BILL, RECORD LECTURE!!!!

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Infinite Ramsey Theorem For Graphs

Exposition by William Gasarch

January 29, 2025

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My apologies to the math majors who are not used to seeing examples.

Examples of The First Few Steps of The Construction

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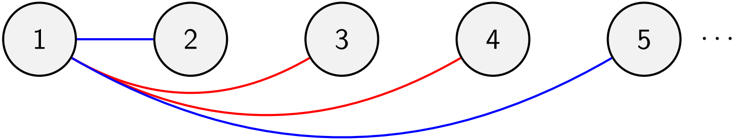

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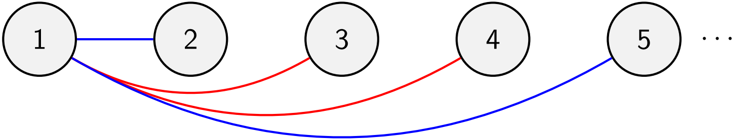

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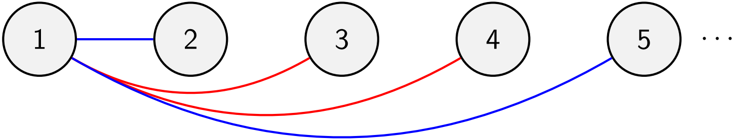

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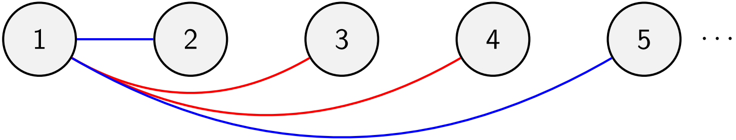

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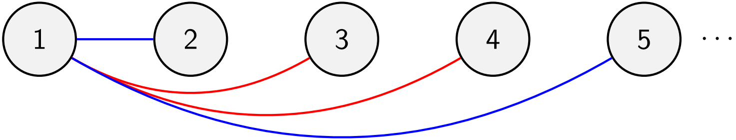

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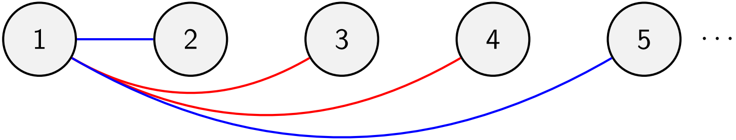

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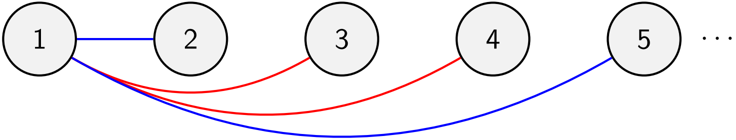

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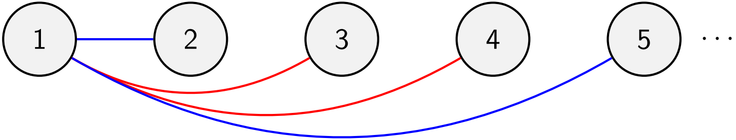

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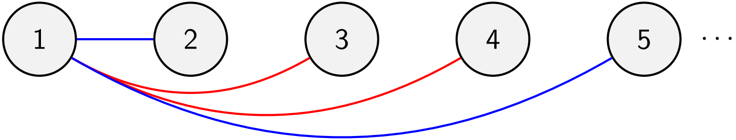

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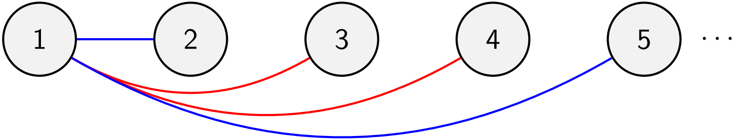

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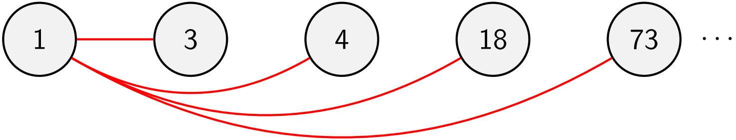

