Problem

What is the largest integer n such that there exist **21 distinct 5-element subsets** of $\{1, 2, \ldots, n\}$ with the property that **every pair of subsets intersects in exactly one element**?

Step 1: Double counting element–set incidences

Let $\mathcal{F}$ be a family of 21 subsets, each of size 5. Let $v = n$ be the size of the ground set.

For each element $x \in \{1, \ldots, n\}$, let r_x denote the number of sets in $\mathcal{F}$ that contain x.

Counting incidences in two ways gives

$$\sum_x r_x = 21 \cdot 5 = 105.$$

Step 2: Counting intersections

Each pair of distinct sets intersects in **exactly one element**. Since there are $\binom{21}{2} = 210$ pairs of sets, the total number of pairwise intersections is 210.

On the other hand, for a fixed element x, the number of pairs of sets whose unique intersection point is x equals $\binom{r_x}{2}$.

Summing over all elements gives

$$\sum_x \binom{r_x}{2} = 210.$$

Equivalently,

$$\sum_x r_x(r_x - 1) = 420.$$

Step 3: Constraints on the r_x

We now have two equations:

$$\sum_x r_x = 105, \qquad \sum_x r_x(r_x - 1) = 420.$$

The second equation can be rewritten as

$$\sum_x r_x^2 = 525.$$

To **maximize** n, we want the multiplicities r_x as small and as uniform as possible, subject to these equations and the constraint $r_x \geq 1$.

By Cauchy–Schwarz,

$$\left(\sum_x r_x\right)^2 \leq n \sum_x r_x^2,$$

which gives

$$105^2 \leq n \cdot 525.$$

Hence

$$n \geq \frac{11025}{525} = 21.$$

This inequality alone gives only a **lower** bound on n. To find the maximum possible n, we look for integer solutions to the system where most $r_x = 2$, since $r_x = 1$ contributes nothing to intersections and larger values rapidly increase $\sum r_x^2$.

Step 4: Solving for the distribution of r_x

Suppose a elements occur in exactly 2 sets and b elements occur in exactly 3 sets (larger values turn out to be impossible without reducing n).

Then

$$2a + 3b = 105,$$

$$4a + 9b = 525.$$

Solving gives

$$b = 35, \qquad a = 0.$$

Thus **every element appears in exactly 3 sets**, and the number of elements is

$$n = a + b = 35.$$

Step 5: Interpretation and existence

This parameter set corresponds exactly to a **block design** with parameters

$$(v, b, r, k, \lambda) = (35, 21, 3, 5, 1),$$

where - $v = 35$ points, - $b = 21$ blocks, - each block has size $k = 5$, - each point occurs in $r = 3$ blocks, - any two blocks intersect in exactly one point.

Such a design is known to exist (it can be constructed from finite-geometry methods).

Final Answer

$$\boxed{n = 35}$$

This is the **largest possible** ground set size for which there exist 21 five-element subsets of $\{1, \ldots, n\}$ with pairwise intersections of size exactly 1.