Application of Ramsey Theory to Multiparty Comm Complexity

Exposition by William Gasarch

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Credit where Credit is Due

The results in this talk are due to Chandra, Furst, Lipton.

Multi-Party Protocols

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The Problem

Alice is A, Bob is B, Carol is C.
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1. A, B, and C have a string of length $n$ on their foreheads.
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Alice is A, Bob is B, Carol is C.

1. A, B, and C have a string of length $n$ on their foreheads.
2. A’s forehead has $a$, B’s has $b$, C’s has $c$. 

Solution

A says $b$, B then computes $a + b + c$ and then says YES if $a + b + c = 2^n + 1$, NO if not.

Solution uses $n + 1$ bits of comm. Can do better?
The Problem

Alice is A, Bob is B, Carol is C.

1. A, B, and C have a string of length \( n \) on their foreheads.
2. A’s forehead has \( a \), B’s has \( b \), C’s has \( c \).
3. They want to know if \( a + b + c = 2^{n+1} - 1 \).
Alice is A, Bob is B, Carol is C.

1. A, B, and C have a string of length \( n \) on their foreheads.
2. A’s forehead has \( a \), B’s has \( b \), C’s has \( c \).
3. They want to know if \( a + b + c = 2^{n+1} - 1 \).
4. **Solution** A says \( b \), B then computes \( a + b + c \) and then says YES if \( a + b + c = 2^{n+1} - 1 \), NO if not.
The Problem

Alice is A, Bob is B, Carol is C.

1. A, B, and C have a string of length $n$ on their foreheads.
2. A’s forehead has $a$, B’s has $b$, C’s has $c$.
3. They want to know if $a + b + c = 2^n + 1 - 1$.
4. **Solution** A says $b$, B then computes $a + b + c$ and then says YES if $a + b + c = 2^{n+1} - 1$, NO if not.
5. **Solution** uses $n + 1$ bits of comm. Can do better?
1. Any protocol requires $n + 1$ bits, hence the one given that takes $n + 1$ is the best you can do. The proof uses Theorems that could be in this course.

2. There is a protocol that takes $\alpha n$ bits for some $\alpha < 1$ but any protocol requires $\Omega(n)$ bits. Either the proof of the upper bound or the proof of the lower bound or both use Theorems that could be in this course.

3. There is a protocol that takes $\ll n$ bits. The proof uses Theorems that could be in this course.
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STUDENTS WORK IN GROUPS
Protocol in $\frac{n}{2} + O(1)$ bits

1. A: $a_0 \cdots a_{n-1}$, B: $b_0 \cdots b_{n-1}$, C: $c_0 \cdots c_{n-1}$.
2. A says: $b_{n-1} \oplus c_0$, $b_{n-2} \oplus c_1$, \ldots , $b_{n/2} \oplus c_{n/2-1}$.
3. Bob knows $c_i$'s so he now knows $b_{n/2}, \ldots, b_{n-1}$.
4. Carol knows $b_i$'s so she now knows $c_0, \ldots, c_{n/2-1}$.
5. Carol knows $a_0, \ldots, a_{n/2-1}$, $b_0, \ldots, b_{n/2-1}$, $c_0, \ldots, c_{n/2-1}$.
   Hence she can compute
   $$a_{n/2-1} \cdots a_0 + b_{n/2-1} \cdots b_0 + c_{n/2-1} \cdots c_0.$$  
   View this as an $(n/2)$-bit string $s$ and a carry bit $z$.
6. $s = 1^{n/2}$: Carol says (MAYBE,$z$). Otherwise: Carol says NO.
7. Bob knows $a_{n/2}, \ldots, a_{n-1}$, $b_{n/2}, \ldots, b_{n-1}$, $c_{n/2}, \ldots, c_{n-1}$ and $z$ so he can compute $a + b + c$. If $= M$ then say YES, if not then say NO.
Vote Again

Vote

There is a protocol that uses $\ll n$ bits and I use Ramsey Theory to prove it.

