BILL, RECORD LECTURE!!!!

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Finite Ramsey Theorem For 3-Hypergraph: Better Bounds

Exposition by William Gasarch

February 20, 2025

Credit Where Credit is Due

The main theorem in these slides, in fact not just the 3-ary case but also the *a*-ary case, appeared in

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Here is a link

`https://www.cs.umd.edu/users/gasarch/TOPICS/canramsey/ErdosRado2.pdf`

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We will do much better.

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We will later see how big we need n to be.

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Hence H is homog for COL.

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d) See next slide for an example of how timing and luck played a major role.

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It is said **correctly** that the result is due to RDPM.