

BILL, RECORD LECTURE!!!!

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Convex Points Thm Known as Happy Ending Thm

Exposition by William Gasarch

Convex Sets And Convex Hulls

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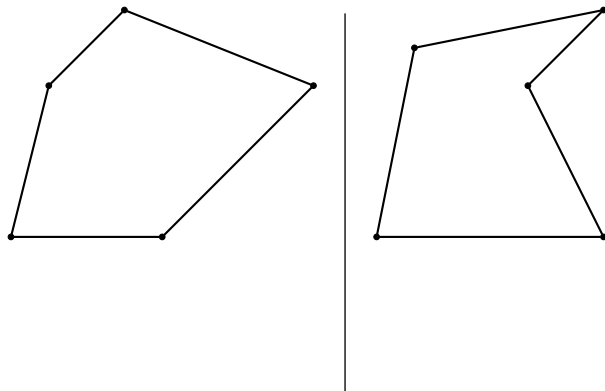
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Convex and Non-Convex Sets on Next Slide.

Convex Set / Non-Convex Set



Left Region is Convex. Right Region is Not Convex.

Definition of A Convex Hull

Def Let $X \subseteq \mathbb{R}^2$ (it will always be a finite set). The **Convex Hull of X** is the smallest convex set that contains all the points in X .

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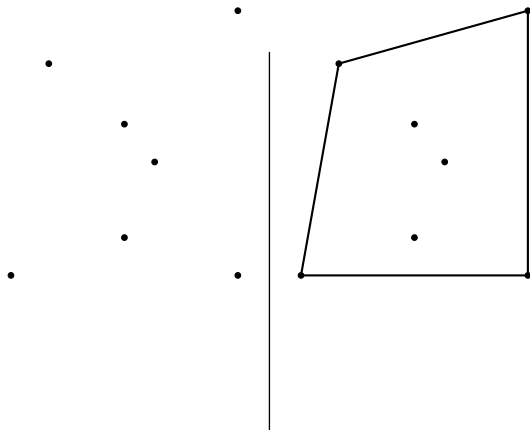
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An Example is on Next Slide.

Size of a Convex Hull

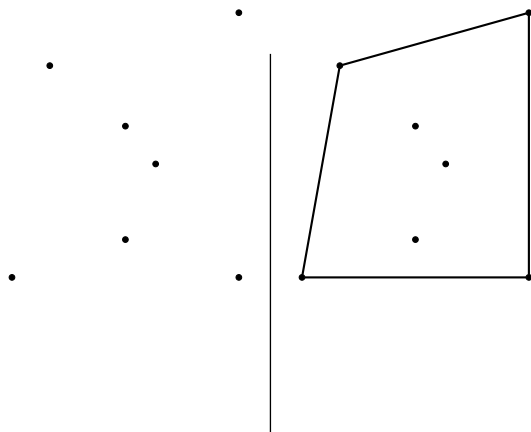
Def The **Size of a Convex Hull** is how many sides it has.

Example of A Convex Hull



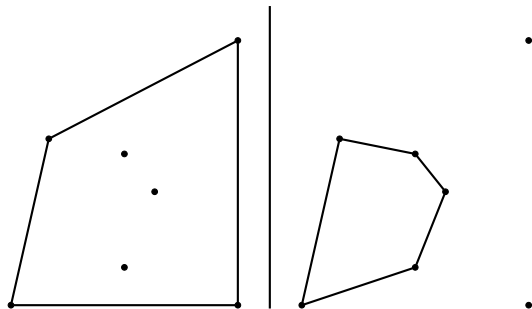
Region In Right Picture is Convex Hull of Points in Left Picture.

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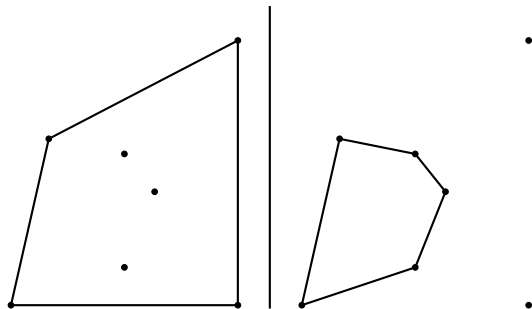


Region In Right Picture is Convex Hull of Points in Left Picture.
RHS is a convex hull of size 4.