There exists a $0 < \beta < \frac{2}{2}$ such that any protocol requires $\geq \beta n$ bits and I use Ramsey Theory to prove it.

I will show a $\sqrt{n} \ll n$ protocol, which will use 3-free sets so will indeed use Ramsey Theory.
Vote Again

Vote

- There is a protocol that uses $\ll n$ bits AND I use Ramsey Theory to prove it.
Vote Again

Vote

- There is a protocol that uses $\ll n$ bits AND I use Ramsey Theory to prove it.
- There exists a $0 < \beta < \frac{1}{2}$ such that any protocol requires $\geq \beta n$ bits AND I use Ramsey Theory to prove it.
Vote Again

Vote

- There is a protocol that uses $\ll n$ bits AND I use Ramsey Theory to prove it.

- There exists a $0 < \beta < \frac{1}{2}$ such that any protocol requires $\geq \beta n$ bits AND I use Ramsey Theory to prove it.

I will show a $\sqrt{n} \ll n$ protocol, which will use 3-free sets so will indeed use Ramsey Theory.
We Look At the $L$-Theorem Backwards

**Notation** $M$ will be $2^{n+1} - 1$ which is $1^{n+1}$ in binary.

**L-Theorem** For all $c$ there exists $M$ such that for all $c$-colorings of $[M] \times [M]$ there exists a mono $L$ or $\downarrow$.
We Look At the L-Theorem Backwards

**Notation** $M$ will be $2^{n+1} - 1$ which is $1^{n+1}$ in binary.

**L-Theorem** For all $c$ there exists $M$ such that for all $c$-colorings of $[M] \times [M]$ there exists a mono $L$ or $\lnot$.

Fix $M$.

**Q** $(\exists c)$: $[M] \times [M]$ can be $c$-colored w/o mono $L$ or $\lnot$?
We Look At the $L$-Theorem Backwards

**Notation** $M$ will be $2^{n+1} - 1$ which is $1^{n+1}$ in binary.

**$L$-Theorem** For all $c$ there exists $M$ such that for all $c$-colorings of $[M] \times [M]$ there exists a mono $L$ or $\bar{L}$.

Fix $M$.

$Q \ (\exists c): [M] \times [M]$ can be $c$-colored w/o mono $L$ or $\bar{L}$?

**Yes** $c = M^2$, color every point differently.
We Look At the $L$-Theorem Backwards

**Notation** $M$ will be $2^{n+1} - 1$ which is $1^{n+1}$ in binary.

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**Q** ($\exists c \ll M^2$): $[M] \times [M]$ can be $c$-colored w/o mono $L$ or $\lnot$?
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**Q** ($\exists c$): $[M] \times [M]$ can be $c$-colored w/o mono $L$ or $\dashv$?

**Yes** $c = M^2$, color every point differently.

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**Yes**, $c = M$, color every row differently.
We Look At the $L$-Theorem Backwards

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**$L$-Theorem** For all $c$ there exists $M$ such that for all $c$-colorings of $[M] \times [M]$ there exists a mono $L$ or $\perp$.

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**Yes** $c = M^2$, color every point differently.

**Q** ($\exists c \ll M^2$): $[M] \times [M]$ can be $c$-colored w/o mono $L$ or $\perp$?

**Yes**, $c = M$, color every row differently.

**Q** ($\exists c$): ALL $c$-colorings of $[M] \times [M]$ there is a mono $L$ or $\perp$?
We Look At the \( L \)-Theorem Backwards

**Notation** \( M \) will be \( 2^{n+1} - 1 \) which is \( 1^n + 1 \) in binary.

**L-Theorem** For all \( c \) there exists \( M \) such that for all \( c \)-colorings of \([M] \times [M]\) there exists a mono \( L \) or \( \not \).

Fix \( M \).

