Poly Van Der Warden’s (PVDW) Theorem

Exposition by William Gasarch

April 28, 2022
Whenever I write $a, d$ or $a, d_1$ or anything of that sort we are assuming $a, d \in \mathbb{N}$ and $a, d \geq 1$. 
Recall VDW’s Theorem

VDW’s Theorem For all $k, c$ there exists $W = W(k, c)$ such that for all $\text{COL}: [W] \to [c]$ there exists $a, d$ such that

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Can we generalize and use $f_1(d), \ldots, f_k(d)$ for some $f_1, \ldots, f_k$?
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Notation $\mathbb{Z}[x]$ is set of polynomials with coefficients in $\mathbb{Z}$. 
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**Notation** $\mathbb{Z}[x]$ is set of polynomials with coefficients in $\mathbb{Z}$.

**Poly VDW Theorem** For all $p_1, \ldots, p_k \in \mathbb{Z}[x]$ and $c \in \mathbb{N}$ there exists $W = W(p_1, \ldots, p_k; c)$ such that for all COL: $[W] \to [c]$ there exists a $a, d$, such that

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True? or is Bill lying to us?
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**True? or is Bill lying to us?** Try to find counterexamples.
Counterexample and Reformulation
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Stupid Counterexample $p_1(d) = 1, c = 2$. The coloring $RBRBRB \cdots$ has no two naturals 1-apart that have same color.
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Poly VDW Theorem For all $p_1, \ldots, p_k \in \mathbb{Z}[x]$ with $(\forall i)[p_i(0) = 0]$, and $c \in \mathbb{N}$, there exists $W = W(p_1, \ldots, p_k; c)$ such that for all $\text{COL}: [W] \to [c]$ there exists a $a, d$ such that

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Credit Where Credit is Due


Used hard math and did not give bounds on PVDW numbers.

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The first Elementary proof was by Walters in *Combinatorial proofs of the Polynomial Van Der Waerden Theorem and the Polynomial Hales-Jewitt Theorem* Journal of the London Math Soc., Vol 61, 2000.

His paper is here: https://www.cs.umd.edu/~gasarch/TOPICS/vdw/walters.pdf

We present his proof.
PVDW\((p_1(x), \ldots, p_k(x); c)\) means
There exists \(W = W(p_1, \ldots, p_k; c)\) such that for all
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\[a, a + p_1(d), \ldots, a + p_k(d)\] same col.
**Notation**

\[ \text{PVDW}(p_1(x), \ldots, p_k(x); c) \] means
There exists \( W = W(p_1, \ldots, p_k; c) \) such that for all \( \text{COL}: [W] \rightarrow [c] \) there exists a \( a, d \) such that

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Easy Cases

\[ \text{PVDW}(x, 2x, 3x, \ldots, (k - 1)x) \]. This is VDW's Thm.
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PVDW\((x, 2x, 3x, \ldots, (k - 1)x)\). This is VDW’s Thm.

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PVDW\((x^2; 2)\): \(W(x^2; 2)\)
Easy Cases

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PVDW\((x^2; 2)\): \(W(x^2; 2) = 5\). Booktalk/GKKZ.
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PVDW(\(x^2; 3\)): \(W(x^2; 3)\)
Easy Cases

PVDW(x, 2x, 3x, ..., (k − 1)x)). This is VDW’s Thm.

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PVDW(x^2 + x; 2): W(x^2 + x; 2) ≤ 13. HW 8. (Liam Numbs).
PVDW(ax^2 + bx; 2): W(ax^2 + bx; 2) ≤ 12|a| + 6|b|. GKKZ.
PVDW(x^2; 3): W(x^2; 3) ≤ 59. Booktalk/GKKZ.
Easy Cases

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PVDW\((x^2; 4)\): \(W(x^2; 4) \leq 1 + 290,085,289^2\). Booktalk/GKKZ.
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PVDW\((ax^2 + bx; 4)\): \(W(ax^2 + bx; 4)\) A few bds known. GKKZ.
PVDW$(x, 2x, 3x, \ldots, (k-1)x))$. This is VDW's Thm.


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First hard case: PVDW$(x^2; 5)$.
Poly Van Der Warden’s (PVDW) Theorem: $\text{PVDW}(x^2)$

Exposition by William Gasarch

April 28, 2022
We Begin Proof of PVDW($x^2$)

$W(x^2; 5)$: The low value of 5 does not help us. We will prove PVDW($x^2$).
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We Begin Proof of $\text{PVDW}(x^2)$

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Recall that this is:

**Thm** For all $c \in \mathbb{N}$ there exists $W = W(x^2; c)$ such that for all $\text{COL}: [W] \to [c]$, there exists $a, d$ such that

$$a, a + d^2 \text{ are same color.}$$
We Begin Proof of $\text{PVDW}(x^2)$

$\mathcal{W}(x^2; 5)$: The low value of 5 does not help us. We will prove $\text{PVDW}(x^2)$.

Recall that this is:

**Thm** For all $c \in \mathbb{N}$ there exists $\mathcal{W} = \mathcal{W}(x^2; c)$ such that for all $\text{COL}: [\mathcal{W}] \rightarrow [c]$, there exists $a, d$ such that

$$a, a + d^2 \text{ are same color.}$$

**Note** None of the results or techniques for $\mathcal{W}(ax^2 + bx; c)$ for $c \leq 4$ will help at all. Oh well.
We Prove a Lemma Which Implies Theorem

Want:

**Thm** For all $c \in \mathbb{N}$ there exists $W = W(x^2; c)$ st for all

$\text{COL} : [W] \rightarrow [c]$ 

$(\exists a, d)[a, a + d^2 \text{ same color}].$

Lemma proves Theorem by taking $r = c$. Second part can't

happen, so first part does.
We Prove a Lemma Which Implies Theorem

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Will prove:

**Lemma** Fix $c \in \mathbb{N}$. For all $r$ there exists $U = U(r)$ st for all $\text{COL}: [U] \to [c]$ EITHER
We Prove a Lemma Which Implies Theorem

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**Thm** For all \( c \in \mathbb{N} \) there exists \( W = W(x^2; c) \) st for all \( \text{COL}: [W] \rightarrow [c] \) 
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**Lemma** Fix \( c \in \mathbb{N} \). For all \( r \) there exists \( U = U(r) \) st for all \( \text{COL}: [U] \rightarrow [c] \) EITHER

\( i) \ (\exists a, d)[a, a + d^2 \) same color], OR
We Prove a Lemma Which Implies Theorem

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**Thm** For all \( c \in \mathbb{N} \) there exists \( W = W(x^2; c) \) st for all \( \text{COL}: [W] \rightarrow [c] \)

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Will prove:

**Lemma** Fix \( c \in \mathbb{N} \). For all \( r \) there exists \( U = U(r) \) st for all \( \text{COL}: [U] \rightarrow [c] \) EITHER

\[
\begin{align*}
&i) \ (\exists a, d)[a, a + d^2 \text{ same color}], \ OR \\
&ii) \ (\exists a, d_1, \ldots, d_r)[a, a + d_1^2, \ldots, a + d_r^2 \text{ all diff cols}].
\end{align*}
\]
We Prove a Lemma Which Implies Theorem

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**Thm** For all \(c \in \mathbb{N}\) there exists \(W = W(x^2; c)\) st for all \(\text{COL}: [W] \rightarrow [c]\)

\((\exists a, d)[a, a + d^2 \text{ same color}].\)

Will prove:

**Lemma** Fix \(c \in \mathbb{N}\). For all \(r\) there exists \(U = U(r)\) st for all \(\text{COL}: [U] \rightarrow [c]\) EITHER

\(i) (\exists a, d)[a, a + d^2 \text{ same color}], \text{ OR}\)

\(ii) (\exists a, d_1, \ldots, d_r)[a, a + d_1^2, \ldots, a + d_r^2 \text{ all diff cols}].\)

Lemma proves Theorem by taking \(r = c\). Second part can't happen, so first part does.
Proof of Base Case of Lemma

Proof of Lemma is by induction on $r$.

