$\mathsf{R}^n \rightarrow (\ell_2, \ell_{2^{dn}})$, $\mathsf{R}^n \not\rightarrow (\ell_2, \ell_{d'n})$

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1 Introduction

In this paper we present the following theorems:

1. (Szlam [5]) There exists d such that $\mathsf{R}^n \rightarrow (\ell_2, \ell_{2^{dn}})$. (He proved a more general theorem. See his paper for details.)

2. (Conlon & Fox [1]) There exists a constant d' such that $\mathsf{R}^n \not\rightarrow (\ell_2, \ell_{2^{d'n}})$. We will just prove the $n = 2$ case in this paper. (They proved a more general theorem. See their paper for details.)

2 There exists d Such That $\mathsf{R}^n \rightarrow (\ell_2, \ell_{2^{dn}})$

Notation 2.1 Let $G_n = (V, E)$ be the graph with $V = \mathsf{R}^n$ and $E = \{(x, y) \colon d(x, y) = 1\}$. Let $c(n)$ be the chromatic number of G_n.

It is well known that $5 \le c(2) \le 7$.
The following are known:

Theorem 2.2

1. *(Larman and Rogers [3])* $c(n) \le (3 + o(1))^n$

2. *(Raigorodskii [4])* $c(n) \le (1.239\ldots + o(1))^n$

3. *(Frankl and Wilson [2])* $c(n) \ge (1 + o(1))(1.2)^n$.

We give the proof by Frankl and Wilson and then use the result to obtain $\mathsf{R}^n \rightarrow (\ell_2, \ell_{2^{dn}})$.
KELIN-LOOK AT THE OTHER PAPERS AND SEE IF THEY ARE WORTH INCLUDING AS WELL.

2.1 Set System Lemma that Frankl and Wilson Used

Theorem 2.3 *Let $k, n, s \in \mathsf{N}$. Let p be a prime. Let $\mu_0, \ldots, \mu_s$ be distinct element of $\{0, \ldots, p-1\}$. Assume $k \equiv \mu_0 \pmod p$. Let $F_1, \ldots, F_L \in \binom{[n]}{k}$ be such that:*

$$\forall 1 \le i < j \le s)(\exists \mu \in \{\mu_1, \ldots, \mu_s\})|F_i \cap F_j| \equiv \mu \pmod p)].$$

Then $L \le \binom{n}{s}$.

Proof:

KELIN: DURING THE PROOF (PAGE 361) THEY HAVE THE FOLLOWING:

Let us choose $0 \le a_i < p$ for $0 \le i \le s_0$ [I THINK s_0 SHOULD BE S, BUT PLEASE CHECK] in such a way that for every integer x we have

$$\prod_{i=1}^{s}(x-\mu_i) \equiv \sum_{i=1}^{s} a_i \binom{x}{i} \pmod{p}.$$

THIS DOES NOT LOOK POSSIBLE. THE LEFT HAND SIDE HAS CONSTANT TERM $\mu_1\mu_2\cdots\mu_s$.

THE RIGHT HAND SIDE HAS CONSTANT TERM 0.

It is quite possible that none of the μ are 0.

MY FIRST THOUGHT WAS: MAYBE THERE IS A TYPO AND THEY MEANT TO BEGIN THE SUMS AND PRODUCTS AT 0. THIS DOES NOT WORK SINCE THEN THE LHS HAS HIGHER DEGREE THAN THE RHS.

$\blacksquare$

The following theorem is obtained by a modification of Theorem 2.3.

KELIN- WE'LL PROVE THEOREM 2.4 ONCE WE FIGURE OUT THEOREM 2.3.

Theorem 2.4 *Let $k, n \in \mathsf{N}$. Let q be a prime power.*

1. *Let $F_1, \ldots, F_L \in \binom{[n]}{k}$ be such that:*

$$(\forall 1 \le i < j \le s)[|F_i \cap F_j| \not\equiv k \pmod{q}].$$

   Then $L \le \binom{n}{q-1}$.

