VDW’s Theorem

**Definition** Let $W, k, c \in \mathbb{N}$. Let $\text{COL}: [W] \to [c]$. A **mono $k$-AP** is an arithmetic progression of length $k$ where every element has the same color. We often say

$$a, a + d, \ldots, a + (k - 1)d$$

are all the same color.
**VDW’s Theorem**

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**VDW’s Theorem** For all $k, c$ there exists $W = W(k, c)$ such that for all $\text{COL} : [W] \rightarrow [c]$ there exists a mono $k$-AP.
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$W(1, c) =$
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$W(1, c) = 1$. 

A mono $1$-AP is just $1$ number.

$W(2, c) = c + 1$. 

By Pigeon Hole Principle.

$W(3, 2)$ = Hmmm, this is the first non-trivial one.
VDW Easy Cases

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$W(3, 2) = \text{Hmmm, this is the first non-trivial one.}$
$W(3, 2)$ exists

We will determine $W$ later.
W(3, 2) exists

We will determine W later. Let \( \text{COL}: [W] \rightarrow [2] \).

We break \([W]\) into blocks of 5: \(B_1, \ldots, B_{\lfloor W \rfloor / 5}\).
$W(3, 2)$ exists

We will determine $W$ later.

We break $[W]$ into blocks of 5: $B_1, \ldots, B_{|W|/5}$.

**We view the 2-coloring of $[W]$ as a $2^5$-coloring of the $B_i$’s**
$W(3, 2)$ exists

We will determine $W$ later.

We break $[W]$ into blocks of 5: $B_1, \ldots, B_{|W|/5}$.

**We view the 2-coloring of $[W]$ as a $2^5$-coloring of the $B_i$’s**

We take enough blocks so that

- Two of the blocks are the same color, say $B_i$ and $B_j$.
- If $B_i$ and $B_j$ are the same color then there exists $B_k$ such that $B_i, B_j, B_k$ are a 3-AP.

If there are 33 blocks then 2 are the same color. So need $B_{65}$ to exist.
\(W(3, 2)\) exists

We will determine \(W\) later.
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If there are 33 blocks then 2 are the same color.

**Worst Case Scenario** $B_1$ and $B_{33}$ same color. So need $B_{65}$ to exist.
Side Note: Can Get By With Less Blocks

Warning This Slide is NOT important.
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If a block is colored **RRRBB**, we are done.

So we don’t really have to look at 32 colorings.
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If a block is colored RRRBB we are done.

So we don’t really have to look at 32 colorings.

How many colorings of a block already have a mono 3-AP.
RRRXY with $X, Y \in \{R, B\}$. 4 colorings.
BBBXY with $X, Y \in \{R, B\}$. 4 colorings.
RBRRR
RBRBR
BRBBB
BRBBB
RBBBX with $X \in \{R, B\}$. 2 colorings.
BRRRX with $X \in \{R, B\}$. 2 colorings.
RRBBB
BBRRR
Side Note: Can Get By With Less Blocks (cont)

RRRXY with $X, Y \in \{R, B\}$. 4 colorings.
BBBXY with $X, Y \in \{R, B\}$. 4 colorings.
RBRRR
RBRBR
BRBBB
BRBRB
RBBB
RBBBX with $X \in \{R, B\}$. 2 colorings.
BRRRX with $X \in \{R, B\}$. 2 colorings.
RRBBB
BBRRR

I have 16 blocks which already have a mono 3-AP. I might have missed some. but if not then can replace 32 with 18.
Side Note: Can Get By With Less Blocks (cont)

**RRR**XY with $X, Y \in \{R, B\}$. 4 colorings.

**BBB**XY with $X, Y \in \{R, B\}$. 4 colorings.

**RBRRR**

**RBRBR**

**BRBBB**

**BRBRB**

**RBBBB**X with $X \in \{R, B\}$. 2 colorings.

**BRRRX** with $X \in \{R, B\}$. 2 colorings.

**RRBBB**

**BBRRR**

I have 16 blocks which already have a mono 3-AP. I might have missed some. but if not then can replace 32 with 18.

I really do not care.
Back to $W(3, 2)$


Break $[W]$ into 65 blocks of size 5.

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- Exists $i,j,k$ such that $B_i, B_j$ same color and $B_k$ such that $B_i, B_j, B_k$ is 3-AP exists.

Break $[W]$ into 65 blocks of size 5.

- Exists $i, j, k$ such that $B_i, B_j$ same color and $B_k$ such that $B_i, B_j, B_k$ is 3-AP exists.

- In every block there exists $x, y$ same color and $z$ such that $x, y, z$ are 3-AP in same block. (This is why blocks-of-5.)
Let \( \text{COL} : [W] \rightarrow [2] \).

Break \([W]\) into 65 blocks of size 5.

