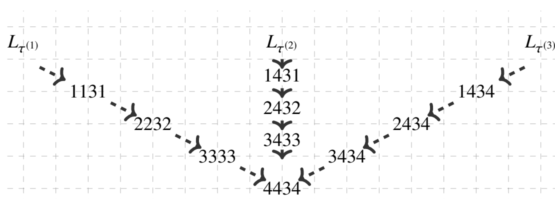


Figure 5.2.4 Three focussed lines in $[1, 4]^4$.

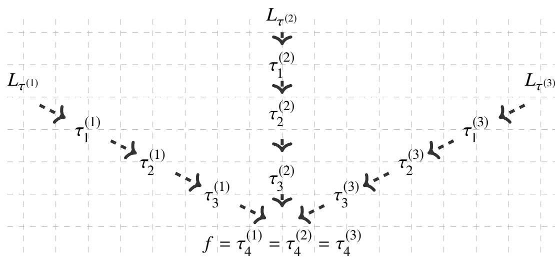


Figure 5.2.5 Three lines in $[1, 4]^4$ focussed at f.

- Let c be a k-colouring of $[1, m]^n$ and let and $\tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(r)} \in [1, m]^n_*$ be r roots. See Figures 5.2.6.

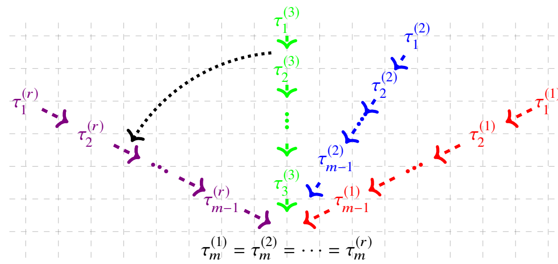


Figure 5.2.6 r colour-focussed lines: different colours and $\tau_m^{(1)} = \tau_m^{(2)} = \cdots = \tau_m^{(r)}$.

Strategy. Induction on m.

Reminder: The Hales-Jewett Theorem. Let $m, k \in \mathbb{N}$ and let A be an alphabet on m symbols. There exists an $n \in \mathbb{N}$ such that whenever A^n is k-coloured there exists a monochromatic line.

Base Case. If $m = 1$ then $H(1, k) = 1$ for any number of colours k.

Inductive step. Given $m > 1$, we assume that $HJ(m - 1, k)$ exists for all k.

(a) *Claim.* For all $1 \leq r \leq k$, there exists n such that whenever $[1, m]^n$ is k coloured, there exists *either* a monochromatic line *or* r colour-focussed lines.

(b) *Base Case.* Let $k \in \mathbb{N}$ and let $r = 1$. We take $n = HJ(m - 1, k)$.
Let c be a k-colouring of $[1, m]^n$. See Figure 5.2.7.

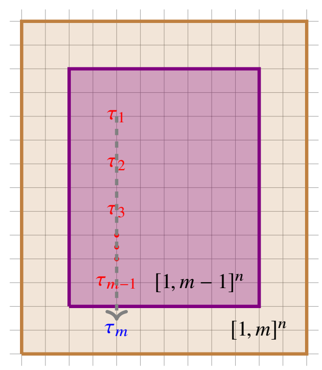


Figure 5.2.7 The colouring c of $[1, m]^n$ induces a k-colouring of $[1, m - 1]^n$. Our choice of n guarantees the existence of a monochromatic line in $[1, m - 1]^n$.

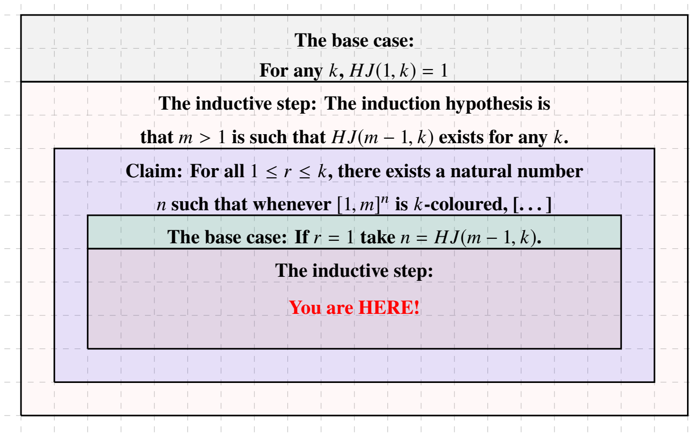


Figure 5.2.8 Where are you?

(c) *Inductive Step.* Let $r \in [1, k-1]$ and let $n = n(r)$ be such that whenever $[1,m]^n$ is k coloured, there exists *either* a monochromatic line *or* r colour-focussed lines. Let $n' = HJ(m-1, k^{m^n})$ and let $N = n + n'$. Let c be a k-colouring of $[1,m]^N = [1,m]^{n+n'}$ *without a monochromatic line*. See Figure 5.2.9.

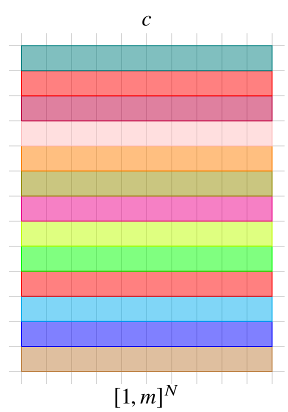


Figure 5.2.9 The k-colouring c of $[1,m]^N = [1,m]^{n+n'}$ *without a monochromatic line*.

i. *A c induced k^{m^n}-colouring of* $[1, m-1]^{n'}$*: Step 1.* See Figure 5.2.10.

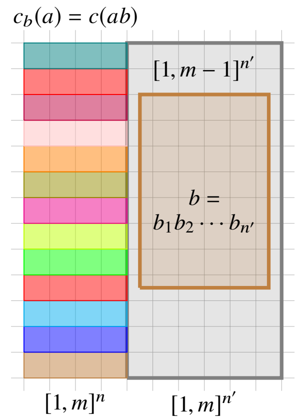


Figure 5.2.10 Choose $b = b_1 b_2 \cdots b_{n'} \in [1, m-1]^{n'}$. Consider c_b, a k-colouring of $[1, m]^n$ such that for $a \in [1, m]^n$, $c_b(a) = c(ab)$.

Step 2. Note that there are k^{m^n} k-colourings of $[1, m]^n$. See Figur 5.2.11.

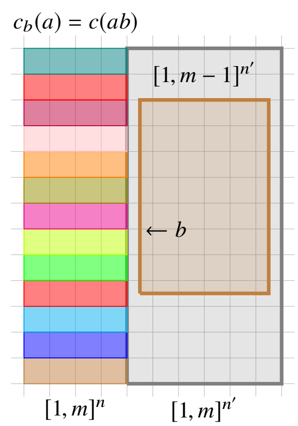
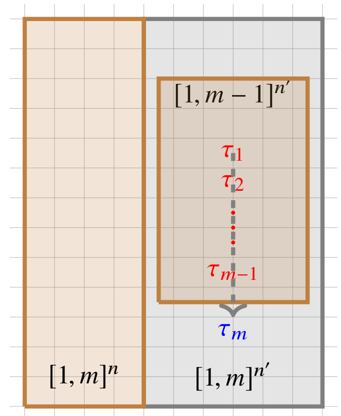


Figure 5.2.11 The mapping $\chi : b \mapsto c_b$ is a k^{m^n}-colouring of $[1, m-1]^{n'}$.

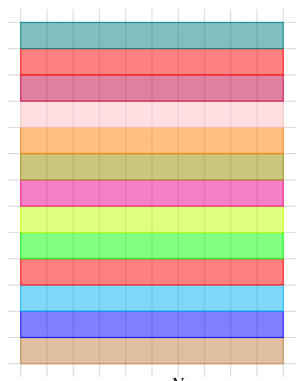
Step 3. There is a χ-monochromatic line in $[1, m-1]^{n'}$. See Figure 5.2.12.



$[1, m]^N = [1, m]^{n+n'} = [1, m]^n \times [1, m]^{n'}$

Figure 5.2.12 There is a χ-monochromatic line L_τ in $[1, m-1]^{n'}$.

ii. *Reminder - Inductive Step.* Let $r \in [1, k-1]$ and let $n = n(r)$ be such that whenever $[1, m]^n$ is k–coloured, there exists *either* a monochromatic line *or* r colour-focussed lines. Let $n' = HJ(m-1, k^{m^n})$ and let $N = n + n'$. Let c be a k-colouring of $[1, m]^N = [1, m]^{n+n'}$ *without a monochromatic line*. See Figure 5.2.13.



$[1, m]^N$

Figure 5.2.13 The k-colouring c of $[1, m]^N = [1, m]^{n+n'}$ *without a monochromatic line*.

iii. *A c induced k-colouring of* $[1, m]^n$*: Step 1* There is a χ-monochromatic line in $[1, m-1]^{n'}$. See Figure 5.2.14.

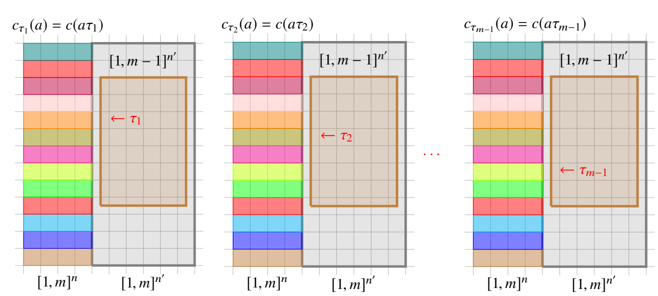


Figure 5.2.14 L_τ is monochromatic: $c_{\tau_1} = c_{\tau_2} = \cdots = c_{\tau_{m-1}}$.

Step 2. A k-colouring c_τ of $[1, m]^n$ emerges. See Figure 5.2.15.

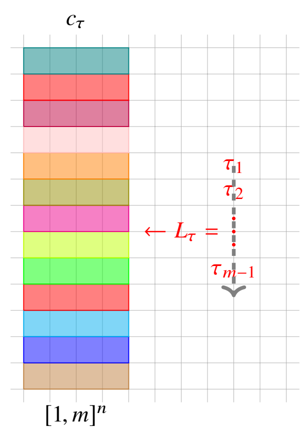


Figure 5.2.15 The k-colouring c_τ of $[1,m]^n$ is with the property that, for any $a \in [1,m]^n$ and any $i \in [1,m-1]$, $c_\tau(a) = c(a\tau_i)$.

Step 3. Back to colour-focussed lines. See Figure 5.2.16.

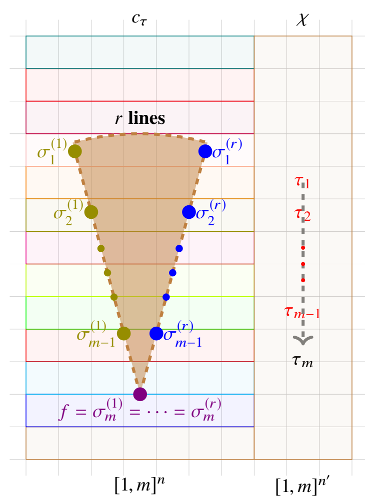


Figure 5.2.16 There are r c_τ-coloured-focussed lines $L_{\sigma^{(1)}}, \ldots, L_{\sigma^{(r)}}$ in $[1,m]^n$ with the focus f and one χ-monochromatic line L_τ in $[1,m]^{n'}$ with the focus τ_m. None of the lines $L_{\sigma^{(1)}}, \ldots, L_{\sigma^{(r)}}$ is monochromatic.

iv. *Making new roots from old:* We define $r+1$ roots in $[1,m]_*^N$ as follows (see Figure 5.2.17):

$$\tau^{(1)} = \sigma^{(1)}\tau,\ \tau^{(2)} = \sigma^{(2)}\tau, \ldots, \tau^{(r)} = \sigma^{(r)}\tau,\ \tau^{(r+1)} = f\tau.$$

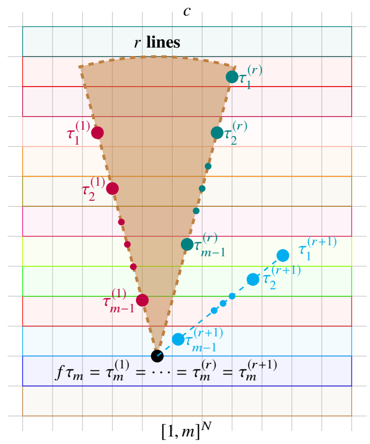


Figure 5.2.17 There are $r + 1$ c-coloured-focussed lines $L_{\tau^{(1)}}, \ldots, L_{\tau^{(r+1)}}$ in $[1, m]^N$ with the focus $f\tau_m$.

v. Where Are You?

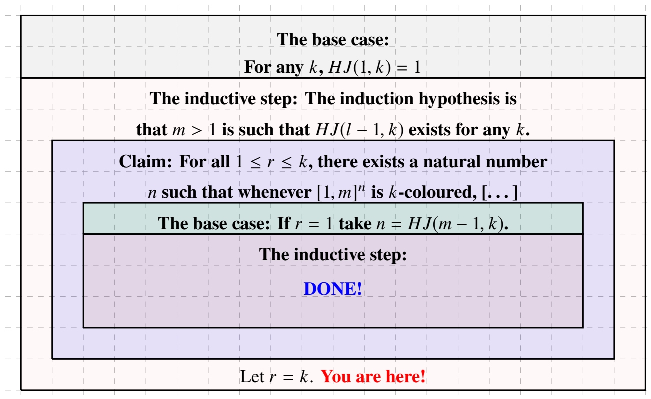


Figure 5.2.18 Where are you?

vi. *Let $r = k$.* See Figure 5.2.19.

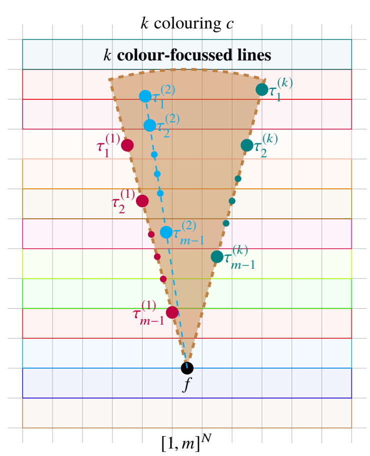


Figure 5.2.19 What is the colour of the focus f? There is a monochromatic line!

vii. *Done!*

$$HJ(m - 1, k) \text{ exists } \Rightarrow HJ(m, k) \text{ exists}$$

-

$\blacksquare$

Resources.

1. See [7], pp. 517-518.
2. Wikipedia
3. Ramsey Theory - by I. Leader - pp 8 -10
4. The Hales-Jewett Theorem - by Andreas Razen
5. The Hales-Jewett Theorem - Blog post by Jay Cumings
6. Blogging, Tic Tac Toe and the Future of Math - by Steve Landsburg

5.3 Exercises

These exercises are based on the material covered in Chapter 5.

Exercise 5.3.1 van der Waerden's theorem. Use the Hales-Jewett theorem to prove van der Waerden's theorem.

Solution. Let $l, k \in \mathbb{N}$ be given. Let $c : \mathbb{N} \to \{1, 2, \ldots, k\}$ be a k-colouring of the set of natural numbers. Let $N = HJ(l, k)$.

We define a k-colouring of the N-cube $[1, l]^N$ as follows

$$c'(x_1 x_2 \cdots x_N) = c(x_1 + x_2 + \ldots + x_N),\ x_1 x_2 \cdots x_N \in [1, l]^N.$$

By the Hales-Jewett theorem there is a c'-monochromatic line rooted in the root $\tau \in [1, l]^N_*$. Let $S \subset [1, N]$ be such that

$$\tau(i) \in [1, l] \text{ if } i \in S \text{ and } \tau(i) = * \text{ if } i \in [1, N]\backslash S.$$

Let

$$a = \sum_{i \in S} \tau(i) \text{ and } d = |[1, N]\backslash S|.$$

Note that

$$\begin{aligned}
\sum_{i=1}^N \tau_1(i) &= \sum_{i \in S} \tau_1(i) + \sum_{i \in [1,N]\backslash S} \tau_1(i) = a + \sum_{i \in [1,N]\backslash S} 1 = a + d \\
\sum_{i=1}^N \tau_2(i) &= \sum_{i \in S} \tau_2(i) + \sum_{i \in [1,N]\backslash S} \tau_2(i) = a + \sum_{i \in [1,N]\backslash S} 2 = a + 2d \\
&\vdots \\
\sum_{i=1}^N \tau_l(i) &= \sum_{i \in S} \tau_l(i) + \sum_{i \in [1,N]\backslash S} \tau_l(i) = a + \sum_{i \in [1,N]\backslash S} l = a + ld.
\end{aligned}$$

On the other hand

$$c'(\tau_1) = c'(\tau_2) = \cdots = c'(\tau_l)$$

which together with

$$c'(\tau_j) = c\left(\sum_{i=1}^N \tau_j(i)\right) = c(a + jd),\ \text{for each } j \in [1, l],$$

implies that

$$c(a + d) = c(a + 2d) = \cdots = (a + ld).$$

Thus, there is a c-monochromatic l-term arithmetic progression.

Exercise 5.3.2 Combinatorial lines. Let $A = \{a, b, c, d\}$. Find all combinatorial lines in A^2.

Solution. Since all roots in A^2_* are given by

$$*a, *b, *c, *d, a*, b*, c*, d*, **,$$

all combinatorial lines given by

aa	ab	ac	ad	aa	ba	ca	da	aa
ba	bb	bc	bd	ab	bb	cb	db	bb
ca	cb	cc	cd	ac	bc	cc	dc	cc
da	db	dc	dd	ad	bd	cd	dd	dd

Exercise 5.3.3 Combinatorial lines.

1. Draw the 2-dimensional cube corresponding to $A = \{1, 2, 3\}$. What does each word in the cube represent if we are playing tic-tac-toe?