2. *If $F_1, \ldots, F_{\binom{[n]}{q-1}+1} \in \binom{[n]}{k}$ then there exists $1 \le i < j \le \binom{[n]}{q-1}+1$ with $|F_i \cap F_j| \equiv k \pmod{q}$. (This is the contrapositive of Part 1.)*

3. *If $F_1, \ldots, F_{\binom{[n]}{q-1}+1} \in \binom{[n]}{2q-1}$ then there exists $1 \le i < j \le \binom{[n]}{q-1}+1$ with $|F_i \cap F_j| = q-1$. (This is Part 2 with $k = 2q-1$ coupled with the observation that if $|F_i \cap F_j| \equiv 2q-1 \pmod{q}$ then $|F_i \cap F_j| = q-1$ since otherwise $F_i = F_j$.)*

2.2 The Chromatic Number of R^n

Theorem 2.5

1. *For all n,*

$$c(n) > \max_{q \text{ prime power}} \frac{\binom{n}{2q-1}}{\binom{n}{q-1}+1}$$

   KELIN: WRITE A PROGRAM THAT COMPUTES c FOR $10 \le n \le 100$. PRE-COMPUT THE BINOMIAL COEFFICIENTS WITH THE RECURRENCE. DO SOME CURVE FITTING TO FIND d SUCH THAT $c \sim 2^{dn}$.

2. *There exists d such that, for all n, $c(n) \ge 2^{dn}$.*

Proof:
1) Let $S \subseteq \mathsf{R}^n$ be all of the vectors such that

- $n - 2q - 1$ of the components are 0.
- $2q - 1$ of the components are $\frac{1}{\sqrt{2q}}$.

Let $F\colon S \to \binom{[n]}{2q-1}$ by viewing each vector in S as a bit vector though with $\frac{1}{\sqrt{2q}}$ instead of 1.
Claim Let $u, v \in S$. If $|F(u) \cap F(v)| = f$ then $d(u,v) = 2 - \frac{f+1}{q}$. Hence $d(u,v) = 1$ iff $|F(u) \cap F(v)| = q - 1$.
Proof of Claim: Assume $|F(u) \cap F(v)| = f$ then:

- There are f coordinates where u and v both have $\frac{1}{\sqrt{2q}}$.
- There are $2q - 1 - f$ coordinates where u has $\frac{1}{2q}$ and v has 0.
- There are $2q - 1 - f$ coordinates where v has $\frac{1}{2q}$ and u has 0.
- There are $n - 4q + f + 2$ coordinates where u and v are both 0.

Hence $d(u,v) = 2 \times (2q - 1 - f) \times \frac{1}{2q} = \frac{2q-1-f}{q} = 2 - \frac{f+1}{q}$.
End of Proof of Claim

Restrict COL to S. Since $|S| = \binom{n}{2q-1}$ and there are c colors, some color must occur $\ge \binom{n}{2q-1}/c = \binom{n}{q-1} + 1$ times. Let S' be the subset of S that has that color. Since $S' \subseteq \binom{[n]}{2q-1}$ and $|S'| \ge \binom{n}{q-1} + 1$, by Theorem 2.4.3, there exists two elements of S with intersection of size $q - 1$. Let those two elements be $F(u)$ and $F(v)$. Since $|F(u) \cap F(v)| = q - 1$, by the Claim, $d(u,v) = 1$.

2) We obtain an approximation to the optimal value of c.

KELIN- THE PAPER TAKES q TO BE $(1+o(1))\frac{2-\sqrt{2}}{2}n$ I WANT TO KNOW HOW HE GETS A PRIME POWER LIKE THAT. PLEASE EXPAND ON THAT. ∎

2.3 There exists d Such That $\mathsf{R}^n \to (\ell_2, \ell_{2^{dn}})$)

Theorem 2.6 *(Szlam [5]) There exists d such that $\mathsf{R}^n \to (\ell_2, \ell_{2^{dn}})$.*

Proof: We will need the following notation: $\vec{1}$ is the vector $(1, 0, \ldots, 0)$ in R^n.

Let COL: $\mathsf{R}^n \to [2]$.
Case 1 There is a BLUE ℓ_m. Done
Case 2 There is no BLUE ℓ_m. We form a coloring COL: $\mathsf{R}^n \to [m]$ as follows:

Given point $p \in \mathsf{R}^n$ look at

$$p + \vec{1}, p + 2\vec{1}, \ldots, p + m\vec{1}.$$

Since there is no BLUE ℓ_m, there exists i such that COL$(p + i\vec{1})$ is RED. Color p with the least such i.