Convex Hull Not the Largest Convex Hull

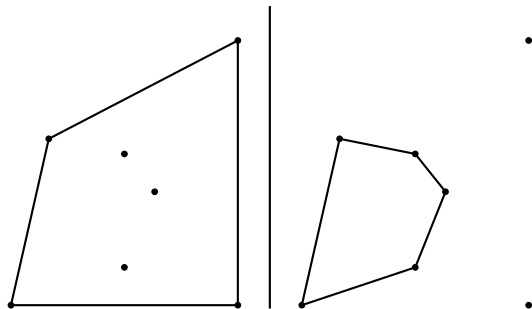


Convex Hull Not the Largest Convex Hull



LHS: 7 points have a convex hull of size 4.

Convex Hull Not the Largest Convex Hull



LHS: 7 points have a convex hull of size 4.

RHS: 5 of those 7 point have a convex hull of size 5.

We Want Large Convex Hulls

Concrete Examples

Given n Points in \mathbb{R}^2 Want Large Convex Hull

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Answer on Next Slides.

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Nothing is known beyond that but note that fits $f(k) = 2^{k-2} + 1$
conjecture.

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$k = 4$ There exists 4 points which are not a convex hull:
A triangle with one point inside.

Hence $f(4) \geq 5$.

The Cases $k = 4$

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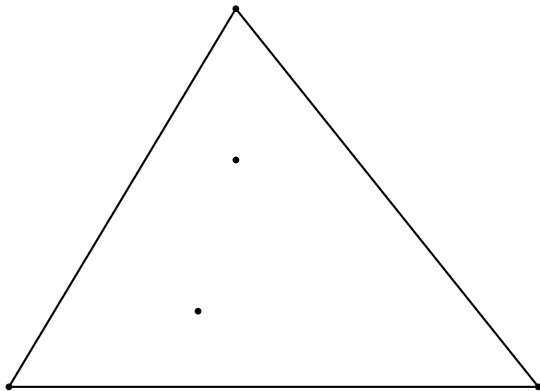
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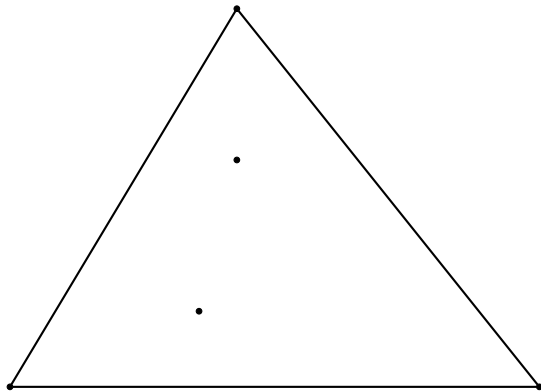
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General proof left to the reader.

Two Points Inside a 3-gon

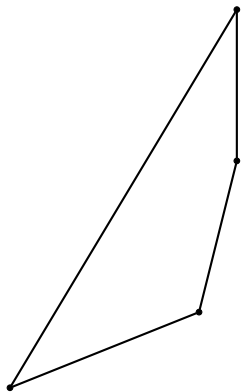


Two Points Inside a 3-gon



See Next Slide for the Amazing 4-Gon!

The Amazing 4-Gon



We Want Large Convex Hulls For General k

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Thm For all k there exists n such that the following holds:

For all sets $X \subseteq \mathbb{R}^2$, $|X| = n$, there exists a subset of k points whose convex hull is a k -gon.

Erdős and Szekeres proved it.

Esther was so impressed that she married Szekeres.

That is why Erdős calls it **The Happy Ending Theorem**.

Their marriage lasted 68 years and they died on the same day.

We prove this 3 ways.

How many of the 3 proofs use Ramsey's Theorem? Vote: 0,1,2,3.

The answer is all 3.

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Proof Using $R_4(5, k)$

This proof is by Erdős-Szekeres

Recall Let $n = R_4(5, k)$. Then for all COL: $\binom{[n]}{4} \rightarrow [2]$ there is RED homog set of size 5 OR a BLUE homog set of size k .

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Given $X \subseteq \mathbb{R}^2$, $|X| = n$, define the following coloring:

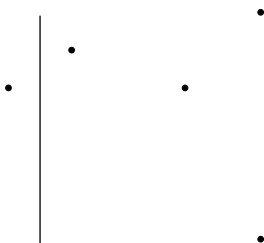
Let $Z \in \binom{X}{4}$.

$$\text{COL}(Z) = \begin{cases} \text{CONV} & \text{if } Z \text{ forms a convex quadrilateral} \\ \text{NOTCONV} & \text{otherwise} \end{cases} \quad (1)$$

Example Of the Coloring



Colored CONV.



Colored NOTCONV.

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Exercise Show that if every 4-subset of H is a convex 4-gon then H is a convex hull of size k .