**Q** (\( \exists c \)): \([M] \times [M]\) can be \( c \)-colored w/o mono \( L \) or \( \not \)?

**Yes** \( c = M^2 \), color every point differently.

**Q** (\( \exists c \ll M^2 \)): \([M] \times [M]\) can be \( c \)-colored w/o mono \( L \) or \( \not \)?

**Yes**, \( c = M \), color every row differently.

**Q** (\( \exists c \)): ALL \( c \)-colorings of \([M] \times [M]\) there is a mono \( L \) or \( \not \)?

**Yes** \( c = 1 \). Stupid but true.
We Look At the \textit{L}-Theorem Backwards

\textbf{Notation} \( M \) will be \( 2^{n+1} - 1 \) which is \( 1^{n+1} \) in binary.

\textbf{L-Theorem} For all \( c \) there exists \( M \) such that for all \( c \)-colorings of \( [M] \times [M] \) there exists a mono \( L \) or \( \lnot \).

Fix \( M \).

\textbf{Q} (\( \exists c \)): \([M] \times [M]\) can be \( c \)-colored w/o mono \( L \) or \( \lnot \)?

\textbf{Yes} \( c = M^2 \), color every point differently.

\textbf{Q} (\( \exists c \ll M^2 \)): \([M] \times [M]\) can be \( c \)-colored w/o mono \( L \) or \( \lnot \)?

\textbf{Yes}, \( c = M \), color every row differently.

\textbf{Q} (\( \exists c \)): ALL \( c \)-colorings of \([M] \times [M]\) there is a mono \( L \) or \( \lnot \)?

\textbf{Yes} \( c = 1 \). Stupid but true.

We actually need a stronger condition:

\textbf{Definition} \( \Gamma(M) \) is the least \( c \) such that there is a \( c \)-coloring of \([M] \times [M]\) w/o mono \( L \) or \( \lnot \).
We Look At the $L$-Theorem Backwards

**Notation** $M$ will be $2^{n+1} - 1$ which is $1^{n+1}$ in binary.

**$L$-Theorem** For all $c$ there exists $M$ such that for all $c$-colorings of $[M] \times [M]$ there exists a mono $L$ or $\bot$.

Fix $M$.

**Q** $(\exists c)$: $[M] \times [M]$ can be $c$-colored w/o mono $L$ or $\bot$?

**Yes** $c = M^2$, color every point differently.

**Q** $(\exists c \ll M^2)$: $[M] \times [M]$ can be $c$-colored w/o mono $L$ or $\bot$?

**Yes**, $c = M$, color every row differently.

**Q** $(\exists c)$: ALL $c$-colorings of $[M] \times [M]$ there is a mono $L$ or $\bot$?

**Yes** $c = 1$. Stupid but true.

We actually need a stronger condition:

**Definition** $\Gamma(M)$ is the least $c$ such that there is a $c$-coloring of $[M] \times [M]$ w/o mono $L$ or $\bot$.

We give a $3 \lg(\Gamma(M)) + O(1)$ bit protocol and then bound $\Gamma(M)$. 
Protocol

\( M = 2^{n+1} - 1 \) throughout.

1. Pre-step: A, B, and C agree on a \( \Gamma(M) \)-coloring \( \chi \) of \([M] \times [M]\) that has no mono \( L \) or \( \neg \).

2. A: \( b, c \), B: \( a, c \), C: \( a, b \). \( a, b, c \in \{0, 1\}^n \) numbers in binary.

3. If A sees \( b + c > M \), says NO and protocol stops. B,C, sim.

4. A finds \( a' \), s.t. \( a' + b + c = M \) and says \( \chi(a', b) \).

5. B finds \( b' \) s.t. \( a + b' + c = M \) and says \( \chi(a, b') \).

6. C says Y if both colors agree with \( \chi(a, b) \), no otherwise.

7. If they all broadcast the same color A says Y, else A says NO.
\( M = 2^{n+1} - 1 \) throughout.

1. Pre-step: A, B, and C agree on a \( \Gamma(M) \)-coloring \( \chi \) of \( [M] \times [M] \) that has no mono \( L \) or \( \top \).

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6. C says Y if both colors agree with \( \chi(a, b) \), no otherwise.