2. First, list all the combinatorial lines in A^2. Explain why a combinatorial line is a winning line in tic-tac-toe. Are there any winning lines in tic-tac-toe that is not a combinatorial line?

3. In Figure 5.3.4 you will see 4 lines on a 4x4x4 cube. For each line, write the combinatorial line and the root associated with the combinatorial line. If it is not a combinatorial line, briefly explain why.

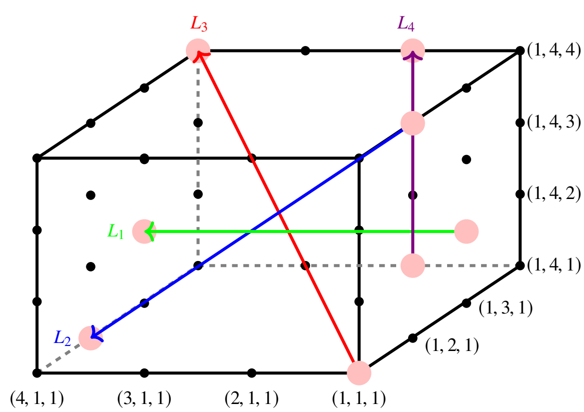


Figure 5.3.4 Three combinatorial lines in $[1, 3]^4$.

Solution.

1. Each word on the 2-dimensional cube corresponds to a position on the tic-tac-toe board.

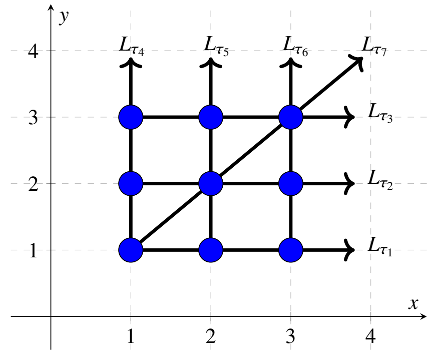


Figure 5.3.5 All combinatorial lines in $[1, 3]^2$.

2. Below is the list of all the roots in A^2 together with the corresponding combinatorial line:

$$\tau_1 = *1 \implies L_{\tau_1} = \{1\ 1, 2\ 1, 3\ 1\}$$
$$\tau_2 = *2 \implies L_{\tau_2} = \{1\ 2, 2\ 2, 3\ 2\}$$
$$\tau_3 = *3 \implies L_{\tau_3} = \{1\ 3, 2\ 3, 3\ 3\}$$
$$\tau_4 = 1* \implies L_{\tau_4} = \{1\ 1, 1\ 2, 1\ 3\}$$
$$\tau_5 = 2* \implies L_{\tau_5} = \{2\ 1, 2\ 2, 2\ 3\}$$
$$\tau_6 = 3* \implies L_{\tau_6} = \{3\ 1, 3\ 2, 3\ 3\}$$
$$\tau_7 = ** \implies L_{\tau_7} = \{1\ 1, 2\ 2, 3\ 3\}$$

Based on the tic-tac-toe board, we can see that τ_1, τ_2, τ_3 correspond to a horizontal winning line in each row, τ_4, τ_5, τ_6 correspond to a vertical winning line in each column, and τ_7 is the diagonal winning line starting at 1 1 and finishing at 3 3.

One winning line that cannot be represented by a combinatorial line is $\{31, 22, 13\}$. This is because there is no root to represent this line. We can see that each word is changing in more than one position and to different letters.

3. We can see that L_2 is not a combinatorial line. Observe that the line L_2 begins at $(1, 2, 4)$ and ends at $(4, 2, 1)$. This means L_2 contains the points $\{1\ 2\ 4, 2\ 2\ 3, 3\ 2\ 2, 4\ 2\ 1\}$. Since the first and third letter in each word change to different letters at different times, this set of words cannot be obtained from a root. It follows that L_2 is not a combinatorial line.

The corresponding roots and lines for the rest are:

$$\tau_1 = *32 \implies L_{\tau_1} = L_1 = \{1\ 3\ 2, 2\ 3\ 2, 3\ 3\ 2, 4\ 3\ 2\}$$
$$\tau_3 = *** \implies L_{\tau_3} = L_3 = \{1\ 1\ 1, 2\ 2\ 2, 3\ 3\ 3, 4\ 4\ 4\}$$
$$\tau_4 = 24* \implies L_{\tau_4} = L_4 == \{2\ 4\ 1, 2\ 4\ 2, 2\ 4\ 3, 2\ 4\ 4\}$$

Exercise 5.3.6 Combinatorial lines. Let $A = \{a, b, c, d\}$.

Can you find a 2-colouring of A^2 that does not contain a monochromatic combinatorial line? If yes, does this contradict the claim of Hales-Jewett theorem?

Justify your answer.

Solution. A possible red/black colouring is given by:

aa	ab	ac	ad	aa	ba	ca	da	aa
ba	bb	bc	bd	ab	bb	cb	db	bb
ca	cb	cc	cd	ac	bc	cc	dc	cc
da	db	dc	dd	ad	bd	cd	dd	dd

This does not contradict the Hales-Jewett theorem. It just shows that $HJ(4,2) > 2$.

Exercise 5.3.7 Combinatorial lines. Let $m, n \in \mathbb{N}$ and let $|A| = m$, i.e. let A be an alphabet on m symbols.

Prove that the number of combinatorial lines in A^n equals $(m+1)^n - m^n$.

Solution. Observe that

$$\#\text{ of combinatorial lines} = \#\text{ of roots in } (A \cup \{*\})^n.$$

The number of all words of length n on the alphabet $A \cup \{*\}$ equals to $(m+1)^n$. Since the number of all all words of length n on the alphabet A equals to m^n it follows that

$$\#\text{ of combinatorial lines} = \#\text{ of roots in } (A \cup \{*\})^n = (m+1)^n - m^n.$$

Exercise 5.3.8 Arithmetic progressions. Use the Hales-Jewett theorem to prove that any 2-colouring of positive integers contains a monochromatic 5-term arithmetic progression.

Solution. Let $c : \mathbb{N} \to \{1,2\}$ be a 2-colouring of the set of natural numbers. Let $N = HJ(5,2)$.

We define a 2-colouring of the N-cube $[1,5]^N$ as follows

$$c'(x_1 x_2 \cdots x_N) = c(x_1 + x_2 + \ldots + x_N),\ x_1 x_2 \cdots x_N \in [1,5]^N.$$

By the Hales-Jewett theorem there is a c'-monochromatic line rooted in the root $\tau \in [1,5]^N_*$. Let $S \subset [1,N]$ be such that

$$\tau(i) \in [1,N] \text{ if } i \in S \text{ and } \tau(i) = * \text{ if } i \in [1,N] \backslash S.$$

Let

$$a = \sum_{i \in S} \tau(i) \text{ and } d = |[1,N] \backslash S|.$$

Note that

$$\begin{aligned}
\sum_{i=1}^N \tau_1(i) &= \sum_{i \in S} \tau_1(i) + \sum_{i \in [1,N]\backslash S} \tau_1(i) = a + \sum_{i \in [1,N]\backslash S} 1 = a + d \\
\sum_{i=1}^N \tau_2(i) &= \sum_{i \in S} \tau_2(i) + \sum_{i \in [1,N]\backslash S} \tau_2(i) = a + \sum_{i \in [1,N]\backslash S} 2 = a + 2d \\
&\ \vdots \\
\sum_{i=1}^N \tau_5(i) &= \sum_{i \in S} \tau_5(i) + \sum_{i \in [1,N]\backslash S} \tau_5(i) = a + \sum_{i \in [1,N]\backslash S} 5 = a + 5d.
\end{aligned}$$

On the other hand

$$c'(\tau_1) = c'(\tau_2) = \cdots = c'(\tau_5)$$

which together with

$$c'(\tau_j) = c\left(\sum_{i=1}^N \tau_j(i)\right) = c(a + jd),\ \text{for each } j \in [1,5],$$

implies that

$$c(a+d) = c(a+2d) = \cdots = (a+5d).$$

Thus, there is a c-monochromatic 5-term arithmetic progression.

Exercise 5.3.9 Hilbert's Cube Lemma. In 1892, David Hilbert, one of the most prominent mathematicians of the 19th and early 20th centuries, proved a statement known as Hilbert's Cube Lemma:

> For any r-colouring χ of $\mathbb{N}$ and for any $m \in \mathbb{N}$ there exist $a, a_1, \ldots, a_m \in \mathbb{N}$ such that the m-cube, i.e. the set
>
> $$Q_m(a, a_1, \ldots, a_m) = \{a + \sum_{i=0}^{m} \epsilon_i a_i : \epsilon_i \in \{0, 1\}\}$$
>
> is monochromatic.

Together with Schur's theorem, van der Waerden's theorem and Ramsey's theorem, Hilbert's Cube Lemma is considered as one of the early pillars of Ramsey Theory.

The purpose of this exercise is to establish the proof of Hilbert's Cube Lemma in the case $m = 3$. Those students interested to prove the general case should use the ideas presented below and mathematical induction.

1. Determine all elements of the 3-cube $Q(1, 2, 3, 4)$.

2. Recall that χ is an r-colouring of $\mathbb{N}$. Prove that the interval $[k+1, k+(r+1)]$, where k is a non-negative integer, contains a χ-monochromatic 1-cube.

3. We say that a χ-monochromatic 1-cube $Q(a, a_1) \subset [k+1, k+(r+1)]$, where k is a non-negative integer, is of the type (a_1, i) if $\chi(Q(a, a_1)) = i$.

   How many different types of χ-monochromatic 1-cubes in $[k+1, k+(r+1)]$ are possible?

4. Consider the interval $[1, (r^2+1)(r+1)]$ and observe that this interval is the union of r^2+1 consecutive intervals of length $r+1$: $[1, (r^2+1)(r+1)] = [1, r+1] \cup [(r+1)+1, 2(r+1)] \cup \cdots \cup [r^2(r+1)+1, (r^2+1)(r+1)]$.

   - Prove that there are p, q, $0 \leq p \leq q \leq r^2$ such that the intervals $[p(r+1)+1, (p+1)(r+1)]$ and $[q(r+1)+1, (q+1)(r+1)]$ contain χ-monochromatic 1-cubes of the same type.
   - Prove that the interval $[1, (r^2+1)(r+1)]$ contains a χ-monochromatic 2-cube.

5. Observe that from (4) it follows that any interval $[k+1, k+(r^2+1)(r+1)]$, where k is a non-negative integer, contains a χ-monochromatic 2-cube $Q(a, a_1, a_2)$ with $a_1 \in [1, r]$.

   Say that a χ-monochromatic 2-cube $Q(a, a_1, a_2) \subset [k+1, k+(r^2+1)(r+1)]$, where k is a non-negative integer and $a_1 \in [1, r]$, is of the type (a_1, a_2, i) if $\chi(Q(a, a_1, a_2)) = i$.

   Establish that the number of possible types of χ-monochromatic 2-cubes in $[k+1, k+(r^2+1)(r+1)]$ is less than $(1+r)^5$.

6. Prove that the interval $[1, (r^2+1)(r+1)^6] = [1, (r+1)^5 \cdot (r^2+1)(r+1)]$ contains a χ-monochromatic 3-cube.

Solution.

1. By definition

   $Q(1, 2, 3, 4) = \{1 + \epsilon_1 \cdot 2 + \epsilon_2 \cdot 3 + \epsilon_3 \cdot 4 : \epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}\} = \{1, 1 + 2, 1 + 3, 1 + 4, 1 + 2 + 3, 1 + 2 + 4, 1 + 3 + 4, 1 + 2 + 3 + 4\} = \{1, 3, 4, 5, 6, 7, 10\}$.

2. By the Pigeonhole Principle, there are $a, b \in [k + 1, k + (r + 1)]$, $a \leq b$, such that $\chi(a) = \chi(b)$. Hence the 1-cube $Q(a, b - a) = \{a, b\}$ is monochromatic.

3. Since $a_1, i \in [1, r]$, there are r^2 possible types of χ-monochromatic 1-cubes in $[k + 1, k + (r + 1)]$.

4.
   - Since there are r^2 possible types of χ-monochromatic cubes and since there are $r^2 + 1$ intervals, by the Pigeonhole Principle, there are p, q, $0 \leq p \leq q \leq r^2$ such that the intervals $[p(r + 1) + 1, (p + 1)(r + 1)]$ and $[q(r + 1) + 1, (q + 1)(r + 1)]$ contain χ-monochromatic 1-cubes of the same type.
   - Let $p, q, 0 \leq p \leq q \leq r^2$ be such that the intervals $[p(r+1)+1, (p+1)(r+1)]$ and $[q(r + 1) + 1, (q + 1)(r + 1)]$ contain χ-monochromatic 1-cubes of the same type (a_1, i). Hence there are monochromatic 1-cubes $Q(a, a_1) \subseteq [p(r + 1) + 1, (p + 1)(r + 1)]$ and $Q(b, a_1) \subseteq [q(r + 1) + 1, (q + 1)(r + 1)]$ coloured by the same colour i.

     Next, we consider the 2-cube $Q(a, a_1, b - a)$.

     Observe that

     $$Q(a, a_1, b - a) = \{a, a + a_1, a + (b - a) = b,$$

     $$a + a_1 + (b - a) = b + a_1\} = Q(a, a_1) \cup Q(b, a_1)$$

     implies that the 2-cube $Q(a, a_1, b - a)$ is χ-monochromatic.

5. Recall that $a_1, r \in [1, r]$. Note that $(k + 1) + 1 + a_2 \leq k + (r^2 + 1)(r + 1)$ implies that $1 \leq a_2 \leq (r^2 + 1)(r + 1) - 2$. Hence the number of possible types (a_1, a_2, i) is

   $$r \cdot ((r^2 + 1)(r + 1) - 2) \cdot r \leq r^2(r^2 + 1)(r + 1).$$

   This together with

   $$r^2 \leq (r + 1)^2 \text{ and } r^2 + 1 \leq (r + 1)^2$$

   establishes that there are less than $(r + 1)^5$ different (a_1, a_2, i) types.

6. Observe that the interval $[1, (r^2 + 1)(r + 1)^6] = [1, (r + 1)^5 \cdot (r^2 + 1)(r + 1)]$ contains $(r + 1)^5$ consecutive intervals of length $(r^2 + 1)(r + 1)$.

   By (5) (and the Pigeonhole Principle) there are $p, q \in [0, (r + 1)^2 - 1]$, $p < q$, such that the intervals $[p + 1, p + (r^2 + 1)(r + 1)]$ and $[q + 1, q + (r^2 + 1)(r + 1)]$ contain χ-monochromatic 2-cubes of the same type, say $Q(a, a_1, a_2) \subseteq [p + 1, p + (r^2 + 1)(r + 1)]$ and $Q(b, a_1, a_2) \subseteq [q + 1, q + (r^2 + 1)(r + 1)]$.

   Next, we consider the 3-cube $Q(a, a_1, a_2, b - a)$.

   Observe that

   $$Q(a, a_1, a_2, b - a) = \{a, a + a_1, a + a_2, a + (b - a) = b, a + a_1 + a_2,$$

   $$a + a_1 + (b - a) = b + a_1, a + a_2 + (b - a) = b + a_2,$$

   $$a + a_1 + a_2 + (b - a) = b + a_1 + a_2\}$$

   $$= Q(a, a_1, a_2) \cup Q(b, a_1, a_2)$$

   implies that the 3-cube $Q(a, a_1, a_2, b - a)$ is χ-monochromatic.

Exercise 5.3.10 Folkman's theorem. The purpose of this exercise is to establish a proof of Folkman's theorem:

For all $r, k \in \mathbb{N}$ there exists a natural number $M(r, k)$ such that foe every r -colouring of $[1, M]$ there exist $a_1, a_2, \ldots, a_k \in [1, M]$ with all a_i distinct, such that the set

$$F_k(a_1, \ldots, a_m) = \left\{ \sum_{i=0}^{m} \epsilon_i a_i : \epsilon_i \in \{0, 1\} \text{ and } \varepsilon_1^2 + \ldots + \varepsilon_k^2 \neq 0 \right\}$$

is monochromatic.

1. What is the set $F_3(1, 2, 5)$?
2. Show that Folkman's theorem is a generalization of Schur's theorem.
3. Next we prove the following claim by induction on k:

   For all $r, k \in \mathbb{N}$ there exists a natural number $n(r, k)$ such that for any r-colouring χ of $[1, n]$ there exist $a_1, a_2, \ldots, a_k$ such that for any nonempty subset I of the set $[1, k]$

   $$a(I) = \sum_{i \in I} a_i \in [1, n] \text{ and } \chi(a(I)) = \chi(a_{\max(I)}).$$

   - Prove the base case, i.e prove that that $n(r, 1)$ exists.
   - For the inductive step suppose that $k \geq 1$ is such that there exists a natural number $n(r, k)$ such that for any r-colouring χ of $[1, n]$ there exist $a_1, a_2, \ldots, a_k$ such that for any nonempty subset I of the set $[1, k]$

     $$a(I) = \sum_{i \in I} a_i \in [1, n] \text{ and } \chi(a(I)) = \chi(a_{\max(I)}).$$

     Let $N = 2 \cdot W(r, n(r, k) + 1)$, where $W(r, n(r, k) + 1)$ is the van der Waerden number that guarantees the existence of a monochromatic $(n(r, k) + 1)$-term arithmetic progression whenever is an interval that contains $W(r, n(r, k) + 1)$ consecutive positive integers r-coloured.
     Fix an r-colouring ξ of the interval $[1, N]$.

     - Part 1: Prove that there are $a_{k+1} \in \left[\frac{N}{2} + 1, N\right]$ and $d \in \mathbb{N}$ such that the arithmetic progression $\{a_{k+1} + j \cdot d : 0 \leq j \leq n(r, k)\} \subset \left[\frac{N}{2} + 1, N\right]$ is ξ- monochromatic.
     - Part 2: Let d be as above. Explain why there exist $a_1, a_2, \ldots, a_k \in \{d, 2d, \ldots, n(r, k) \cdot d\}$ such that for any nonempty subset I of the set $[1, k]$:

       $$a(I) = \sum_{i \in I} a_i \in \{d, 2d, \ldots, n(r, k) \cdot d\} \text{ and } \xi(a(I)) = \xi(a_{\max(I)}).$$

     - Part 3: Complete the proof of the inductive step.

