PH(2) \leq 14

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March 8, 2022
Review of $\text{PH}(k)$

Def $A \subseteq \mathbb{N}$ is \textbf{large} if $|A| > \min(A)$. 
Review of \( \text{PH}(k) \)

**Def** \( A \subseteq \mathbb{N} \) is **large** if \( |A| > \min(A) \).

**Def** \( \text{PH}(k) \) is the least \( n \) such that for all 2-colorings of \( \binom{\{k,\ldots,n\}}{2} \) there exists a large homog set.

(\( \text{PH} \) stands for Paris-Harrington.)
Review of PH($k$)

**Def** $A \subseteq \mathbb{N}$ is **large** if $|A| > \min(A)$.

**Def** PH($k$) is the least $n$ such that for all 2-colorings of $\binom{\{k,\ldots,n\}}{2}$ there exists a large homog set. (PH stands for Paris-Harrington.)

**Def** PH(2) is the least $n$ such that for all 2-colorings of $\binom{\{2,\ldots,n\}}{2}$ there exists a large homog set.
Let $\text{COL} : \binom{\{2,\ldots,14\}}{2} \rightarrow [2]$. We show there is a large homog set.
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Let \( \text{COL}: \binom{\{2,\ldots,14\}}{2} \rightarrow [2] \). We show there is a large homog set. 

**Note** The graph has 13 vertices so every point has degree 12.
\[ \text{deg}_R(2) \geq 8, \text{ so } \text{deg}_B(2) \leq 4 \]

**Case 1** \( \text{deg}_R(2) \geq 8 \). Let the 8 smallest \( R \)-neighbors of 2 be \( x_1 < \cdots < x_8 \).
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- There exists \( 1 \leq i < j \leq 8 \) such that \( \text{COL}(x_i, x_j) = R \).
  - Large homog set: \( \{2, x_i, x_j\} \).
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- For all \( 1 \leq i < j \leq 8 \), \( \text{COL}(x_i, x_j) = B \) AND \( x_1 \leq 7 \). Large homog set: \( \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\} \).
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- For all $1 \leq i < j \leq 8$, $\text{COL}(x_i, x_j) = B$ AND $x_1 \geq 8$.
  $x_2 \geq 9$, $x_3 \geq 10$, $\cdots$, $x_8 \geq 15$. 

Contradiction since we are coloring $\{2, \ldots, 14\}$.
$\deg_{\text{R}}(2) \geq 8$, so $\deg_{\text{B}}(2) \leq 4$

**Case 1** $\deg_{\text{R}}(2) \geq 8$. Let the 8 smallest $R$-neighbors of 2 be $x_1 < \cdots < x_8$.

- There exists $1 \leq i < j \leq 8$ such that $\text{COL}(x_i, x_j) = \text{R}$.
  - Large homog set: $\{2, x_i, x_j\}$.
- For all $1 \leq i < j \leq 8$, $\text{COL}(x_i, x_j) = \text{B}$ AND $x_1 \leq 7$.
  - Large homog set: $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$.
- For all $1 \leq i < j \leq 8$, $\text{COL}(x_i, x_j) = \text{B}$ AND $x_1 \geq 8$.
  - $x_2 \geq 9$, $x_3 \geq 10$, $\cdots$, $x_8 \geq 15$.
  - Contradiction since we are coloring $\binom{\{2,\ldots,14\}}{2}$.
\[ \deg_R(2) = 7, \text{ so } \deg_B(2) = 5 \]

**Case 2** \( \deg_R(2) = 7 \). Let the 7 smallest \( R \)-neighbors of 2 be \( x_1 < \cdots < x_7 \).
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- For all $1 \leq i < j \leq 7$, $\text{COL}(x_i, x_j) = B$ AND $x_1 \leq 6$.
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Last Case on Next Slide.
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\[ \text{deg}_R(2) = 7, \text{ so } \text{deg}_B(2) = 5 \] (cont)

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**Remaining Case**
$\deg_R(2) = 7$, so $\deg_B(2) = 5$ (cont)

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**Remaining Case**

For all $1 \leq i < j \leq 7$, $\text{COL}(x_i, x_j) = B$ AND $x_1 \geq 7$.

Hence $\{x_1, \ldots, x_7\} \subseteq \{7, 8, 9, 10, 11, 12, 13, 14\}$.