$r = 1$ For all $\text{COL} : [U] \to [c]$ EITHER
Proof of Lemma is by induction on $r$.

$r = 1$ For all $\text{COL}: [U] \rightarrow [c]$ EITHER

i) $(\exists a, d)[a, a + d^2 \text{ same color}]$ OR
Proof of Base Case of Lemma

Proof of Lemma is by induction on $r$.

$r = 1$ For all $\text{COL} : [U] \rightarrow [c]$ EITHER

\begin{enumerate}
  \item $(\exists a, d)[a, a + d^2 \text{ same color}]$ OR
  \item $(\exists a, d_1)[a, a + d_1^2 \text{ diff cols}]$.
\end{enumerate}
Proof of Base Case of Lemma

Proof of Lemma is by induction on $r$.

$r = 1$ For all COL: $[U] \rightarrow [c]$ EITHER

i) $(\exists a, d)[a, a + d^2$ same color] OR

ii) $(\exists a, d_1)[a, a + d_1^2$ diff cols].

$U(1) = 2$. Take $a = d = d_1 = 1$.

$a = 1$

$a + d^2 = 1 + 1^2 = 2$. 

Proof of Base Case of Lemma

Proof of Lemma is by induction on $r$.

$r = 1$ For all $\text{COL}: [U] \to [c]$ EITHER

i) $(\exists a, d)[a, a + d^2 \text{ same color}]$ OR

ii) $(\exists a, d_1)[a, a + d_1^2 \text{ diff cols}]$.

$U(1) = 2$. Take $a = d = d_1 = 1$.

$a = 1$

$a + d^2 = 1 + 1^2 = 2$.

So they have the same color.
Proof of Base Case of Lemma

Proof of Lemma is by induction on $r$.

$r = 1$ For all COL: $[U] \rightarrow [c]$ EITHER

1) $(\exists a, d)[a, a + d^2$ same color] OR
2) $(\exists a, d_1)[a, a + d_1^2$ diff cols].

$U(1) = 2$. Take $a = d = d_1 = 1$.

$a = 1$

$a + d^2 = 1 + 1^2 = 2$.

So they have the same color.

If $a, d$ same col have 1. If $a, d$ diff col have 2.
Proof of Ind Step of Lemma

Assume that there exists \( U = U(r) \) st

\( (\exists a, d)[a, a + d^2 \text{ same color}], \) OR
Proof of Ind Step of Lemma

Assume that there exists \( U = U(r) \) st

- \( (\exists a, d)[a, a + d^2 \text{ same color}], \text{ OR} \)
- \( (\exists a, d_1, \ldots, d_r)[a, a + d_1^2, \ldots, a + d_r^2 \text{ all diff cols}] \).
Proof of Ind Step of Lemma

Assume that there exists $U = U(r)$ st

- $(\exists a, d)[a, a + d^2$ same color], OR
- $(\exists a, d_1, \ldots, d_r)[a, a + d_1^2, \ldots, a + d_r^2$ all diff cols].

We need to prove $U(r + 1)$ exists.

GOTO WHITE BOARD to prove

$$U(r + 1) \leq$$

$$(U(r)W(2U(r), c^{U(r)}))^2 + U(r)W(2U(r), c^{U(r)}).$$
We used VDW to prove $\text{PVDW}(x^2)$. 

---

**Note What we Used**

We used VDW to prove $\text{PVDW}(x^2)$. 

(This is not quite right since we only use a finite VDW theorem, and in fact the infinite one is false.)

Keep that in mind.
Note What we Used

We used VDW to prove PVDW($x^2$).

We denote that informally as:

\[ \text{PVDW}(x, 2x, 3x, \ldots) \implies \text{PVDW}(x^2). \]

(This is not quite right since we only use a FINITE VDW theorem, and in fact the infinite one is false.)
We used VDW to prove $PVDW(x^2)$.

We denote that informally as:
$PVDW(x, 2x, 3x, \ldots) \implies PVDW(x^2)$.
(This is not quite right since we only use a FINITE VDW theorem, and in fact the infinite one is false.)

Keep that in mind.
Poly Van Der Warden’s (PVDW) Theorem: $\text{PVDW}(x^2 + x)$

Exposition by William Gasarch

April 28, 2022
We Begin Proof of $\text{PVDW}(x^2 + x)$

**Thm** For all $c \in \mathbb{N}$ there exists $W = W(x^2 + x; c)$ st, for all $\text{COL}: [W] \rightarrow [c]$, there exists $a, d$ st

$$a, a + d^2 + d \text{ are same color.}$$
We Prove a Lemma Which Implies Theorem

Want:

**Thm** For all \( c \in \mathbb{N} \) there exists \( W = W(x^2 + x; c) \) st for all \( \text{COL}: [W] \to [c] \),

\[(\exists a, d)[a, a + d^2 + d \text{ same color }].\]
We Prove a Lemma Which Implies Theorem

Want:

**Thm** For all \( c \in \mathbb{N} \) there exists \( W = W(x^2 + x; c) \) st for all \( \text{COL} : [W] \rightarrow [c] \),

\((\exists a, d)[a, a + d^2 + d \text{ same color }]\).

Will prove:

**Lemma** Fix \( c \in \mathbb{N} \). For all \( r \) there exists \( U = U(r) \) st for all \( \text{COL} : [U] \rightarrow [c] \) EITHER
We Prove a Lemma Which Implies Theorem

Want:

**Thm** For all $c \in \mathbb{N}$ there exists $W = W(x^2 + x; c)$ st for all $\text{COL} : [W] \to [c]$, $(\exists a, d)[a, a + d^2 + d \text{ same color }]$.

