CMSC 752 Homework 13
Morally Due Tue May 6, 2025
Dead Cat May 8

1. (50 points) For this problem you can assume the Gallai-Witt Theorem in two dimensions which we state here for your convinence.

   For all k, for all c, there exists $GW = GW(k, c)$ such that for all

   $$\mathrm{COL}\colon [GW] \times [GW] \rightarrow [c]$$

   there exists a monochromatic $k \times k$ equally spaced grid. More preciesly there exists $a, b, d)$ such that the following are all the same color:

   $$\{(a + id, b + jd)\colon 0 \le i, j \le k - 1\}.$$

   Prove the following which is the Can VDW theorem.

   For all k there exists $C = C(k)$ such that, for all $\mathrm{COL}\colon [C] \rightarrow [\omega]$ one of the two occurs

   - *There exists a, d such that $a, a + d, \ldots, a + (k - 1)d$ are the same color.*
   - *There exists a, d such that $a, a+d, \ldots, a+(k-1)d$ are all different colors.*

   Hint given $\mathrm{COL}\colon [W] \rightarrow [\omega]$ form

   $\mathrm{COL}'\colon [W'] \times [W'] \rightarrow [c]$ (W' and c' to be deterined by you) as follows:

   $\mathrm{COL}'(a, d)$ is determined by looking at

   $X = \{\mathrm{COL}(a), \mathrm{COL}(a + d), \ldots, \mathrm{COL}(a + kd)\}$

   and outputing the $k + 1$-tuple

   Number of times $\mathrm{COL}(a)$ appears in X.

   Number of times $\mathrm{COL}(a + d)$ appears in X.

   $\vdots \quad \vdots \quad \vdots$

Number of times $\mathrm{COL}(a + kd)$ appears in X.

For example, if the

$\mathrm{COL}(a) = R$

$\mathrm{COL}(a + d) = R$

$\mathrm{COL}(a + 2d) = B$

$\mathrm{COL}(a + 3d) = Y$

$\mathrm{COL}(a + 4d) = R$

$\mathrm{COL}(a + 5d) = B$

Then

$\mathrm{COL}'(a, d) = (3, 3, 2, 1, 3, 2)$

2. (50 points) Use the Prob Method to get a lower bound on $W(k, c)$. (This DOES NOT use the fancy Lovasz Local Lemma.)