2-Color Rado's Theorem Over The Reals:
A Case Where it Does Not Hold
Exposition by William Gasarch

1 Pre Introduction

The following is well known.

Theorem 1.1 *For all* COL: $\mathsf{R}^2 \rightarrow [2]$ *there exists 2 points, same color, 1 inch apart.*

We rephrase this but first need some definitions.

Definition 1.2

1. ℓ_2 is 2 points in the plane an inch apart.
2. ℓ_3 is three colinear points p_1, p_2, p_3 where $d(p_1, p_2) = d(p_2, p_3) = 1$.
3. You can define ℓ_k.
4. Given COL: $\mathsf{R}^2 \rightarrow [2]$, a *RED* ℓ_k is an ℓ_k where all the points in it are RED. Similar for a BLUE ℓ_k.

Notation 1.3 Let $n, a, b \geq 2$. $\mathsf{R}^n \rightarrow (\ell_a, \ell_b)$ means that, for all COL: $\mathsf{R}^n \rightarrow [2]$, either there is a RED ℓ_a or a BLUE ℓ_b.

Many results are known about when $\mathsf{R}^n \rightarrow (\ell_a, \ell_b)$ and when $\mathsf{R}^n \not\rightarrow (\ell_a, \ell_b)$. We *do not* summarize them here. (When this document becomes part of a larger document we will.)

Conlon & Wu [2] showed that there exists m (around 10^{50}) such that

$$(\forall n)[\mathsf{R}^n \not\rightarrow (\ell_3, \ell_m)].$$

Implicit in their proof was a result in (what we call) Rado's theorem over the reals. They did not present it that way, nor did they isolate it from the rest of the proof. They also only proved the case that they needed.

In this document we present a generalization of that result and propose some new questions inspired by it.

2 Introduction

Recall Rado's Theorem:

Theorem 2.1 *Let $a_1, \ldots, a_n \in \mathsf{Z}$. The following are equivalent.*

- *For all finite colorings of N^+ there exists a mono solution to $\sum_{i=1}^{n} a_i x_i = 0$.*
- *There exists $I \subseteq \{1, \ldots, n\}$, $I \neq \emptyset$, such that $\sum_{i \in I} a_i = 0$.*

Rado's Theorem is not the end of the story. For example, the following questions are not resolved by Rado's Theorem:

1. *True or False: Every 2-coloring of* N^+ *has a mono solution to* $x + y - 4z = 0$.
2. *True or False: Every 2-coloring of* N^+ *has either a RED solution to* $x + y - 4z = 0$ *or a BLUE solution to* $x + y - 5z = 0$.
3. *Characterize the set of pairs of sets of equations* (E_1, E_2) *such that for every 2-coloring of* N^+ *either there is a RED solution to* E_1 *or a BLUE solution to* E_2.
4. *True or False: Every 2-coloring of* R^+ *has a mono solution to* $x + y - 4z = 0$.
5. *True or False: Every 2-coloring of* R^+ *has either a RED solution to* $x+y-4z = 0$ *or a BLUE solution to* $x + y - 5z = 0$.
6. *Characterize the set of pairs of sets of equations* (E_1, E_2) *such that for every 2-coloring of* R^+ *either there is a RED solution to* E_1 *or a BLUE solution to* E_2.

We suspect that questions 1,2,4,5 are not to hard.

In this document we given a condition on sets of equations E_1 (which will be just one linear equation) and E_2 (which will be a set of many linear equations) such that there is a 2-coloring of R^+ with no RED solution to A and no BLUE solution to B.

We will first need two lemmas that are interesting in their own right.

3 Theorems About Intersection

What does

$$f(x) = x^2 + \pi x + e \pmod{13}$$

mean? More concretely, what is

$$f(10) = 100 + 10\pi + e \pmod{13}?$$

We define it similar to mod13 over Z: subtract multiples of 13 until the result is in $[0, 13)$. For example $f(10) \sim 134.1325$, so $f(10) \bmod 13 \sim 4.1324$.

Definition 3.1 Let $f\colon \mathsf{R} \rightarrow \mathsf{R}$. Let $q \in \mathsf{N}$. Then $f \pmod{q}\colon \mathsf{R} \rightarrow [0, q)$ is the function that, on input x, returns the element of

$$\{f(x) + kq\colon k \in \mathsf{Z}\}$$

that is in $[0, q)$.

