BILL, RECORD LECTURE!!!!

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There Is a 2-Coloring Of the Plane Without a mono Red 3-Stick or a mono Blue Big-Stick

Exposition by William Gasarch-U of MD

Credit Where Credit is Due

The main result in these slides is due to Conlon and Wu (2022).

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Main Theorem

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Open Find an easier proof of $\mathbb{R}^2$ case.

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And we know
$1 = d(\vec{a}_1, \vec{a}_2)$,
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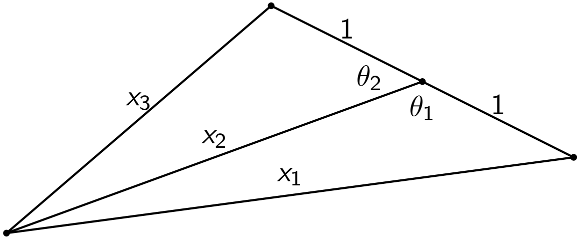
Algebra and Trigonometry



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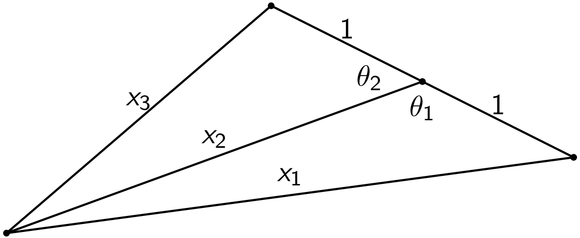
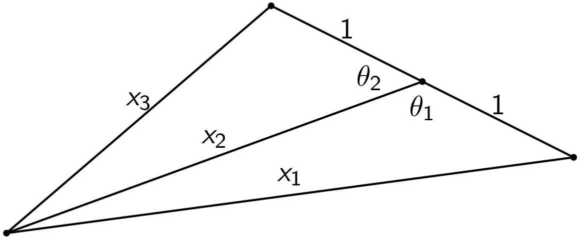


Bottom Triangle:

Algebra and Trigonometry



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Law of cosines: $x_1^2 = x_2^2 + 1 - 2x_2 \cos(\theta_1)$.

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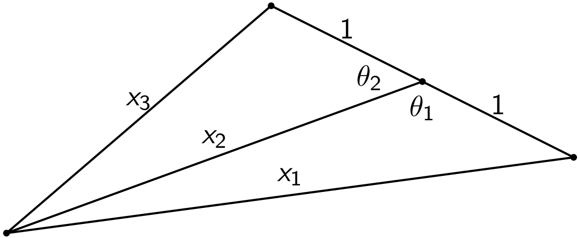

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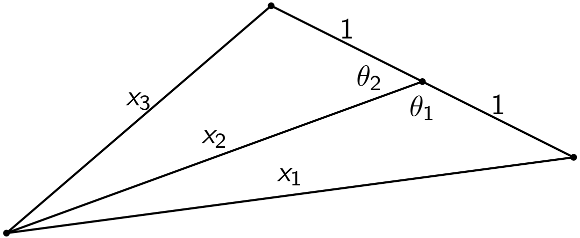

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Add to get

$$x_1^2 + x_3^2 = 2x_2^2 + 2.$$

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We will also have a condition on COL' that will make $\mathrm{COL}(\vec{a}) = \mathrm{COL}'(d(\vec{0}, \vec{a}))$ not have any **B** ℓ_m

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For all $1 \le i \le m$ let $x_i = d(\vec{0}, \vec{a}_i)$.
For all $1 \le i \le m-1$ we know $1 = d(\vec{a}_i, \vec{a}_{i+1})$.