4. To complete the proof of Folkman's theorem show that one can take $M = M(r, k) = n(r, r \cdot (k - 1) + 1)$, where $n(r, r \cdot (k - 1) + 1)$ is the number guaranteed by the lemma proved in (3).

   Fix an r-colouring χ of $[1, M]$.

   - Justify the following claim: There exist $a_1, a_2, \ldots, a_{r(k-1)+1}$ such that for any nonempty subset I of the set $[1, r(k - 1) + 1]$

     $$a(I) = \sum_{i \in I} a_i \in [1, M] \text{ and } \chi(a(I)) = \chi(a_{\max(I)})..$$

- Define the r-colouring η of the set $[1, r(k-1)+1]$ in the following way: For any $j \in [1, r(k-1)+1]$

$$\eta(j) = \chi(a(I)) = \chi\left(\sum_{i \in I} a_i\right)$$

where $\emptyset \neq I \subseteq [1, r(k-1)+1]$ and $\max(I) = j$.
Is the colouring η well defined? Why yes, or why not?
- Prove that there is an η-monochromatic set $S \subset [1, r(k-1)+1]$ such that $|S| = k$.
- Finish the proof of Folkman's theorem by proving that the set of all sums of the elements of the set $A = \{a_i : i \in C\}$ is χ-monochromatic.

Solution.

1. $F_3(1,2,5) = \{1, 1+2, 1+5, 1+2+5, 2, 2+5, 5\} = \{1,2,3,5,6,7,8\}$.
2. Observe that Folkman's theorem guaranties the existence of a monochromatic $F_2(a,b) = \{a, b, a+b\}$ in any r-colouring of the set of natural numbers.
3. .

We take $n(r,1) = 1$. Observe that $I = \{1\}$ is the only nonempty subset of the set $[1] = \{1\}$ and $a(I) = F_1(1) = \{1\}$.

- ◦ Part 1: Apply van der Waerden's theorem.
  ◦ Part 2: Define an r-colouring ξ' of $[1, n(r,k)]$ by

$$\xi'(j) = \xi(j \cdot d), \text{ for any } j \in [1, n(r,k)].$$

Observe that $n(r,k) \leq W(r, n(r,k)+1) = \frac{N}{2}$. Thus we can apply the inductive hypothesis and find $b_1, b_2, \ldots, b_k \in [1, n(r,k)]$ such that for any nonempty subset I of the set $[1,k]$

$$b(I) = \sum_{i \in I} b_i \in [1, n(r,k)]] \text{ and } \xi'(b(I)) = \xi'(b_{\max(I)}).$$

This implies that for $a_i = i \cdot d$, $i \in [1,k]$, we have, for any nonempty subset I of the set $[1,k]$,

$$a(I) = \sum_{i \in I} a_i \in \{d, 2d, \ldots, n(r,k) \cdot d\} \text{ and } \xi(a(I)) = \xi(a_{\max(I)}).$$

  ◦ Consider the sequence $a_1, a_2, \ldots, a_k, a_{k+1}$, where a_i, $i \in [1, k+1]$, are established in Part 1 and Part 2.
Let I be a non-empty subset of the set $[1, k+1]$.
If $\max(I) < k+1$ then, by Part 2,

$$a(I) = \sum_{i \in I} a_i \in [1, N] \text{ and } \xi(a(I)) = \xi(a_{\max(I)}).$$

If $\max(I) = k+1$ then, by Part 2,

$$\sum_{i \in I \setminus \{k+1\}} a_i \in \{d, 2d, \ldots, n(r,k) \cdot d\}$$

which implies that

$$a(I) = \sum_{i \in I} a_i = a_{k+1} + \sum_{i \in I \setminus \{k+1\}} a_i$$

is a term in the ξ-monochromatic arithmetic progression that we established in Part 1.
Hence

$$\xi(a(I)) = \xi(a_{k+1}),$$

which completes the proof of the inductive step.

4. - This follows from the lemma and our choice of M.
   - Yes, it is. By our choice of $a_1, a_2, \ldots, a_{r(k-1)+1}$, if $I, J \subseteq [1, r(k-1)+1]$ and $\max(I) = \max(J) = j$ then

     $$chi(a(I)) = \chi(a(J)) = \chi(a_j).$$

   - By the Pigeonhole Principle, at least one of the r colours (pigeonholes) must contain at least k elements (pigeons) of the set $[1, r(k-1)+1]$.
   - This follows from the lemma and our choice of M.
   - Let $S = \{s_1, s_2, \ldots, s_k\}$ with $s_i < s_j$ if $i < j$.
     For $i \in [1, k]$, let $a'_i = a_{s_i}$.
     Let $I, J \subseteq [1, k]$ be two non-empty subsets with $p = \max(I)$ and $q = \max(J)$.
     Then

     $$\chi\left(\sum_{i \in I} a'_i\right) = \chi\left(\sum_{i \in I} a_{s_i}\right) = \chi(a_{s_p})$$

     and

     $$\chi\left(\sum_{i \in J} a'_i\right) = \chi\left(\sum_{i \in J} a_{s_i}\right) = \chi(a_{s_q}).$$

     But since $s_p, s_q \in S$ it follows that $\eta(s_p) = \eta(s_q)$ which is the same as $\chi(a_{s_p}) = \chi(a_{s_q})$. Therefore for any $\emptyset \neq I, J \subseteq [1, k]$ we have that

     $$\chi\left(\sum_{i \in I} a'_i\right) = \chi\left(\sum_{i \in J} a'_i\right).$$

Chapter 6

Colourings of the Plane

6.1 Erdős-Szekeres Problem of Convex Polygons

> *Where there is love there is life.* — Mahatma Gandhi, Indian leader, 1869 — 1948

Warm Up. Consider a finite set of points S in the plane, and ask, for example, this question: Is it true that there will always be a set of three points in S that are the vertices of a triangle?

Points in General Position in Plane. We say that the set of points A in the plane is in general position if there is no line that contains three points from A. See Figure 6.1.1.

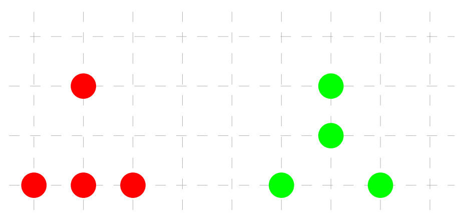


Figure 6.1.1 Which of the two sets is a set of points in general position?

Problem. For any integer $n \geq 3$, determine the smallest positive integer $N(n)$ such that any set of at least $N(n)$ points in general position in the plane (i.e., no three of the points are on a line) contains n points that are the vertices of a convex n-gon.

Convex n-gon. A convex n-gon is an n-gon with the property that if two points A and B are inside of the n-gon then the whole segment $\overline{AB}$ is inside of the n-gon. See Figure 6.1.2.

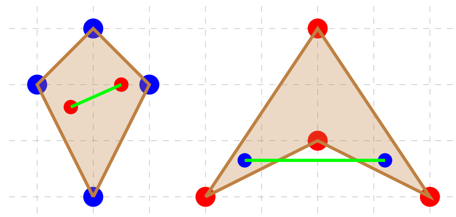


Figure 6.1.2 A convex quadrilateral and a non-convex quadrilateral

Example 6.1.3 $N(3) = 3$. □

Example 6.1.4 $n = 4$. In 1932 Esther Klein made the following observation: Among any five points in general position in the Euclidean plane, it is always possible to select four points that form the vertices of a convex quadrilateral. □

Question 6.1.5 Is it possible to find four points in the plane that do not form a convex quadrilateral? □

Therefore. . . $N(4) =$

$n = 5$. See Figures 6.1.6 an d 6.1.7.

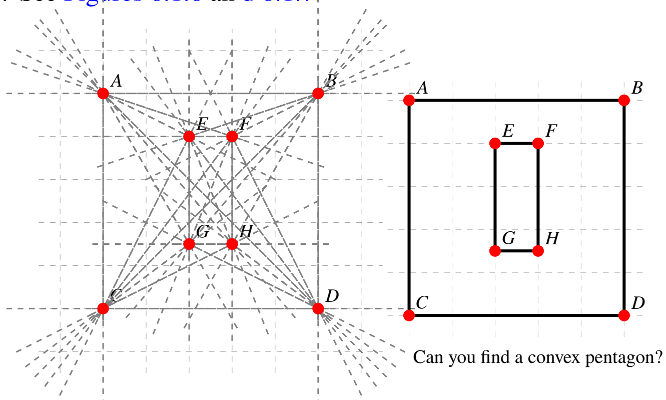


Figure 6.1.6 Eight points in general position.

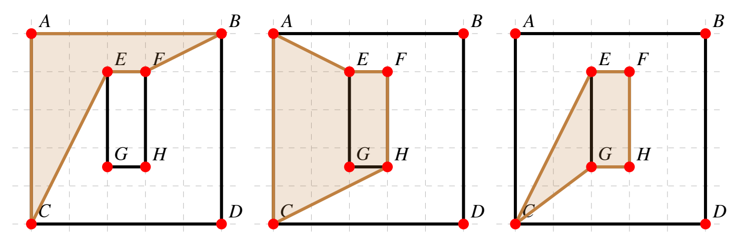


Figure 6.1.7 No! - A few cases.

Therefore . . . $N(5) \geq 9$.

$n = 5 \ldots$ *Part II* Let S be a set of nine points in the plane in general position. Let $\overline{S}$ be the convex hull of S.

1. If $\overline{S}$ has *five or more* vertices, we are done. See Figure 6.1.8.

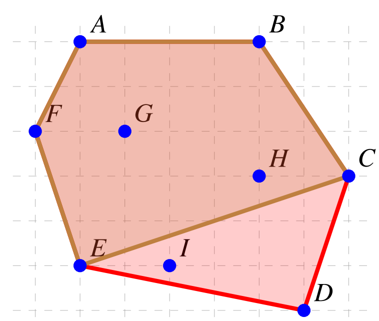


Figure 6.1.8 The convex hull of S has six points.

2. Let the convex hull $\overline{S}$, the convex hull of S, has three or four vertices. Then the set $T = S \backslash \overline{S}$ contains six or five (remaining) points of S and they are all inside of $\overline{S}$. Let $\overline{T}$ be the convex hull of T.

3. If $|\overline{T}| = 5$ or $|\overline{T}| = 6 \ldots$ Done! See Figure 6.1.9.

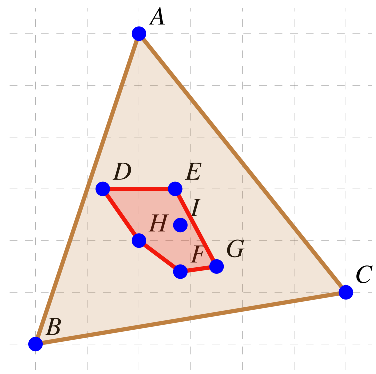


Figure 6.1.9 Example: $|S| = |\{A, B, \ldots, I\}| = 9$, $|\overline{S}| = |\{A, B, C\}| = 3$, $|T| = |\{D, E, \ldots, I\}| = 6$, and $|\overline{T}| = |\{D, E, F, G, H\}| = 5$.

4. For the remaining cases see Figure 6.1.10.

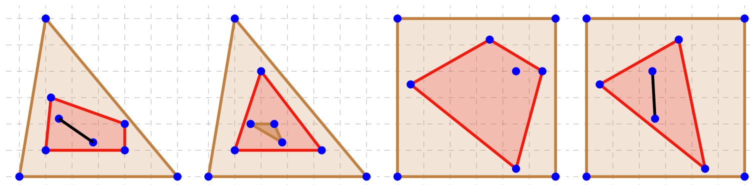


Figure 6.1.10 Four remaining cases.

Configuration of the type (3, 3, 2).

1. *Consider the inside triangle and the line segment.*
   - The line that contains the line segment intersects two sides of the triangle.
   - Notice the vertex where those two sides of the triangle intersect.
   - Draw rays starting at the end points of the line segment as on Figure 6.1.11

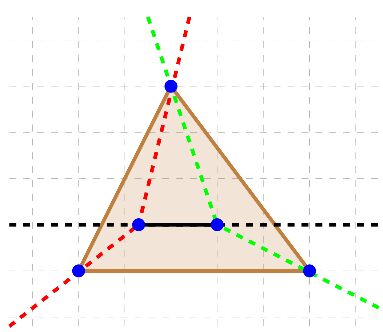


Figure 6.1.11 A triangle, a line segment, and four rays.

Three regions. Notice the three open regions in the plane on the Figure 6.1.12:

- None of the three regions intersects the interior of the triangle
- Region 1 and Region 2 intersect (part of the plane ‘above’ the top vertex.)
- Region 3 does not intersect either Region 1 or Region 2.

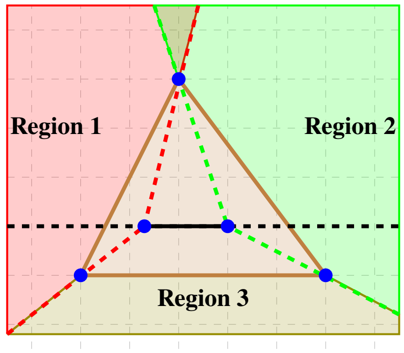
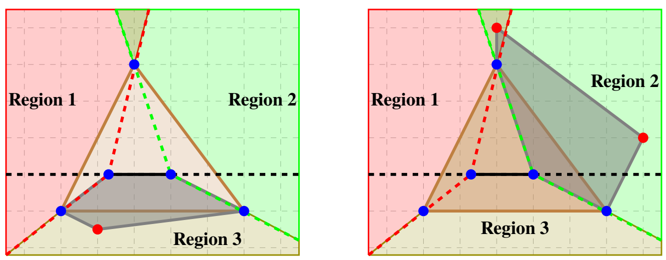


Figure 6.1.12 Three regions.

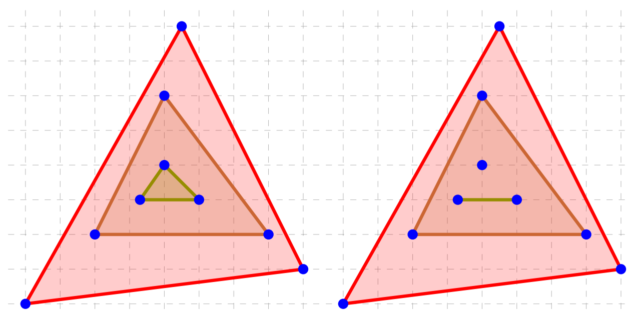
Three points outside of the triangle. Note that the remaining three points in the configuration $3-3-2$ cannot be on the boundary of any of Regions 1-3. (Why?) See Figure 6.1.13.



One of the *outside points* belongs to Region 3. None of the *outside points* belongs to Region 3.

Figure 6.1.13 There is a convex pentagon!

Configuration of the type $(3,3,3)$. Note that the configuration the 8-point configuration $(3,3,2)$ is contained in the configuration $(3,3,3)$. See Figure 6.1.14.



Type $(3,3,3)$ **Type** $(3,3,2)$

Figure 6.1.14 Type $(3,3,3)$ contains Type $(3,3,2)$.

Therefore the configuration of the type $(3,3,3)$ contains a convex pentagon.

Configuration of the type $(4,3,1)$.

1. Consider a triangle and a single point inside of it, and note three regions, Figure 6.1.15.

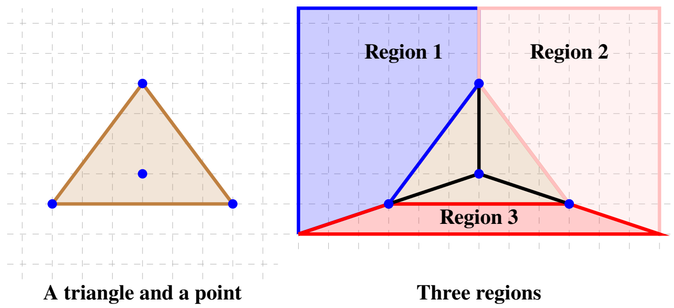


Figure 6.1.15 Type $(*, 3, 1)$.

2. By the Pigeonhole Principle, at least two of the remaining four points must belong to the same region, say Region 2. See Figure 6.1.16.

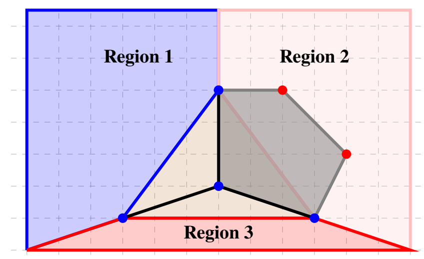


Figure 6.1.16 There is a convex pentagon!

Configuration of the type $(4, 4, 1)$. Note that the configuration $(4, 4, 1)$ contains the configuration $(4, 3, 1)$. See Figure 6.1.17.

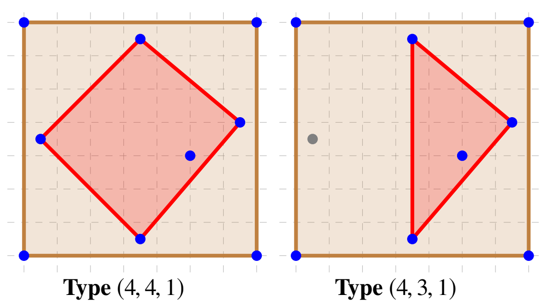


Figure 6.1.17 The configuration of the type $(4, 4, 1)$ contains a convex pentagon..

Configuration of the type $(4, 3, 2)$. Note that the configuration $(4, 3, 2)$ contains the configuration $(4, 3, 1)$. See Figure 6.1.18.