Note 3.2 Let $f \in \mathsf{R}[x]$. Let q be a prime. All $\equiv$ are mod q. We wonder when the following is true:

$$(1) \qquad a \equiv b \implies f(a) \equiv f(b).$$

1. If $a, b \in \mathsf{Z}$ and $f \in \mathsf{Z}[x]$ then (1) is TRUE.

2. If $a, b \in \mathsf{Z}$ and $f \in \mathsf{R}[x]$ then (1) is FALSE. Let $q = 13$, $a = 10$, $b = 23$, $f(x) = 0.5x$.

   $f(10) \bmod 13 = 5 \bmod 13 = 5$

   $f(23) \bmod 13 = 11.5 \bmod 13 = 11.5$

3. If $a, b \in \mathsf{R}$ and $f \in \mathsf{Z}[x]$ then (1) is FALSE. Let $q = 13$, $a = 13 + \frac{1}{13}$, $b = 130 + \frac{1}{13}$, and $f(x) = x^2$

   $f(13 + \frac{1}{13}) = 13^2 + 2 + \frac{1}{169} \equiv 2 + \frac{1}{169} \sim 2.005617$

   $f(130 + \frac{1}{13}) = 130^2 + 20 + \frac{1}{169} \equiv 7 + \frac{1}{169} \sim 7.005617$

4. If $a, b \in \mathsf{R}$ and $f \in \mathsf{R}[x]$ then (1) is FALSE. Either the second or third example on this list suffices.

Consider again

$$f(x) = x^2 + \pi x + e \pmod{13}$$

Let $m \in \mathsf{N}$ (we are thinking of m large). Each element of

$$X = \{f(1), f(2), \ldots, f(m)\}$$

is in one of $[0,1)$, $[1,2)$, ..., $[12,13)$. We wonder how the elements of X are distributed in those intervals. For example, how many of the intervals $[0,1), [1,2), \ldots, [12,13)$ intersect X.

More generally, Let

1. $\alpha, \beta \in \mathsf{R}$

2. q be a prime.

3. $f(x) = x^2 + \alpha x + \beta \pmod{q}$.

4. $m \in \mathsf{N}$. We think of m has large.

5. $X = \{f(1), f(2), \ldots, f(m)\}$.

Every element of X is in in one of $[0,1)$, $[1,2)$, ..., $[q-1,q)$. We wonder how many of the intervals $[0,1), [1,2), \ldots, [q-1,q)$ intersect X. Is it possible that most of the elements of X are in just a few intervals? In the appendix we have some empirical results on this question. The next lemmas answers the question for quadratics over Z, Q, and R. The answer is NO.

Theorem 3.3 *Let $\alpha \in \mathsf{Z}$, q a prime, and $m \geq q$.*

1. *Let $f(x) = x^2 + \alpha x \pmod{q}$.*

   Let

   $$X = \{f(1), f(2), \ldots, f(m)\}.$$

   Then at $\frac{q+1}{2}$ of the intervals $[0,1)$, $[1,2)$, ..., $[q-1,q)$ intersect with X. (Since $a \in \mathsf{Z}$ in reality $f(x) \in \{0, 1, \ldots, q-1\}$. We use intervals to be consistent with the versions over Q and R.)

2. *Let* $\beta \in \mathsf{R}$. *Let* $p(x) = x^2 + \alpha x + \beta$.

   Let $f(x) = p(x) \pmod q$, *and*

$$X = \{f(1), f(2), \ldots, f(m)\}.$$

   Then $\frac{q+1}{2}$ *of the intervals* $[0,1)$, $[1,2)$, $\ldots$, $[q-1,q)$ *intersect with* X. *(Part 2 follows from Part 1.)*

Proof:

We show that f is a $\le$ 2-to-1 map.

Assume $i \ne j$. All $\equiv$ are mod q.

$$f(i) = f(j)$$

$$i^2 + \alpha i \equiv j^2 + \alpha j$$

$$i^2 - j^2 \equiv \alpha j - \alpha i$$

$$(i-j)(i+j) \equiv \alpha(j-i)$$

Since $i \ne j$ we can divide by $i - j$.

$$(i+j) \equiv -\alpha$$

$$j \equiv -i - \alpha.$$

For all j except the case where $j \equiv -j - \alpha$ ($j \equiv -\alpha/2$) there is an $i \ne j$ such that $f(i) = f(j)$. Hence

$$|X| = |\{f(1), \ldots, f(q-1)\}| = \frac{q+1}{2}.$$

▮

Theorem 3.4 *Let* $r, s \in \mathsf{Z}$ *such that* $\gcd(r,s) = 1$, q *a prime such that* $s \not\equiv 0 \pmod q$, *and* $m \ge sq$.

1. *Let* $f(x) = x^2 + \frac{r}{s}x \pmod q$.