Proof Using $R_3(k, k)$ (by P. Johnson (1986))

Recall Let $n = R_3(k, k)$. Then for all COL: $\binom{[n]}{3} \rightarrow [2]$ there is a **R** homog set of size k OR a **B** homog set of size k .

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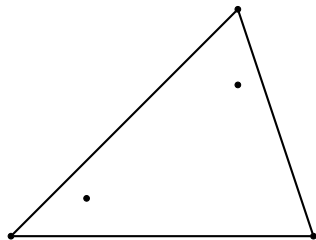
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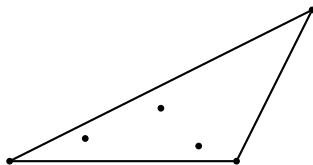
Given $X \subseteq \mathbb{R}^2$, $|X| = n$, define the following coloring:
Let $Z \in \binom{X}{3}$.

$$\text{COL}(Z) = \begin{cases} \text{EVEN} & \text{if the numb of points of } X \text{ in triangle } Z \text{ is even} \\ \text{ODD} & \text{if the numb of points of } X \text{ in triangle } Z \text{ is odd} \end{cases}$$

Example of The Coloring



Colored EVEN.



Colored ODD.

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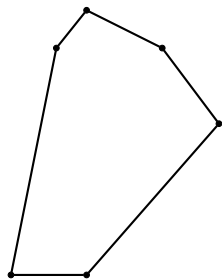
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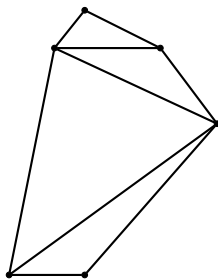
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We Proof there are no points of H in the convex hull of H and hence we have a convex hull of size k .

Convex Hull of H and its Triangulation



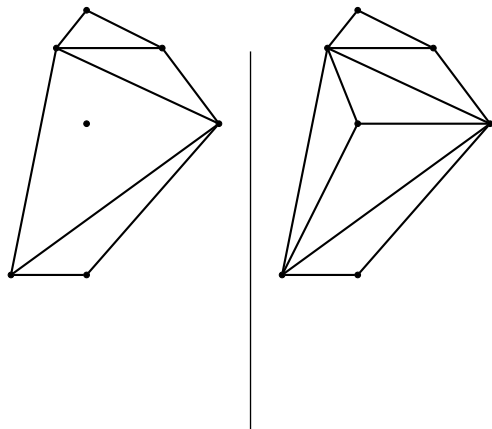
Convex Hull of H .



Triangulation.

We need to prove that there are no points of H in the convex hull.

Assume There is a Point of H in the Convex Hull

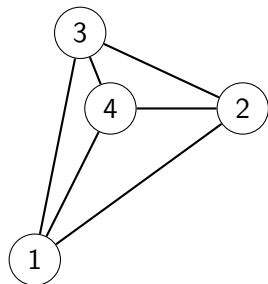


Which \triangle point is in.

More Triangulation.

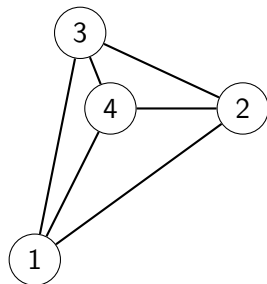
Need just the new point and its neighbors, and need labels.

Parity Argument



All these points are in H .

Parity Argument



All these points are in H . Assume all \triangle colored EVEN.

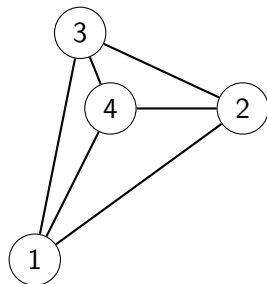
1 – 2 – 4 has an even number of points

2 – 3 – 4 has an even number of points

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Parity Argument



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$1 - 2 - 4$ has an even number of points

$2 - 3 - 4$ has an even number of points

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$1 - 2 - 3$ has an even number of points.

Not possible. $1 - 2 - 3$ has the points of $1 - 2 - 4$ AND $2 - 3 - 4$ AND $1 - 3 - 4$ AND the point 4. Thats Odd!

Third Proof

The Third Proof will be on the HW

APPLICATION?

Vote

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Thm $(\forall k)(\exists n)(\forall X \subseteq \mathbb{R}^n)(\exists Y \subseteq X)$ such that $|Y| = k$ and the convex hull of Y has size k .

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- 2) “Application”
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The proof that Erdős-Szekeres had was the $R_4(5, k)$.

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Calling the proof **an application of Ramsey Theory** is a bit odd since it is really **one of the two reasons Ramsey Theory was invented** (the other was Ramsey's problem in logic).