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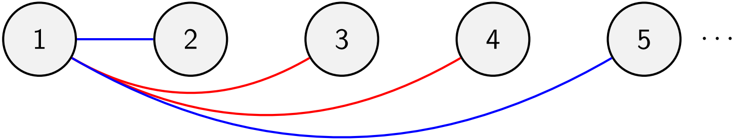

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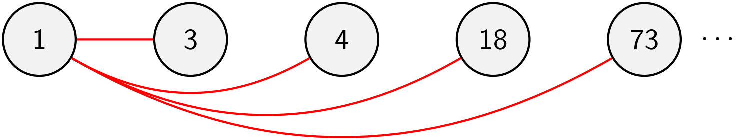

We Omit 1 from future pictures but its **Still Alive and Well.**
`https://www.youtube.com/watch?v=8--jVqaU-G8`.

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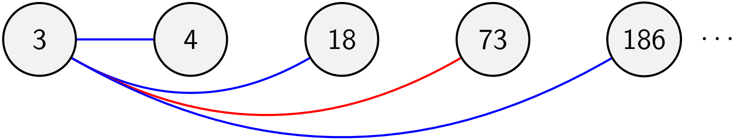

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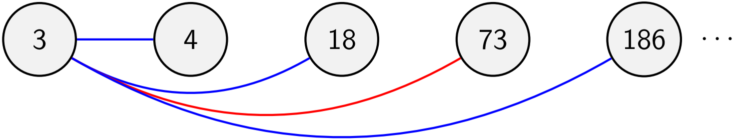

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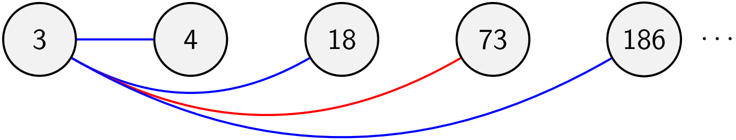

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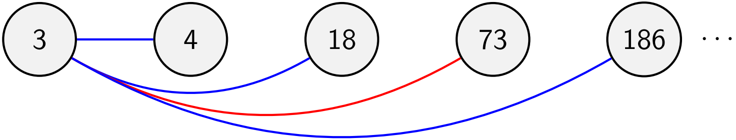

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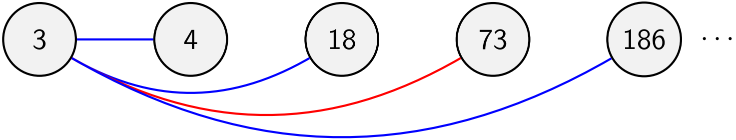

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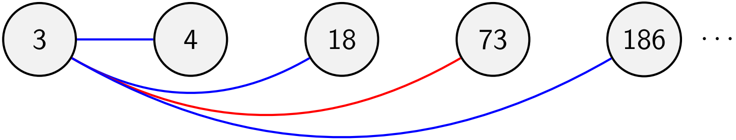

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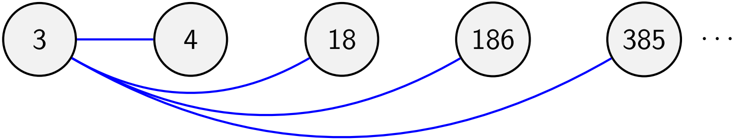

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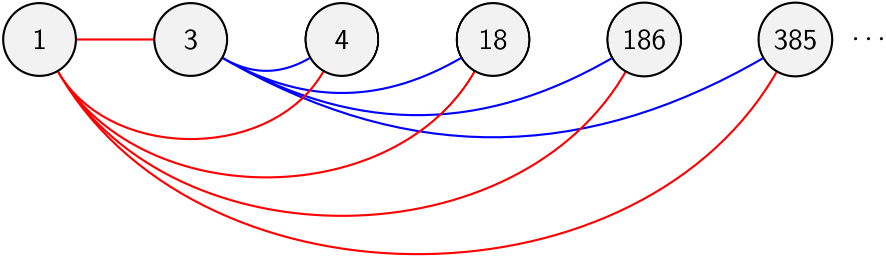

