Exposition by William Gasarch-U of MD

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 $L \subseteq \{a, b\}^*$ is often called **a language**.

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 $SUBSEQ(aaba) = \{e, a, b, aa, ab, ba, aaa, aab, aba, aaba\}.$

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Question L decidable \implies SUBSEQ(L) decidable?

Def (X, \preceq) is a **Quasi Order** if

- ▶ If $x \leq y$ and $y \leq z$ then $x \leq z$ (transitive).
- ▶ For all $x \in X$, $x \leq x$ (reflexive).

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Most wqo are also partial order, but NOT the one on the HW which caused this hot mess.

Def (X, \leq) is a **Well Quasi Order (wqo)** if $(X \leq)$ is a quasi order AND the following holds:

For all infinite sequences $x_1, x_2, ...$

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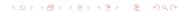
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Use Def One when want to prove (X, \leq) is a wqo.

Use Def Two when you already know (X, \preceq) is a wgo.

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Discuss Prove this is a wqo.

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Very Hard Theorem (We Won't Prove it)

Def H is a **minor** of G (Denoted by $H \leq_m G$) if one can obtain H by taking G and carrying out the following operations in some order:

- 1) Remove a vertex (and all of the edges from it).
- 2) Remove an edge.
- 3) Contract an Edge (so merge vertices at ends).

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We use (\mathcal{G}, \leq_m) as an example of a wqo in the next few slides.



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2) O is an Obstruction Set for Y if

$$(\forall x \notin Y)(\exists o \in O)[o \leq_m x].$$

(Obstruction set for Planar graphs is $\{K_{3,3}, K_5\}$.)

Thm Let (X, \preceq) be a wqo. Let $Y \subseteq X$ be closed downward. Then there exists a **Finite Obstruction Set** for Y.

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1) O is Obstruction: If $z_1 \in X - Y$ then either $z_1 \in O$ (DONE) or $z_1 \notin O$, so there exists $z_2 \in X - Y$ with $z_2 \prec z_1$. Repeat process with z_2 . end up with

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Has to stop or else have infinite descending sequence. Ends at an element of O.



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2) *O* is finite: All elements of *O* are incomparable to each other. If *O* was infinite then would have an infinite antichain.