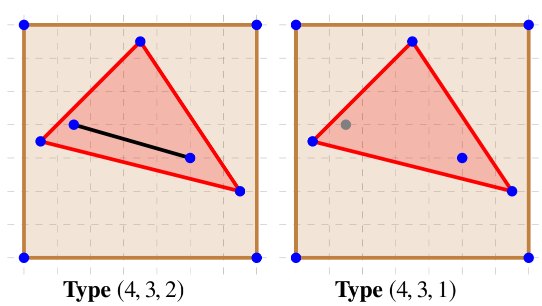
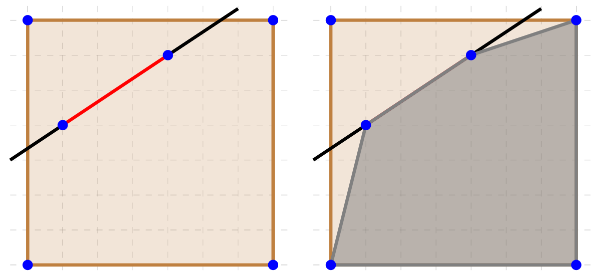


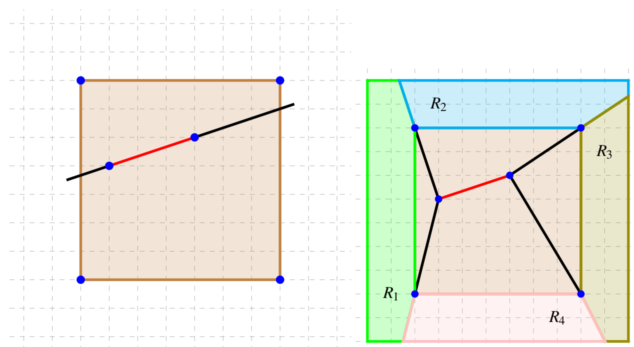
Figure 6.1.18 The configuration of the type $(4, 3, 2)$ contains a convex pentagon..

Configuration of the type $(3, 4, 2)$. Consider the inside quadrilateral and the line segment. See Figures 6.1.19 — Figure 6.1.21.



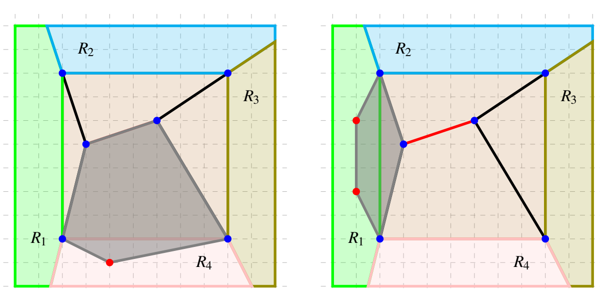
The line intersects the adjacent sides. **There is a convex pentagon!**

Figure 6.1.19 Case 1: The line that contains the line segment intersects the adjacent sides of the quadrilateral.



The line intersects the opposite sides. **Four regions.**

Figure 6.1.20 Case 2: The line that contains the line segment intersects the opposite sides of the quadrilateral.



Case 2.1:
One point belongs to Region 2 or Region 4.

Case 2.2:
Two points belong to Region 1 or Region 3.

Figure 6.1.21 Case 2: There is a convex pentagon!

Therefore

$$N(5) = 9.$$

6.2 Erdős-Szekeres Problem of Convex Polygons - Part Two

All generalizations are false, including this one. — Mark Twain, American author, 1835 — 1910

Reminder. For any integer $n \geq 3$, determine the smallest positive integer $N(n)$ such that any set of at least $N(n)$ points in general position in the plane (i.e. no three of the points are on a line) contains n points that are the vertices of a convex n-gon.

1. $N(3) =$
2. $N(4) =$
3. $N(5) =$

Question 6.2.1

1. Does $N(n)$ exist for any $n \geq 3$?
2. Which values of $N(n)$ are known?
3. Are there any bounds for the size of $N(n)$?

□

Pattern?

1. $N(3) = 3 = 2^1 + 1 = 2^{3-2} + 1$
2. $N(4) = 5 = 2^2 + 1 = 2^{4-2} + 1$
3. $N(5) = 9 = 2^3 + 1 = 2^{5-2} + 1$

Well... Szekeres and Peters, 2006:

$$N(6) = 17 = 2^4 + 1 = 2^{6-2} + 1.$$

Conjecture 6.2.2 *For any $n \geq 3$*

$$N(n) = 2^{n-2} + 1.$$

Not long before his death in 1996, Erdős wrote that he would pay \$500 for a proof of this conjecture.

Still... How do we prove that $N(n)$ exists for all $n \geq 3$?

Two Theorems:

Recall Theorem 2.3.19:

Ramsey's Theorem. For any natural numbers $k, r, l_1, \ldots, l_r$ there exists the least natural number $m_0 = R(k; l_1, l_2, \ldots, l_r)$ such that for any $m \geq m_0$, if the set of all k-element subset of the set S_m, where $|S_m| = m$, is r-coloured then there exists $i \in [1, r]$ and the l_i-element subset $\Delta_{l_i} \subseteq S_m$ such that all its k-element subsets have the colour i.

Lemma 6.2.3 *Let $n \geq 4$ be an integer. Then n points in the plane form a convex polygon if and only if every four of them form a convex quadrilateral.*

For a proof by induction, see Exercise 6.6.9.

Theorem 6.2.4 Erdős-Szekeres' Theorem *$N(n)$ exists for any $n \geq 3$.*

Proof. We already know that $N(3) = 3$, $N(4) = 5$, and $N(5) = 9$.

Let $n \geq 4$. Let $m > R(4; n, 5)$ and let S_m be a set of m points in the plane in general position. Let

$$S_m^{(4)} = \{\{A, B, C, D\} : A, B, C, D \in S_m\},$$

i.e., let $S_m^{(4)}$ be the set of all four-element subsets of S_m.

We define a 2-colouring

$$c : S_m^{(4)} \to \{\bullet, \bullet\}$$

in the following way. For $T \in S_m^{(4)}$

$$c(T) = \begin{cases} \textcolor{red}{\bullet} & \text{if } T \text{ forms a concave quadrilateral} \\ \textcolor{blue}{\bullet} & \text{if } T \text{ forms a convex quadrilateral.} \end{cases}$$

See Figure 6.2.5.

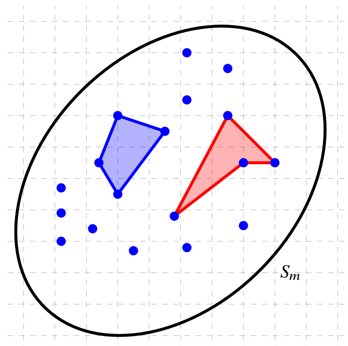


Figure 6.2.5 A convex quadrilateral and a non-convex quadrilateral

By Ramsey's theorem (and our choice of $m > R(4; n, 5)$) there is an n-element set $\Delta_n \subset S_m$ such that all of its four-element subsets are coloured blue or a 5-element set $\Delta_5 \subset S_m$ such that all of its four-element subsets are coloured red.

Since $N(4) = 5$, any set of five points in the plane in general position contains a convex quadrilateral. This implies that it is impossible to find a 5-element set $\Delta_5 \subset S_m$ such that all of its four-element subsets are coloured red.

Hence there must be an n-element set $\Delta_n \subset S_m$ such that all of its four-element subsets are coloured blue. But then, by Lemma, the set Δ_n forms a convex n-gon. ■

Therefore... For any $n \geq 4$, $N(n) \leq R(4; n, 5)$.

Cups and Caps. See Figure 6.2.6.

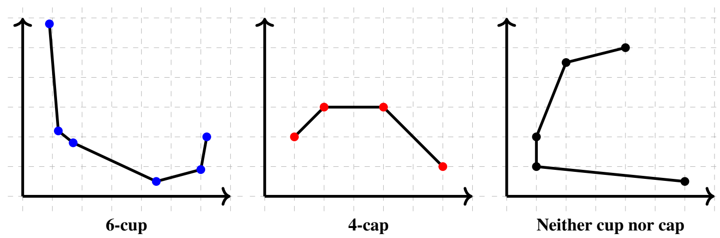


Figure 6.2.6 Cups and caps

Observation 6.2.7 Note that in a k-cup, the sequence of slopes is increasing and that in an l-cap, the sequence of slopes is decreasing.

Observation 6.2.8 It is clear that if we find a cup or a cap in a set S in **some system of coordinates**, then we will also find a convex polygon. See Figure 6.2.9.

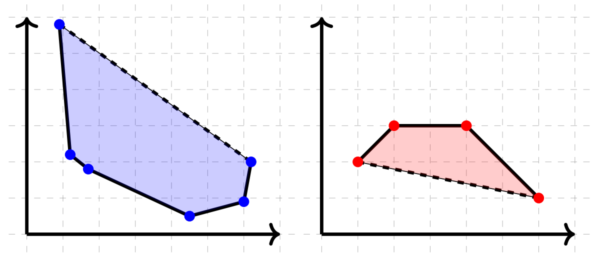


Figure 6.2.9 Convex polygons from cups and caps

Observation 6.2.10 The expression "in some system of coordinates" can be substituted for the expression "in any system of coordinates, in which there are no two points in S that belong to the vertical line". Let us call any system of coordinates with such property **right for** S. In what follows we will always assume that, for a given set S of points in general position, we have chosen a coordinate system that is right for S. See Figure 6.2.11.

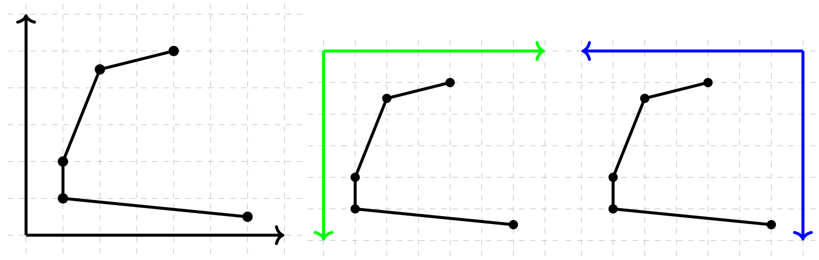


Figure 6.2.11 Making a *right* coordinate system

Definition 6.2.12 For $k, l \geq 3$ we define $f(k, l)$ to be the least positive integers such that any set S of points in the plane (with a given coordinate system that is 'right' for S) in general position such that

$$|S| \geq f(k, l)$$

contains either a k-cup or an l-cap. ◊

Theorem 6.2.13 Theorem about cups and caps: *For $k, l \geq 3$ the number $f(k, l)$ exists and, for $k, l \geq 4$ we have that*

$$f(k, l) \leq f(k-1, l) + f(k, l-1) - 1.$$

Proof. We prove the theorem by induction on $m = k + l$ if $m \geq 6$.

1. Base Case: For any $k \geq 3$,

$$f(k, 3) = f(3, k) = k.$$

   Take a set S with k points in the plane in general position. Let

$$S = \{A_1, A_2, \ldots, A_k\}$$

   where for $i < j$, the x-coordinate of A_i is less than the the x-coordinate of A_j. Let s_i be the slope of the line segment $\overline{A_i A_{i+1}}$, for $i \in [1, k-1]$. If

$$s_1 < s_2 < \cdots < s_{k-1}$$

   then S contains a k-cup.

If, for some $i \in [1, k-2]$,

$$s_i > s_{i+1}$$

then the set S contains a 3-cap $\{A_i, A_{i+1}, A_{i+2}\}$.

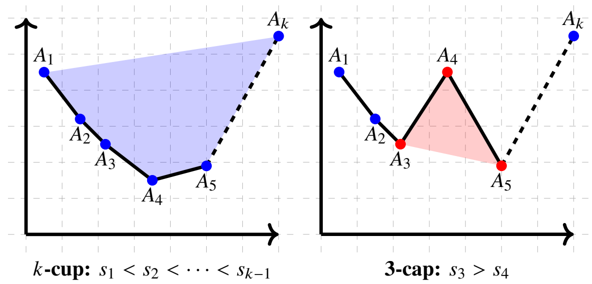


Figure 6.2.14 $f(k, 3) = f(3, k) = k$

2. Note that we have proved that $f(3, 3) = 3$ and $f(4, 3) = f(3, 4) = 4$. In particular this means that if $k, l \geq 3$ are such that $k + l \leq 7$ then $f(k, l)$ exists.

3. Inductive Step: Let $m \geq 7$ and $k, l \geq 3$ be such whenever $k + l = m$ then $f(k, l)$ exists. We choose $k, l \geq 3$ such that

$$k + l = m + 1.$$

This implies that

$$(k-1) + l = k + (l-1) = m$$

and, hence $f(k-1, l)$ and $f(k, l-1)$ exist.

Let

$$n = f(k-1, l) + f(k, l-1) - 1.$$

Let us fix a set S of cardinality n and any right for S system of coordinates. We have to prove that S contains either a k-cup or an l-cap. Let L be the set of all points that are the left ends of $(k-1)$-cups in S.

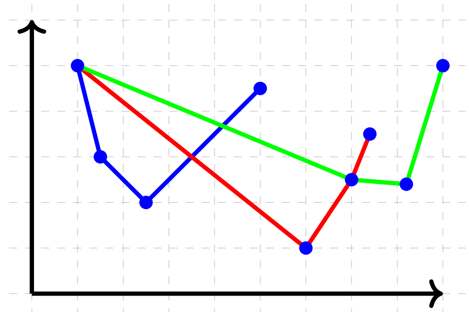


Figure 6.2.15 For $k - 1 = 4$, the top left point belongs to L.

(a) Let us assume first that the set $S \backslash L$ has at least $f(k-1, l)$ points. Then it contains either a $(k-1)$-cup or an l-cap. But taking the set L out of S *destroys* all $(k-1)$-cups in S. Hence $S \backslash L$ does not contain any $(k-1)$-cups and therefore must contain an l-cap.

(b) Suppose then that $|S \backslash L| \leq f(k-1, l) - 1$. It follows

$$|L| = |S| - |S \backslash L| \geq n - f(k-1, l) + 1 = f(k, l-1).$$

Therefore, there exists a k-cup in L (and everything is alright) or there exists a $(l-1)$-cap in L.

Let us consider the point Y, the right end of that cap.

Let X be the point that is immediately to the left of Y in the $(l-1)$-cap in L. Since $Y \in L$, the point Y is a left end of some $(k-1)$-cup in S.

Let Z be the point that is immediately to the right of Y in the $(k-1)$-cup in S. If

$$\text{Slope of } \overline{XY} > \text{ Slope of } \overline{YZ}$$

then adding the point Z to the $(l-1)$-cap in L makes an l-cap in S. (See Figure 6.2.16.)

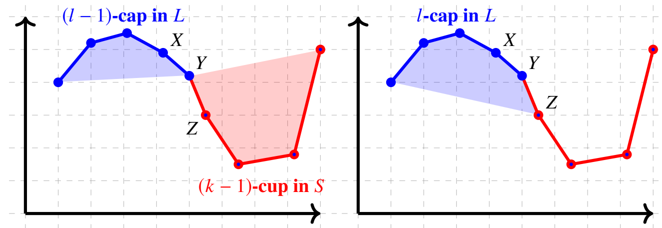


Figure 6.2.16 Getting an l-cap in S from an $(l-1)$-cap in L.

If

$$\text{Slope of } \overline{XY} < \text{ Slope of } \overline{YZ}$$

then adding the point X to the $(k-1)$-cup in S makes an k-cup in S. (See Figure 6.2.17.)

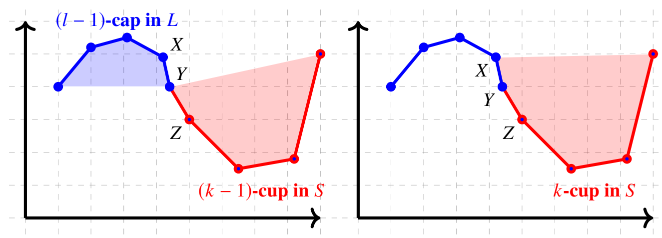


Figure 6.2.17 Getting a k-cup in S from an l-cap in L.

Therefore, any set S such that

$$|S| = n = f(k-1, l) + f(k, l-1) - 1$$

contains either a k-cup or an l-cup which implies that $f(k, l)$ exists and that

$$f(k, l) \leq f(k-1, l) + f(k, l-1) - 1.$$

By the Principle of Mathematical Induction, $f(k, l)$ exists for any $k, l \geq 3$.

■

How big is $f(k, l)$?

Theorem 6.2.18 *If $k, l \in \mathbb{N}$ are such that $k + l \geq 6$ then*

$$f(k, l) \leq \binom{k+l-4}{k-2} + 1.$$

Proof. Proof via induction on $k+l$: If $k+l=6$ then $k=l=3$ and

$$f(3,3)=3 \text{ and } \binom{3+3-4}{3-2}+1=2+1=3.$$

If $k+l=7$ then $k=4$ and $l=3$ or $k=3$ and $l=4$. From

$$f(4,3)=f(3,4)=4 \text{ and } \binom{4+3-4}{4-2}+1=3+1=4 \text{ and } \binom{3+4-4}{3-2}+1=4$$

we conclude that in the case that $k+l=7$ we have

$$f(k,l)\le\binom{k+l-4}{k-2}+1.$$

Suppose that $m\ge 7$ is such that whenever $k,l\ge 3$ are such such that $k+l=m$ then

$$f(k,l)\le\binom{k+l-4}{k-2}+1.$$

Suppose that $k+l=m+1$. Observe that

$$(k-1)+l=k+(l-1)=m$$

together with Theorem 6.2.13 implies that

$$f(k,l)\le f(k-1,l)+f(k,l-1)-1\le\binom{k+l-5}{k-3}+\binom{k+l-5}{k-2}+1.$$

To finish the proof recall that

$$\binom{a}{b-1}+\binom{a}{b}=\binom{a+1}{b}.$$

■

Actually...

$$f(k,l)=\binom{k+l-4}{k-2}+1.$$

Back to $N(n)$:

$$N(n)\le f(n,n)\le\binom{2n-4}{n-2}+1.$$

Also...

$$N(n)\ge 2^{n-2}+1.$$

What is known?