Will prove:

**Lemma** Fix $c \in \mathbb{N}$. For all $r$ there exists $U = U(r)$ st for all $\text{COL} : [U] \to [c]$ EITHER

1. $(\exists a, d)[a, a + d^2 + d \text{ same color }]$, OR
We Prove a Lemma Which Implies Theorem

Want:

**Thm** For all $c \in \mathbb{N}$ there exists $W = W(x^2 + x; c)$ st for all

$\text{COL} : [W] \to [c],$

$(\exists a, d)[a, a + d^2 + d \text{ same color }].$

Will prove:

**Lemma** Fix $c \in \mathbb{N}$. For all $r$ there exists $U = U(r)$ st for all

$\text{COL} : [U] \to [c]$ EITHER

1. $(\exists a, d)[a, a + d^2 + d \text{ same color}], \text{ OR}$
2. $(\exists a, d_1, \ldots, d_r)[a, a + d_1^2 + d_1, \ldots, a + d_r^2 + d_r \text{ diff cols}].$
We Prove a Lemma Which Implies Theorem

Want:

**Thm** For all \( c \in \mathbb{N} \) there exists \( W = W(x^2 + x; c) \) st for all \( \text{COL}: [W] \to [c] \),
\[
(\exists a, d)[a, a + d^2 + d \text{ same color }].
\]

Will prove:

**Lemma** Fix \( c \in \mathbb{N} \). For all \( r \) there exists \( U = U(r) \) st for all \( \text{COL}: [U] \to [c] \) EITHER
\[
\begin{align*}
\text{i)} & \ (\exists a, d)[a, a + d^2 + d \text{ same color}], \text{ OR} \\
\text{ii)} & \ (\exists a, d_1, \ldots, d_r)[a, a + d_1^2 + d_1, \ldots, a + d_r^2 + d_r \text{ diff cols}].
\end{align*}
\]

Lemma proves Theorem by taking \( r = c \). Second part can't happen, so first part does.
Proof of Lemma is by induction on $r$.

$r = 1$ For all $\text{COL} : [U] \to [c]$ EITHER
Proof of Base Case of Lemma

Proof of Lemma is by induction on $r$.

$r = 1$ For all $\text{COL}: [U] \to [c]$ EITHER

1) $(\exists a, d)[a, a + d^2 + d \text{ same color}], \text{ OR}$
Proof of Base Case of Lemma

Proof of Lemma is by induction on \( r \).

\( r = 1 \) For all \( \text{COL} : [U] \rightarrow [c] \) EITHER

1) \( (\exists a, d)[a, a + d^2 + d \text{ same color}] \), OR

2) \( (\exists a, d_1)[a, a + d_1^2 + d_1 \text{ diff colors}] \).
Proof of Base Case of Lemma

Proof of Lemma is by induction on $r$.

$r = 1$ For all $\text{COL} : [U] \rightarrow [c]$ EITHER

i) $(\exists a, d)[a, a + d^2 + d \text{ same color}], \text{ OR}$

ii) $(\exists a, d_1)[a, a + d_1^2 + d_1 \text{ diff colors}]$.

$U(1) = 3$. Take $a = d = d_1 = 1$.

$a = 1$

$a + d^2 + d = 1 + 1^2 + 1 = 3$.

If they are the same col, have $i$, else have $ii$. 

Proof of Ind Step of Lemma

Assume that there exists $U = U(r)$ st for all $\text{COL}: [U] \to [c]$ EITHER

\begin{itemize}
  \item $i)$ ($\exists a, d \left[ a, a + d \right]$ same color],
  \item $ii)$ ($\exists a, d_1, \ldots, d_r \left[ a, a + d_2 + d_1, \ldots, a + d_2 + d_1 + \ldots + d_r \right]$ diff cols]
\end{itemize}

GOTO WHITE BOARD
Proof of Ind Step of Lemma

Assume that there exists $U = U(r)$ st for all $\text{COL} : [U] \to [c]$ EITHER

i) $(\exists a, d)[a, a + d^2 + d \text{ same color}], \ OR$
Proof of Ind Step of Lemma

Assume that there exists $U = U(r)$ st for all $\text{COL}: [U] \rightarrow [c]$ EITHER
\begin{enumerate}
  \item $(\exists a, d)[a, a + d^2 + d \text{ same color}], \text{ OR}$
  \item $(\exists a, d_1, \ldots, d_r)[a, a + d_1^2 + d_1, \ldots, a + d_r^2 + d_r \text{ diff cols}]$.
\end{enumerate}

GOTO WHITE BOARD
Note What we Used

We showed

\[ PVDW(x, 2x, 3x, \ldots) \implies PVDW(x^2 + x) \]
We showed

$$\text{PVDW}(x, 2x, 3x, \ldots) \implies \text{PVDW}(x^2 + x)$$

Note that $\text{PVDW}(x^2)$ did not help get $\text{PVDW}(x^2 + x)$. 
Note What we Used

We showed

\[ \text{PVDW}(x, 2x, 3x, \ldots) \implies \text{PVDW}(x^2 + x) \]

Note that \( \text{PVDW}(x^2) \) did not help get \( \text{PVDW}(x^2 + x) \).

Keep that in mind.
This Generalizes

**Thm** Let $A, B \in \mathbb{Z}$. For all $c \in \mathbb{N}$ there exists

$W = W(Ax^2 + Bx; c)$ st for all $\text{COL} : [W] \rightarrow [c],

$(\exists a, d)[a, a + Ad^2 + Bd \text{ same color}].$
This Generalizes

**Thm** Let $A, B \in \mathbb{Z}$. For all $c \in \mathbb{N}$ there exists $W = W(Ax^2 + Bx; c)$ st for all $\text{COL} : [W] \rightarrow [c], (\exists a, d)[a, a + Ad^2 + Bd \text{ same color}].$

$$PVDW(x, 2x, 3x, \ldots) \implies PVDW(Ax^2 + Bx).$$
This Generalizes

**Thm** Let $A, B \in \mathbb{Z}$. For all $c \in \mathbb{N}$ there exists $W = W(Ax^2 + Bx; c)$ st for all COL: $[W] \rightarrow [c]$, $(\exists a, d)[a, a + Ad^2 + Bd$ same color].

\[
PVDW(x, 2x, 3x, \ldots) \implies PVDW(Ax^2 + Bx).
\]

Proof is similar to
This Generalizes

**Thm** Let $A, B \in \mathbb{Z}$. For all $c \in \mathbb{N}$ there exists $W = W(Ax^2 + Bx; c)$ st for all $\text{COL}: [W] \rightarrow [c]$, $(\exists a, d)[a, a + Ad^2 + Bd \text{ same color}]$.

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Proof is similar to

$$\text{PVDW}(x, 2x, 3x, \ldots) \implies \text{PVDW}(x^2 + x).$$
Poly Van Der Warden’s (PVDW) Theorem:
PVDW\left(x^2, x^2 + x\right)

Exposition by William Gasarch

April 28, 2022
We Begin Proof of $\text{PVDW}(x^2, x^2 + x)$

**Thm** For all $c \in \mathbb{N}$ there exists $W = W(x^2, x^2 + x; c)$ such that, for all $\text{COL}: [W] \rightarrow [c]$, there exists $a, d$ such that

$$a, a + d^2, a + d^2 + d$$

are same col.
We Prove a Lemma Which Implies Theorem

Want:

$\textbf{Thm } (\forall c \in \mathbb{N})(\exists W = W(x^2, x^2 + x; c) \text{ st for all } \text{COL}: [W] \rightarrow [c] (\exists a, d)[a, a + d^2, a + d^2 + d \text{ same col}].$
We Prove a Lemma Which Implies Theorem

Want:

\[ \forall c \in \mathbb{N} \left( \exists W = W(x^2, x^2 + x; c) \text{ st for all } \text{COL} : \right. \]
\[ \left. [W] \rightarrow [c] \left( \exists a, d \right) \left[ a, a + d^2, a + d^2 + d \text{ same col} \right] \right. \]

Think about what the lemma will be with your neighbor.
We Prove a Lemma Which Implies Theorem

Want:

**Thm** \((\forall c \in \mathbb{N})(\exists W = W(x^2, x^2 + x; c) \text{ st for all } \text{COL}: [W] \rightarrow [c] \ (\exists a, d)[a, a + d^2, a + d^2 + d \text{ same col}].\)

Think about what the lemma will be with your neighbor.