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Go to White Board to finish proof.
$W(3, 2)$ Really

We got

$$W(3, 2) \leq 5 \times (2 \times 32 + 1) = 365.$$
\( W(3, 2) \) Really

We got

\[ W(3, 2) \leq 5 \times (2 \times 32 + 1) = 365. \]

If use that 18 of the block colors already get you a 3-AP then

\[ W(3, 2) \leq 5 \times (2 \times 14 + 1) = 145. \]
$W(3, 2)$ Really

We got

$$W(3, 2) \leq 5 \times (2 \times 32 + 1) = 365.$$  

If use that 18 of the block colors already get you a 3-AP then

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What is $W(3, 2)$?
We got

\[ W(3, 2) \leq 5 \times (2 \times 32 + 1) = 365. \]

If use that 18 of the block colors already get you a 3-AP then

\[ W(3, 2) \leq 5 \times (2 \times 14 + 1) = 145. \]

What is \( W(3, 2) \)?

One can work out by hand that

\[ W(3, 2) = 9. \]
$W(3, 3)$


Go to White Board to finish the proof.
\( W(3, 3) \)

**COL:** \([W] \rightarrow [3] \).

How big should the blocks be?
$W(3, 3)$


How big should the blocks be? 7.

Then $(\exists x, y)$ same color with $z$ such that $x, y, z$ is 3-AP all in a block.

We view the 3-coloring of $[W]$ as a 3-coloring of the $B_i$'s.

Need blocks so $B_i, B_j$ same color, $B_i, B_j, B_k$ 3-AP, $B_k$ exists.

Go to White Board to finish the proof.
$W(3, 3)$


How big should the blocks be? 7.

Then $(\exists x, y)$ same color with $z$ such that $x, y, z$ is 3-AP all in a block.

We view the 3-coloring of $[W]$ as a $3^7$-coloring of the $B_i$'s
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How big should the blocks be? 7.
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$2 \times (3^7 + 1)$
$W(3, 3)$


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Go to White Board to finish the proof.
From what you have seen:

\[ W(3, c) \]
From what you have seen:

▶ You COULD do a proof that $W(3, 4)$ exists. You would need to iterate what I did twice.
From what you have seen:

- You **COULD** do a proof that $W(3, 4)$ exists. You would need to iterate what I did twice.
- You can **BELIEVE** that $W(3, c)$ exists though might wonder how to prove it formally.
From what you have seen:

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- There are ways to formalize the proof; however, they are not enlightening.
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▶ There are ways to formalize the proof; however, they are not enlightening.
▶ The Hales-Jewitt Theorem is a general theorem from which VDW is a corollary. We won’t be doing that.
What Did We Use to Prove $W(3, c)$?

$W(2, c) = c + 1$ is just PHP.
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$W(2, c) = c + 1$ is just PHP.

$W(2, 2^5) \Rightarrow W(3, 2)$

$W(2, 3^{2 \times 3^7} + 1) \Rightarrow W(3, 3)$.

$W(2, X) \Rightarrow W(3, 4)$ where $X$ is an Issac-number.
What Did We Use to Prove $W(3, c)$?

$W(2, c) = c + 1$ is just PHP.

$W(2, 2^5) \implies W(3, 2)$

$W(2, 3^{2\times3^7} + 1) \implies W(3, 3)$.

$W(2, X) \implies W(3, 4)$ where $X$ is an Issac-number.

Note that we do not do

$W(3, 2) \implies W(3, 3)$. 
$W(4, 2)$


Go to White Board to finish proof.
$W(4, 2)$


**Key** Take blocks of size $2W(3, 2)$.  
Within a block there will be mono 3-AP and fourth elt exists.
$W(4, 2)$


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Go to White Board to finish proof.
\( W(4, 2) \)

**COL:** \([ W ] \rightarrow [3].\)

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Key Take blocks of size $2W(3, 2)$. Within a block there will be mono 3-AP and fourth elt exists.

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How many blocks?
We want to get a mono 3-AP of blocks and room for a fourth.
$W(4, 2)$


**Key** Take blocks of size $2W(3, 2)$. Within a block there will be mono 3-AP and fourth elt exists.

**Key** Take blocks of size $2W(3, 2)$.

How many blocks?
We want to get a mono 3-AP of blocks and room for a fourth. $W(3, 2^{2W(3, 2)})$. 

Go to White Board to finish proof.
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Induction, But On What?

(2, 2) ≺ (2, 3) ≺ \cdots ≺ (3, 2) ≺ (3, 3) ≺ \cdots ≺ (4, 2) \cdots
Induction, But On What?

\[(2, 2) \prec (2, 3) \prec \cdots \prec (3, 2) \prec (3, 3) \prec \cdots \prec (4, 2) \cdots\]

This is an $\omega^2$ induction. The ordering is well-founded so it works.
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This is an $\omega^2$ induction. That's why the numbers are so large.
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How large?
Induction, But On What?

This is an $\omega^2$ induction. The ordering is well-founded so it works.

This is an $\omega^2$ induction. That's why the numbers are so large.

How large? The bounds are not primitive recursive.
A False Prediction

In 1983 there were two thoughts in the air

1. \((W(k,c))\) is not prime recursive and a logician will prove this deep result. Perhaps like the Large Ramsey Numbers (1977) though not that big.

2. \((W(k,c))\) is surely prime recursive and a combinatorist will prove this perhaps with a clever elementary technique.