Hence $B$ neighbors of 2 are $\supseteq \{3, 4, 5, 6\}$. 
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**Cases**

- There exists $3 \leq i < j \leq 6$ such that $(i, j)$ is $B$.
  Large Homog Set: $\{2, i, j\}$.
Case 2 \( \deg_R(2) = 7 \). Let the 7 smallest \( R \)-neighbors of 2 be \( x_1 < \cdots < x_7 \).

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For all \( 1 \leq i < j \leq 7 \), \( \text{COL}(x_i, x_j) = B \) AND \( x_1 \geq 7 \). Hence \( \{x_1, \ldots, x_7\} \subseteq \{7, 8, 9, 10, 11, 12, 13, 14\} \).

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- For all \( 3 \leq i < j \leq 6 \), \((i, j)\) is \( R \).
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Case 3 \( \deg_R(2) = 6 \). Let the 6 smallest \( R \)-neighbors of 2 be 
\[ x_1 < x_2 < x_3 < x_4 < x_5 < x_6. \]
\[ \text{deg}_R(2) = 6, \text{ so } \text{deg}_B(2) = 6 \]

**Case 3** \( \text{deg}_R(2) = 6 \). Let the 6 smallest \( R \)-neighbors of 2 be \( x_1 < x_2 < x_3 < x_4 < x_5 < x_6 \).

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- For all \( 1 \leq i < j \leq 6 \), \( \text{COL}(x_i, x_j) = B \) AND \( x_1 \leq 5 \).
  
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Last Case on Next Slide
\[ \text{deg}_R(2) = 6, \text{ so } \text{deg}_B(2) = 6 \text{ (cont)} \]

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Remaining Case

For all \( 1 \leq i < j \leq 6 \), \( \text{COL}(x_i, x_j) = B \) AND \( x_1 \geq 6 \).
Hence \( R \) neighbors of 2 are \( \subseteq \{6, 7, 8, 9, 10, 11, 12, 13, 14\} \).
Hence \( B \) neighbors of 2 are 
\[ y_1 = 3 < y_2 = 4 < y_3 = 5 < y_4 < y_5 < y_6. \]
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\( \text{deg}_R(2) = 6, \text{ so } \text{deg}_B(2) = 6 \) (cont)

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For all \( 1 \leq i < j \leq 6 \), \( \text{COL}(x_i, x_j) = B \) AND \( x_1 \geq 6 \). Hence \( R \) neighbors of 2 are \( \subseteq \{6, 7, 8, 9, 10, 11, 12, 13, 14\} \). Hence \( B \) neighbors of 2 are \( y_1 = 3 < y_2 = 4 < y_3 = 5 < y_4 < y_5 < y_6 \).

**Cases**

- There exists \( 1 \leq i < j \leq 6 \) such that \( (y_i, y_j) \) is \( B \). Large Homog Set: \( \{2, y_i, y_j\} \).

- For all \( 1 \leq i < j \leq 6 \), \( (y_i, y_j) \) is \( R \). Large Homog Set: \( \{3, 4, 5, y_4, y_5, y_6\} \).
$\text{deg}_R(2) \leq 5$

**Case 4** $\text{deg}_R(2) \leq 5$. Then $\text{deg}_B(2) \geq 7$.
If $\text{deg}_B(2) = 7$ use the argument used for $\text{deg}_R(2) = 7$.
If $\text{deg}_B(2) \geq 8$ use the argument used for $\text{deg}_R(2) \geq 8$. 
Exact Bound on $\text{PH}(2)$

We have

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*Known*

$$\text{PH}(2) = 8.$$
What About $\text{PH}(3)$? $\text{PH}(4)$?

We know that $\text{PH}(2) = 8$. 

Known $\text{PH}(3) = 13$ Gee, that's not so big.

Known $\text{PH}(4) \leq 687$ Gee, looking bigger.
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Gee, that's not so big.

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The results on $\text{PH}(2), \text{PH}(3), \text{PH}(4)$ were done in 1978.
What About \( \text{PH}(3) \)? \( \text{PH}(4) \)?

We know that \( \text{PH}(2) = 8 \).

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The results on \( \text{PH}(2), \text{PH}(3), \text{PH}(4) \) were done in 1978.

I do not think anyone has looked at actual \( \text{PH} \) numbers since then.
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We know that $\text{PH}(2) = 8$.
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Surely we can make progress now, perhaps with computers.
Yes, but don’t call me Shirley.