**Lemma** Fix \(c \in \mathbb{N}.\) For all \(r\) there exists \(U = U(r)\) st for all \(\text{COL}: [U] \rightarrow [c]\) EITHER
We Prove a Lemma Which Implies Theorem

Want:
Thm \( (\forall c \in \mathbb{N})(\exists W = W(x^2, x^2 + x; c) \text{ st for all } \text{COL}: [W] \rightarrow [c] (\exists a, d)[a, a + d^2, a + d^2 + d \text{ same col}]. \)

Think about what the lemma will be with your neighbor.
Lemma Fix \( c \in \mathbb{N}. \) For all \( r \) there exists \( U = U(r) \) st for all \( \text{COL}: [U] \rightarrow [c] \) EITHER
\[
\text{i) } (\exists a, d)[a, a + d^2, a + d^2 + d \text{ same col}], \text{ OR}
\]
We Prove a Lemma Which Implies Theorem

Want:

**Thm** \((\forall c \in \mathbb{N})(\exists W = W(x^2, x^2 + x; c) \text{ st for all } \text{COL}: [W] \rightarrow [c] (\exists a, d)[a, a + d^2, a + d^2 + d \text{ same col}]).\)

Think about what the lemma will be with your neighbor.

**Lemma** Fix \(c \in \mathbb{N}\). For all \(r\) there exists \(U = U(r)\) st for all \(\text{COL}: [U] \rightarrow [c]\) EITHER

\[ i) \ (\exists a, d)[a, a + d^2, a + d^2 + d \text{ same col}], \text{ OR} \]

\[ ii) \ (\exists a, d_1, \ldots, d_r) \]

\[ [a, \{a + d_1^2, a + d_1^2 + d_1\}, \ldots, \{a + d_r^2, a + d_r^2 + d_r\} \text{ diff colors}]. \]

(The pair in \(\{\}\) are same col.)
We Prove a Lemma Which Implies Theorem

Want:

**Thm** $(\forall c \in \mathbb{N})(\exists W = W(x^2, x^2 + x; c) \text{ st for all } \text{COL}: [W] \rightarrow [c] (\exists a, d)[a, a + d^2, a + d^2 + d \text{ same col}].)$

Think about what the lemma will be with your neighbor.

**Lemma** Fix $c \in \mathbb{N}$. For all $r$ there exists $U = U(r)$ st for all \text{COL}: $[U] \rightarrow [c]$ EITHER

$i$) $(\exists a, d)[a, a + d^2, a + d^2 + d \text{ same col}], \text{ OR}$

$ii$) $(\exists a, d_1, \ldots, d_r)$

$[a, \{a + d_1^2, a + d_1^2 + d_1\}, \ldots, \{a + d_r^2, a + d_r^2 + d_r\} \text{ diff colors}].$

(The pair in $\{\}$ are same col.)

Lemma proves Theorem by taking $r = c$. Second part can’t happen, so first part does.
Proof of Base Case of Lemma

Proof of Lemma is by Induction.

\[ r = 1 \] For all \( \text{COL}: [U] \rightarrow [c] \) EITHER
Proof of Base Case of Lemma

Proof of Lemma is by Induction.

\( r = 1 \) For all \( \text{COL}: [U] \rightarrow [c] \) EITHER

i) \( (\exists a, d)[a, a + d^2, a + d^2 + d \text{ same col}], \text{ OR} \)

There is one thing wrong with this proof. Can you tell?
Proof of Base Case of Lemma

Proof of Lemma is by Induction.

\( r = 1 \) For all \( \text{COL} : [U] \rightarrow [c] \) EITHER

i) \( (\exists a, d)[a, a + d^2, a + d^2 + d \text{ same col}], \) OR

ii) \( (\exists a, d_1)[a, \{a + d_1^2, a + d_1^2 + d_1\} \text{ diff colors}]. \)
Proof of Base Case of Lemma

Proof of Lemma is by Induction.

\( r = 1 \) For all \( \text{COL} : [U] \rightarrow [c] \) EITHER

i) \( (\exists a, d)[a, a + d^2, a + d^2 + d \text{ same col}], \text{ OR} \)

\( \text{ii) } (\exists a, d_1)[a, \{a + d_1^2, a + d_1^2 + d_1\} \text{ diff colors}]. \)

\( U(1) = W(2, c). \)

Will get \( a', d_1 \) st \( a', a' + d_1 \) are same col.

Rewrite: \( a' = (a' - d_1^2) + d_1^2. \) Let \( a = a' - d_1^2. \)
Proof of Base Case of Lemma

Proof of Lemma is by Induction.

\( r = 1 \) For all \( \text{COL} : [U] \rightarrow [c] \) EITHER

i) \( (\exists a, d)[a, a + d^2, a + d^2 + d \text{ same col}], \) OR

ii) \( (\exists a, d_1)[a, \{a + d_1^2, a + d_1^2 + d_1\} \text{ diff colors}]. \)

\( U(1) = W(2, c). \)

Will get \( a', d_1 \) st \( a', a' + d_1 \) are same col.

Rewrite: \( a' = (a' - d_1^2) + d_1^2. \) Let \( a = a' - d_1^2. \)

\( a + d_1^2 = a' \)

\( a + d_1^2 + d_1 = a' + d_1 \)
Proof of Base Case of Lemma

Proof of Lemma is by Induction.

\( r = 1 \) For all \( \text{COL}: [U] \rightarrow [c] \) EITHER

i) \( (\exists a, d)[a, a + d^2, a + d^2 + d \text{ same col}], \text{ OR} \)

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\( U(1) = W(2, c) \).

Will get \( a', d_1 \) st \( a', a' + d_1 \) are same col.

Rewrite: \( a' = (a' - d_1^2) + d_1^2 \). Let \( a = a' - d_1^2 \).

\( a + d_1^2 = a' \)

\( a + d_1^2 + d_1 = a' + d_1 \)

So they have the same color.
Proof of Base Case of Lemma

Proof of Lemma is by Induction.

\( r = 1 \) For all \( \text{COL} : [U] \rightarrow [c] \) EITHER

i) \((\exists a, d)[a, a + d^2, a + d^2 + d \text{ same col}], \text{ OR} \) 

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\( U(1) = W(2, c) \).

Will get \( a', d_1 \) st \( a', a' + d_1 \) are same col.

Rewrite: \( a' = (a' - d_1^2) + d_1^2 \). Let \( a = a' - d_1^2 \).

\( a + d_1^2 = a' \)

\( a + d_1^2 + d_1 = a' + d_1 \)

So they have the same color.

If \( a \) is that col, have i. If \( a \) is diff col, have ii.
Proof of Lemma is by Induction.