1. The next step is if not to prove the hypothesis, then at least to improve the estimation a little bit.
2. The inequality
$$N(n)\le\binom{2n-4}{n-2}+1$$
was proved by Erdős and Szekeres in 1935.
3. And in 1998 there were three improvements at once!
   (a) The first of them was made by F. Chung and R. Graham:
   $$N(n)\le\binom{2n-4}{n-2}.$$

(b) D. Kleitman and L. Pachtler showed that it is true that

$$N(n) \leq \binom{2n-4}{n-2} + 7 - 2n$$

(c) The third improvement was achieved by G. Tot and P. Vultr:

$$N(n) \leq \binom{2n-5}{n-2} + 2.$$

The last result is approximately twice as good as the result of Erdős - Szekeres.

4. The current record also belongs to Tot and Vultr (2005).

$$N(n) \leq \binom{2n-5}{n-2} + 1, \; n \geq 5.$$

5. Chung and Graham offered $100 for the first proof that

$$N(n) \leq c^n,$$

where $c < 4$ is a constant.

Resources.

1. Happy Ending Problem - Wikipedia
2. Happy Ending Problem by Ron Graham
3. Happy Ending Problem by D. Harvey
4. Erdős, P.; Szekeres, G. (1935), "A combinatorial problem in geometry", Compositio Math 2: 463-470.
5. The Erdos-Szekeres problem on points in convex position - a survey by W. Morris and V. Soltan
6. A Puzzle of Clever Connections Nears a Happy End by Kevin Hartnett

6.3 The Chromatic Number of the Plane

It doesn't matter how long my hair is or what colour my skin is or whether I'm a woman or a man. — John Lennon, English musician, singer and songwriter, 1940 - 1980

Problem. What is the smallest number of colours sufficient for colouring the plane in such a way that no two points of the same colour are unit distance apart? — Edward Nelson, 1950

Edward Nelson was born on May 4, 1932, in Decatur, Georgia. He is a professor in the Mathematics Department at Princeton University.

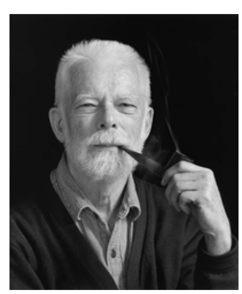


Figure 6.3.1 Edward Nelson (Source: Wikipedia)

Definition 6.3.2 Chromatic Number of the Plane. The smallest number of colours sufficient for colouring the plane in such a way that no two points of the same colour are unit distance apart is called the **chromatic number of the plane** and it is denoted by χ. ◊

Question 6.3.3 Are two colours enough?

Solution.

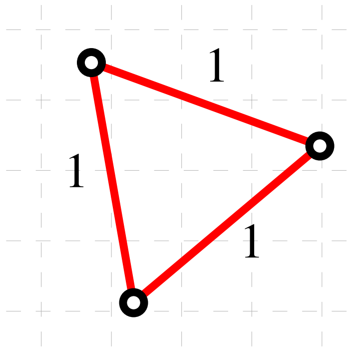


Figure 6.3.4 No, two colours are not enough!

□

Proposition 6.3.5 $\chi \geq 3$.

Question 6.3.6 Are three colours enough? □

Figure 6.3.7 Leo Moser, 1921 — 1970

Leo Moser was a professor of mathematics at the University of Alberta. William Moser a was a professor of Mathematics at the University of Saskatchewan, the University of Manitoba and McGill University.

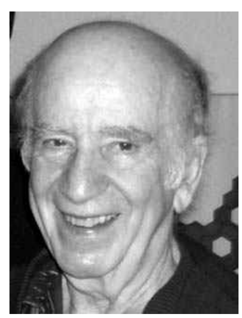
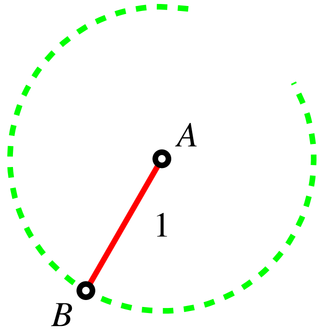


Figure 6.3.8 William Moser, 1927 — 2009

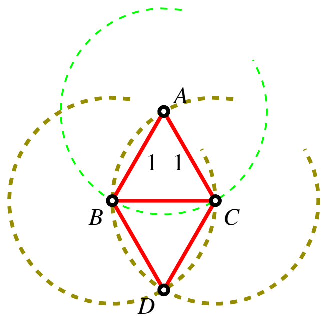
Be generous and patient as teachers, be active in projects which benefit the mathematical community and, above all, have as long and as happy a mathematical life as I have had, and am still having. — W. Moser in 2003. (Source MacTutor.)

The Moser brothers' construction:



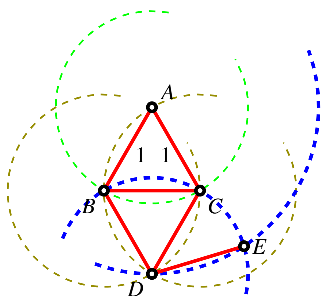
Start by choosing a point$A$ in the plane and then draw a circle with the centre at A and radius 1. Denote this circle by C_1. Next, choose a point B on the circle C_1. Draw the line segment $\overline{AB}$.

Figure 6.3.9 Step 1



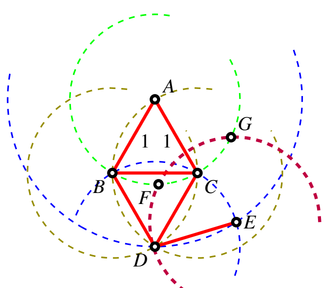
Draw a circle with the centre at B and radius 1. Denote this circle by C_2. Let C be the intersection point of C_1 and C_2. Draw a circle, call it C_3, with the centre at C and radius 1. Observe that the point A belongs to both C_2 and C_3. Let D be the the other intersection point of C_2 and C_3. Draw the line segments $\overline{AC}$, $\overline{BC}$, $\overline{BD}$, and $\overline{CD}$. Observe that all those line segments are of length 1.

Figure 6.3.10 Step 2



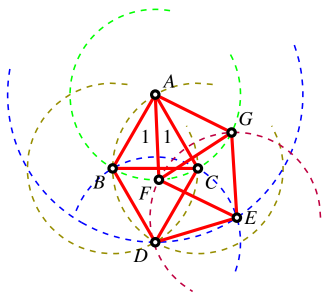
Draw a circle, call it C_4, with the centre at D and radius 1. Draw a circle with the centre at A and passing through the point D. Denote this circle by C_5 Next, choose a point E in the intersection of C_4 and C_5. Draw the line segment $\overline{DE}$. Observe that $|\overline{DE}| = 1$.

Figure 6.3.11 Step 3



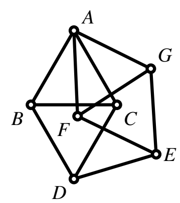
Draw a circle with the centre at E and radius 1. Denote this circle by C_6. Let F and G be the intersection point of C_1 and C_6.

Figure 6.3.12 Step 4



Draw the line segments $\overline{AF}$, $\overline{AG}$, $\overline{EF}$, and $\overline{EG}$. Observe that all those line segments are of length 1.

Figure 6.3.13 Step 5



The Moser Spindle

Figure 6.3.14 Step 6

Reminder: Question. Are three colours enough?

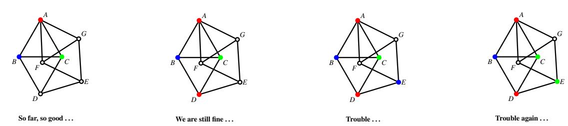


Figure 6.3.15 Toss the Moser Spindle on the red/blue/green coloured plane. Remember that every edge in the Moser Spindle is of length 1.

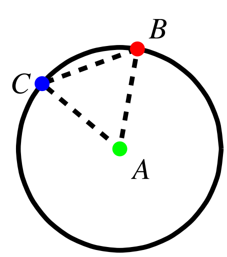
Proposition 6.3.16 $\chi \geq 4$.

Question. Are three colours enough? (Again.)

Hugo Hadwiger in 1961: Consider a three colouring of the plane: $c : \Pi \to \{\bullet, \bullet, \bullet\}$.

Hugo Hadwiger used the following construction to show that there must be two points, say, X and Y, such that

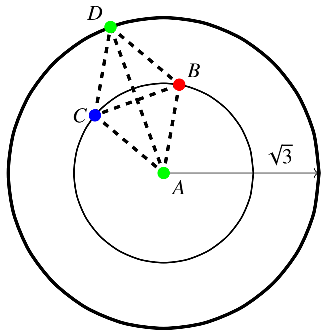
$$|\overline{XY}| = 1 \text{ and } c(X) = C(Y).$$



$|\overline{AB}| = |\overline{AC}| = |\overline{BC}| = 1$

Figure 6.3.17 Step 1

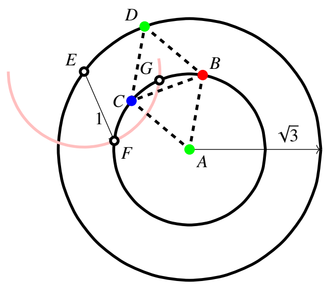
Start by choosing a point A in the plane and then draw a circle with the centre at A and radius 1. Denote this circle by C_1. Suppose that $c(A) = \bullet$. If there is a green point on the circle C_1, then we have two points coloured by the same colour that are one unit apart. Suppose that all points on the circle C_1 are coloured either blue or red. Next, choose a point B on the circle C_1 and suppose that $c(B) = \bullet$. There are two points on C_1 that are one unit apart from B. (Why?) If one of them is red, we are done. Suppose that both of them are blue and pick one of them. Call it C.



$|\overline{BD}| = |\overline{CD}| = 1$

Figure 6.3.18 Step 2

Draw a circle with the centre at A and radius $\sqrt{3}$. Denote this circle by C_2. Let D be the intersection point of the circle C_2 and the line of symmetry of the line segment $\overline{BC}$ that is not on the same side of the line BC as the point A. Observe that $|\overline{BD}| = |\overline{CD}| = 1$. (Why?) If the point D is coloured red or blue then we have two points that are one unit apart and of the same colour. Suppose that $c(D) =$ ●.

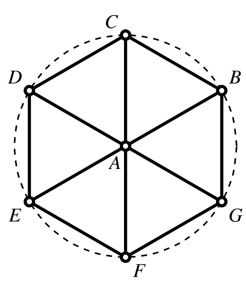


$|\overline{DE}| = |\overline{EF}| = |\overline{EG}| = |\overline{FG}| = 1$

Figure 6.3.19 Step3

Let E be a point on the circle C_2 such that $|\overline{DE}| = 1$. Draw a circle with the centre at E and radius 1. Denote this circle by C_3. Observe that C_3 intersects the circle C_1 at two points, F and G and that $|\overline{FG}| = 1$. (Why?) Recall our assumption that all points on the circle C_1 are coloured or blue or red. Colour E by any of the three colours. What happens?

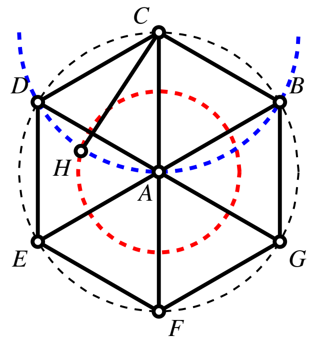
Therefore... $\chi \geq 4$.
Question. Are three colours enough? (Again.)
Golomb Graph. By Solomon Golomb (1965).



$|\overline{AB}| = |\overline{BC}| = |\overline{CD}| = \cdots = |\overline{GB}| = 1$

Figure 6.3.20 Step 1

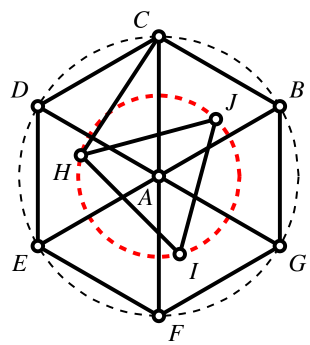
Draw a circle with the centre at A and radius 1. Denote this circle by C_1. Let $BCDEFG$ be a regular hexagon inscribed in the circle C_1.



$|\overline{CH}| = 1$

Figure 6.3.21 Step 2

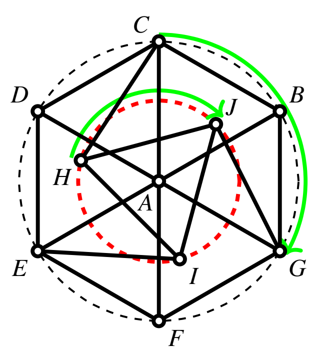
Draw a circle with the centre at A and radius $\frac{\sqrt{3}}{3}$. Denote this circle by C_2. Draw a circle with the centre at C and radius 1. Let H be an intersection point of C_2 and C_3. Observe that $|\overline{CH}| = 1$.



$|\overline{HI}| = |\overline{HJ}| = |\overline{IJ}| = 1$

Figure 6.3.22 Step 3

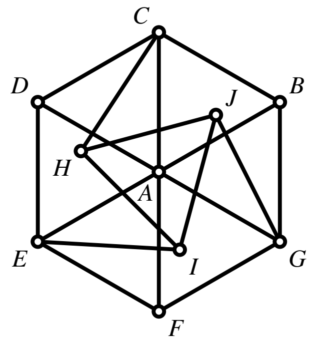
Let $\triangle HIJ$ be an equilateral triangle inscribed in C_2. Observe that, since the radius of C_2 equals to $\frac{\sqrt{3}}{3}$, $|\overline{HI}| = |\overline{HJ}| = |\overline{IJ}| = 1$.



$|\overline{GJ}| = |\overline{EI}| = 1$

Figure 6.3.23 Step 4

Let r be the clockwise rotation by $\frac{2\pi}{3}$ with the centre at A. Then $r(C) = G$ and $r(H) = J$. Since r is an isometry it follows that $|\overline{GJ}| = |\overline{CH}| = 1$. Similarly, $|\overline{EI}| = |\overline{CH}| = 1$.



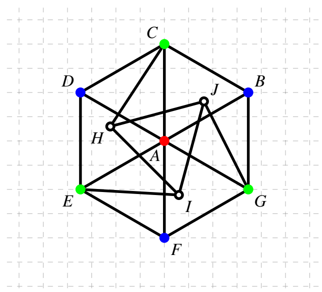
The Golomb Graph.

$|\overline{GJ}| = |\overline{EI}| = 1$

Figure 6.3.24 Step 5

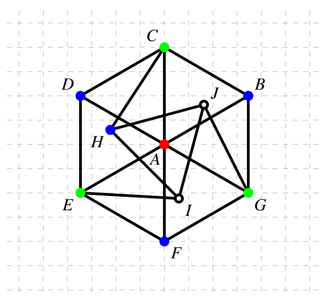
Are three colours enough?

Toss the Golomb graph on the red/blue/green coloured plane. Recall that every edge in the Golomb graph is of length 1.



Suppose that the point A is coloured red and that each triangle that has A as a vertex is a rainbow triangle.

Figure 6.3.25 Start ...

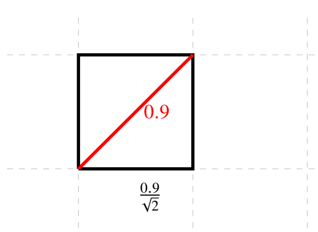


If H is coloured green, then there are two points, C and H, at the unit distance coloured by the same colour. Suppose that the point H is coloured blue. What are our options for colouring I and J? What if we assume that the point H is coloured red?

Figure 6.3.26 ... and finish!

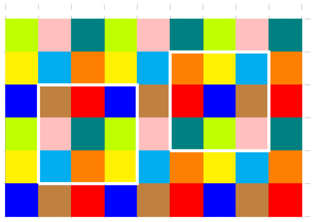
Therefore... $\chi \geq 4$.

What About Upper Bounds?



Consider the grid in which the length of the side of each square in the grid equals $\frac{0.9}{\sqrt{2}}$. Observe that the length of the diagonal of each grid cell is $d = 0.9$.

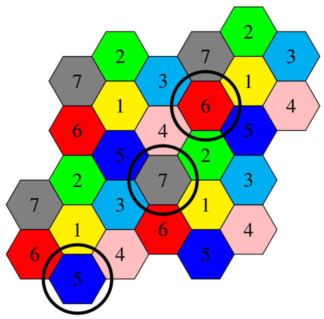
Figure 6.3.27 Step 1



A 9-colouring in which all neighbours of each square are of different colours. Is it possible to find two points of the same colour that are one unit apart?

Figure 6.3.28 Step 2

Theorem 6.3.29 $\chi \leq 7$. *(Hadwiger, 1961)*



A 7-colouring of a tessellation of the plane by regular hexagons, with diameter slightly less than one. Observe that each hexagon is surrounded by hexagons of a different colour.

Figure 6.3.30 Hadwiger, 1961

Proposition 6.3.31 $\chi \leq 7$. *(Szekely, 1983)*

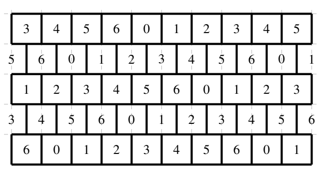
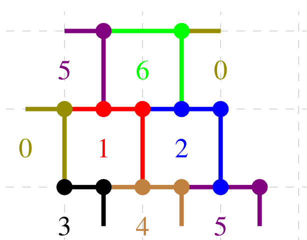


Figure 6.3.32 Szekely, 1983

7-colouring by László Székely: Start with a row of squares of diagonal 1, with cyclically alternating colours from 0 to 6 of the squares. Obtain the consecutive rows of coloured squares by shifting the previous row by 2.5 squares. Upper and right boundaries are included in each square, except for the upper left and lower right corner.



