Lower Bounds on $W(3, c)$

Exposition by William Gasarch

April 15, 2022
VDW’s Theorem

**Theorem (VDW)** For all $k, c$ there exists $W = W(k, c)$ such that, for all $c$-colorings of $[W]$, there exists $a, d$ such that

$$a, a + d, \ldots, a + (k - 1)d \text{ are the same color.}$$
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- Proof gave gross upper bounds on $W(k, c)$. Not Prim. Rec.

Shelah has an alternative proof that gives Prim Rec bounds that some would still call gross. Proof is elementary.

Gowers proved $W(k, c) \leq 2^{2^{2^{k+9}}}$ Proof uses very hard math.
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$$W(k, c) \leq 2^{2c^{2k+9}}$$

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The Only Known VDW Numbers

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Upper bounds are Ginormous!
Actual numbers are small!
Want lower bounds to see how close they are to upper bounds.
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The Usual Approach

Given $c$, find $V$ such that there is a $c$-coloring of $V$ with no mono 3-AP's. Try to make $V$ as big as possible. Then $W(3, c) > V$.

We won't be doing that. We do it backwards.

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**3-free Sets**

**Definition** $A \subseteq [V]$ is **3-free** if there are no 3-AP’s in $A$. Note that if $[V]$ is colored and has no 3-AP’s then every color is 3-free.
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**Shifting** $A$ If $A \subseteq [V]$ and $t \in [V]$ then

$$A + t = \{x + t \pmod{V} : x \in A\}$$

$A + t$ is a **shift of $A$**.
$t$ is called **the shift**.
Ideal World 3-free $A \subseteq [V]$ can be shifted around so that none of the shifts overlap. This would be $\frac{V}{|A|}$ shifts and hence there is a $\frac{V}{|A|}$-coloring with no mono 3-AP’s.
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**Real World** Let $A \subseteq [V]$ be a 3-free set. We want to take a (small) number of shifts to cover $[V]$ There will be some overlap.
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We Use Randomness

We take $c$ random shifts where we determine $c$ later. What is Prob that some element of $[V]$ was NOT covered?
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What is \( \text{Prob} \) that some element of \([V]\) was NOT covered?
Let \( x \in [V] \) and \( t \) be a random shift.
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Let $x \in [V]$ and $t$ be a random shift.

$\Pr(x \in A + t) = \frac{|A|}{V}$. 

We choose $c$ so that this is $< 1$. 

$V \ln(\frac{|V|}{|A|})$ is close to the ideal of $\frac{|V|}{|A|}$. 
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\[
\Pr(x \notin A + t) = 1 - \frac{|A|}{V} \sim e^{-|A|/V}.
\]

\[
\Pr(x \notin A + t_1 \cup \cdots \cup A + t_c) \leq e^{-|A|c/V}.
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\Pr(\exists x \notin A + t_1 \cup \cdots \cup A + t_c) \leq Ve^{-|A|c/V}.
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\[
c = \frac{V \ln(V)}{|A|}
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Note \( \frac{V \ln(V)}{|A|} \) is close to the ideal of \( \frac{V}{|A|} \).
Recap

We have shown the following.

**Theorem** Let $V \in \mathbb{N}$ and let $A \subseteq [V]$ be a 3-free set. Let $c = \frac{V \ln(V)}{|A|}$. Then there is a $c$-coloring of $[V]$ with no mono 3-APs. Hence $W(3, c) > V$. 

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3-Free Set

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3-Free Set Facts

- If $A$ is not 3-free then there exists $a, a + d, a + 2d \in A$.
- If $A$ is not 3-free then there exists $x, y, z \in A$ such that $x + z = 2y$.
- **Notation** The size of the largest 3-free set of $[V]$ is denoted $sz(V)$. 

$\text{sz}(V) \geq V^{0.63}$

View $[V]$ as numbers in base 3.

$$A = \{ w \in [V] : \text{Base 3 rep of } w \text{ only has 0's and 1's} \}$$
\[ \text{sz}(V) \geq V^{0.63} \]

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Key Since base 3 rep of \(x, y, z\) has only 0's and 1's, adding them is carry free.

\[ x = x_L \cdots x_0 \]
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**Size of** \(A\) \([V]\) in base 3 takes \(\log_3(V)\) digits. So

\[ |A| \sim 2^{\log_3(V)} \sim V^{\log_3(2)} = V^{0.63} \]
\( \text{sz}(V) \geq V^{0.68} \)

View \([V]\) as numbers in base 5.
(Attempt- it won’t work)

\[ A = \{ w \in [V] : \text{Base 5 rep of } w \text{ only has 0’s, 1’s, 2’s} \} \]

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\(|A| \sim V^{\log_5(3)} \sim |V|^{0.68}.

3-Free Assume \(x, y, z \in A\) and \(x + z = 2y\).
Key Since base 5 rep of \(x, y, z\) has only 0’s, 1’s, 2’s adding them is carry free.
\(x = x_L \cdots x_0\)
\(z = z_L \cdots z_0\)
\(y = y_L \cdots y_0\)
If \(x + z = 2y\) then, for all \(i\), \(x_i + z_i = 2y_i\).
If \(y_i = 0\) then \(x_i = z_i = 0\).
View \([V]\) as numbers in base 5.