\( r = 1 \) For all \( \text{COL} : [U] \rightarrow [c] \) EITHER

1) \((\exists a, d)[a, a + d^2, a + d^2 + d \text{ same col}], \text{ OR}\)

2) \((\exists a, d_1)[a, \{a + d_1^2, a + d_1^2 + d_1\} \text{ diff colors}].\)

\( U(1) = W(2, c). \)

Will get \( a', d_1 \) st \( a', a' + d_1 \) are same col.

Rewrite: \( a' = (a' - d_1^2) + d_1^2. \) Let \( a = a' - d_1^2. \)

\( a + d_1^2 = a' \)
\( a + d_1^2 + d_1 = a' + d_1 \)

So they have the same color.

If \( a \) is that col, have i. If \( a \) is diff col, have ii.

There is one thing wrong with this proof. Can you tell?
The Issue and Our Convention

\[ U(1) = W(2, c). \]
The Issue and Our Convention

\[ U(1) = W(2, c) \]. Will get \( a', d_1 \) st \( a', a' + d_1 \) are same col.

Rewrite: \( a' = (a' - d_1^2) + d_1^2 \). Let \( a = a' - d_1^2 \).
The Issue and Our Convention

\[ U(1) = W(2, c) \]. Will get \( a', d_1 \) s.t. \( a', a' + d_1 \) are same col.
Rewrite: \( a' = (a' - d_1^2) + d_1^2 \). Let \( a = a' - d_1^2 \).
\[ a + d_1^2 = a' \quad a + d_1^2 + d_1 = a' + d_1 \]

There is one thing wrong with this proof. Can you tell?
What if \( a' - d_1^2 < 0 \)? Then \( a < 0 \).
Can you fix this?
Fix:
\[ U(1) = W(2; c) + W(2; c). \] Do the above in \( W(2; c) \) part.

Convention

We ignore this issue since we know how to fix it.
Hence our bds are a byte lower than bds in real proof.
The bounds are so big that we don't care.
The Issue and Our Convention

\[ U(1) = W(2, c). \] Will get \( a', d_1 \) st \( a', a' + d_1 \) are same col.

Rewrite: \( a' = (a' - d_1^2) + d_1^2 \). Let \( a = a' - d_1^2 \).

\[ a + d_1^2 = a' \quad a + d_1^2 + d_1 = a' + d_1 \]

So they have the same color.
The Issue and Our Convention

\[ U(1) = W(2, c). \] Will get \( a', d_1 \) st \( a' \), \( a' + d_1 \) are same col.

Rewrite: \( a' = (a' - d_1^2) + d_1^2 \). Let \( a = a' - d_1^2 \).

\[ a + d_1^2 = a' \quad a + d_1^2 + d_1 = a' + d_1 \]

So they have the same color.

If \( a \) is that col, have \( i \). If \( a \) is diff col, have \( ii \).
The Issue and Our Convention

\[ U(1) = W(2, c) \]. Will get \( a', d_1 \) st \( a', a' + d_1 \) are same col.

Rewrite: \( a' = (a' - d_1^2) + d_1^2 \). Let \( a = a' - d_1^2 \).

\[ a + d_1^2 = a' \quad a + d_1^2 + d_1 = a' + d_1 \]

So they have the same color.

If \( a \) is that col, have \( i \). If \( a \) is diff col, have \( ii \).

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Rewrite: \( a' = (a' - d_2^2) + d_1^2 \). Let \( a = a' - d_1^2 \).
\[ a + d_1^2 = a' \quad a + d_1^2 + d_1 = a' + d_1 \]
So they have the same color.
If \( a \) is that col, have \( i \). If \( a \) is diff col, have \( ii \).
There is one thing wrong with this proof. Can you tell?

What if \( a' - d_2^2 < 0 \)? Then \( a < 0 \).
The Issue and Our Convention

\[ U(1) = W(2, c). \] Will get \( a', d_1 \) st \( a', a' + d_1 \) are same col.
Rewrite: \( a' = (a' - d_1^2) + d_1^2. \) Let \( a = a' - d_1^2. \)
\[ a + d_1^2 = a' \quad a + d_1^2 + d_1 = a' + d_1 \]
So they have the same color.
If \( a \) is that col, have \( i. \) If \( a \) is diff col, have \( ii. \)
There is one thing wrong with this proof. Can you tell?

**What if \( a' - d_1^2 < 0? \)** Then \( a < 0. \)
Can you fix this?
The Issue and Our Convention

\[ U(1) = W(2, c) \] Will get \( a', d_1 \) st \( a', a' + d_1 \) are same col.
Rewrite: \( a' = (a' - d_1^2) + d_1^2 \). Let \( a = a' - d_1^2 \).
\[ a + d_1 = a' \quad a + d_1^2 + d_1 = a' + d_1 \]
So they have the same color.
If \( a \) is that col, have \( i \). If \( a \) is diff col, have \( ii \).
There is one thing wrong with this proof. Can you tell?

**What if \( a' - d_1^2 < 0 \)?** Then \( a < 0 \).
Can you fix this?
Fix: \( U(1) = W(2; c)^2 + W(2; c) \). Do the above in \( W(2; c) \) part.
The Issue and Our Convention

\[ U(1) = W(2, c). \] Will get \( a', d_1 \) st \( a', a' + d_1 \) are same col.

Rewrite: \( a' = (a' - d_1^2) + d_1^2. \) Let \( a = a' - d_1^2. \)

\[ a + d_1^2 = a' \quad a + d_1^2 + d_1 = a' + d_1 \]

So they have the same color.

If \( a \) is that col, have \( i. \) If \( a \) is diff col, have \( ii. \)

There is one thing wrong with this proof. Can you tell?

**What if \( a' - d_1^2 < 0? \)** Then \( a < 0. \)

Can you fix this?

Fix: \( U(1) = W(2; c)^2 + W(2; c). \) Do the above in \( W(2; c) \) part.

**Convention** We ignore this issue since we know how to fix it.
The Issue and Our Convention

\[ U(1) = W(2, c). \] Will get \( a', d_1 \) st \( a', a' + d_1 \) are same col.

Rewrite: \( a' = (a' - d_1^2) + d_1^2 \). Let \( a = a' - d_1^2 \).

\[ a + d_1^2 = a' \quad \text{and} \quad a + d_1^2 + d_1 = a' + d_1 \]

So they have the same color.

If \( a \) is that col, have \( i \). If \( a \) is diff col, have \( ii \).

There is one thing wrong with this proof. Can you tell?

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**Convention** We ignore this issue since we know how to fix it.

Hence our bds are a byte lower than bds in real proof.