Closer look: Upper and right boundaries are included in each square, except for the upper left and lower right corner.

Figure 6.3.33 Székely: Closer look

Theorem 6.3.34 $\chi \geq 5$. *(de Grey, 2018)*

"In seeking graphs that can serve as M in our construction, we focus on graphs that contain a high density of Moser spindles. The motivation for exploring such graphs is that a spindle contains two pairs of vertices distance $\sqrt{3}$ apart, and these pairs cannot both be monochromatic. Intuitively, therefore, a graph containing a high density of interlocking spindles might be constrained to have its monochromatic $\sqrt{3}$-apart vertex pairs distributed rather uniformly (in some sense) in any 4-colouring. Since such graphs typically also contain regular hexagons of side- length 1, one might be optimistic that they could contain some such hexagon that does not contain a monochromatic triple in any 4-colouring of the overall graph, since such a triple is always an equilateral triangle of edge $\sqrt{3}$ and thus constitutes a locally high density, i.e. a departure from the aforementioned uniformity, of monochromatic $\sqrt{3}$-apart vertex pairs." — de Grey, Aubrey D.N.J. (2018), "The Chromatic Number of the Plane Is at least 5", Geombinatorics, 28: 5–18, arXiv:1804.02385.

Figure 6.3.35 Aubrey de Grey, 1963 –

One of de Grey's tools: Multiple tightly linked Moser spindles:

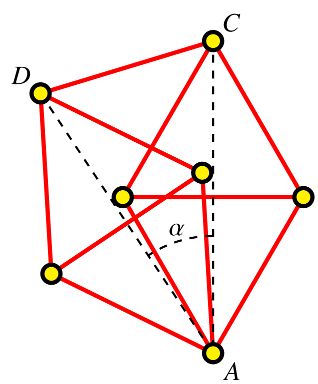


Figure 6.3.36 The angle α

Consider a Moser spindle and observe that (A, C) and (A, D) are two pairs of vertices distance $\sqrt{3}$ apart. Also, observe that the measure of the angle $\angle DAC$ is $\alpha = \arccos\left(\frac{5}{6}\right) \approx 33.56^0$.

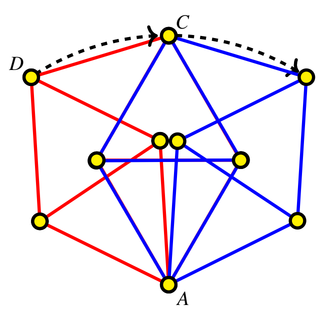


Figure 6.3.37 Rotation by α

Rotate the Moser spindle through α about the point A to obtain another Moser spindle. Observe that the two spindles share four vertices and five edges.

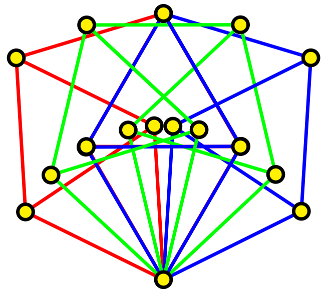


Figure 6.3.38 Three Moser spindles

An additional clockwise rotation by $\frac{\alpha}{2}$ produces "three tightly linked Moser spindles."

de Grey's Proof. Initially, de Grey constructed a graph with 20425 vertices. He "developed a custom program" to test this graph for the existence of monochromatic points with the unit distance under a 4-colouring. In de Grey's words, "This algorithm was implemented in Mathematica 11 on a standard MacBook Air and terminated in only a few minutes."

Follow up. In his original paper, de Gray describes a construction of graph G with 1581 vertices that yields, under any 4-colouring, a pair of monochromatic points one unit apart. Again in de Grey's words, "Happily, G has turned out to be within the reach of standard SAT solvers."

On August 3, 2019, as part of the Polymath16 project, Jaan Parts posted an image of a unit distance graph with 510 vertices and 2508 edges that confirms that $\chi \geq 5$.

Question 6.3.39

1. Is it possible to further reduce the size of the "good" graph?

2. How to find a human-verifiable proof that $\chi \geq 5$?

   In October 2020, Jaan Parts from Kazan, Russia, published an article entitled "The chromatic number of the plane is at least 5 – a human-verifiable proof" (Geombinatorics 30/2 (2020) 77 – 102) accessible at arXiv:2010.12661v1:

   "De Grey's proof [of the fact that $\chi \geq 5$] is not only the first, but, in our opinion, the best of the known ones. But like all others found thus far, it has an annoying flaw: it cannot be verified without using a computer. Here we make an attempt to fill this gap: that is, we present a proof of the known fact $\chi \geq 5$, which can be verified manually in full in a reasonable time."

□

Three facts:

1. Computing has become an instrumental part of mathematical research.

2. Mathematical research has become increasingly collaborative. See, for example, the Polymath Project.

3. Ron Graham offered \$250 for a proof that $\chi \leq 6$. (If you are the first one to prove that $\chi \leq 6$, please contact Steve Buttler to collect the cheque.)

Resources.

1. See [7], pp. 13-20.

2. Hadwiger-Nelson problem – Wikipedia

3. Chromatic Number of the Plane by Cut The Knot

4. Open problems in Euclidean Ramsey Theory by Ron Graham and Eric Tressler

5. The chromatic number of the plane is at least 5 by Aubrey D.N.J. de Grey
6. The Moser Spindle by Evelyn Lamb

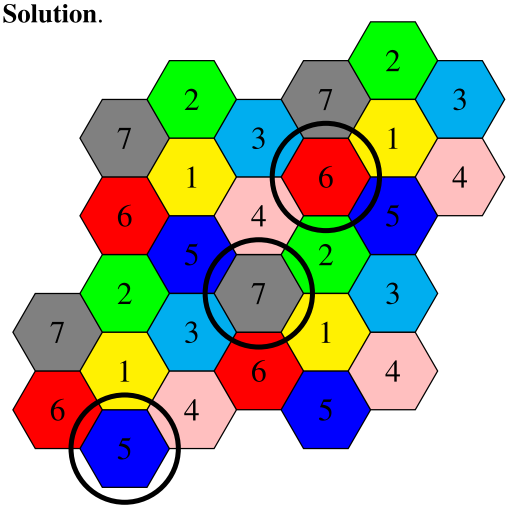
6.4 The Polychromatic Number of the Plane

> *Things forbidden have a secret charm.* — Publius Cornelius Tacitus, a senator and a historian of the Roman Empire, c. 56 — 117

Problem. What is the smallest number of colours needed for colouring the plane in such a way that no colour realizes all distances? (Paul Erdős, 1958)

Example 6.4.1 A 7-colouring that avoids the distance 1 in each colour:

Solution.



A 7-colouring of a tessellation of the plane by regular hexagons, with diameter slightly less than one. Observe that each hexagon is surrounded by hexagons of a different colour.

Figure 6.4.2 Hugo Hadwiger in 1961:

□

Definition 6.4.3 The smallest number of colours sufficient for colouring the plane in such a way that no colour realizes all distances is called the **polychromatic number of the plane** and it is denoted by χ_p. ◊

Observation 6.4.4 $\chi_p \leq \chi$

The Lower Bound: $4 \leq \chi_p$. (Established by Dmitry E. Raiskii in 1970. This proof is by Alexei Merkov from 1997.)

Proof.

1. Assume that there is a 3-colouring of the plane

$$c : \mathbb{E}^2 \to \{\bullet, \bullet, \bullet\}$$

such that

- There are no two points coloured red at the distance r;
- There are no two points coloured blue at the distance b;
- There are no two points coloured green at the distance g.

2. Let a Cartesian coordinate system in $\mathbb{E}^2$ be given.
3. We construct three Moser spindles like on Figure 6.4.5:

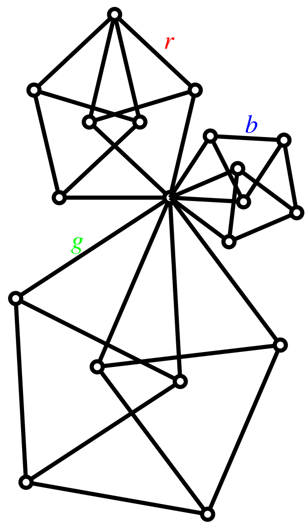


Figure 6.4.5 Three Moser spindles share the origin O as a common point and with the edges of lengths r, b, and g.

4. Consider 18 vectors, each of them with its initial point at the origin and the terminal point being a vertex in one of the three Moser spindles.

   Call those vectors

$$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_6, \vec{v}_7, \vec{v}_8, \ldots, \vec{v}_{12}, \vec{v}_{13}, \vec{v}_{14}, \ldots, \vec{v}_{18}.$$

   Here the terminal points of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_6$ belong to the Moser spindle with all edges of length r, the terminal points of the vectors $\vec{v}_7, \vec{v}_8, \ldots, \vec{v}_{12}$ belong to the Mosers spindle with all edges of length b, and the terminal points of the vectors $\vec{v}_{13}, \vec{v}_{14}, \ldots, \vec{v}_{18}$ belong to the Moser spindle with all edges of length g. See Figure 6.4.6.

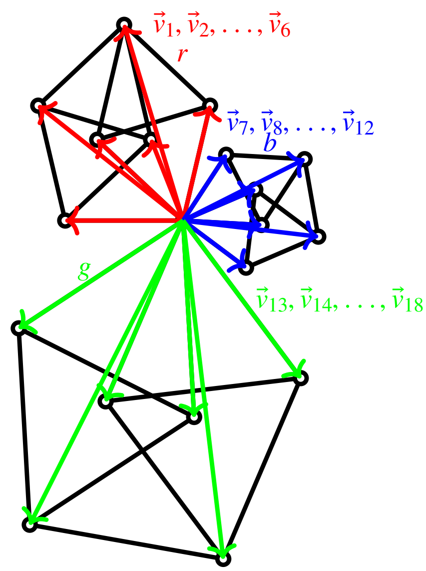


Figure 6.4.6 Eighteen vectors with the same initial point.

5. Next we define a 3-colouring c' of the vector space

$$\mathbb{E}^{18} = \{(a_1, a_2, \ldots, a_{18}) : a_1, a_2, \ldots, a_{18} \in \mathbb{R}\}$$

by

$$c'(a_1, a_2, \ldots, a_{18}) = c(P)$$

where P is the terminal point of the vector

$$a_1 \cdot \vec{v}_1 + \cdots + a_6 \cdot \vec{v}_6 + a_7 \cdot \vec{v}_7 + \cdots + a_{12} \cdot \vec{v}_{12} + a_{13} \cdot \vec{v}_{13} + \cdots + a_{18} \cdot \vec{v}_{18}.$$

6. Let $M \subset \mathbb{E}^{18}$ be the set of all 18-tuples such that $(a_1, a_2, \ldots, a_{18}) \in M$ if and only if all of the following conditions are satisfied:

   (a) $a_i \in \{0, 1\}$ for all $i \in \{1, 2, \ldots, 18\}$;

   (b) $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \in \{0, 1\}$

   (c) $a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} \in \{0, 1\}$

   (d) $a_{13} + a_{14} + a_{15} + a_{16} + a_{17} + a_{18} \in \{0, 1\}$

   For example

$$(\underbrace{1, 0, 0, 0, 0, 0,}_{1 \le i \le 6} \underbrace{0, 0, 0, 0, 0, 1,}_{7 \le i \le 12} \underbrace{1, 0, 0, 0, 0, 0}_{13 \le i \le 18}) \in M$$

   but

$$(\underbrace{1, 0, 0, 0, 0, 0,}_{1 \le i \le 6} \underbrace{0, 0, 0, 0, 0, 0,}_{7 \le i \le 12} \underbrace{1, 1, 0, 0, 0, 0}_{13 \le i \le 18}) \notin M.$$

7. Note that

$$|M| = 7^3.$$

8. Consider the set

$$M_r = \{(a_1, a_2, a_3, a_4, a_5, a_6, \underbrace{0, 0, \cdots, 0}_{\text{All 0's}}) \in M : a_1, \ldots, a_6 \in \{0, 1\}\}$$

   and note that $|M_r| = 7$.

9. *Two observations and a conclusion:*

   (a) If $(a_1, a_2, a_3, a_4, a_5, a_6, \underbrace{0, 0, \cdots, 0}_{\text{All 0's}}) \in M_r$ and $a_i \neq 0$ for some $i \in \{1, \ldots, 6\}$, then

$$\vec{OP} = a_1 \cdot \vec{v}_1 + \cdots + a_6 \cdot \vec{v}_6 + 0 \cdot \vec{v}_7 + \cdots + 0 \cdot \vec{v}_{12} + 0 \cdot \vec{v}_{13} + \cdots + 0 \cdot \vec{v}_{18} = \vec{v}_i$$

   and P is one of the points in the Moser spindle that has all edges of length r.

   (b) The Moser spindle that has all edges of length r cannot have three red vertices:

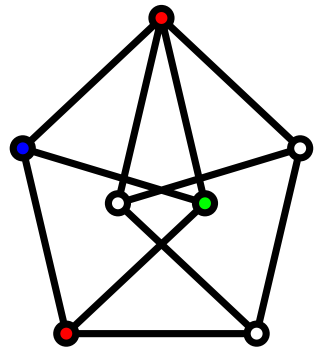


Figure 6.4.7 If there are three r vertices then two of them are r units apart.

(c) The set M_r can have at most two elements coloured r by the colouring c'.

Another observation:

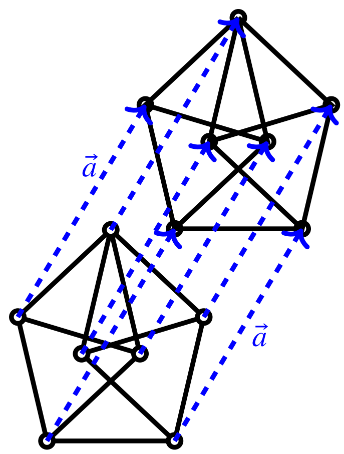


Figure 6.4.8 A translate of the Moser spindle is the Moser spindle.

For each of the 49 elements of the set

$$M_{bg} = \{(0, 0, 0, 0, 0, 0, a_7, a_8, \ldots, a_{18}) \in M : a_7, \ldots, a_{18} \in \{0, 1\}\}$$

we make a translate of M_r in $\mathbb{E}^{18}$:

$$M_r{}^a = a + M_r, \; a \in M_{bg}.$$

Clearly

$$M = \cup_{a \in M_{bg}} M_r^a$$

and, for all $a, b \in M_{bg}$,

$$a \neq b \Rightarrow M_r{}^a \cap M_r{}^b = \emptyset.$$

In other words we have divided the set M into $7^2 = 49$ mutually disjunct copies of M_r.

How many elements in $M_r{}^a$, $a \in M_{bg}$, are coloured red by c'?

10. Let $(0, 0, 0, 0, 0, 0, a_7, a_8, \ldots, a_{18}) \in M_{bg}$ and let

$$\vec{a} = 0 \cdot \vec{v}_1 + \cdots + 0 \cdot \vec{v}_6 + a_7 \cdot \vec{v}_7 + \cdots + a_{12} \cdot \vec{v}_{12} + a_{13} \cdot \vec{v}_{13} + \cdots + a_{18} \cdot \vec{v}_{18}.$$

Then the elements of $M_r{}^a$ are coloured by c' in the same way that c colours the vertices of the Moser spindle that is obtained as the translate of the original Moser spindle by $\vec{a}$!

Therefore, for each $a \in M_{bg}$, the set $M_r{}^a$ can have at most TWO red elements.

11.

$$\begin{aligned} \text{\# of red elements of } M &= \sum_{a \in M_{bg}} \text{\# of red elements of } M_r{}^a \\ &\le \sum_{a \in M_{bg}} 2 = 2 \cdot 49 = 98. \end{aligned}$$

12. Similarly

$$\text{\# of blue elements of } M \le 98$$

and

$$\text{\# of green elements of } M \le 98.$$

Therefore

$$\begin{aligned} 7^3 = {} & (\text{\# of red elements of } M) + (\text{\# of blue elements of } M) \\ + {} & (\text{\# of green elements of } M) \le 3 \cdot 98 = 3 \cdot (2 \cdot 7^2) = 6 \cdot 7^2. \end{aligned}$$

Contradiction!

13. Therefore, our assumption that there is a 3-colouring of the plane

$$c : \mathbb{E}^2 \to \{\bullet, \bullet, \bullet\}$$

such that

- There are no two points coloured red at the distance r;
- There are no two points coloured blue at the distance b;
- There are no two points coloured green at the distance g;

led to a contradiction!

14. There is at least one colour in every 3-colouring of the plane that realizes all distances. This implies

$$4 \le \chi_P.$$

■

The Upper Bound. $\chi_P \le 6$. (S.B. Stechkin, 1970)

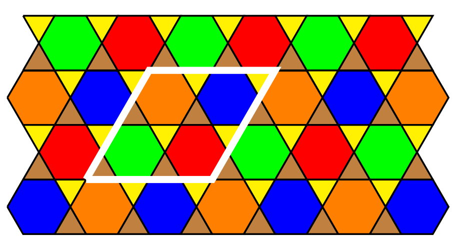


Figure 6.4.9 Steichkin's 6-coloring of the plane.

Take a Closer Look.

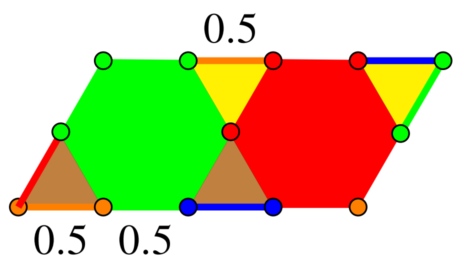


Figure 6.4.10 Steichkin's 6-coloring of the plane - a closer look.

Note:

- All sides of all triangles and hexagons are of length 0.5.
- Every hexagon includes its boundary except its rightmost and two lowest vertices.
- Triangles do not include their boundaries.

Which Distances are Avoided?

