1 Erdős-Rado Proof of 3-hypergraph Ramsey

Def 1.1 $R_a(k)$ is the least number $n$ such that, for all $\text{COL} : \binom{[n]}{a} \rightarrow [2]$ there exists a homog set of size $k$.

In our proof in class that $R_3(k)$ exists the construction used 2-ary Ramsey many times, and then 1-ary Ramsey once. Each use of 2-ary Ramsey cuts the set from size $X$ to size $\log X$. The one use of 1-ary Ramsey cuts the set from size $X$ to size $X^2$. All of those uses of 2-ary Ramsey force us to start with a very big set.

To be able to start with a smaller set we will do a different construction. This one will use 1-ary Ramsey many times, but 2-ary Ramsey only once.

Theorem 1.2 For almost all $k$, $R_3(k) \leq 2^{2^{4k}}$.

Proof: Let $n$ be a number to be determined. Let COL be a 2-coloring of $\binom{[n]}{3}$. We define a sequence of vertices, 

$$x_1, x_2, \ldots, x_{R_2(k)}.$$

Here is the intuition: Let $x_1 = 1$. Let $x_2 = 2$. The vertices $x_1, x_2$ induces the following coloring of $\{3, \ldots, n\}$.

$$\text{COL}^*(y) = \text{COL}(x_1, x_2, y).$$

Let $V_1$ be a 1-homogeneous for $\text{COL}^*$ of size at least $\frac{n-2}{2}$. Let $\text{COL}^{**}(x_1, x_2)$ be the color of $V_1$. Let $x_3$ be the least vertex left (bigger than $x_2$).

The number $x_3$ induces two colorings of $V_1 - \{x_3\}$:

$$(\forall y \in V_1 - \{x_3\})[\text{COL}^*_1(y) = \text{COL}(x_1, x_3, y)]$$

$$(\forall y \in V_1 - \{x_3\})[\text{COL}^*_2(y) = \text{COL}(x_2, x_3, y)]$$
Let $V_2$ be a 1-homogeneous for $\text{COL}_1^*$ of size $\frac{|V_1|-1}{2}$. Let $\text{COL}^{**}(x_1, x_3)$ be the color of $V_2$. Restrict $\text{COL}_2^*$ to elements of $V_2$, though still call it $\text{COL}_2^*$. We reuse the variable name $V_2$ to be a 1-homogeneous for $\text{COL}_2^*$ of size at least $\frac{|V_2|}{2}$. Let $\text{COL}^{**}(x_1, x_3)$ be the color of $V_2$. Let $x_4$ be the least element of $V_2$. Repeat the process.

We describe the construction formally.

CONSTRUCTION

\[
\begin{align*}
x_1 &= 1 \\
V_1 &= \{n\} - \{x_1\}
\end{align*}
\]

Let $2 \leq i \leq R_2(k)$. Assume that $x_1, \ldots, x_{i-1}, V_{i-1}$, and

\[
\text{COL}^{**}: \left(\frac{x_1, \ldots, x_{i-1}}{2}\right) \rightarrow [2]
\]

are defined.

\[
\begin{align*}
x_i &= \text{the least element of } V_{i-1} \\
V_i &= V_{i-1} - \{x_i\} \text{ (We will change this set without changing its name).}
\end{align*}
\]

We define $\text{COL}^{**}(x_1, x_i)$, $\text{COL}^{**}(x_2, x_i)$, \ldots, $\text{COL}^{**}(x_{i-1}, x_i)$. We will also define smaller and smaller sets $V_i$. We will keep the variable name $V_i$ throughout.

For $j = 1$ to $i - 1$

1. $\text{COL}^*: V_i \rightarrow [2]$ is defined by $\text{COL}^*(y) = \text{COL}(x_j, x_i, y)$.

2. Let $V_i$ be redefined as the largest 1-homogeneous set for $\text{COL}^*$. Note that $|V_i|$ decreases by at most half.

3. $\text{COL}^{**}(x_j, x_i)$ is the color of $V_i$.

KEY: For all $1 \leq i_1 < i_2 \leq i$, for all $y \in V_i$, $\text{COL}(x_{i_1}, x_{i_2}, y) = \text{COL}^{**}(x_{i_1}, x_{i_2})$.

END OF CONSTRUCTION
When we derive upper bounds on $n$ we will show that the construction can be carried out for $R_2(k)$ stages. For now assume the construction ends.

We have vertices

$$X = \{x_1, x_2, \ldots, x_{R_2(k)}\}$$

and a 2-coloring $\text{COL}^*$ of $\binom{X}{2}$. By the definition of $R_2(k)$ there exists a Homog-for-$\text{COL}^*$:

$$H = \{x_{i_1}, \ldots, x_{i_k}\}.$$ 

Let the color of this homogenous set be 1. We show that $H$ is a homogenous set for $\text{COL}$. Let $1 \leq i_1 < i_2 < i_3 \leq k$. We show that $\text{COL}(x_{i_1}, x_{i_2}, x_{i_3}) = 1$.

By the definition of $\text{COL}^*$ for all $y \in V_{i_2}$, $\text{COL}(x_{i_1}, x_{i_2}, y) = \text{COL}^*(x_{i_1}, x_{i_2}) = 1$. In particular $\text{COL}(x_{i_1}, x_{i_2}, x_{i_3}) = 1$.

We now see how large $n$ must be so that the construction be carried out. Note that in stage $i$ $|V_i|$ be decreases by at most half, $i$ times. Hence $|V_{i+1}| \geq \frac{|V_i|}{2^i}$.

Therefore

$$|V_i| \geq \frac{|V_i|}{2^1 + 2^2 + \cdots + 2^i} \geq \frac{n - 1}{2^{(i-1)/2}}.$$ 

We want $|V_{R_2(k)}| \geq 1$. It suffice so take $n = 2^{R_2(k)^2}$.

We know that $R_2(k) \leq 2^{2k}$. Hence

$$R_3(k) \leq 2^{R_2(k)^2} \leq 2^{2^{2k}}.$$