HW 01 Some Solutions

William Gasarch-U of MD
Problem 2

1. Prove that for every $c$, for every $c$ coloring of $\binom{\mathbb{N}}{2}$, there is a homogenous set USING a proof similar to what I did in class.

SKETCH

When processing a node $x_i$ instead of saying "Either an inf numb of R or B edges come out of $x_i$." say "Either an inf numb of $R_1$ or $\cdots$ or $R_c$ edges come out of $x_i$.

2. Prove that for every $c$, for every $c$ coloring of $\binom{\mathbb{N}}{2}$, there is an inf homogenous set USING induction on $c$.

SKETCH

c = 1 trivial.
c = 2 is the proof your saw in class.

Assume $c \geq 3$ Assume theorem true for all $c' < c$. We will only be using $c - 1$ and 2.

When $c$-color $\binom{\mathbb{N}}{2}$ with colors $\{1, \ldots, c\}$ view it as $c - 1$ colors: 1, 2, $\ldots$, $c - 2$ and color $\{c - 1, c\}$ for those edges colored EITHER. Get homog set.

If it's homog with color 1 or $\cdots$ $c - 2$ then done.

If it's homog color $\{c - 1, c\}$ then use 2-color case.

VOTE Which proof did you like better.
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**VOTE** Which proof did you like better.
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When doing the case where color \(\{c-1, c\}\) occurs infinitely often we use 2-ary Ramsey.

So I am using the theorem

\[(\forall) \text{ COL: } (N) \rightarrow [2] (\exists) \text{ inf homog set.}\]
Problem 2- A Subtle Point

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When doing the case where color \{c − 1, c\} occurs \textit{inf} often we use 2-ary Ramsey.

So I am using the theorem
\((\forall) \text{COL} : \binom{\mathbb{N}}{2} \rightarrow [2] (\exists) \text{inf homog set.}\)

NO, I am not using that! The set I am coloring is an infinite subset of \(\mathbb{N}\). So I am really using the following trivial corollary of the above theorem:
\((\forall) \text{inf } A \subseteq \mathbb{N}, (\forall) \text{COL} : \binom{A}{2} \rightarrow [2] (\exists) \text{inf homog set.}\)
Problem 3

Proof for $a$-ary $c$-color Ramsey.

**SKETCH** Given $\text{COL} : \binom{\mathbb{N}}{a} \to [c]$, form $\text{COL}' : \binom{\mathbb{N}}{a-1} \to [c]$ via

$$\text{COL}'(z_1, \ldots, z_{a-1}) = \text{COL}(x_1, z_1, \ldots, z_{a-1}).$$
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Find homog set inductively and kill all vertices not in that set.
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Lather, Rinse, Repeat to get $x_1, x_2, \ldots$. 
Problem 4 (slightly modified)

\( x_1, x_2, x_3, \ldots \) is an inf seq of reals.
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\(x_1, x_2, x_3, \ldots\) is an inf seq of reals.
For \(i < j\).

\[
COL(i, j) = \begin{cases} 
  \text{RED} & \text{if } x_i < x_j \\
  \text{BLUE} & \text{if } x_i > x_j \\
  \text{GREEN} & \text{if } x_i = x_j 
\end{cases}
\]

Apply Ramsey Theory to get a theorem.
If homog RED then get subseq set \(x_{i_1} < x_{i_2} < \ldots\)
If homog BLUE then get subseq set \(x_{i_1} > x_{i_2} > \ldots\)
If homog GREEN then get subseq set \(x_{i_1} = x_{i_2} = \ldots\)

Thm
Every inf seq of \(\mathbb{R}\) has either an inf \(\uparrow\) seq, an inf \(\downarrow\) seq, or an inf = seq.
Problem 4 (slightly modified)

\[ x_1, x_2, x_3, \ldots \text{ is an inf seq of reals.} \]
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**Thm** Every inf seq of \(R\) has either an inf \(\uparrow\) seq, an inf \(\downarrow\) seq, or an inf = seq.
Problem 4 Extra

Can generalize to $\mathbb{R}^n$ by either applying Ramsey with 3-colors $n$ times, or applying Ramsey with $3^n$ colors.

**Thm** Every inf seq of $\mathbb{R}^n$ has an inf subseq where, for each coordinate, either $\uparrow$ seq, or $\downarrow$ or $\equiv$. 
Problem 4 Extra

Can generalize to $\mathbb{R}^n$ by either applying Ramsey with 3-colors $n$ times, or applying Ramsey with $3^n$ colors.