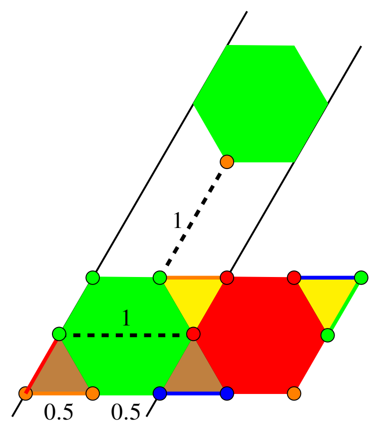


Figure 6.4.11 No two green points that are 1 unit apart.

Note:

- Four colours used to colour hexagons do not realize the distance 1.
- Two colours used to colour triangles do not realize the distance 0.5.

Notation. Steichkin's colouring is of the type $(1, 1, 1, 1, \frac{1}{2}, \frac{1}{2})$.

Theorem 6.4.12 $4 \leq \chi_p \leq 6$.

Resources.

1. See [7], pp 32-44.
2. Open problems in Euclidean Ramsey Theory by R. Graham and E. Tressler

6.5 Fractional Chromatic Number

Perhaps I am doomed to retrace my steps under the illusion that I am exploring, doomed to try and learn what I should simply recognize, learning a mere fraction of what I have forgotten. — André Breton, a French writer and poet,1896 — 1966.

Definition 6.5.1 For a given graph $G = (V, E)$ and the positive integers m and n, $m \leq n$, a proper n/m-colouring with n colours of the graph G is a function that assigns to each vertex a set of m distinct colours, in such a way that adjacent vertices are assigned mutually disjoint sets.

The fractional chromatic number of G is defined as

$$\chi_f(G) = \inf\left\{\frac{n}{m} : \text{there is a proper } n/m \text{ colouring of } G\right\}.$$

Let m and n be positive integers with $m \leq n$. An n/m-colouring is a function with n colours of the plane that assigns a set of m distinct colours to each point in the plane so that any two points that are one unit apart are assigned mutually disjoint sets.

The fractional chromatic number of the plane is:

$$\chi_f(\mathbb{R}^2) = \inf\left\{\frac{n}{m} : \text{there is a proper } n/m \text{ colouring of the plane}\right\}.$$

◊

The current lower bound for the fractional chromatic number of the plane, $\chi_f(\mathbb{R}^2) \geq \frac{1999983}{512933} \approx 3.8991$, was established in 2020 by Bellitto, Pêcher, and Sédillot, and the upper bound, 4.3599, was established in 1993 by Hochberg and O'Donnell.

Example 6.5.2 The Moser spindle (MS) is used to establish an early lower bound of the chromatic number of the plane. Four colours are required to colour the vertices, so that no two adjacent vertices are of the same colour.

We can apply the definition of the fractional colouring to the above fact to establish that $\chi_f(MS) \leq 4/1 = 4$.

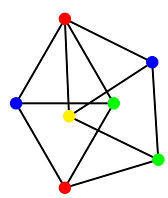


Figure 6.5.3 A 4/1-fractional colouring of the vertices of the Moser spindle.

But can we do better? If we place three colours in each vertex, we find that we only need eleven colours to colour the Moser spindle. Which gives us $\chi_f(MS) \leq \frac{11}{3} \approx 3.66$.

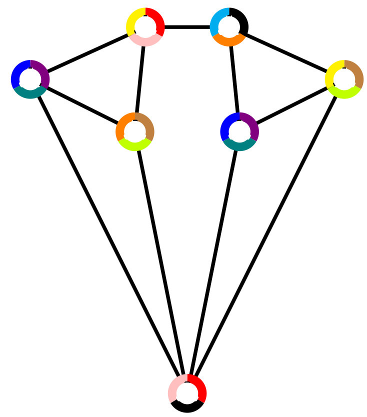


Figure 6.5.4 A graph isomorphic to the Moser spindle together with its proper 11/3 colouring

In fact, it is known that the fractional chromatic number of the Moserr spindle is 3.5. This is because the Moser spindle has 7 vertices and the independence number 2. This was first observed in 1992 by David Fisher and Daniel Ullman.

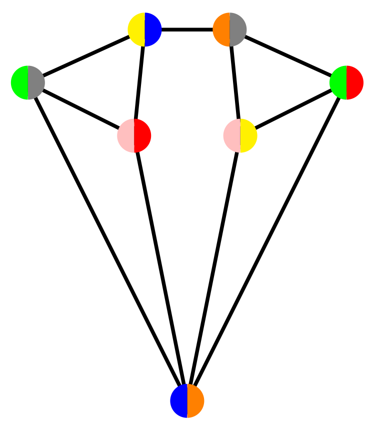


Figure 6.5.5 A graph isomorphic to the Moser spindle together with its proper 7/2 colouring

Observe that the fact that $\chi_f(MS) = 3.5$ implies that $\chi_f(\mathbb{R}^2) \geq 3.5$. □

Resources:

Bellitto, T., Pêcher, A., and Sédillot, A. (2020). On the density of sets of the Euclidean plane avoiding distance 1. https://arxiv.org/abs/1810.00960

Cranston, D. W., Rabern, L. (2017). The fractional chromatic number of the plane. Combinatorica. 37(5): 837-861.

Hochberg, R., O'Donnell, P. (1993). A large independent set in the unit distance graph. Geombinatorics. 2(4):83-84.

D. Fisher and D. Ullman. (1992). The fractional chromatic number of the plane. Geombinatorics, 2(1):8-12.

6.6 Exercises

Exercise 6.6.1 Convex quadrilaterals. Show that, given five points in the plane with no three collinear, the number of convex quadrilaterals formed by these points is odd.

Solution. We consider three cases.

- Case 1: Suppose that the five vertices form a convex pentagon. Then any four of them form a convex quadrilateral.

There are five convex quadrilaterals in this case.

- Case 2: Suppose that four points form a convex quadrilateral (say A_1, A_2, A_3, and A_4 in clockwise order) which contains the fifth point A_5 in its interior. Let S be the intersection of the diagonals $\overline{A_1A_3}$ and $\overline{A_2A_4}$. Then the point A_5 lies in one of the four triangles into which diagonals dissect the quadrilateral, say $\triangle SA_1A2$. Observe that the quadrilaterals $A_1A_5A_3A_4$ and $A_2A_3A_4A_5$ are convex but the other two are not.

  There are three convex quadrilaterals in this case.

- Case 3: Suppose that three points (say A_1, A_2, A_3) form a triangle with A_4 and A_5 in its interior. The line A_4A_5 intersects two of the three sides, say $\overline{A_1A_2}$ and $\overline{A_1A_3}$. Then $A_2, A_3A_4A_5$ is the only convex quadrilateral.

Exercise 6.6.2 Convex hexagon. Find 10 points in the plane in general position that do not contain a convex hexagon. Justify your answer!

Solution. Consider the following configuration:

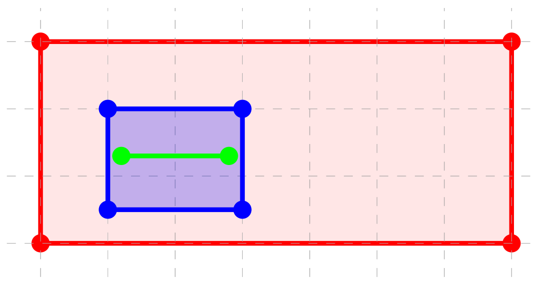
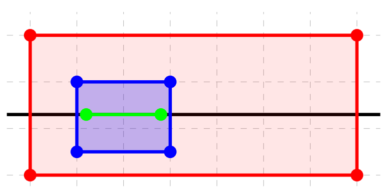


Figure 6.6.3 Ten points in general position.

This is a $(4, 4, 2)$ configuration. As we already know, the eight points in the (4,4) part of the given configuration do not contain a convex pentagon.

Suppose that there is a convex hexagon in this configuration. Then the hexagon has to contains both green points inside of the blue quadrilateral. (If the convex hexagon contains only one green point then the remaining five points belong to the set of red and blue points, which contradicts the fact that those points do not contain a convex pentagon.)

Since the line determined by the two green points divides the plane in the way that each of the two half-planes contains only two red and two blue points, the convex hexagon must have two red, two blue, and two green vertices. But this hexagon is not convex because the blue points are inside of the quadrilateral determined by the red and green points. See Figure 6.6.4.



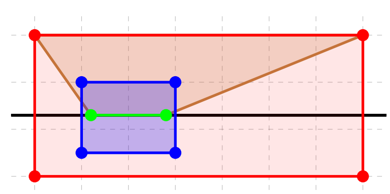


Figure 6.6.4 There is no convex hexagon.

Exercise 6.6.5 Convex quadrilaterals. Prove that any five points in the plane in general position contain an *empty* convex quadrilateral, i.e. a convex quadrilateral that does not contain in its interior the remaining point.

Is this true if you take more than five points in general position?

Solution. If the given five points are vertices of a convex pentagon then any four points form and empty convex quadrilateral.

By Esther Klein's result we know that any five points in the plane in general position contain convex quadrilateral, Say that the given five points S do not determine a convex pentagon. Then the convex hull of the set S has three or four points. In both cases there is an empty convex quadrilateral. See Figure 6.6.6

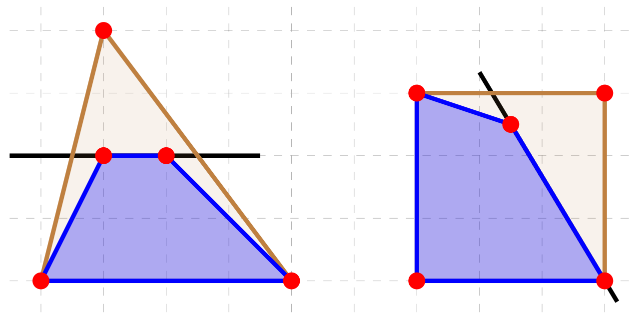


Figure 6.6.6 Empty convex quadrilaterals.

Let S be a set of n points in the plane in general position, with $n \geq 6$. Let

$$L = \{l_i : i \in [1, n(n-1)/2]\}$$

be the set of all lines in the plane determined by pairs of points from the set S. Let p be a line that is not parallel to any line from the set L and such that all points that belong to the set S are on the same side of p. (In other words, none of the line segments with the end points from S intersects the line p.)

Move the line p towards the set S. Let p' be the line that contains a point from S, that is parallel to the line p and that there is no element form S between lines p and p'. (See Figure 6.6.7.)

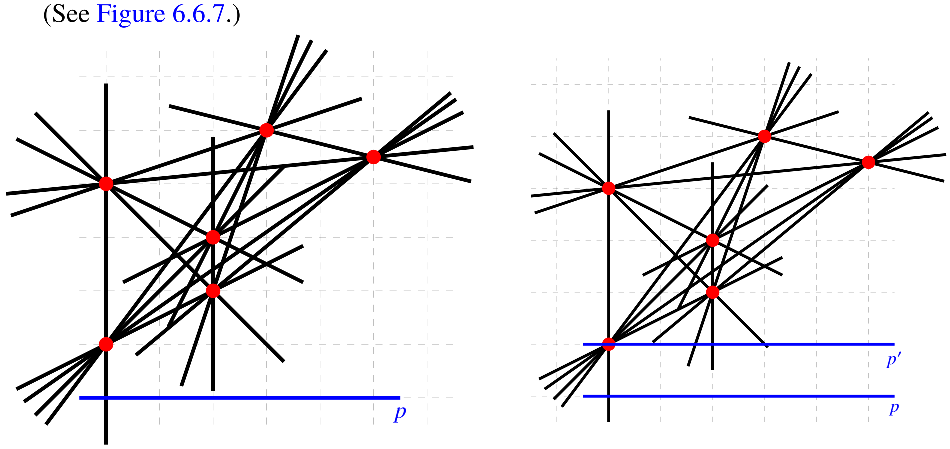


Figure 6.6.7 Note that p' contains only one point from S. Why?

Continue moving the line p till it reaches five points from S. (See Figure 6.6.8.)

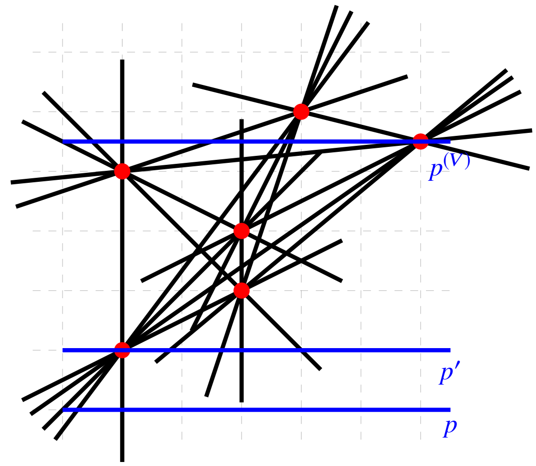


Figure 6.6.8 Note that the line $p^{(V)}$ divides the set S so that one point from S is on the $p^{(V)}$, only four points from S are on the same side of the line $p^{(V)}$ as it is the line p.

What should we do next?

Exercise 6.6.9 Convex polygons. Use mathematical induction to prove that if $n \geq 4$ then n points in the plane in general position form a convex polygon if and only if every four of them form a convex quadrilateral.

Solution. The base case is trivial.

Suppose that the claim is true for some fixed $n \geq 4$, i.e. suppose that $n \geq 4$ is such that any n points in the plane in general position form a convex polygon if and only if every four of them form a convex quadrilateral.

Let S be the set of $n+1$ points in the plane in the general position. If the points from S form a convex $(n+1)$-gon then any four points from S are vertices on that convex polygon and thus form a convex quadrilateral.

Suppose that any four points from S form a convex quadrilateral. We note that by the inductive hypothesis this means that any n points from S form a convex n-gon.

Let $\overline{S}$ be the convex hull of S. The $|\overline{S}| = n+1$ or $|\overline{S}| = n$. (Is it possible that $|\overline{S}| < n$? Why?)

If $|\overline{S}| = n+1$ then $S = \overline{S}$ and S is convex $(n+1)$-gon.

Let $|\overline{S}| = n$. Then the point A such that $\{A\} = S \backslash \overline{S}$ is inside the n-gon determined by the points from $\overline{S}$. Let $X \in \overline{S}$ and consider the line through the points A and X. This line intersects the line segment with the endpoints $Y, Z \in S$. See Figure 6.6.10.

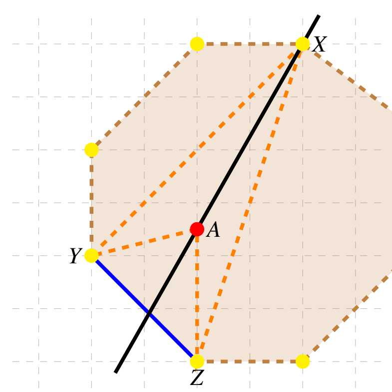


Figure 6.6.10 Is the quadrilateral determined by the points A, X, Y, and Z convex?

Observe that the quadrilateral determined by the points A, X, Y, and Z is not convex which contradicts our assumption that any four points from S form a convex quadrilateral.

Hence $|\overline{S}| = n + 1$ and S is a convex $(n + 1)$-gon, which completes the inductive step.

Exercise 6.6.11 Tarsi's proof. Michael Tarsi proved Erdős-Szekeres theorem on convex polygons in the following way:

1. Take $n \geq 4$.
2. Take a set S of points in the plane in general position of the size $m = R(3; n, n)$.
3. Enumerate the elements of the set S by numbers from 1 to m.
4. Colour 3-element subsets of S in the following way:
   - Colour the set $\{i, j, k\}$, $i < j < k$, red if we travel from i to j to k in a clockwise direction.
   - Colour the set $\{i, j, k\}$, $i < j < k$, blue if we travel from i to j to k in a counterclockwise direction.

Complete Tarsi's proof!

Solution. By Ramsey's theorem there is a set T, $T \subset S$, and $|T| = n$, such that for any $i, j, k \in T$, $i < j < k$, the set $\{i, j, k\}$ is always of the same colour, say blue. But this implies that any four points from T form a convex quadrilateral. To see this take $x, y, z, w \in T$ and suppose that x, y, z, and w determine a concave quadrilateral. Suppose that the point w is inside of the triangle determined by x, y, and z. Also, suppose that $x < y < z$.

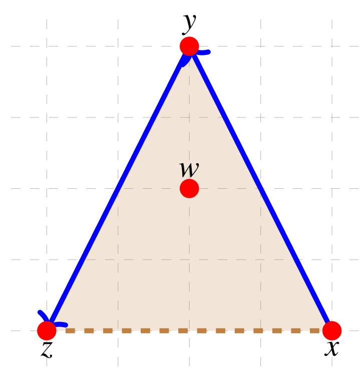


Figure 6.6.12 $\{x, y, z\}$

We consider four cases given on Figure 6.6.13:

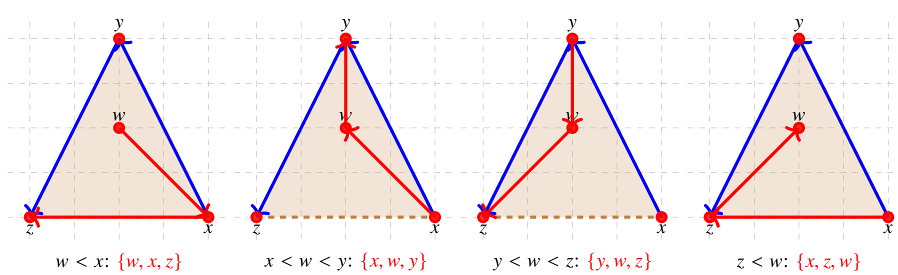


Figure 6.6.13 Four cases.

Exercise 6.6.14 $f(5, 5)$**.** Prove that $f(5, 5) = 21$.

Directions: Adopt the proof of Theorem 2.5 in The Erdős-Szekeres problem on points in convex position - a survey by W. Morris and V. Soltan for the case $k = l = 5$.