By Theorem 2.5 there exists points $u, v \in \mathsf{R}^n$ and $1 \le i \le m$ such that $d(u,v) = 1$ and u, v are the same color. Hence $u + i\vec{1}$ and $v + i\vec{1}$ are both RED. Since $d(u,v) = 1$, $d(u + i\vec{1}, v + i\vec{1}) = 1$. Hence $u + i\vec{1}$ and $v + i\vec{1}$ form a RED ℓ_2. ∎

2.4 While We're Here, Constructive Ramsey Lower Bounds

KELIN- BEFORE ADDING THIS SECTION TO THE MONOGRAPH I WILL GIVE IT MORE CONTEXT.

Frankl and Wilson also used Theorem 2.4 to obtain a constructive lower bound on the Ramsey number $R(k)$.

Theorem 2.7

1. *Let $n \in \mathsf{N}$. Let p be a prime (though we will also use that its a prime power).*

   KELIN- THE PAPER USES q A PRIME POWER, BUT LATER SETS IT TO p WHICH I ASSUME IS A PRIME. SO I JUST SET IT TO A PRIME IN THE FIRST PLACE.

   Let $G = (V, E)$ be defined as follows.

   - *V is $\{F \subseteq [n] : |F| = p^2 - 1\}$. Note that $|V| = \binom{n}{p^2-1}$.*
   - *E is (F, F') such that $|F \cap F'| \not\equiv -1 \pmod{p}$.*

   Then G contains no $\binom{n}{p-1}$-clique or $\binom{n}{p-1}$-ind. set.

2. *$R(k) \ge 2^{(1+o(1)\log^2 k/4\log\log k}$ with a constructive proof. (Note that this use of k is different than the use of k in Theorem 2.3, 2.4, and the first part of this theorem.)*

Proof:

1a) Let $F_1, \ldots, F_L$ be a complete subgraph of G. By the definition of G,

- $F_1, \ldots, F_L \in \binom{[n]}{p^2-1}$.
- $(\forall 1 \le i < j \le L)[|F_i \cap F_j| \not\equiv -1 \pmod{p}$.

By Theorem 2.4, with $q = p$ and $k = p^2 - 1$, we obtain $L \le \binom{n}{p-1}$.

1b) Let $F_1, \ldots, F_L$ be an independent set in G.

By the definition of G,

- $F_1, \ldots, F_L \in \binom{[n]}{p^2-1}$.
- $(\forall 1 \le i < j \le L)[|F_i \cap F_j| \equiv -1 \equiv p - 1 \pmod{p}$.

By Theorem 2.3, with $k = p^2 - 1$, $\mu_0 = p - 1$, $\mu_1 = p - 1$, we obtain $L \le \binom{n}{2}$.

KELIN: THE ABOVE LINE DOES NOT WORK SINCE $\mu_0 = \mu_1$. THE PAPER DOES NOT USE THEOREM 1. THE PAPER INSTEAD REFERS TO EQUATION (2) WHICH IS FROM A RESULT BY RAY-CHADHUIRI AND WILSON. BUT THEY LATER SAY, RIGHT AFTER THEOREM 1 *Clearly Theorem 1 generalizes (2).* HENCE I THOUGHT I COULD AVOID USING RAY-CHAD... .

ALSO, $\binom{n}{2}$ SEEMS WAY TO GOOD A BOUND TO GET- FAR MORE THAN WE NEED. I SUSPECT MY TWO CONFUSIONS ARE RELATED AND WHEN YOU FIGURE OUT ONE, YOU WILL FIGURE OUT THE OTHER.

2) Setting $n = p^3$ we obtain the result.

KELIN- WORK THIS OUT. ∎

3 Lemmas Needed To Show there exists d, $\mathsf{R}^n \not\to (\ell_2, \ell_{2^{dn}})$

We will be 2-coloring the $m \times m$ square and then use that to form a periodic coloring of R^2. Hence we think of coloring the $m \times m$ square with the two horizontal sides identified and the new vertical sides identified. We denote this T_m^2. (The T is for Taurus.)

BILL- THE PAPER USES $m \times m$. I WILL LATER SAY WHY I USE $m \times m$.

KELIN: WE NEED A PICTURE FOR AN EXAMPLE. YOU CAN DO A COLOR PICTURE OF A colored square.

We need several lemmas.

Definition 3.1 Let $t \in \mathsf{R}^+$. Let $P \subseteq T_m^2$.

1. P is t-separated if, for all $p, q \in P$, $d(p, q) \geq t$.
2. P is *maximally t-separated* (1) if P is t-separated and (2) for all $r \notin P$, $P \cup \{r\}$ is not t-separated.