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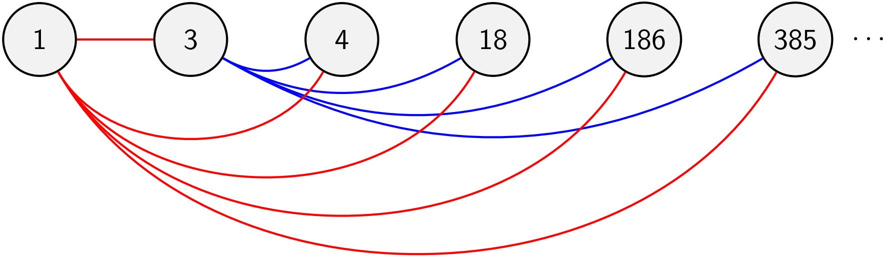

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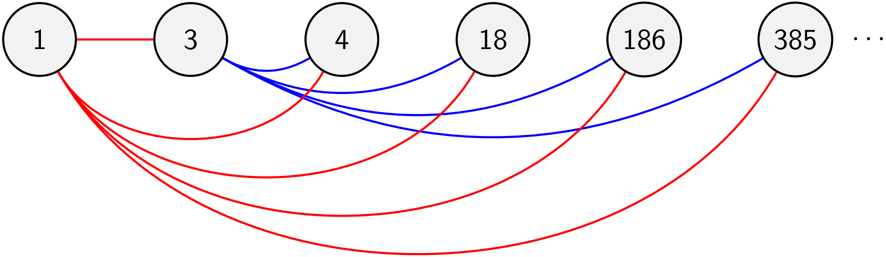

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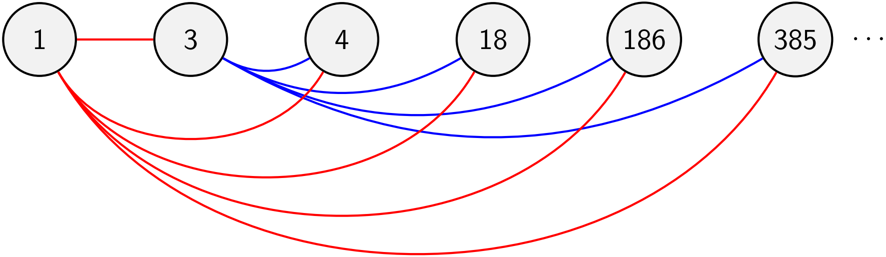

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We said earlier

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Still have an Infinite Number of Nodes In Play.

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$c_1 =$ **R** if $|\{y \in H_1 \colon \mathrm{COL}(x_1, y) =$ **R**$\}| = \infty$, **B** otherwise.

$H_2 = \{x_1\} \cup \{y \in H_1 \colon \mathrm{COL}(x_1, y) = c_1\}$
$x_2 =$ the least element of $H_2 - \{x_1\}$.
$c_2 =$ **R** if $|\{y \in H_2 \colon \mathrm{COL}(x_2, y) =$ **R**$\}| = \infty$, **B** otherwise.

Assume H_s, x_s, c_s are defined.
$H_{s+1} = \{x_1, \ldots, x_s\} \cup \{y \in H_s \colon \mathrm{COL}(x_s, y) = c_s\}$
$x_{s+1} =$ the least element of $H_{s+1} - \{x_1, \ldots, x_s\}$.
$c_{s+1} =$ **R** if $|\{y \in H_{s+1} \colon \mathrm{COL}(x_{s+1}, y) =$ **R**$\}| = \infty$, **B** otherwise.

$$X = \{x_1, x_2, \ldots\}$$

The Coloring of the Nodes

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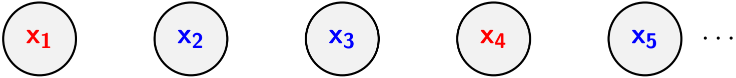

All of the edges from x_1 to the right are **R**.

