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BILL RECORD LECTURE!!!



Euclidean Ramsey Theory Chromatic Number of the Plane

Exposition by William Gasarch

April 29, 2025

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B) We do not care about the geometric size. For example, the Square can be any size.

In Euclidean Ramsey Theory we will be seek an object of a certain size, for example the unit square.

Thm \forall COL: $\mathbb{R}^2 \rightarrow [2] \exists 2$ points, same color, 1 inch apart.

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Discuss Try to proof it, what are your thoughts.



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Proof on the next page.

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Look at an equilateral triangle in the plane

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Look at an equilateral triangle in the plane

3 vertices and 2 colors. So 2 of the vertices are the same color.

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Vertices 1 and 2 are an inch apart.

Consider the graph G = (V, E) where

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Consider the graph G = (V, E) where $V = \mathbb{R}^2$ $E = \{(x, y) : d(x, y) = 1\}.$

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The Theorem **Thm** \forall COL: $\mathbb{R}^2 \rightarrow [2] \exists 2$ points, same color, 1 inch apart. Can be rephrased as

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1) \forall COL: $\mathbb{R}^2 \rightarrow$ [3] \exists 2 points, same color, 1 inch apart. (So $\chi \ge 4$)

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3) Unknown to Science!

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 ∀ COL: ℝ² → [3] ∃ 2 points, same color, 1 inch apart. (So χ ≥ 4)
∃ COL: ℝ² → [3] no 2 points, same color, 1 inch apart. (So χ = 3.)
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Ánswer on next slide

Thm \forall COL: $\mathbb{R}^2 \rightarrow [3] \exists 2$ points, same color, 1 inch apart. (So $\chi \geq 4$.)

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 $\operatorname{COL}(1) = \mathbb{R}$, so $\operatorname{COL}(2) \neq \mathbb{R}$ and $\operatorname{COL}(3) \neq \mathbb{R}$.

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Distance from 1 to 4 is $\sqrt{3}$.





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Upshot 1 If p, q are $\sqrt{3}$ apart then COL(p) = COL(q).



Distance from 1 to 4 is $\sqrt{3}$.

Upshot 1 If p, q are $\sqrt{3}$ apart then COL(p) = COL(q). **Upshot 2** If $COL(p) = \mathbb{R}$ then circle of radius $\sqrt{3}$ around p is \mathbb{R} .







Look at a chord of the circle of length 1.





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Recall The proof that $\chi \ge 2$ is to restrict the coloring to a 3-point set, the equilateral triangle.

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Is there a finite subset of the plane that shows $\chi \ge 3$?

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Recap 2-Coloring and 3-Coloring the Plane

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▶ There is a 3-point subset of the plane that is NOT 2-colorable.



Recap 2-Coloring and 3-Coloring the Plane

- There is a 3-point subset of the plane that is NOT 2-colorable.
- ▶ There is a 7-point subset of the plane that is NOT 3-colorable.

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1) \forall COL: $\mathbb{R}^2 \rightarrow$ [4] \exists 2 points, same color, 1 inch apart. (So $\chi \ge 5$.)

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 ∀ COL: ℝ² → [4] ∃ 2 points, same color, 1 inch apart. (So χ ≥ 5.)
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3) Unknown to Science!

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Answer on next slide

Thm (Aubrey de Grey, 2018) \forall COL: $\mathbb{R}^2 \rightarrow [4] \exists 2$ points, same color, 1 inch apart. (So $\chi \geq 5$.)

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Thm (Aubrey de Grey, 2018) \forall COL: $\mathbb{R}^2 \rightarrow [4] \exists 2$ points, same color, 1 inch apart. (So $\chi \ge 5$.) de Grey did this with a construction of a 1581-vertex unit-distance graph that is not 4-colorable.

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As of 2021 this was gotten down to a 509-vertex graph.

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So we know that $\chi \geq 5$.
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1) \forall COL: $\mathbb{R}^2 \rightarrow [5] \exists 2$ points, same color, 1 inch apart. (So $\chi \ge 6$.)

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3) Unknown to Science!

The status of



The status of $\forall \operatorname{COL} \colon \mathbb{R}^2 \to [5] \exists 2 \text{ points, same color, } 1 \text{ inch apart.}$

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The status of $\forall \text{ COL} \colon \mathbb{R}^2 \to [5] \exists 2 \text{ points, same color, } 1 \text{ inch apart.}$ is **Unknown to Science!**

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Upper bound on χ

Thm $\chi \leq 7$.



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Thm $\chi \leq 7$. There is a 7-coloring of the plane, so $5 \leq \chi \leq 7$.

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Upper bound on χ

Thm $\chi \leq 7$. There is a 7-coloring of the plane, so $5 \leq \chi \leq 7$. Here is the 7-coloring: https://thatsmaths.com/2022/03/24/ the-chromatic-number-of-the-plane/