The bounds are so big that we don’t care.
Proof of Ind Step of Lemma

Assume that there exists $U = U(r)$ st for all $\text{COL}: [U] \rightarrow [c]$ EITHER

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Proof of Ind Step of Lemma

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i) \( (\exists a, d)[a, a + d^2, a + d^2 + d \text{ same col}], \text{ OR} \)

ii) \( (\exists a, d_1, \ldots, d_r) \)

\[ a, \{a + d_1^2, a + d_1 + d\}, \ldots, \{a + d_r^2, a + d_r^2 + d_r\} \text{ diff colors}]. \]

GOTO WHITE BOARD
Poly Van Der Warden’s (PVDW) Theorem: 
\[ \text{PVDW}(x^2, x) \]

Exposition by William Gasarch

April 28, 2022
We Begin Proof of $\text{PVDW}(x^2, x)$

**Thm** For all $c \in \mathbb{N}$ there exists $W = W(x, x^2; c)$ st, for all $\text{COL} : [W] \to [c]$, there exists $a, d$ st

$$a, a + d, a + d^2 \text{ are same col.}$$
We Prove a Lemma Which Implies Theorem

Want:

**Thm** For all $c \in \mathbb{N}$ there exists $W = W(x, x^2; c)$ st for all $COL: [W] \rightarrow [c]$, 
$(\exists a, d)[a, a + d, a + d^2 \text{ same col}]$.
We Prove a Lemma Which Implies Theorem

Want:

**Thm** For all $c \in \mathbb{N}$ there exists $W = W(x, x^2; c)$ st for all $COL: [W] \to [c]$, $(\exists a, d)[a, a + d, a + d^2 \text{ same col }]$.

Think about what the lemma will be with your neighbor.
We Prove a Lemma Which Implies Theorem

Want:

**Thm** For all \( c \in \mathbb{N} \) there exists \( W = W(x, x^2; c) \) st for all

\[ \text{COL}: [W] \rightarrow [c], \]

\[ (\exists a, d)[a, a + d, a + d^2 \text{ same col }]. \]

Think about what the lemma will be with your neighbor.

Will prove:

**Lemma** Fix \( c \in \mathbb{N} \). For all \( r \) there exists \( U = U(r) \) st for all

\[ \text{COL}: [U] \rightarrow [c] \text{ EITHER } \]

\[ (\exists a, d)[a, a + d, a + d^2 \text{ same col }], \]

\[ \text{OR } (\exists a, d_1, \ldots, d_r)[a, \{a + d_1, a + d_2\}, \ldots, \{a + d_r, a + d_2r\} \text{ diff cols}]. \]

(The pair in \{\} are same col.)

Lemma proves Theorem by taking \( r = c \). Second part can't happen, so first part does.
We Prove a Lemma Which Implies Theorem

Want:

**Thm** For all \( c \in \mathbb{N} \) there exists \( W = W(x, x^2; c) \) st for all
COL: \([W] \rightarrow [c] \),
(\( \exists a, d \))[a, a + d, a + d^2 \) same col \].

Think about what the lemma will be with your neighbor.

Will prove:

**Lemma** Fix \( c \in \mathbb{N} \). For all \( r \) there exists \( U = U(r) \) st for all
COL: \([U] \rightarrow [c] \) EITHER

i) \( (\exists a, d)[a, a + d, a + d^2 \) same col \], OR
We Prove a Lemma Which Implies Theorem

Want:

**Thm** For all $c \in \mathbb{N}$ there exists $W = W(x, x^2; c)$ st for all $\text{COL}: [W] \rightarrow [c],\ 
(\exists a, d)[a, a + d, a + d^2 \text{ same col }].$

Think about what the lemma will be with your neighbor.

Will prove:

**Lemma** Fix $c \in \mathbb{N}$. For all $r$ there exists $U = U(r)$ st for all $\text{COL}: [U] \rightarrow [c]$ EITHER

i) $(\exists a, d)[a, a + d, a + d^2 \text{ same col}], \ OR$

ii) $(\exists a, d_1, \ldots, d_r)[a, \{a + d_1, a + d_1^2\}, \ldots, \{a + d_r, a + d_r^2\} \text{ diff cols}].$

(The pair in $\{\}$ are same col.)
We Prove a Lemma Which Implies Theorem

Want:

**Thm** For all $c \in \mathbb{N}$ there exists $W = W(x, x^2; c)$ st for all

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Think about what the lemma will be with your neighbor.

Will prove:

**Lemma** Fix $c \in \mathbb{N}$. For all $r$ there exists $U = U(r)$ st for all

$\text{COL} : [U] \rightarrow [c]$ EITHER

1) $(\exists a, d)[a, a + d, a + d^2 \text{ same col}], \text{ OR}$

2) $(\exists a, d_1, \ldots, d_r)[a, \{a + d_1, a + d_1^2\}, \ldots, \{a + d_r, a + d_r^2\} \text{ diff cols}].$

(The pair in $\{}$ are same col.)

Lemma proves Theorem by taking $r = c$. Second part can’t happen, so first part does.
Proof of Base Case of Lemma

Proof of Lemma is by induction on $r$.

$r = 1$ For all $\text{COL} : \{U\} \to \{c\}$ EITHER
Proof of Base Case of Lemma

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Proof of Base Case of Lemma

Proof of Lemma is by induction on $r$.

$r = 1$ For all $\text{COL} : [U] \rightarrow [c]$ EITHER

i) $(\exists a, d)[a, a + d, a + d^2$ same color], OR

ii) $(\exists a, d_1)[a, \{a + d_1, a + d_1^2\}$ diff colors].
Proof of Base Case of Lemma

Proof of Lemma is by induction on $r$.

$r = 1$ For all $\text{COL}: [U] \to [c]$ EITHER

i) $(\exists a, d)[a, a + d, a + d^2 \text{ same color}], \text{ OR}$

ii) $(\exists a, d_1)[a, \{a + d_1, a + d_1^2\} \text{ diff colors}]$.

Let $U(1) = W(x^2 - x; c)$. For a $c$-colorings of $[U(1)]$ get $a', d_1$ st $a', a' + d_1^2 - d_1$ are same col.

Rewrite: $a' = (a' - d_1) + d_1$. Let $a = a' - d_1$. 
Proof of Base Case of Lemma

Proof of Lemma is by induction on $r$.

**$r = 1$** For all $\text{COL} : [U] \rightarrow [c]$ EITHER

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ii) \((\exists a, d_1)[a, \{a + d_1, a + d_1^2\} \text{ diff colors}]\).

Let \( U(1) = W(x^2 - x; c) \). For a \( c \)-colorings of \([U(1)]\) get \( a', d_1 \) st \( a', a' + d_1^2 - d_1 \) are same col.

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\[ a' = a + d_1 \]
\[ a' + d_1^2 - d_1 = a + d_1^2. \]
So they have the same color.
Proof of Base Case of Lemma

Proof of Lemma is by induction on $r$.

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Let $U(1) = W(x^2 - x; c)$. For a $c$-colorings of $[U(1)]$ get $a', d_1$ st $a', a' + d_1^2 - d_1$ are same col.
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$a' + d_1^2 - d_1 = a + d_1^2$.

So they have the same color.
If $a$ is that col, have i. If $a$ is diff col, have ii.
Assume that there exists $U = U(r)$ st for all $\text{COL}: [U] \rightarrow [c]$ EITHER
Proof of Ind Step of Lemma

Assume that there exists $U = U(r)$ st for all $\text{COL} : [U] \rightarrow [c]$ EITHER

- $(\exists a, d)[a, a + d^2 + d \text{ same col}], \text{ OR}$

GOTO WHITE BOARD
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Assume that there exists \( U = U(r) \) st for all \( \text{COL}: [U] \rightarrow [c] \) EITHER

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\[ (\exists a, d_1, \ldots, d_r)[a, \{a + d_1, a + d_1^2\}, \ldots, \{a + d_r, a + d_r^2\} \text{ diff cols}] \]

GOTO WHITE BOARD
A Powerful Notation and a General Approach

Exposition by William Gasarch

April 28, 2022
Which Proofs were Similar?