Solution. We take the following facts as given:

1. $f(k, l) = f(l, k)$

2. From
$$f(k, l) \le \binom{k + l - 4}{k - 2} + 1$$
it follows that $f(4, 4) \le 7$, $f(5, 4) \le 11$, and $f(5, 5) \le 21$.

3. $f(k, 3) = f(3, k) = k$.

Consider the set S of 6 points in the plane in general position given on Figure 6.6.15.

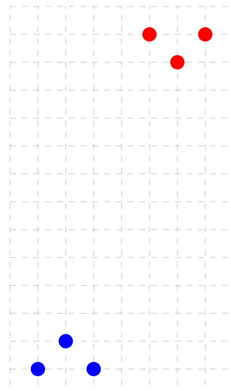


Figure 6.6.15 Six points in general position with no a 4-cup or a 4-cap.

Note that there is NO a 4-cup or a 4-cap in this configuration. Hence $f(4, 4) \ge 7$, which implies that $f(4, 4) = 7$.

Next we consider a configuration of 6 points with no 4-cup or 4-cap, like one above. We call this configuration A. Let B be a 4-cup. We place the configurations A and B in the way that the following conditions are satisfied:

1. Every point in B has greater first coordinate than the first coordinates of any point in A.

2. The slope of any line connecting a point in A with a point in B is greater than the slope of any line connecting two points in the same configuration.

Let $S = A \cup B$. See Figure 6.6.16.

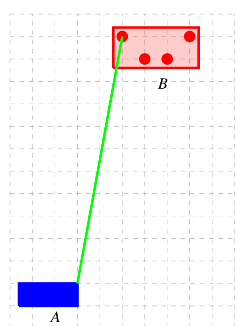


Figure 6.6.16 The 10 point configuration S with no a 5-cup or a 4-cap.

Also note

1. Any cup in S that contains both points from A and B can contain only ONE point from B. (Why?)

2. Therefore the configuration S can contain at most a 4-cup.

3. The configuration S does not contain a 4 cap. (Why?)

Hence, the configuration S contains neither a 5-cap nor a 4-cap. It follows that

$$f(5,4) \geq |S| + 1 = |A| + |B| + 1 = 6 + 4 + 1 = 11.$$

Thus, $f(5,4) = f(4,5) = 11$.

Finally we consider two configurations, X and Y, that satisfy the following conditions:

1. $|X| = |Y| = 10$.

2. The configuration X does not contain a 4-cap or a 5-cap. This is possible because $10 < f(4,5)$.

3. The configuration Y does not contains a 5-cup or a 4-cap. This is possible because $10 < f(5,4)$.

4. Every point in Y has greater first coordinate than the first coordinates of any point in X.

5. The slope of any line connecting a point in X with a point in Y is greater than the slope of any line connecting two points in the same configuration.

Let $Z = X \cup Y$. See Figure 6.6.17.

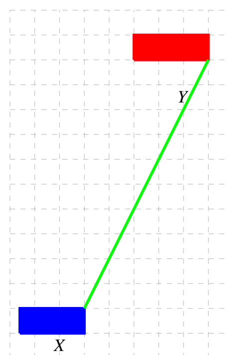


Figure 6.6.17 The 20 point configuration Z with no a 5-cup or a 5-cap.

Also note

1. Any cup in Z that contains both points from X and Y can contain only ONE point from Y. (Why?)

2. Therefore the configuration Z can contain at most a 4-cup.

3. Any cap in Z that contains both points from X and Y can contain only ONE point from X. (Why?)

4. The configuration Z can contain at most a 4-cap. (Why?)

Hence, the configuration S contains neither a 5-cap nor a 5-cap. It follows that

$$f(5,5) \geq |Z| + 1 = |X| + |Y| + 1 = 10 + 10 + 1 = 21.$$

Thus, $f(5,5) = 21$.

Exercise 6.6.18 Moser spindle. In the May 1961 issue of the Canadian Math Bulletin, Leo and William Moser posted their solution to the following problem:

1. Prove that every set of six points in the plane can be coloured in three colours in such a way that no two points a unit distance apart have the same colour.
2. Show that in (a) six cannot be replaced by seven.

In other words, Leo and William Moser established that the chromatic number of the plane $\chi \geq 4$.

In this exercise we will follow Leo and William Moser's solution to the problem above.

Let two points which are a unit distance apart be called "friends", otherwise they are "strangers". If a finite set of points can be coloured by k colours so that no pair of friends have the same colour we say that this set permits a proper k-colouring. In the rest of this note we assume that any set of four or more points is a subset of a plane.

1. Prove that $\chi \geq 3$,in other words prove that if the points in the plane are coloured by one of the two given colours then there must exist two points that are at a unit distance and coloured by the same colour.
2. Show that four points in the plane cannot be friends to each other and that two points cannot have three common friends.
3. Show that any set of four points permits a proper 3-colouring.
4. Show that any set of five points permits a proper 3-colouring.
5. Show that any set of six points permits a proper 3-colouring.
6. Use the Moser spindle to prove that $\chi \geq 4$.

Solution.

1. Consider an equilateral triangle.
2. Observe that the existence of four points that are friends with each other would imply the existence of a triangle inscribed in the unit circle with all its sides equal to 1. A mutual friend of two friends, say A and B, must belong to the intersection of circles with their centres at A and B and radii equal to 1.
3. Say that A and B are friends. Use two colours to colour those two points. Only one of C ad D can be a mutual friend to A and B, say C. Use the third colour to colour C. How should we colour the point D?
4. Observe that from the second part of (2) it follows that not all of the points A, B, C, D, and E can have exactly three friends each. Suppose that the point A does not have three friends. Say that A has two friends, B and C. Use (3) to properly 3-colour points B, C, D, and E. How should we colour the point A? If A has four friends then those four points belong to a circle with the centre at A and we can colour them properly with two colours.
5. Let A, B, C, D, E, and F be six points in the plane. If there is a point with two or four or five friends then we can use a similar argument as in (4). Suppose that all points have exactly three friends. Say that A is a friend with B, C, and D. Observe that A, E, and F have a common friend, say B. If E and F are strangers then there is a proper 2-colouring of the six points. If E and F are friends, consider the following two cases: (1) B is the only common friend for E and F; (2) E and F have another common friend.

6. Use three colours and try to avoid colouring two friends with the same colour.

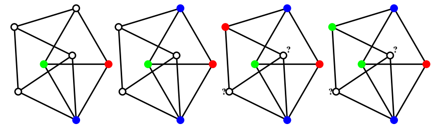


Figure 6.6.19 Three colours are not enough!

Exercise 6.6.20 Density and chromatic number. The "density version" of the problem of finding the chromatic number of the plane was attributed to Leo Moser in H. T. Croft, Incidence incidents, Eureka (Cambridge) **30** (1967), 22 – 26:

"What is $m_1(\mathbb{R}^2)$, the maximum density of a measurable set in the plane that does not contain a unit-distance pair?"

In more casual terms, the question is to determine the size (in terms of the portion of the plane) that a set that avoids the unit distance cannot exceed.

Erdős said that "it seems very likely" that $m_1(\mathbb{R}^2) < 0.25$.

The current best upper upper bound for $m_1(\mathbb{R}^2)$ is 0.25688 and it is obtained by G. Ambrus and M. Matolcsi ("Density estimates of 1-avoiding sets via higher order correlations," arXiv:1809.05453).

In 1967, Croft established that $0.2293 < m_1(\mathbb{R}^2) \leq \frac{2}{7} = 0.2857$ and commented that "the bounds are surprisingly close."

In this exercise we will follow Croft's argument that $m_1(\mathbb{R}^2) > 0.2293$. Interestingly enough this is still the best known lower bound for $m_1(\mathbb{R}^2)$.

Start by covering the plane with an infinite equilateral triangular lattice. The length of the side of the triangles in the lattice will be determined by the following construction.

Say that $\triangle XYZ$ is one of the triangles in the lattice. We construct three mutually congruent "lumps" in $\triangle XYZ$ in the following way:

- Choose an angle $\theta \in \left(0, \frac{\pi}{6}\right)$.
- Let B and C be points inside of $\triangle XYZ$ such that $\overline{XB} = \overline{XC} = \dfrac{1}{2}$ and that $\angle(YXC) = \angle(ZXB) = \theta$.
- Draw the arc BC on the circle with the centre at X and radius equal to $\dfrac{1}{2}$.
- Let A be the point on the line segment $\overline{XZ}$ such that $\overline{AB} \perp \overline{XZ}$.
- Let D be the point on the line segment $\overline{XY}$ such that $\overline{CD} \perp \overline{XY}$.
- Call the "interior" of the region in the plane bounded by the line segments $\overline{XA}$, $\overline{XD}$, $\overline{AB}$, $\overline{CD}$, and the arch BC, a "lump" centred at X with the angle θ.
- Do the same construction for the vertices Y and Z. Keep the angle θ same as above to obtain two more congruent lumps in $\triangle XYZ$.
- Choose $|\overline{XY}|$, the length of the side of the equilateral triangle so that the distance between the corresponding points (obtained in this construction) on the line segment $\overline{XY}$ equals to 1.
- Do the same construction for all triangles in the lattice.

See Figure 6.6.21.

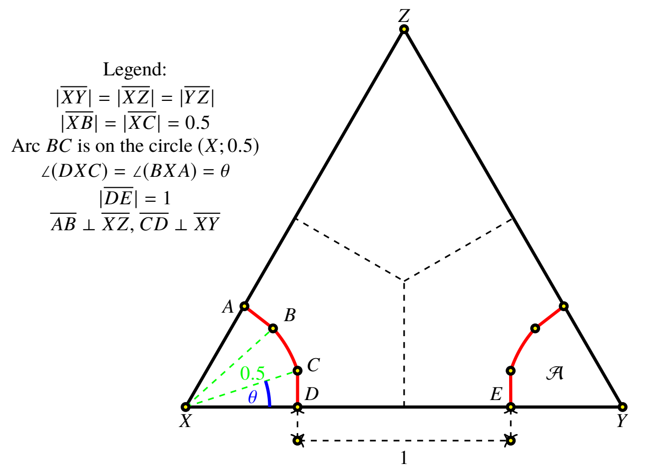


Figure 6.6.21 Croft's construction of "three lumps"

Let $S = S(\theta)$ be the set of all points in the plane that belong to one of the lumps obtained by the above construction for some (fixed) θ.

1. What is the length of the line segment $\overline{XY}$?
2. Show that if $M, N \in S$ then $|\overline{MN}| \neq 1$.
3. Find the area $\mathcal{A} = \mathcal{A}(\theta)$ of a lump.
4. Determine the area $A_\triangle = A_\triangle(\theta)$ of a triangle in the lattice.
5. Find the density $\delta(S) = \delta(S(\theta))$ of the set $S = S(\theta)$.
6. Use your knowledge of calculus to find the maximum value of $\delta(S(\theta))$. Use technology to determine the maximum value.
7. Are you convinced that $m_1(\mathbb{R}^2) > 0.2293$? Why yes or why not?

Solution.

1. From the $\triangle XDC$ it follows that

$$|\overline{XD}| = |\overline{XC}| \cdot \cos\theta = \frac{\cos\theta}{2}.$$

Hence

$$|\overline{XY}| = |\overline{XD}| + |\overline{DE}| + |\overline{EY}| = \frac{\cos\theta}{2} + 1 + \frac{\cos\theta}{2} = 1 + \cos\theta.$$

2. We distinguish two cases:

   - Case 1: There is a vertex X in the lattice such that M and N belong to the same lump or that they belong to two different lumps that are associated to X (i.e M and N are in two different triangles in the lattice that share the vertex X).

     In this case M and N belong to the interior of a circle with radius equal to $\frac{1}{2}$ which implies that $|\overline{MN}| < 1$.

- Case 2: There are two different vertices in the lattice, X and X', such that M belongs to a lump centred at X and N belongs to a lump centred at X'. By construction, the distance between a point that belongs to the boundary of the lump centred at X and a point that belongs to the boundary of the lump centred at X' is greater than or equal to 1. Hence, in this case $|\overline{MN}| > 1$.

3. Consider the lump centred at the vertex X on Figure 6.6.21 and observe that

$$\mathcal{A} = (\text{Area of } \triangle XDC) + (\text{Area of circular sector } XBC) + (\text{Area of } \triangle XAB).$$

Next, from

$$(\text{Area of } \triangle XDC) = (\text{Area of } \triangle XAB) = \frac{1}{2} \cdot \left(\frac{1}{2} \cdot \sin\theta\right) \cdot \left(\frac{1}{2} \cdot \cos\theta\right) = \frac{\sin(2\theta)}{16}$$

and

$$(\text{Area of circular sector } XBC) = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{\pi}{3} - 2\theta\right) = \frac{1}{8} \cdot \left(\frac{\pi}{3} - 2\theta\right)$$

it follows that

$$\mathcal{A} = \mathcal{A}(\theta) = \frac{\sin(2\theta)}{8} + \frac{1}{8} \cdot \left(\frac{\pi}{3} - 2\theta\right) = \frac{1}{8} \cdot \left(\sin(2\theta) + \frac{\pi}{3} - 2\theta\right).$$

4. Since the side of the equilateral triangle equals to $1 + \cos\theta$ it follows that

$$A_\triangle = \frac{\sqrt{3} \cdot (1 + \cos\theta)^2}{4}.$$

5. Recall that the question is to determine the portion of the plane that the set $S = S(\theta)$ occupies. Because all triangles in the lattice are congruent, the density of the set S in the plane will be equal to the portion of a single triangle in the lattice that the three lumps occupy. Hence

$$\text{Density of } S(\theta) = \delta(S(\theta)) = \frac{3 \cdot \mathcal{A}}{A_\triangle}$$

$$= 3 \cdot \frac{\frac{1}{8} \cdot \left(\sin(2\theta) + \frac{\pi}{3} - 2\theta\right)}{\frac{\sqrt{3} \cdot (1+\cos\theta)^2}{4}} = \frac{\sqrt{3}}{2} \cdot \frac{\sin(2\theta) + \frac{\pi}{3} - 2\theta}{(1 + \cos\theta)^2}.$$

6. From

$$\frac{d\delta}{d\theta} = \frac{\sqrt{3}}{2} \cdot \frac{(2\cos(2\theta) - 2)(1 + \cos\theta)^2 - 2(\sin(2\theta) + \frac{\pi}{3} - 2\theta)(1 + \cos\theta) \cdot (-\sin\theta)}{(1 + \cos\theta)^4}$$

$$= \sqrt{3} \cdot \frac{(\cos(2\theta) - 1)(1 + \cos\theta) + (\sin(2\theta) + \frac{\pi}{3} - 2\theta) \cdot \sin\theta}{(1 + \cos\theta)^3}$$

$$= \sqrt{3} \cdot \frac{-2\sin^2\theta \cdot (1 + \cos\theta) + (\sin(2\theta) + \frac{\pi}{3} - 2\theta) \cdot \sin\theta}{(1 + \cos\theta)^3}$$

$$= \sqrt{3} \cdot \frac{\sin\theta \cdot (-2\sin\theta - 2\sin\theta\cos\theta + \sin(2\theta) + \frac{\pi}{3} - 2\theta)}{(1 + \cos\theta)^3}$$

$$= \sqrt{3} \cdot \frac{\sin\theta \cdot (-2\sin\theta + \frac{\pi}{3} - 2\theta)}{(1 + \cos\theta)^3}$$

$$= 2\sqrt{3} \cdot \frac{\sin\theta \cdot (\frac{\pi}{6} - \sin\theta - \theta)}{(1+\cos\theta)^3}$$

it follows that

$$\frac{d\delta}{d\theta} = 0 \Leftrightarrow \sin\theta + \theta = \frac{\pi}{6}.$$

Observe that by the Intermediate Value Theorem there is $\theta_0 \in \left(0, \frac{\pi}{6}\right)$ such that

$$\sin\theta_0 + \theta_0 = \frac{\pi}{6}.$$

WolframAlpha gives $\theta_0 \approx 0.263316$ radians.

To decide if the critical number $\theta = \theta_0$ yields the absolute maximum value of $\delta = \delta(S(\theta))$ we use the First Derivative Test.

Observe that

$$\frac{d\delta}{d\theta} > 0 \Leftrightarrow \eta(\theta) = \frac{\pi}{6} - \sin\theta - \theta > 0.$$

But

$$\frac{d\eta}{d\theta} = -\cos\theta - 1 < 0, \theta \in \left(0, \frac{\pi}{6}\right)$$

which means that the function $\eta(\theta)$ is decreasing from $\eta(0) = \frac{\pi}{6}$ to $\eta(\frac{\pi}{6}) = -\frac{1}{2}$. Also, this implies that θ_0 is the only only critical number of the function δ.

Hence, when passing through $\theta = \theta_0$, $\dfrac{d\delta}{d\theta}$ changes its sign from positive to negative, and by the First Derivative Test the number $\delta = \delta(S(\theta_0))$ is the local and absolute maximum value of the function $\delta = \delta(S(\theta))$.

From

$$\delta(S(\theta)) = \frac{\sqrt{3}}{2} \cdot \frac{\sin(2\theta) + \frac{\pi}{3} - 2\theta}{(1+\cos\theta)^2}$$

it follows that

$$\delta(S(\theta_0)) \approx \delta(S(0.263316)) \approx \frac{\sqrt{3}}{2} \cdot \frac{\sin(0.526632) + \frac{\pi}{3} - 0.526632}{(1+\cos(0.263316))^2} \approx 0.229365.$$

7. Let $\mathcal{T}$ be the family of all sets in the plane that avoid the unit distance. By definition

$$m_1(\mathbb{R}^2) = \sup\{\delta(T) : T \in \mathcal{T}\}.$$

Since $S(\theta_0) \in \mathcal{T}$, it follows that

$$m_1(\mathbb{R}^2) \geq \delta(S(\theta_0)) > 0.2293.$$

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