The Infinite $a$-ary Can Ramsey Thm

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Min-Homog, Max-Homog, Rainbow

Def: Let $\text{COL} : \binom{\mathbb{N}}{2} \rightarrow \omega$. Let $V \subseteq \mathbb{N}$. Assume $a < b$ and $c < d$.

- $V$ is homog if $\text{COL}(a, b) = \text{COL}(c, d)$ iff TRUE.
- $V$ is min-homog if $\text{COL}(a, b) = \text{COL}(c, d)$ iff $a = c$.
- $V$ is max-homog if $\text{COL}(a, b) = \text{COL}(c, d)$ iff $b = d$.
- $V$ is rainb if $\text{COL}(a, b) = \text{COL}(c, d)$ iff $a = c$ and $b = d$.

Can Ramsey Thm for $\binom{\mathbb{N}}{2}$: For all $\text{COL} : \binom{\mathbb{N}}{2} \rightarrow \omega$, there exists an infinite set $V$ such that $V$ is homog OR min-homog OR max-homog OR rainb.
Restate So We Can Generalize

**Def:** Let \( COL : \binom{\mathbb{N}}{2} \to \omega \). Let \( V \subseteq \mathbb{N} \). Assume \( a_1 < a_2 \) and \( b_1 < b_2 \).

- \( V \) is *homog* if \( COL(a_1, a_2) = COL(b_1, b_2) \) iff \( TRUE \). So \( COL(x, y) \) does not depend on the first or second coordinate. We call this \( \emptyset \)-homog.

- \( V \) is *min-homog* if \( COL(a_1, a_2) = COL(b_1, b_2) \) iff \( a_1 = b_1 \). So \( COL(x, y) \) depend on the first coordinate only. We call this \( \{1\}\)-homog.

- \( V \) is *max-homog* if \( COL(a_1, a_2) = COL(b_1, b_2) \) iff \( a_2 = b_2 \). So \( COL(x, y) \) depend on the second coordinate only. Can call this \( \{2\}\)-homog.

- \( V \) is *rainb* if
  \[\text{COL}(a_1, a_2) = \text{COL}(b_1, b_2) \text{ iff } a_1 = b_1 \text{ and } a_2 = b_2.\]
  So \( COL(x, y) \) depend on the first and second coordinate only. Can call this \( \{1, 2\}\)-homog.

**Can Ramsey Thm for \( \binom{\mathbb{N}}{2} \):** For all \( COL : \binom{\mathbb{N}}{2} \to \omega \), there exists \( A \subseteq \{1, 2\} \) and an infinite set \( V \) such that \( V \) is \( A \)-homog.
All 8 Cases For 3-Ary Can Ramsey

\[ \text{COL} : \binom{\mathbb{N}}{3} \to \omega. \ V \subseteq \mathbb{N}. \ a_1 < a_2 < a_3 \text{ and } b_1 < b_2 < b_3. \]
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\( V \) is \( \emptyset \)-homog if \( \text{COL}(a_1, a_2, a_3) = \text{COL}(b_1, b_2, b_3) \) iff TRUE.
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\( V \) is \( \{3\}\)-homog if \( \text{COL}(a_1, a_2, a_3) = \text{COL}(b_1, b_2, b_3) \) iff \( a_3 = b_3 \).
All 8 Cases For 3-Ary Can Ramsey

\[ \text{COL} : \left( \mathbb{N} \right)^3 \to \omega. \] 
\[ V \subseteq \mathbb{N}. \ a_1 < a_2 < a_3 \text{ and } b_1 < b_2 < b_3. \]

\[ V \text{ is } \emptyset\text{-homog if } \text{COL}(a_1, a_2, a_3) = \text{COL}(b_1, b_2, b_3) \text{ iff } \text{TRUE}. \]

\[ V \text{ is } \{1\}\text{-homog if } \text{COL}(a_1, a_2, a_3) = \text{COL}(b_1, b_2, b_3) \text{ iff } a_1 = b_1. \]

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\[ V \text{ is } \{1, 2\}\text{-homog if } \text{COL}(a_1, a_2, a_3) = \text{COL}(b_1, b_2, b_3) \text{ iff } (a_1 = b_1) \land (a_2 = b_2). \]
COL : \((\mathbb{N}_3) \rightarrow \omega\). \(V \subseteq \mathbb{N}\). \(a_1 < a_2 < a_3\) and \(b_1 < b_2 < b_3\).

\(V\) is \(\emptyset\)-homog if \(\text{COL}(a_1, a_2, a_3) = \text{COL}(b_1, b_2, b_3)\) iff \(TRUE\).

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All 8 Cases For 3-Ary Can Ramsey

COL : \( (\mathbb{N})^3 \rightarrow \omega \). \( V \subseteq \mathbb{N} \). \( a_1 < a_2 < a_3 \) and \( b_1 < b_2 < b_3 \).

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\( V \) is \( \{2, 3\} \)-homog if
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Can Ramsey Thm for $\binom{\mathbb{N}}{3}$: For all $\text{COL} : \binom{\mathbb{N}}{3} \to \omega$, there exists $A \subseteq \{1, 2, 3\}$ and an infinite set $V$ such that $V$ is $A$-homog.
All 8 Types are Possible

Define $\text{COL} : \left( ^{3} \mathbb{N} \right) \rightarrow \omega$ by

$$\text{COL}(x < y < z) = (x, z)$$

Then $\mathbb{N}$ is a $(1, 3)$-homog set.
All 8 Types are Possible

Define $COL : \binom{\mathbb{N}}{3} \to \omega$ by

$$COL(x < y < z) = (x, z)$$

Then $\mathbb{N}$ is a $(1, 3)$-homog set.

The rest of the cases are similar.
Proofs of 3-ary Can Ramsey

There are three proofs of 3-ary Ramsey.

1. One is similar to the proof of 2-ary Ramsey that used 4-ary. It uses 6-ary.
2. One is similar to the proof of 2-ary Ramsey that used 3-ary. It uses 5-ary (I think).
3. One is Mileti-Style.

Doing these is extra credit on hw02.
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I leave it to you to state and prove $a$-ary Can Ramsey.