7. If they all broadcast the same color A says Y, else A says NO.

Number of bits: \( 2 \lg(\Gamma(M)) + O(1) \). We show this is \( \leq O(\sqrt{n}) \).

But first we show that it works.
Protocol

\[ M = 2^{n+1} - 1 \text{ throughout.} \]

1. Pre-step: A, B, and C agree on a \( \Gamma(M) \)-coloring \( \chi \) of \([M] \times [M]\) that has no mono \( L \) or \( \overline{L} \).
2. A: \( b, c \), B: \( a, c \), C: \( a, b \). \( a, b, c \in \{0, 1\}^n \) numbers in binary.
3. If A sees \( b + c > M \), says NO and protocol stops. B, C, sim.
4. A finds \( a' \), s.t. \( a' + b + c = M \) and says \( \chi(a', b) \).
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Why Does This Work?

Assume $a + b + c = M - \lambda$ where $\lambda \in \mathbb{Z}$.
Why Does This Work?

Assume \( a + b + c = M - \lambda \) where \( \lambda \in \mathbb{Z} \).

\[ a' = M - b - c = M - (a + b + c) + (a + b + c) - b - c = M - (M - \lambda) + a = a + \lambda \]

\[ b' = b + \lambda \] (similar reasoning)

\[ (a', b') = (a + \lambda, b) \]

\[ (a, b') = (a, b + \lambda) \]

If protocol says YES then \( \chi(a + \lambda, b) = \chi(a, b + \lambda) = \chi(a, b) \).

Since \( \chi \) has no mono \( L \) or \( \dashv \), \( \lambda = 0 \) so \( a + b + c = M \).

If protocol says NO then either \( \chi(a + \lambda, b) \neq \chi(a, b + \lambda) \): so \( \lambda \neq 0 \).

\( \chi(a + \lambda, b) \neq \chi(a, b + \lambda) \): so \( \lambda \neq 0 \).

\( \chi(a, b + \lambda) \neq \chi(a, b) \): so \( \lambda \neq 0 \).

In all cases \( \lambda \neq 0 \) so \( a + b + c \neq M \).
Why Does This Work?

Assume $a + b + c = M - \lambda$ where $\lambda \in \mathbb{Z}$.

$$a' = M - b - c = M - (a + b + c) + (a + b + c) - b - c = M - (M - \lambda) + a = a + \lambda$$

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b' = b + \lambda \text{ (similar reasoning)}
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\[(a', b) = (a + \lambda, b)\]
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(a', b) = (a + \lambda, b)
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If protocol says YES then \( \chi(a + \lambda, b) = \chi(a, b + \lambda) = \chi(a, b) \).

Since \( \chi \) has no mono \( L \) or \( \uparrow \), \( \lambda = 0 \) so \( a + b + c = M \).
Why Does This Work?

Assume $a + b + c = M - \lambda$ where $\lambda \in \mathbb{Z}$.

$a' = M - b - c = M - (a + b + c) + (a + b + c) - b - c = M - (M - \lambda) + a = a + \lambda$

$b' = b + \lambda$ (similar reasoning)

$(a', b) = (a + \lambda, b)$

$(a, b') = (a, b + \lambda)$

If protocol says YES then $\chi(a + \lambda, b) = \chi(a, b + \lambda) = \chi(a, b)$. Since $\chi$ has no mono $L$ or $\updownarrow$, $\lambda = 0$ so $a + b + c = M$.

If protocol says NO then either $\chi(a + \lambda, b) \neq \chi(a, b + \lambda)$: so $\lambda \neq 0$.

$\chi(a + \lambda, b) \neq \chi(a, b)$: so $\lambda \neq 0$.

$\chi(a, b + \lambda) \neq \chi(a, b)$: so $\lambda \neq 0$. 
Why Does This Work?

Assume \( a + b + c = M - \lambda \) where \( \lambda \in \mathbb{Z} \).