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An Example of A Coloring with q = 5

$\mathrm{COL}''(0) = $ **R**
$\mathrm{COL}''(1) = $ **B**
$\mathrm{COL}''(2) = $ **B**
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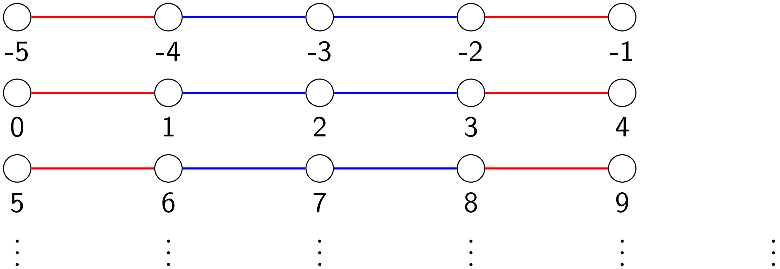

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The next slide recaps where we are and says why COL'' helps us.

COL'' and COL' and COL

Assume $\mathrm{COL}'' \colon \mathbb{Z}_q \to [2]$:

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Assume $\mathrm{COL}'' \colon \mathbb{Z}_q \to [2]$:

1) COL'' has no **R** solution (in $\mathbb{Z}_q$) to $y_1 + y_3 = 2y_2 + 2$.

COL'' and COL' and COL

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Lemmas and a Theorem of Independent Interest

Consider $\mathbf{p}(\mathbf{x}) \in \mathbb{R}[\mathbf{x}] \pmod{\mathbf{q}}$

What does $p(x) = x^2 + \pi x + e \pmod{13}$ mean?

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So it makes sense to consider $p(x) \pmod{q}$ where $p(x) \in \mathbb{R}[x]$.

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X hits at least $q/6$ of the intervals $[0, 1)$, $[1, 2)$, ..., $[q - 1, q)$.

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$\mathrm{SQ}_q = \{1^2 \pmod{q}, 2^2 \pmod{q}, \ldots, q^2 \pmod{q}\}$.

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1) Show $x^2 + \beta \pmod{q}$ hits $\ge (q+1)/2$ intervals.
2) Show that adding αx has a small effect since $|k\alpha \pmod{q}| \le \frac{1}{q}$.

We consider several sets and see how many intervals they hit.

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q is a prime so squaring is 2-to-1. Hence $|\mathrm{SQ}_q| = (q+1)/2$.

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Since every element in SQ_q is an integer, hits $(q+1)/2$ intervals.

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Since $k \not\equiv 0 \pmod{q}$, $\{k, 2k, \ldots, qk\} = \{1, 2, \ldots, q\}$. Hence $X = Y$.

Why $\mathbf{m = q^3}$?

We have shown that

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We will do this on the next slide.

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$\{f(k), f(2k), \ldots, f(qk)\}$:

$f(k) = f_1(k) + k\alpha$. **Key** Recall $k\alpha \pmod{q})| \leq \frac{1}{q} \leq 1$.

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Recap The set $Y = \{f_1(k), \ldots, f_1(qk)\}$ hits $(q+1)/2$ intervals of length 1.
$Z = \{f(k), \ldots, f(qk)\}$ can be viewed as taking every element in Y and adding or subtracting ≤ 1 to it. It is easy to show that Z hits $\geq q/6$ intervals.

Case 2: $k \equiv 0 \pmod{q}$

OMITTED FOR NOW.

Another Lemma Of Independent Interest

The Sign Function and Other Notation

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Notation If $\eta \in \{-1, 0, 1\}^*$ then $\eta(i)$ is the ith character in η.

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(x,y)	$(p_1(x,y), p_2(x,y), p_3(x,y))$	sign pattern
$(0,0)$	$(-3,-7,0)$	$(-,-,0)$
$(10,0)$	$(7,-27,40)$	$(+,-,+)$
$(0,10)$	$(17,23,-10)$	$(+,+,-)$
$(1,1)$	$(0,-6,3)$	$(0,-,+)$
$(5,10)$	$(22,13,30)$	$(+,+,+)$

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I doubt it has anywhere near 27.

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Lemma About Sign Patterns

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Proof Omitted. (It is difficult.)

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Proof Omitted. (It is difficult.)

Lemma is a corollary of a more general theorem by Olenik-Petrovsky-Thom-Milnor.

Big Theorem

Do on Whiteboard.