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sz(\mathcal{V}) \geq \mathcal{V}^{0.68}

View \mathcal{V} as numbers in base 5.
(Attempt- it won’t work)

$$A = \{ w \in \mathcal{V} : \text{Base 5 rep of } w \text{ only has 0’s, 1’s, 2’s} \}$$

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Shucky Darns!
\[ \text{sz}(V) \geq V^{0.68} \]

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Shucky Darns! Need to add one more condition.
The Real Set $A$

$A$ is the set of all $w \in [V]$ such that

- Base 5 rep of $w$ only has 0’s, 1’s, 2’s.
- Base 5 rep of $w$ exactly $1/3$ of the digits are 0.

3-free

$x = x_L \cdots x_0$
$z = z_L \cdots z_0$
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FIRST look at the $L/3$ places where $y_i = 0$. Then $x_i = z_i = 0$.

Key For all other places $x_i \neq 0$, $z_i \neq 0$. 
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SECOND look at the places where $y_i = 1$. $x_i + z_i = 2$ and $x_i \neq 0$, $y_i \neq 0$ Hence $x_i = z_i = 1$. 
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FIRST look at the $L/3$ places where $y_i = 0$. Then $x_i = z_i = 0$.

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THIRD look at the places where $y_i = 2$. $x_i + z_i = 4$, so $x_i = z_i = 2$. 
The Real Set $A$

$A$ is the set of all $w \in [V]$ such that

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So $x = y = z$. 
What is $|A|$?

Choose $L/3$ of the digits to be 0. \( \binom{L}{L/3} \sim L^{L/3} \)
What is $|A|$?

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For the remainder use 1’s or 2’s, so $2^{2L/3}$
What is $|A|$?

Choose $L/3$ of the digits to be 0. \( \binom{L}{L/3} \sim L^{L/3} \)

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Leave it to the reader to work it out.
Let $r$ be such that $2^{r(r+1)/2} - 1 \leq V \leq 2^{(r+1)(r+2)/2} - 1$.

Note that $r \sim \sqrt{2 \lg(V)}$. 

\[ \text{sz}(V) \geq V^{1 - \frac{1}{\sqrt{\lg V}}} \]
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Write the numbers in \([V]\) in base 2.
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Write the numbers in \([V]\) in base 2.

Break the numbers into \( r \) blocks of bits.
\[ \text{sz}(V) \geq V^{1 - \frac{1}{\sqrt{\log V}}} \]

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The first (rightmost) block is one 1 long.
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Write the numbers in \([V]\) in base 2.

Break the numbers into \( r \) blocks of bits.

The first (rightmost) block is one 1 long.

The second block is 2 bits long.
Let \( r \) be such that \( 2^{r(r+1)/2} - 1 \leq V \leq 2^{(r+1)(r+2)/2} - 1 \). Note that \( r \sim \sqrt{2 \lg(V)} \).

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The first (rightmost) block is one 1 long.

The second block is 2 bits long.

The \( r \)th block is \( r \) bits long.

We denote the \( i \)th block as \( B_i \), a number.
An Example!

991746118991 in binary is

1110011011101000101011001101010101001111

B₁ = 1
B₂ = 3
B₃ = 1
B₄ = 5
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991746118991 in binary is

1110011011101000101011001101010101001111

We write it as:

000001110; 01101110; 1000101; 011001; 10101; 0101; 001; 11; 1
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$B_1 = 1$
$B_2 = 3$
$B_3 = 1$
$B_4 = 5$
The Set $A$

$A$ is the set of all $B_r B_{r-1} \cdots B_1$ such that:

1. For $1 \leq i \leq r - 2$ the leftmost bit of $B_i$ is 0. This leads to carry-free addition.

2. $\sum_{i=1}^{r-2} B_i^2 = B_r B_{r-1}$ (The $B_r B_{r-1}$ is concatenation.)

We leave it to the reader to prove that $|A|$ is as big as we said (this is easy) and that the set is 3-free (This requires some thought.)
Recall that we prove:

**Thm** Let $V \in \mathbb{N}$ and let $A \subseteq [V]$ be a 3-free set. Then there is a $\frac{V \ln(V)}{|A|}$-coloring of $[V]$ with no mono 3-APs. Hence $\mathcal{W}(3, \frac{V \ln(V)}{|A|}) \geq V$. 
Recall that we prove:

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Recall that we sketched:

**Thm** There exists a 3-free subset of $[V]$ of size $\geq V^{1 - \frac{1}{\sqrt{\lg V}}}$.
Recall that we prove:

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$$W(3, \frac{V \ln(V)}{|A|}) \geq V.$$ 

Recall that we sketched:

**Thm** There exists a 3-free subset of $[V]$ of size $\geq V^{1 - \frac{1}{\sqrt{\ln V}}}$

Combine these two to get:

**Thm** Let $V \in \mathbb{N}$. Then there is a $V^{\frac{1}{\sqrt{\ln V}}} \ln(V)$-coloring of $[V]$ with no mono 3-APs. Hence

$$W(3, V^{\frac{1}{\sqrt{\ln V}}} \ln(V)) \geq V.$$