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a' = M - b - c = M - (a + b + c) + (a + b + c) - b - c = M - (M - \lambda) + a = a + \lambda
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\( b' = b + \lambda \) (similar reasoning)

\( (a', b) = (a + \lambda, b) \)

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If protocol says YES then \( \chi(a + \lambda, b) = \chi(a, b + \lambda) = \chi(a, b) \).

Since \( \chi \) has no mono \( L \) or \( \nabla \), \( \lambda = 0 \) so \( a + b + c = M \).

If protocol says NO then either

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\( \chi(a, b + \lambda) \neq \chi(a, b) \): so \( \lambda \neq 0 \).

In all cases \( \lambda \neq 0 \) so \( a + b + c \neq M \).
Relating $\Gamma(M)$ with VDW

We need to bound $\lg(\Gamma(M))$. 

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**Lemma** Let $Z$ be such that $3M < W(3, Z)$. Then $\Gamma(M) \leq Z$. 
Relating $\Gamma(M)$ with VDW

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**Lemma** Let $Z$ be such that $3M < W(3, Z)$. Then $\Gamma(M) \leq Z$.

**Proof**
Relating $\Gamma(M)$ with VDW

We need to bound $\lg(\Gamma(M))$.

**Lemma** Let $Z$ be such that $3M < W(3, Z)$. Then $\Gamma(M) \leq Z$.

**Proof**
Let $COL$ be an $Z$-coloring of $\{1, \ldots, 3M\}$ with no mono 3-AP’s.
We need to bound $\log(\Gamma(M))$.

**Lemma** Let $Z$ be such that $3M < W(3, Z)$. Then $\Gamma(M) \leq Z$.

**Proof**

Let $COL$ be an $Z$-coloring of $\{1, \ldots, 3M\}$ with no mono 3-AP's. Define $COL': [M] \times [M] \rightarrow [Z]$

$$COL'(x, y) = COL(x + 2y)$$
Relating $\Gamma(M)$ with VDW

We need to bound $\log(\Gamma(M))$.

**Lemma** Let $Z$ be such that $3M < W(3, Z)$. Then $\Gamma(M) \leq Z$.

**Proof**

Let $COL$ be an $Z$-coloring of $\{1, \ldots, 3M\}$ with no mono 3-AP’s. Define $COL' : [M] \times [M] \to [Z]$ by

$$COL'(x, y) = COL(x + 2y)$$

**Claim** $COL'$ has no mono $L$’s or $\perp$.
Relating $\Gamma(M)$ with VDW

We need to bound $\lg(\Gamma(M))$.

**Lemma** Let $Z$ be such that $3M < W(3, Z)$. Then $\Gamma(M) \leq Z$.

**Proof**

Let $COL$ be an $Z$-coloring of $\{1, \ldots, 3M\}$ with no mono 3-AP's. Define $COL' : [M] \times [M] \rightarrow [Z]$

$$COL'(x, y) = COL(x + 2y)$$

**Claim** $COL'$ has no mono $L$'s or $\exists$.

If $COL'$ has a mono $L$ or $\exists$ then there exists $x, y \in [M], \lambda \in \mathbb{Z}$:

$$COL'(x, y) = COL'(x + \lambda, y) = COL'(x, y + \lambda)$$
Relating $\Gamma(M)$ with VDW

We need to bound $\lg(\Gamma(M))$.

**Lemma** Let $Z$ be such that $3M < W(3, Z)$. Then $\Gamma(M) \leq Z$.

**Proof**
Let $\text{COL}$ be an $Z$-coloring of $\{1, \ldots, 3M\}$ with no mono 3-AP’s.
Define $\text{COL}' : [M] \times [M] \rightarrow [Z]$

$$\text{COL}'(x, y) = \text{COL}(x + 2y)$$

**Claim** $\text{COL}'$ has no mono $L$’s or $\dag$.
If $\text{COL}'$ has a mono $L$ or $\dag$ then there exists $x, y \in [M], \lambda \in \mathbb{Z}$:

$$\text{COL}'(x, y) = \text{COL}'(x + \lambda, y) = \text{COL}'(x, y + \lambda) \text{ hence}$$
Relating $\Gamma(M)$ with VDW

We need to bound $\log(\Gamma(M))$.

