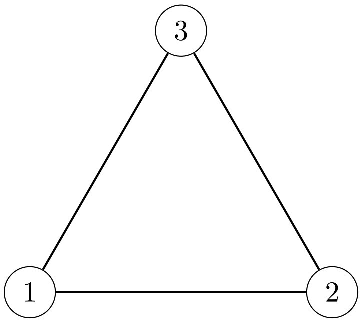


# 1 Our Starting Point

We start with a theorem that is easy; however, it is the starting point for our line of research.

**Theorem 1.1** . *For all* COL: $\mathsf{R}^2 \rightarrow [2]$ *there exists 2 points, same color, 1 inch apart.*

**Proof:**
Look at an equilateral triangle in the plane with side length 1.

Since there are 3 vertices and 2 colors, 2 of the vertices are the same color. these two points are the same color and 1 inch apart. $\blacksquare$

There are two directions (actually many more) to investigate as variants of Theorem 1.1.

1. What if we use more colors?
2. What if we want three monochromatic colinear points $p, q, r$ such that $d(p,q) = d(q,r) = 1$?

The second question, with more parameters, is the theme of this book. We will consider the first question as well since it will hel us with the second question.

# 2 The Chromatic Number of the Plane

Consider the graph $G = (V, E)$ where

$$\begin{aligned} V = &\ \mathsf{R}^2 \\ E = &\ \{(x,y) \colon d(x,y) = 1\} \end{aligned}$$

**Notation 2.1** $\chi$ is the chromatic number of this graph.

Theorem 1.1 can be rephrased as

**Theorem 2.2** $\chi \geq 3$.

We investigate what $\chi$ can be.

**Theorem 2.3** $\chi \geq 4$.

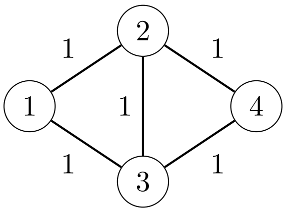


Figure 1: A Graph Gadget For 3-Coloring the Plane

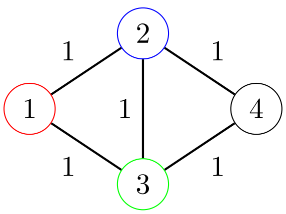


Figure 2: Coloring of Graph Gadget for 3-coloring

**Proof:**

Assume, by way of contradiction, that COL is a proper 3-coloring of the plane. Glue together two unit equilateral triangles to form the configurtion in Figure 1

We can assume that COL(1) = RED. Since COL is a proper 3-coloring

$$\mathrm{COL}(2) \neq \mathrm{RED} \text{ and } \mathrm{COL}(3) \neq \mathrm{RED}.$$

Since $d(2,3) = 1$, COL(2) $\neq$ COL(3). Hence we can assume COL(2) = BLUE and COL(3) = GREEN. Figure 2 shows all that we know at this point.

Since COL(2) = BLUE, COL(3) = GREEN, $d(2,4) = 3(3,4) = 1$, we have COL(4) = RED. Figure 3 shows all that we know at this point.

Note that $d(1,4) = \sqrt{3}$. Hence

- If $p, q$ are $\sqrt{3}$ apart then COL($p$) = COL($q$).
- If COL($p$) = RED then circle of radius $\sqrt{3}$ around $p$ is RED.

∎

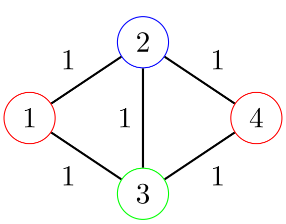


Figure 3: Coloring of Graph Gadget for 3-coloring: All Four Points