Proofs of all $PVDW(x^2 - \Box x \ldots, x^2, \ldots, x^2 + \Box x)$ are similar.
Which Proofs were Similar?

Proofs of all $PVDW(x^2 - \Box x \ldots, x^2, \ldots, x^2 + \Box x)$ are similar. Proof used VDW for Base and for Ind.
Which Proofs were Similar?

Proofs of all \( \text{PVDW}(x^2 - □x \ldots, x^2, \ldots, x^2 + □x) \) are similar. Proof used VDW for Base and for Ind.

**Key** There is one lead coefficient and its for quadratic-degree 2. We will denote this \((1, 0): 1\) quad lead coeff, 0 linear lead coeffs.
Which Proofs were Similar?

Proofs of all $PVDW(x^2 - □x, \ldots, x^2, \ldots, x^2 + □x)$ are similar. Proof used VDW for Base and for Ind.

**Key** There is one lead coefficient and its for quadratic-degree 2. We will denote this $(1, 0)$: 1 quad lead coeff, 0 linear lead coeffs.

Proofs of all $PVDW(x, x^2 - □x, \ldots, x^2, \ldots, x^2 + □x)$ are similar.
Which Proofs were Similar?

Proofs of all $\text{PVDW}(x^2 - \Box x \ldots, x^2, \ldots, x^2 + \Box x)$ are similar. Proof used VDW for Base and for Ind.

**Key** There is one lead coefficient and its for quadratic-degree 2. We will denote this $(1, 0)$: 1 quad lead coeff, 0 linear lead coeffs.

Proofs of all $\text{PVDW}(x, x^2 - \Box x, \ldots, x^2, \ldots, x^2 + \Box x)$ are similar. Proofs used $\text{PVDW}(x^2 - \Box x \ldots, x^2, \ldots, x^2 + \Box x)$ for Base and Ind.

**Key** There are two lead coefficients and they are for quadratic-degree 2 and linear-degree 1. We will denote this $(1, 1)$: 1 quad lead coeff, 1 linear lead coeffs.
Notation Let $P$ be a finite subset of $\mathbb{Z}[x]$ such that $(\forall p \in P)[p(0) = 0]$. Assume the max degree of a poly is $d$. For $1 \leq i \leq d$ let $n_i$ be the number of lead coefficients of polys in $P$ of degree $i$. 
Associate to Each Set of Poly’s an Index

Notation Let $P$ be a finite subset of $\mathbb{Z}[x]$ such that $(\forall p \in P)[p(0) = 0]$. Assume the max degree of a poly is $d$. For $1 \leq i \leq d$ let $n_i$ be the number of lead coefficients of polys in $P$ of degree $i$. The index of $P$ is $(n_d, n_{d-1}, \ldots, n_1)$. 

Examples

\{ $x^3$, $x^3 + \square x^2 + \square$, $x^2 + \square x$, $3x$, $4x$, $10x$ \} has index $(1, 1, 3)$.

\{ $x^4$, $2x^4 + \square x^3$, $x^2$, $2x^2$, $100x^2$, $x$, $100000x$ \} has index $(2, 0, 3, 2)$. 
Associate to Each Set of Poly’s an Index

**Notation** Let $P$ be a finite subset of $\mathbb{Z}[x]$ such that $(\forall p \in P)[p(0) = 0]$. Assume the max degree of a poly is $d$. For $1 \leq i \leq d$ let $n_i$ be the number of lead coefficients of polys in $P$ of degree $i$. The **index** of $P$ is $(n_d, n_{d-1}, \ldots, n_1)$.

**Examples**
Associate to Each Set of Poly’s an Index

**Notation** Let $P$ be a finite subset of $\mathbb{Z}[x]$ such that 
$(\forall p \in P)[p(0) = 0].$
Assume the max degree of a poly is $d$. 
For $1 \leq i \leq d$ let $n_i$ be the number of lead coefficients of polys in 
$P$ of degree $i$.
The **index** of $P$ is $(n_d, n_{d-1}, \ldots, n_1)$.

**Examples**
{$x^3, x^3 + \Box x^2 + \Box x, x^2 + \Box x, 3x, 4x, 10x$} has index $(1, 1, 3)$. 
Notation Let \( P \) be a finite subset of \( \mathbb{Z}[x] \) such that 
\[(\forall p \in P)[p(0) = 0].\]
Assume the max degree of a poly is \( d \).
For \( 1 \leq i \leq d \) let \( n_i \) be the number of lead coefficients of polys in \( P \) of degree \( i \).
The **index** of \( P \) is \((n_d, n_{d-1}, \ldots, n_1)\).

**Examples**

\( \{x^3, x^3 + \Box x^2 + \Box x, x^2 + \Box x, 3x, 4x, 10x\} \) has index \((1, 1, 3)\).

\( \{x^4, 2x^4 + \Box x^3, x^2, 2x^2, 100x^2, x, 100000x\} \) has index \((2, 0, 3, 2)\).
A Powerful Notation

PVDW(1, 0) means 
$(\forall P \subseteq \mathbb{Z}[x]), P$ of index $(1, 0)$, PVDW$(P)$ is true.
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From what we did you could easily prove \(P(1, 0)\).
A Powerful Notation

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PVDW(n_d, . . . , n_1) means
(∀P ⊆ ℤ[x]), P of index (n_d, . . . , n_1), PVDW(P) is true.
A Powerful Notation

PVDW(1, 0) means
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PVDW\((n_d, \ldots, n_1)\) means
\((\forall P \subseteq \mathbb{Z}[x]), P\) of index \((n_d, \ldots, n_1)\), PVDW\((P)\) is true.

We showed PVDW\((1, 0) \implies PVDW(1, 1)\).
A Powerful Notation

PVDW\((1, 0)\) means \\
\((\forall P \subseteq \mathbb{Z}[x]), P \text{ of index } (1, 0), \text{ PVDW}(P) \text{ is true.}\) \\
From what we did you could easily prove \(P(1, 0)\).

PVDW\((n_d, \ldots, n_1)\) means \\
\((\forall P \subseteq \mathbb{Z}[x]), P \text{ of index } (n_d, \ldots, n_1), \text{ PVDW}(P) \text{ is true.}\)

We showed PVDW\((1, 0) \implies \text{ PVDW}(1, 1)\).

But what about PVDW\((1, 0)\)? That was proven by VDW.
Can we Express VDW in our Powerful Notation?

PVDW(4) would include PVDW(x, 2x, 3x, 4x) which is $(\forall c)[VDW(5, c)]$. 
Can we Express VDW in our Powerful Notation?

PVDW(4) would include PVDW(x, 2x, 3x, 4x) which is 
(∀c)[VDW(5, c)].

Our notation is not so powerful after all! It cannot express VDW!
Can we Express VDW in our Powerful Notation?