So what happened? Logician (Shelah) proved \((W(k,c))\) prime recursive: clever!

- Proof is elementary. Can be in this class but won't.
- Bounds still large. Not able to write down.
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Deep Math From Search for Better Upper Bounds on VDW Numbers

Exposition by William Gasarch

April 21, 2022
A Man, A Plan, A Canal: Panama!

Well, a plan anyway. We outline a plan for getting better upper bounds on $W(k,c)$. On the one hand, it lead to very deep mathematics. On the other hand, It DID succeed! (Oh! Thats a good thing!)
A Man, A Plan, A Canal: Panama!

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A Man, A Plan, A Canal: Panama!

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Upper Density

**Definition** Let $A \subseteq \mathbb{N}$ The **upper density of $A$** is

$$\limsup_{n \to \infty} \frac{|A \cap [n]|}{n}$$
**Upper Density**

**Definition** Let $A \subseteq \mathbb{N}$ The upper density of $A$ is

$$\limsup_{n \to \infty} \frac{|A \cap [n]|}{n}$$

**Definition** Positive upper density means that the upper density is $> 0$. 

Examples

1. For all $k$, $\{x : x \equiv 0 \pmod{k}\}$ has upper den $1/k$.

2. $\{x^2 : x \in \mathbb{N}\}$ has upper den 0.
Definition Let $A \subseteq \mathbb{N}$ The upper density of $A$ is

$$\limsup_{n \to \infty} \frac{|A \cap [n]|}{n}$$

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Examples

1. For all $k$, $\{x : x \equiv 0 \pmod{k}\}$ has upper den $\frac{1}{k}$. 
Definition Let $A \subseteq \mathbb{N}$ The upper density of $A$ is

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Definition Positive upper density means that the upper density is $> 0$.

Examples

1. For all $k$, $\{x : x \equiv 0 \pmod{k}\}$ has upper den $\frac{1}{k}$.
2. $\{x^2 : x \in \mathbb{N}\}$ has upper den $0$. 
A Conjecture, 1936

**Conjecture** If $A \subseteq \mathbb{N}$ has positive upper density then, for all $k$, $A$ has a $k$-AP.
**A Conjecture, 1936**

**Conjecture** If $A \subseteq \mathbb{N}$ has positive upper density then, for all $k$, $A$ has a $k$-AP.

**Theorem** Conj implies VDW’s Theorem. HW or Final.
A Conjecture, 1936

**Conjecture** If $A \subseteq \mathbb{N}$ has positive upper density then, for all $k$, $A$ has a $k$-AP.

**Theorem** Conj implies VDW’s Theorem. HW or Final.

The hope was that the proof of Conj would require a new proof of VDW’s Theorem that would lead to better bounds.
Roth’s Theorem, 1952

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- Roth won the Fields Medal in 1958 for his work on Diophantine approximation (so not for this work).
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- Proof is elementary but strains the use of the word *elementary*. 

The theorem is known as **Szemeredi’s Theorem**. 

Szemeredi should have won Fields Medal ($15,000) but did not since combinatorics was not seen as deep math.

Szemeredi won the Abel Prize ($700,000) in 2012 for his work in combinatorics.

So there!

What is better financially: Fields Medal when you are 40 or Abel prize when you are 70?

Fields Medal can lead to better jobs and pay while you are still young. 

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Gowers won the Fields Medal ($15,000) in 1998 for this work.

Why did Gowers win the Fields Medal but not Szemeredi?

Gowers work used traditional deep math. Szemeredi's used new deep math that was not appreciated.

Combinatorics was less respected in 1975 than in 1998.

Causes of change: (1) combinatorics using deep math, (2) CS inspired new problems in combinatorics.
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None of these results used mathematics of interest.
Known Lower Bounds

1. Easy Use of Prob Method (was on HW) $W(k, 2) \geq \sqrt{k}2^{k/2}$ (Easy extension to 3 colors)

2. Very sophisticated use yields $W(k, 2) \geq \frac{2^k}{k^c}$ (Does not extend to 3 colors.)

3. If $p$ is prime then $W(p, 2) \geq p(2^p - 1)$. Constructive! (Does not extend to 3 colors.)
The Green-Tao Theorem

Green-Tao proved the following in 2004.

Theorem
For all \( k \) there is a \( k \)-AP of primes.

▶ Does not follow from Sz Thm, primes do have upper density 0.

▶ Tao won the Field’s Medal ($15,000) in 2006, a MacArthur Genius award ($500,000) in 2006, and a Breakthrough Prize ($3,000,000 but not as much prestige) in 2014.

▶ Green won the ConservaMath Medal ($0) in 2006.

The ConservaMath Medal is a merit-based alternative to the Field’s Medal. Deserving recipients should solve a real longstanding problem, rather than an invented problem.

Green earned this award in 2006 for the Green-Tao Thm to dim the star of Obama-supporter Tao, making Tao less effectively politically

▶ There is also a ConservaMedical Medal—an alternative to the Nobel Prize in Medicine. It went to Donald Trump for his Medical Advice on Coronavirus.

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