**CMSC 752 Homework 4**
**Morally Due Tue Feb 25, 2025**
**Dead Cat Feb 27**

**Notation** $R_a(k)$ is the $a$-ary Ramsey number.

1. (30 points) RECALL:
   We used $R_2(k) \le 2^{2k}$ to obtain $R_3(k) \le 2^{2^{4k}}$.

   What if Arushi, Danesh, and Sharma show $R_2(k) \le 2^{1.9k}$?

   What if Soren and Rishi show $R_2(k) \le 2^{1.7k}$?

   What if Larry, Moe, and Curly show $R_2(k) \le 2^{1.5n}$?

   How will these new bounds on $R_2(k)$ help get better bounds on $R_3(k)$? That is the point of this problem.

   Find a function $f$ that has the following properties (and prove them):

   - $f(2) = 4$
   - If $x < 2$ then $f(x) < 4$.
   - *IF* $R_2(k) \le 2^{ck}$ *THEN* $R_3(k) \le 2^{2^{f(c)k}}$.
     (Note: You may get an upper bound like (I am making this up) $2^{2^{3n+7}}$. Ignore the additive contant 7.)

     I am asking you to reprove the finite 3-hypergraph Ramsey Theorem with the parameter $c$ so that you can see how an improvement on the upper bound for $R_2(k)$ will result in an improvement on the upper bound for $R_3(k)$.

2. (30 points) Find a function $f$ that has the following properties (and prove them):

   *IF* $R_3(k) \le 2^{2^{ck}}$ *THEN* $R_4(k) \le f(c,k)$

   I am asking you to prove the finite 4-hypergraph Ramsey Theorem with the parameter $c$ so that you can

   (a) see what the upper bound on $R_4(k)$ was when the upper bound on $R_3(k)$ was $2^{2^{4k}}$.

   (b) see what the upper bound on $R_4(k)$ was now that the upper bound on $R_3(k)$ was $2^{2^{2k}}$.

   (c) be prepared for the future when Danesh-1, Danesh-2, and Danesh-3 team up to show $R_3(k) \le 2^{2^{1.99k}}$.

3. (40 points)

(a) (40 points) Prove the following: For all $n \ge 2$, for all functions

$$f \colon \mathsf{Z}^n \to \mathsf{Z}$$

there exists infinite $\mathsf{D} \subseteq \mathsf{Z}$ such that

$$f \colon \mathsf{D}^n \to \mathsf{Z}$$

is NOT onto.

(b) (0 points) THINK ABOUT: Let $T(n)$ be the number of times you used Ramsey's Theorem in the proof of Part 1. What was your $T(n)$?

4. (0 points, Extra Credit)

   (a) Give your name (this will not get you any extra credit, but since I grade this one by putting the names of who got it right into a file, this makes my life easier.)

   (b) Recall that a coloring of the edges of $K_{n,n}$ (the complete bipartite graph) is a coloring of $[n] \times [n]$.

   Prove the following

   *For all $k$ there exists $n = B(k)$ such that for all*

$$\mathrm{COL}\colon [n] \times [n] \rightarrow [2]$$

   *there exists $A \subseteq [n]$ and $B \subseteq [n]$ such that $|A| = |B| = k$ and* COL *restricted to $A \times B$ is constant. Provide bounds on $B(k)$.*