**Thm** Every inf seq of $\mathbb{R}^n$ has an inf subseq where, for each coordinate, either ↑ seq, or ↓ or =$.$

This is a part of the proof of the Bolzano-Weierstrass Theorem. Next Slide.
Bolzano-Weierstrass Theorem

**Lemma**

1. Any increasing sequence bounded sequence of reals converges to a real.
2. Any decreasing sequence bounded sequence of reals converges to a real.

This is not obvious. This depends on the construction of the Reals.
Bolzano-Weierstrass Theorem

**Lemma**

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This is not obvious. This depends on the construction of the Reals.

**BW Thm** If $p_1, p_2, p_3, \ldots$ is an inf sequence of points in $R^n$ that is contained in a box, then there exists a subsequence that converges to a point in $R^n$. 
Bolzano-Weierstrass Theorem

Lemma

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BW Thm If $p_1, p_2, p_3, \ldots$ is an inf sequence of points in $R^n$ that is contained in a box, then there exists a subsequence that converges to a point in $R^n$.

Proof

Problem 4 yields that there is a subsequence in each coordinate that is either ↓, ↑, or =. Lemma yields each coord converges.
Problem 5- History

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It is the worst math novelty song ever. Listen for yourself: https://www.youtube.com/watch?v=df018klwKHg
Problem 5

\[ p_1, p_2, p_3, \ldots, \]

be an infinite sequence of points in \( \mathbb{R}^2 \).
Consider the following coloring of \( \binom{N}{2} \).

\[
COL(i, j) = \begin{cases} 
RED & \text{if } d(p_i, p_j) > 1 \\
BLUE & \text{if } d(p_i, p_j) < 1 
\end{cases}
\] (2)

Apply Ramsey Theorem. What do you get?

**SOLUTION**

**Thm** Given an infinite sequence of points in \( \mathbb{R}^2 \) there exists an infinite subset so that either (a) they are all within 1 of each other, or (b) they are all more than 1 apart.
Problem 4 and 5 thoughts

The proofs of the theorems in Problem 4 and 5 are FAR EASIER with Ramsey Theory. The proofs without Ramsey end up doing Ramsey in context.
Prove or disprove:

For every 2-coloring of the edges of $K_{\mathbb{N},\mathbb{N}}$ there exists $H_1$, $H_2$ infinite such that $(H_1, H_2)$ is a homog set.
Problem 6 (Extra Credit)

Prove or disprove:

For every 2-coloring of the edges of $K_{\mathbb{N}, \mathbb{N}}$ there exists $H_1, H_2$ infinite such that $(H_1, H_2)$ is a homog set.

Discuss and Vote

SOLUTION FALSE. Color with $COL(i, j) = \begin{cases} \text{RED} & \text{if } i < j \\ \text{BLUE} & \text{if } i \geq j \end{cases}$
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Thought What if we use 100 colors? The same counterexample works but you end up with an \((H_1, H_2)\) homog set that only has TWO colors. We will call that a 2-homog set.
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Prove or disprove:

For every 100-coloring of the edges of \(K_{\mathbb{N},\mathbb{N}}\) there exists \(H_1, H_2\) infinite such that \((H_1, H_2)\) is a 2-homog set. 3-homog set(?)

Some \(c\)-homog with \(c < 100\)?
Problem 7 (Extra Credit)

Prove or disprove:

For all colorings \(\text{COL} : (\mathbb{Z}/2) \rightarrow [2]\) there exists a set \(H \subseteq \mathbb{Z}\) that is order-equiv to \(\mathbb{Z}\) and is homogenous.
Problem 7 (Extra Credit)

Prove or disprove:

For all colorings $\text{COL} : \left( \mathbb{Z} \right) \rightarrow [2]$ there exists a set $H \subseteq \mathbb{Z}$ that is order-equiv to $\mathbb{Z}$ and is homogenous.

Discuss and Vote

Solution

FALSE. Color with $\text{COL}(i, j) = \begin{cases} 
\text{RED} & \text{if } i, j \geq 0 \\
\text{BLUE} & \text{if } i, j < 0 \\
\text{BLUE} & \text{if one is } \geq 0 \text{ and the other is } < 0 
\end{cases}$.
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