Lemma 3.2 *Let $t \in \mathsf{R}^+$ and $m \in \mathsf{N}$.*

1. *There exists $P \subseteq T_m^2$ that is maximally t-seperated.*
2. *If $P \subseteq T_m^2$ is maximally t-seperated then $|P| \leq \frac{(m/t)^2}{\pi}$.*
3. *If $P \subseteq T_m^2$ is maximally $\frac{1}{3}$-seperated then $|P| \leq (1.7m)^2$. This follows from Part 2.*

Proof:
1) A greedy algorithm forms a maximally t-seperated set.

BILL: How fast is this? Can we get a faster algorithm?

2) Let $p \in P$. Then there is no element of P inside the circle centered at p of radius t. This circle has area πt^2. The set T_m^2 has area m^2. Hence

$|P| \times \pi t^2 \leq m^2$, so $|P| \leq \frac{(m/t)^2}{\pi}$. ∎

Lemma 3.3 *Let $t \in \mathsf{R}^+$. Let $S \subseteq \mathsf{R}^2$ be t-seperated. Let $\vec{p} \in \mathsf{R}^2$. Let $s \geq 0$. The number of points of S within s of $\vec{p}$ is at most $(2s/t + 1)^2$.*

Proof: Let T be the set of points within t of $\vec{p}$. For every $\vec{q} \in T$ we look at the circle centered at $\vec{q}$ of radius $t/2$ (we can't use radius t since then the circles would not be disjoint). These circles have no other points of T in them and are disjoint. These circles have area $\pi(t/2)^2$. The union of these circles is contained in the circle around $\vec{p}$ of radius $s + t/2$ which has area $\pi(s + t/2)^2$. Hence

$|T| \times \pi t^2/4 \leq \pi(s + t/2)^2$
$|T| \times (t/2)^2 \leq (s + t/2)^2$
$|T| \leq (\frac{s+t/2}{t/2})^2 = (2s/t + 1)^2$. ∎

Definition 3.4 Assume $S \subseteq \mathsf{R}^2$ or $S \subseteq T_2^m$. If $p \in S$ then V_p is the set of points of R^2 or T_2^m that are closer (or tied) to p then to any other point of S. The *Voronoi Diagram of S* is the set of all the V_p's.

BILL- DO EXAMPLES

1. A NORMAL EXAMPLE
2. AN EXAMPLE WHERE THE VORONOI CELL IS A POLYGON WITH LOTS OF SIDES. I THINK IF THE SET OF POINTS IS A p AND m POINTS ON THE CIRCLE OF RADIUS 1 AROUND x THEN V_p would be a m-sided convex polygon.

Note 3.5 There exists $S \subseteq \mathsf{R}^n$ and an $s \in S$ such that V_p is a convex $|S|$-gon. See BILL-WILL NEED FIGURE NUMBER.

Lemma 3.6 *Let $S \subseteq \mathsf{R}^2$ be a maximal t-separated set. We form the Voronoi diagram of S. The Voronoi cells are $\{V_p\}_{p \in S}$.*

1. *If $x \in V_p$ then $d(x,p) \le t$.*
2. *If $p, p' \in V_p$ then $d(p,p') \le 2t$. (This follows from Part 1.)*
3. *If $p, p' \in S$ and V_p, $V_{p;}$ share a boundary then $d(p,p') \le 2t$.*
4. *V_p is convex polygon with ≤ 25 sides.*

Proof:
1) Assume, by way of contradiction, that there is an $x \in V_p$ and $d(x,p) > t$. Since $x \in V_p$, $d(x,p)$ is the smallest distance from x to a point of S. Hence x is greater than t away from any point in S. Since S is maximal, $x \in S$ which is a contradiction.
3) Draw a line from p to p'. It will hit a point x that is on both the boundary of V_p and the boundary of $V_{p'}$. By Part 1
$d(p,p') = d(p,x) + d(x,p') \le t + t = 2t$.
4) V_p is a convex polygon. Map each side of V_p to the p' such that V_p and $V_{p'}$ share that side. Using Part 2 we get that the number of sides is bounded above by the number of points of $p' \in S$ such that $d(p,p') \le 2t$. By Lemma 3.3 the number of such points is $\le ((2 \times 2t)/t + 1)^2 = 5^2 = 25$. ∎