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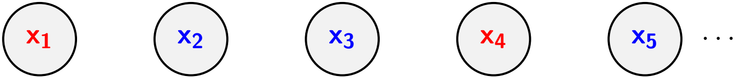

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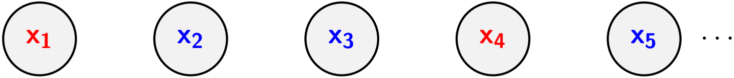

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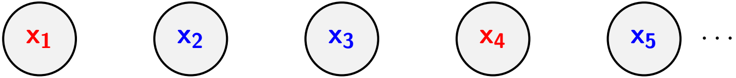

All of the edges from x_1 to the right are **R**.

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All of the edges from x_4 to the right are **R**.

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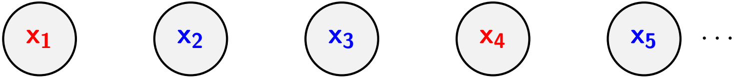

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All of the edges from x_5 to the right are **B**.

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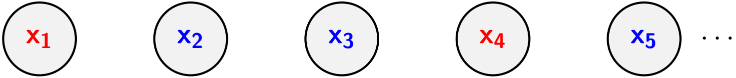

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All of the edges from x_s to the right are c_s.

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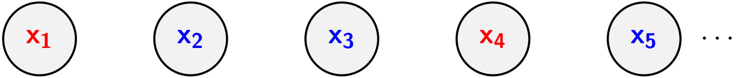

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What do you think our next step is?

Some Color Appears Infinitely Often

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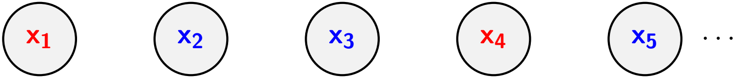

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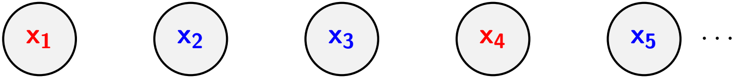

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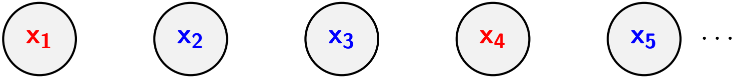

All of the edges from $\mathbf{x_s}$ to the right are $\mathbf{c_s}$.

R or **B** (0r both) appears infinitely often. We will assume **R**.

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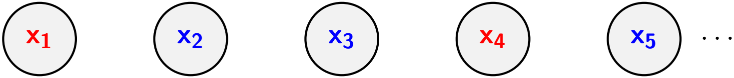

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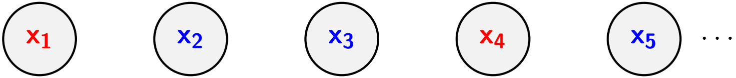

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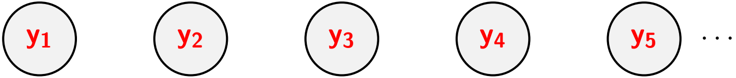

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All of the edges from $\mathbf{y_s}$ to the right are **R**.

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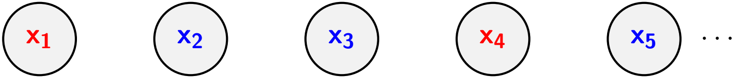

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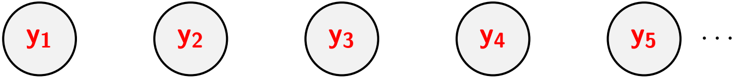

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DONE!

Variants Of
The
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This is easy to prove using the same technique we used for the $c = 2$ case.

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Thm For all k there exists $n = R(k)$ such that for all
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