No Monochromatic Right Triangles under CH

Proof Presentation with Gen's Notes

Theorem

Theorem:
If the Continuum Hypothesis (CH) holds, then there exists a coloring $COL : \mathbb{R}^2 \to [\omega]$ *such that there is no monochromatic right triangle.*

Goal: Assign countably many colors to $\mathbb{R}^2$ such that no right triangle has all points the same color.

Proof Strategy

- Assume CH holds: then $\mathbb{R}^2$ has a well-ordering of type ω_1.
- Build a transfinite sequence of countable sets:
  - H_α: countable sets of points
  - E_α: associated lines and circles
- Define a coloring function f and a constraint function φ to control color choices on geometric structures.
- Ensure geometric constraints prevent right-angled monochromatic triangles.

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We'll build up the plane and coloring incrementally using transfinite recursion.

Constructing H_α and E_α

Define H_α, E_α by recursion:

1. $H_\alpha \subset H_\beta$, $E_\alpha \subset E_\beta$ for $\alpha < \beta$
2. For limit λ: $H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha$, same for E_λ
3. $\bigcup_{\alpha<\omega_1} H_\alpha = \mathbb{R}^2$

Constructing H_α and E_α cont...

4. If $x, y \in H_\alpha$ are distinct then their connecting line as well as their Thales circle [1] is in E_α.

[1] Thales' theorem states that if A, B, and C are distinct points on a circle where the line between A and C is a diameter, then the angle $\angle ABC$ is right. In the context of our proof, I think the Thales circle is referring to the circle whose diameter is the line connecting x and y,

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9. If $L \in E_\alpha$ is a line, $x \in H_\alpha \cap L$, then the line perpendicular to L ar x is also in E_α.

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Constructing f and φ

Skolem function φ and coloring function f:

$$\varphi : \bigcup E_\alpha \to [\omega]^\omega, \quad f : \bigcup H_\alpha \to \omega$$

Constraints:

10. $|\omega - \varphi(C)| \leq 2$.

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14. If $x \in L_1 \cap L_2$, then $f(x) \notin \varphi(L_1) \cap \varphi(L_2)$.

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φ acts like a Skolem function to constrain valid colorings for geometric objects.

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Key idea: no circle (like a Thales circle) can contain 3 same-colored points under the rules.

Ensuring Conditions Hold

Can we always color new points to satisfy conditions? Yes, because:

- Every new $x \in H_{\alpha+1} - H_\alpha$ lies on only one new $e \in E_\alpha$.

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"Skolem-type closure" $=$ inductively picking from ω-many options to satisfy local constraints.

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New Lines

Let $L \in E_{\alpha+1} - E_\alpha$ be a new line. For each line $g_i \in E_\alpha$ perpendicular to L:

$$x_i = L \cap g_i, \quad x_i \in H_{\alpha+1} - H_\alpha$$

Then:

- $f(x_i) \in \varphi(g_i) \cap \varphi(L)$ by (14) and (16).

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Handling New Circles in $E_{\alpha+1} - E_\alpha$

Let $\alpha < \omega_1$ and suppose φ, f are already defined on H_α, E_α. Let

$$C \in E_{\alpha+1} - E_\alpha$$

be a new circle containing six points:

$$x_1, y_1, x_2, y_2, x_3, y_3 \quad \text{with} \quad f(x_t) = f(y_t) \text{ for } t = 1, 2, 3$$

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- So at least 3 of the 6 points must lie in H_α (new distinct points can't share a color).
- Thus, C is determined by 3 points in $H_\alpha \Rightarrow C \in E_\alpha$ by (5).

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- This way, no more than 2 points on C have the same color

Why the Coloring Step Always Works

This part of the proof makes sure the coloring step can always be completed while obeying conditions 10–16. The constraints are "local" in nature, and we have ω-many colors to choose from.

Key Insight from Condition 6:

- Every new point $x \in H_{\alpha+1} - H_\alpha$ lies on exactly one $e \in E_\alpha$.
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This recursive coloring approach ensures every step completes without conflict.

Summary and Key Takeaways

- CH allows us to well-order $\mathbb{R}^2$ in type ω_1
- Build up the plane through transfinite recursion with geometric closure
- Define coloring f and constraint φ to avoid monochromatic right triangles
- Coloring choices remain infinite at each step due to constraints being local
- Result: $\mathbb{R}^2 \to \omega$ coloring with no monochromatic right triangle

The power of CH lets us build complicated global structures by managing local rules.