Van Der Warden's (VDW) Theorem

Exposition by William Gasarch

April 21, 2022

Definition Let $W, k, c \in \mathbb{N}$. Let COL: $[W] \rightarrow [c]$. A mono k-AP is an arithmetic progression of length k where every elements has the same color. We often say

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VDW's Theorem For all k, c there exists W = W(k, c) such that for all COL: $[W] \rightarrow [c]$ there exists a mono k-AP.

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If there are 33 blocks then 2 are the same color.

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Worst Case Scenario B_1 and B_{33} same color. So need B_{65} to exist.

Side Note: Can Get By With Less Blocks

Warning This Slide is NOT important.



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How many colorings of a block already have a mono 3-AP.

Side Note: Can Get By With Less Blocks (cont)

```
RRRXY with X, Y \in \{R, B\}. 4 colorings.

BBBXY with X, Y \in \{R, B\}. 4 colorings.

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I have 16 blocks which already have a mono 3-AP. I might have missed some. but if not then can replace 32 with 18.

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Let $COL \colon [W] \to [2]$.



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Go to White Board to finish proof.

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We got

$$W(3,2) \le 5 \times (2 \times 32 + 1) = 365.$$

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What is W(3,2)?

One can work out by hand that

W(3,2) = 9.

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What Did We Use to Prove W(3, c)?

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$$W(2,c) = c + 1$$
 is just PHP.

What Did We Use to Prove W(3, c)?

$$\begin{split} & \mathcal{W}(2,c) = c+1 \text{ is just PHP.} \\ & \mathcal{W}(2,2^5) \implies \mathcal{W}(3,2) \\ & \mathcal{W}(2,3^{2\times 3^7}+1) \implies \mathcal{W}(3,3). \\ & \mathcal{W}(2,X) \implies \mathcal{W}(3,4) \text{ where } X \text{ is an Issac-number.} \end{split}$$

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Note that we **do not** do $W(3,2) \implies W(3,3).$

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Key Take blocks of size 2W(3,2). Within a block there will be mono 3-AP and fourth elt exists.

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Induction, But On What?

$(2,2) \prec (2,3) \prec \cdots \prec (3,2) \prec (3,3) \prec \cdots \prec (4,2) \cdots$

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This is an ω^2 induction. The ordering is well-founded so it works.

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In 1983 there were two thoughts in the air

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So what happened?

Logician (Shelah) proved W(k, c) prim rec: clever!

- Proof is elementary. Can be in a this class but won't.
- Bounds still large. Not able to write down.

Deep Math From Search for Better Upper Bounds on VDW Numbers

Exposition by William Gasarch

April 21, 2022

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It DID succeed! (Oh! Thats a good thing!)

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$$\limsup_{n\to\infty}\frac{|A\cap[n]|}{n}$$

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Examples

- 1. For all k, $\{x : x \equiv 0 \pmod{k}\}$ has upper den $\frac{1}{k}$.
- 2. $\{x^2 : x \in \mathbb{N}\}$ has upper den 0.

A Conjecture, 1936

Conjecture If $A \subseteq \mathbb{N}$ has positive upper density then, for all k, A has a k-AP.



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The hope was that the proof of Conj would require a new proof of VDW's Theorem that would lead to better bounds.

Roth's Theorem, 1952

Theorem If $A \subseteq \mathbb{N}$ has positive upper density then A has a 3-AP.

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- The proof used Fourier Analysis so not elementary
- Roth won the Fields Medal in 1958 for his work on Diophantine approximation (so not for this work).

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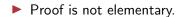
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 - Causes of change: (1) combinatorics using deep math, (2) CS inspired new problems in combinatorics.

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None of these results used mathematics of interest.

Known Lower Bounds

- 1. Easy Use of Prob Method (was on HW) $W(k,2) \ge \sqrt{k}2^{k/2}$ (Easy extension to 3 colors)
- 2. Very sophisticated use yields $W(k,2) \ge \frac{2^k}{k^{\epsilon}}$ (Does not extend to 3 colors.)
- 3. If p is prime then $W(p,2) \ge p(2^p 1)$. Constructive! (Does not extend to 3 colors.)

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