**Lemma** Let $Z$ be such that $3M < W(3, Z)$. Then $\Gamma(M) \leq Z$.

**Proof**

Let $COL$ be an $Z$-coloring of $\{1, \ldots, 3M\}$ with no mono 3-AP’s. Define $COL' : [M] \times [M] \rightarrow [Z]$

$$COL'(x, y) = COL(x + 2y)$$

**Claim** $COL'$ has no mono $L$’s or $\neg$.

If $COL'$ has a mono $L$ or $\neg$ then there exists $x, y \in [M], \lambda \in \mathbb{Z}$:

$$COL'(x, y) = COL'(x + \lambda, y) = COL'(x, y + \lambda)$$

hence

$$COL(x + 2y) = COL(x + 2y + \lambda) = COL(x + 2y + 2\lambda)$$: a mono 3-AP

(If $\lambda < 0$ then $x + 2y + 2\lambda, x + 2y + \lambda, x + 2y$ is the 3-AP.)
In talk on $W(3, c)$ we proved:

**Thm** Let $V \in \mathbb{N}$ and let $A \subseteq [V]$ be a 3-free set. Then there is a $\frac{V \ln(V)}{|A|}$-coloring of $[V]$ with no mono 3-APs. Hence $W(3, \frac{V \ln(V)}{|A|}) \geq V$.  

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$W(3, \frac{V \ln(V)}{|A|}) \geq V.$

In talk on $W(3, c)$ we sketched:

**Thm** There exists a 3-free subset of $[V]$ of size $\geq V^{1 - \frac{1}{\sqrt{\ln V}}}$
In talk on $W(3, c)$ we proved:

**Thm** Let $V \in \mathbb{N}$ and let $A \subseteq [V]$ be a 3-free set. Then there is a $\frac{V \ln(V)}{|A|}$-coloring of $[V]$ with no mono 3-APs. Hence

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W(3, \frac{V \ln(V)}{|A|}) \geq V.
\]

In talk on $W(3, c)$ we sketched:

**Thm** There exists a 3-free subset of $[V]$ of size $\geq V^{1-\frac{1}{\sqrt{\lg V}}}$

We combine these two to get:

**Thm** Let $V \in \mathbb{N}$. Then there is a $V^{\frac{1}{\sqrt{\lg V}} \ln(V)}$-coloring of $[V]$ with no mono 3-APs. Hence

\[
W(3, V^{\frac{1}{\sqrt{\lg V}} \ln(V)}) \geq V.
\]
Just Plug in \( V = 3M \)

**Thm** Let \( V \in \mathbb{N} \). Then there is a \( V^{\frac{1}{\sqrt{\lg V} \ln(V)}} \)-coloring of \([V]\) with no mono 3-APs. Hence

\[
W(3, V^{\frac{1}{\sqrt{\lg V} \ln(V)}}) \geq V.
\]

Hence \( W(3, (3M)^{\frac{1}{\sqrt{\lg 3M} \ln(3M)}}) \geq 3M. \)

Hence \( \Gamma(M) \leq (3M)^{\frac{1}{\sqrt{\lg 3M} \ln(3M)}} \)

Hence \( \lg(\Gamma(M)) \leq \frac{1}{\sqrt{\lg 3M}} \lg(3M) + \lg(\ln(3M)) = O(\sqrt{\log(M)}) \)

\[
M = 2^{n+1} - 1 \sim 2^n \text{ so } \lg(\Gamma(M)) \leq O(\sqrt{n})
\]
We showed our protocol uses \( \leq 3 \lg(\Gamma(M)) \leq O(\sqrt{n}) \).

Known: lower bound of \( \Omega(\lg(\Gamma(M)) \).

Original paper had lower bound of \( \Omega(1) \) which is all they needed for their goal which was non-linear lower bounds on branching programs.

Gasarch showed lower bound of \( \Omega(\log \log n) \).

\( k \)-player version of this game has also been studied.