BILL- I DO NOT THINK I NEED THE LEMMA BELOW FOR THE THEOREM. THEY NEED TO USE A SET OF SIZE $m/5$ THAT HAS POINTS 5 APART. WE WILL JUST NEED THAT ℓ_m DOES NOT HIT TWO ANALOGOUS VORONOI CELLS FROM DIFF TILES. THIS WILL BE ACCOMPLISHED BY MAKING THE TILES $m \times m$ SINCE THE MAX DISTANCE BETWEEN POINTS OF ℓ_m IS $m - 1$. THE PAPER DOES MORE COMPLICATED THINGS

Lemma 3.7 *Let K be a 1-seperated set. Let $s \ge 1$. There is a set $K' \subseteq K$ that is s-separated such that $|K'| \ge |K|/(2s+1)^2$.*

4 There exists d', $\mathsf{R}^n \not\to (\ell_2, \ell_{2^{d'n}})$

Theorem 4.1 *There exists d' such that $\mathsf{R}^2 \not\to (\ell_2, \ell_{2^{d'n}})$.*

Proof: Let P be a maximal $\frac{1}{3}$-separated subset of T_2^m. We create the Voronoi diagram of P.

Let $Q \subseteq P$ be formed by, for each $p \in P$, choose it with probability x (we will determine x later).

Let $S \subseteq Q$ be the set of points $s \in Q$ such that, for all $s' \in Q$, $d(s, s') > 5/3$.

Recall that we have a Voronoi diagram formed by the points in P. Let the Voronoi cells that have a point of S in them be denoted $V_1, \ldots, V_{|S|}$.

We will color each V_i, including boundary, RED. We will color every other point in T_2^m BLUE. We will then use this to periodically color R^2. We view this as tiling the plane with $m \times m$ tiles and coloring all the tiles the same.

We will show that if you take a nine tiles arrange 3×3 then there is no RED ℓ_2 or BLUE ℓ_m with a point in the middle tile. This will suffice.

No RED ℓ_2 This part does not use probability.

Let q, q' both be RED.

Case 1: q, q' are in the same Voronoi cell. By Lemma 3.6.2 $d(q, q') \le 1/3$.

Case 2: q, q' are in the same tile but in different Voronoi cells. Let the Voronoi cells have centers p, p'. Then

$$d(p, p') \le d(p, q) + d(q, q') + d(q', p') \le \frac{1}{3} + 1 + \frac{1}{3} = \frac{5}{3}.$$

But by definition of S, $d(p, p') > \frac{5}{3}$.

Case 3: q, q' are in different tiles but in the analogous Voronoi cells. Let the Voronoi cells have centers p, p'. Since $d(p, p') = m$, $d(q, q') \ge m - \frac{1}{3} > 1$.

Case 4: q, q' are in different tiles and non-analogous Voronoi cells. Since the Voronoi diagram was on a Taurus this is identical to Case 2.

No BLUE ℓ_m

Let $L = (q_1, \ldots, q_m)$ be an ℓ_m. We bound the probability that L is BLUE.

Let $\{p_i\}_{i=0}^{m'-1}$ be such that, for $0 \le i \le m'-1$, $q_i \in V_{p_i}$. We need to bound the probability that V_{p_i} is BLUE. Not so fast! We need to show that all of the V_{p_i} are distinct.

Let $q, q' \in \{q_0, \ldots, q_{m'-1}\}$. Let $\{p, p'\}$ be such that $q \in V_p$ and $q' \in V_{p'}$.

Case 1 q, q' are in the same tile and in the same Voronoi cell. This cannot happen since $d(q, q') \ge 1$ and by Lemma 3.6.2 the diameter of these cells is $2/3$.

Case 2 q, q' are in the different tiles but in analogous Voronoi cells. Two points in analogous cells are at least $m - \frac{2}{3}$ apart. Since $d(q, q') \le m - 1$, q, q' cannot be in different tiles but in analogous Voronoi cells.

The probability that L is BLUE is the prob that V_{p_1}, V_{p_2}, …, V_{p_m} are all BLUE.

Let $p \in P$. We determine a lower bound on the probability that V_p is RED. Recall that V_p is RED iff $p \in S$.

BILL TO BILL- I NEED TO FINISH THIS. IT REQUIRES THAT LEMMA ABOUT SIGN PATTERNS. ∎

References

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