PVDW(4) would include PVDW(x, 2x, 3x, 4x) which is $(\forall c)[VDW(5, c)]$. Our notation is not so powerful after all! It cannot express VDW!

We extend our notation. We want $(\forall k)[PVDW(k)]$. We use PVDW(ω).
Can we Express VDW in our Powerful Notation?

PVDW(4) would include PVDW(x, 2x, 3x, 4x) which is $(\forall c)[VDW(5, c)]$.

Our notation is not so powerful after all! It cannot express VDW!

We extend our notation. We want $(\forall k)[PVDW(k)]$.

We use PVDW($\omega$).

**Example**

PVDW(7, $\omega$, 12) means $(\forall k)[PVDW(7, k, 12)]$.

We rewrite that as

$(\forall k_3 \leq 7)(\forall k_2 \leq \omega)(\forall k_1 \leq 12)[PVDW(k_1, k_2, k_3)]$.

(When we write $k_2 \leq \omega$ we also mean $k_2 \in \mathbb{N}$ so it can’t be $\omega$.)

Notation

Let $\mathbb{N}^+ = \mathbb{N} \cup \{\omega\}$.

Let $n_d, \ldots, n_1 \in \mathbb{N}^+$.

PVDW($n_d$, $\omega$, $n_1$) means

$(\forall k_d \leq n_d) \cdots (\forall k_1 \leq n_1)[PVDW(k_d, \ldots, k_1)]$. 
Can we Express VDW in our Powerful Notation?

PVDW(4) would include PVDW(x, 2x, 3x, 4x) which is 
(∀c)[VDW(5, c)].

Our notation is not so powerful after all! It cannot express VDW!

We extend our notation. We want (∀k)[PVDW(k)].
We use PVDW(ω).

Example
PVDW(7, ω, 12) means (∀k)[PVDW(7, k, 12)].
We rewrite that as
(∀k₃ ≤ 7)(∀k₂ ≤ ω)(∀k₁ ≤ 12)[PVDW(k₁, k₂, k₃)].
(When we write k₂ ≤ ω we also mean k₂ ∈ N so it can’t be ω.

Notation Let N⁺ be N ∪ {ω}.
Let nₐ, . . . , n₁ ∈ N⁺.
PVDW(nₐ, ω, n₁) means
(∀kₐ ≤ nₐ) ⋯ (∀k₁ ≤ n₁)[PVDW(kₐ, . . . , k₁)].
What Did We Prove?

Our proof of $PVDW(x^2)$ has all the ideas to prove $PVDW(\omega) \implies PVDW(1, 0)$. 
What Did We Prove?

Our proof of $PVDW(x^2)$ has all the ideas to prove $PVDW(\omega) \implies PVDW(1,0)$.

Our proof of $PVDW(x, x^2)$ has all the ideas to prove $PVDW(1,0) \implies PVDW(1,1)$. 
Poly VDW thm proven by ind on the indexes of sets. Ordering:

\[ (1) \preceq (2) \preceq \cdots \preceq (\omega) \preceq (1, 0) \preceq (1, 1) \preceq \cdots \preceq (1, \omega) \]

\[ \preceq (2, 0) \preceq (2, 1) \preceq \cdots (2, \omega) \cdots \preceq (1, 0, 0) \preceq \cdots \cdots \]

This is an $\omega^\omega$ ind. **Contrast** VDW was a $\omega^2$ ind.

We do this in two parts.
Actual Proof of Poly VDW Theorem

Poly VDW thm proven by ind on the indexes of sets. Ordering:

\[(1) \prec (2) \prec \cdots \prec (\omega) \prec (1, 0) \prec (1, 1) \prec \cdots \prec (1, \omega)\]

\[\prec (2, 0) \prec (2, 1) \prec \cdots (2, \omega) \cdots \prec (1, 0, 0) \prec \cdots \cdots\]

This is an \(\omega^\omega\) ind. **Contrast** VDW was a \(\omega^2\) ind.

We do this in two parts.

1. Let \(0 \leq i \leq d\). Let \(n_d, \ldots, n_i \in \mathbb{N}^+\) with \(n_i \in \mathbb{N}\).

\[
PVDW(n_d, \ldots, n_i, \omega, \ldots, \omega) \implies PVDW(n_d, \ldots, n_i + 1, \omega, \ldots, \omega).
\]
Actual Proof of Poly VDW Theorem

Poly VDW thm proven by ind on the indexes of sets. Ordering:

(1) ≺ (2) ≺ · · · ≺ (ω) ≺ (1, 0) ≺ (1, 1) ≺ · · · ≺ (1, ω)
≺ (2, 0) ≺ (2, 1) ≺ · · · (2, ω) · · · ≺ (1, 0, 0) ≺ · · ·

This is an ωω ind. Contrast VDW was a ω2 ind.
We do this in two parts.

1. Let 0 ≤ i ≤ d. Let nd, . . . , ni ∈ N+ with ni ∈ N.

   PVDW(nd, . . . , ni, ω, . . . , ω) ⇒ PVDW(nd, . . . , ni+1, ω, . . . , ω).

2. PVDW(ω, . . . , ω) ⇒ PVDW(1, 0, . . . , 0).
   d ω’s in the 1st part; d 0’s in the 2nd part.
Bounds on Poly VDW Numbers

1. The bounds given by this proof are not primitive recursive.
2. The bounds given by this proof are bigger than those for VDW's Theorem.
   The Prim Rec hierarchy had functions of levels 1, 2, 3, ...
   The bounds from proof of VDW theorem are at level $\omega$.
   The bounds from proof of POLVDW theorem are at level $\omega^\omega$.
3. Are better bounds known? See next slide.
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   The Prim Rec hierarchy had functions of levels 1, 2, 3, . . . .
   The bounds from proof of VDW theorem are at level $\omega^2$.
   The bounds from proof of POLVDW theorem are at level $\omega^\omega$.
3. Are better bounds known? See next slide.
A False Prediction

In 1999 there were two thoughts in the air

1. $\text{PVDW}(\vec{n})$ is not prim rec and a logician will prove this deep result. Perhaps like the Large Ramsey Numbers (1977) though not that big.

2. $\text{PVDW}(\vec{n})$ is surely prim rec and a combinatorist will prove this perhaps with a clever elementary technique.

The above dichotomy is false. The Poly VDW theorem is just not that well known, even now. So there were no thoughts in the air. (More on that Later.)

Even so, are there better bounds? VOTE: BETTER BOUNDS KNOWN, BETTER BOUNDS UNKNOWN.

Logician (Shelah) proved $\text{PVDW}(\vec{n})$ prim rec: clever!

▶ Proof is elementary. Can be in this class but won't.

▶ Bounds still large. Not able to write down.

▶ Proof badly needs someone to write it up better.

▶ Bill—remember to tell them how you learned of Shelah's result.
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Looking Back to VDW Theorem

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\[ \text{PVDW}(\omega) \implies \text{PVDW}(1, 0). \]
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So we can obtain a proof of VDW that you can write down nicely.

1. The proof really is the proof I already showed you.
2. While one COULD obtain a clean proof of VDW nobody has bothered writing this up (except me).