BILL, RECORD LECTURE!!!!

BILL RECORD LECTURE!!!

History of Upper Bounds On: VDW Numbers

Exposition by William Gasarch

January 23, 2025

Recall VDW's Theorem

Thm $(\forall k, c \in \mathbb{N})(\exists W = W(k,c))(\forall COL\colon W(k,c) \rightarrow [c]$
there exists a, d such that

$$\mathrm{COL}(a) = \mathrm{COL}(a+d) = \mathrm{COL}(a+2d) = \cdots = \mathrm{COL}(a+(k-1)d)$$

Recall VDW's Theorem

Thm $(\forall k, c \in \mathbb{N})(\exists W = W(k,c))(\forall COL\colon W(k,c) \rightarrow [c]$
there exists a, d such that

$$\mathrm{COL}(a) = \mathrm{COL}(a+d) = \mathrm{COL}(a+2d) = \cdots = \mathrm{COL}(a+(k-1)d)$$

VDW's proof gave bounds on $W(k,c)$ that were not prim. rec.

Recall VDW's Theorem

Thm $(\forall k, c \in \mathbb{N})(\exists W = W(k,c))(\forall COL\colon W(k,c) \to [c]$
there exists a, d such that

$$\mathrm{COL}(a) = \mathrm{COL}(a+d) = \mathrm{COL}(a+2d) = \cdots = \mathrm{COL}(a+(k-1)d)$$

VDW's proof gave bounds on $W(k,c)$ that were not prim. rec.

Erdös and Turan had an idea!

The Erdös-Turan Conjecture

The Erdös-Turan Conjecture

Speculation

The Erdös-Turan Conjecture

Speculation If $A \geq W(k,c)$ and $\mathrm{COL}\colon [A] \rightarrow [c]$, and **R** is the most common color then we suspect there is a **R** k-AP.

The Erdös-Turan Conjecture

Speculation If $A \geq W(k, c)$ and $\mathrm{COL}\colon [A] \rightarrow [c]$, and **R** is the most common color then we suspect there is a **R** k-AP.

Skip The Coloring

The Erdös-Turan Conjecture

Speculation If $A \ge W(k,c)$ and $\mathrm{COL}\colon [A] \to [c]$, and **R** is the most common color then we suspect there is a **R** k-AP.

Skip The Coloring If $A \subseteq [N]$ and is **"dense enough"** then we suspect that A has a k-AP.

The Erdös-Turan Conjecture

Speculation If $A \geq W(k,c)$ and $\mathrm{COL}\colon [A] \to [c]$, and **R** is the most common color then we suspect there is a **R** k-AP.

Skip The Coloring If $A \subseteq [N]$ and is **"dense enough"** then we suspect that A has a k-AP.

Def If $A \subseteq [N]$ then the **density** of A is $|A|/N$.

The Erdös-Turan Conjecture

Speculation If $A \geq W(k, c)$ and $\mathrm{COL}\colon [A] \to [c]$, and **R** is the most common color then we suspect there is a **R** k-AP.

Skip The Coloring If $A \subseteq [N]$ and is **"dense enough"** then we suspect that A has a k-AP.

Def If $A \subseteq [N]$ then the **density** of A is $|A|/N$.

Erdös-Turan Conjecture For all $\delta > 0$, for all k, there exists $N = N(\delta, k)$ such that the following holds:

The Erdös-Turan Conjecture

Speculation If $A \geq W(k, c)$ and $\mathrm{COL}\colon [A] \rightarrow [c]$, and **R** is the most common color then we suspect there is a **R** k-AP.

Skip The Coloring If $A \subseteq [N]$ and is **"dense enough"** then we suspect that A has a k-AP.

Def If $A \subseteq [N]$ then the **density** of A is $|A|/N$.

Erdös-Turan Conjecture For all $\delta > 0$, for all k, there exists $N = N(\delta, k)$ such that the following holds:

If $A \subseteq [N]$ and A has density $\geq \delta$ then A has a k-AP.

The Erdös-Turan Conjecture

Speculation If $A \geq W(k,c)$ and $\mathrm{COL}\colon [A] \to [c]$, and **R** is the most common color then we suspect there is a **R** k-AP.

Skip The Coloring If $A \subseteq [N]$ and is **"dense enough"** then we suspect that A has a k-AP.

Def If $A \subseteq [N]$ then the **density** of A is $|A|/N$.

Erdös-Turan Conjecture For all $\delta > 0$, for all k, there exists $N = N(\delta, k)$ such that the following holds:

If $A \subseteq [N]$ and A has density $\geq \delta$ then A has a k-AP.

Easy ET-conj implies VDW's Theorem.

The Erdös-Turan Conjecture

Speculation If $A \geq W(k,c)$ and $\mathrm{COL}\colon [A] \rightarrow [c]$, and **R** is the most common color then we suspect there is a **R** k-AP.

Skip The Coloring If $A \subseteq [N]$ and is **"dense enough"** then we suspect that A has a k-AP.

Def If $A \subseteq [N]$ then the **density** of A is $|A|/N$.

Erdös-Turan Conjecture For all $\delta > 0$, for all k, there exists $N = N(\delta, k)$ such that the following holds:

If $A \subseteq [N]$ and A has density $\geq \delta$ then A has a k-AP.

Easy ET-conj implies VDW's Theorem.

Hope The proof of the ET-conj will be a **diff** proof of VDW's theorem that gives better bounds on the VDW numbers.

Upper Bounds on VDW Numbers: 1927-1953

Upper Bounds on VDW Numbers: 1927-1953

1) 1927: VDW's Proof. Bounds are not prim. rec.

Upper Bounds on VDW Numbers: 1927-1953

1) 1927: VDW's Proof. Bounds are not prim. rec.

2) 1936: Erdös-Turan make their conjecture.

Upper Bounds on VDW Numbers: 1927-1953

1) 1927: VDW's Proof. Bounds are not prim. rec.

2) 1936: Erdös-Turan make their conjecture.

3) 1953: Roth proves ET-conj for $k = 3$. Used Fourier Analysis.

Upper Bounds on VDW Numbers: 1927-1953

1) 1927: VDW's Proof. Bounds are not prim. rec.

2) 1936: Erdös-Turan make their conjecture.

3) 1953: Roth proves ET-conj for $k = 3$. Used Fourier Analysis. We will prove this later in the course.

Upper Bounds on VDW Numbers: 1974-1988

Upper Bounds on VDW Numbers: 1974-1988

4) 1974: Szemerédi proves ET-conj for $k = 4$. Proof is combinatorial but difficult. Uses VDW theorem so does not give better bounds on $W(4, c)$.

Upper Bounds on VDW Numbers: 1974-1988

4) 1974: Szemerédi proves ET-conj for $k = 4$. Proof is combinatorial but difficult. Uses VDW theorem so does not give better bounds on $W(4, c)$.

5) 1975: Szemerédi proves ET-conj. Proof is combinatorial but difficult. Uses VDW's theorem so does not give better bounds on $W(k, c)$. The ET-Conj is now called Szemerédi's Theorem.

Upper Bounds on VDW Numbers: 1974-1988

4) 1974: Szemerédi proves ET-conj for $k = 4$. Proof is combinatorial but difficult. Uses VDW theorem so does not give better bounds on $W(4, c)$.

5) 1975: Szemerédi proves ET-conj. Proof is combinatorial but difficult. Uses VDW's theorem so does not give better bounds on $W(k, c)$. The ET-Conj is now called Szemerédi's Theorem.

6) 1977: Furstenberg proves ET-conj using Ergodic methods. Proof does not gives bounds on $W(k, c)$. Later proof theorists extract out bounds from the proof. They are worse than VDW's bounds.

Upper Bounds on VDW Numbers: 1974-1988

4) 1974: Szemerédi proves ET-conj for $k = 4$. Proof is combinatorial but difficult. Uses VDW theorem so does not give better bounds on $W(4, c)$.

5) 1975: Szemerédi proves ET-conj. Proof is combinatorial but difficult. Uses VDW's theorem so does not give better bounds on $W(k, c)$. The ET-Conj is now called Szemerédi's Theorem.

6) 1977: Furstenberg proves ET-conj using Ergodic methods. Proof does not gives bounds on $W(k, c)$. Later proof theorists extract out bounds from the proof. They are worse than VDW's bounds.

7) 1988: The Hales-Jewitt Thm implies VDW's Theorem. Shelah gives a new proof of the HJ Thm which gives primitive recursive (though still quite large) bounds on the VDW numbers. The proof is elementary and does not use any of the ET stuff.

Upper Bounds on VDW Numbers: 2001

Upper Bounds on VDW Numbers: 2001

8) 2001: Gowers proves Szemerédi's Thm a new way using combinatorics and Fourier Analysis to obtain the following:

Upper Bounds on VDW Numbers: 2001

8) 2001: Gowers proves Szemerédi's Thm a new way using combinatorics and Fourier Analysis to obtain the following:

$$W(k,c) \leq 2^{2^{c^{2^{2^{k+9}}}}}.$$