   Let

$$X = \{f(1), f(2), \ldots, f(m)\}.$$

   Then at least $q/2$ *of the intervals* $[0,1)$, $[1,2)$, $\ldots$, $[q-1,q)$ *intersect with* X.

2. *Let* $\beta \in \mathsf{R}$. *Let* $p(x) = x^2 + \frac{r}{s}x + \beta$.

   Let $f(x) = p(x) \pmod q$, *and*

$$X = \{f(1), f(2), \ldots, f(m)\}.$$

   Then at least $q/2$ *of the intervals* $[0,1)$, $[1,2)$, $\ldots$, $[q-1,q)$ *intersect with* X. *(Part 2 follows from Part 1.)*

Proof:

We show that f is $\le$ 2-to-1 when restricted to

$$X_1 = \{f(s), f(2s), f(3s), \ldots f(qs)\}.$$

Assume $i \ne j$. All $\equiv$ are mod q.

$$f(is) = f(js)$$

$$(is)^2 + \frac{r}{s}is \equiv (js)^2 + \frac{r}{s}js$$

$$i^2s^2 + ir \equiv j^2s^2 + jr$$

$$(i^2 - j^2)s^2 \equiv (j-i)r$$

$$(i-j)(i+j)s^2 \equiv (j-i)r$$

Since $i \ne j$ we can divide by $i - j$.

$$(i+j)s^2 \equiv -r \pmod q$$

$$(i+j) \equiv -\frac{r}{s^2}$$

$$i = -j + \frac{r}{s^2}.$$

Hence, for every i, there is at most one $j \ne i$ such that $f(si) = f(sj)$ (there will be no such i if $i \equiv -j - \frac{r}{s^2}$).

Therefore

$$|X| \le |\{f(s), \ldots, f(sq)\}| \le \frac{q}{2}.$$

∎

Theorem 3.5 *Let $\alpha \in \mathsf{R}$, q a prime, and $m \geq q^3$.*

1. *Let $p(x) = x^2 + \alpha x$.*
   Let $f(x) = p(x) \pmod q$, and
   $$X = \{f(1), f(2), \ldots, f(m)\}.$$
   Then at least $q/6$ of the intervals $[0,1)$, $[1,2)$, …, $[q-1,q)$ intersect with X.

2. *Let $\beta \in \mathsf{R}$. Let $p(x) = x^2 + \alpha x + \beta$.*
   Let $f(x) = p(x) \pmod q$, and
   $$X = \{f(1), f(2), \ldots, f(m)\}.$$
   Then at least $q/6$ of the intervals $[0,1)$, $[1,2)$, …, $[q-1,q)$ intersect with X. (Part 2 is the same proof as Part 1 just a bit messier.)

Proof: Consider

$$\alpha \bmod q, \quad 2\alpha \bmod q, \quad \ldots, \quad q^2\alpha \bmod q.$$

Map each one to which interval $[0,1)$, …, $[q-1,q)$ that it is in. Some interval has $\geq q$ of these values. Two of those values are $\leq 1/q$ apart. So there exists i, j such that

$$|i\alpha \pmod q - j\alpha \pmod q| \leq \frac{1}{q}.$$

Hence there exists k with $|k| \leq q^2$ such that $|k\alpha \pmod q| \leq \frac{1}{q}$. We will assume $k > 0$. the case where $k < 0$ is similar. There are two case depending on if $k \equiv 0 \pmod q$ or not.

Case 1: $k \not\equiv 0 \pmod q$.

We consider $f_1(x) = x^2 \pmod q$. Let

$$X_1 = \{f_1(1), f_1(2), \ldots, f_1(q)\} = \{1^2 \bmod q, 2^2 \bmod q, \ldots, q^2 \bmod q\}$$

By Theorem 3.3, X_1 intersects $\frac{q+1}{2}$ intervals.

We now look at f on multiples of k.

$$Y_1 = \{f_1(k), f_1(2k), \ldots, f_1(qk)\} = \{k^2, (2k)^2, \ldots, (qk)^2\}$$

(Notice that since $k \leq q^2$, $qk \leq q^3 \leq m$. Hence qk is in the range of f that we care about. That is why we need the premise $m \geq q^3$.)

Since $k \not\equiv 0 \pmod q$, $\{k, 2k, \ldots, qk\} = \{1, 2, \ldots, q\}$. Hence $X_1 = Y_1$.

We have shown that

$$\{f_1(k), f_1(2k), \ldots, f_1(qk)\}.$$

hits $(q+1)/2$ intervals. We need to show that $Z = \{f(1), f(2), \ldots, f(q^3)\}$ hits $\geq q/6$ intervals.

We show that $\{f(1), \ldots, f(q^3)\}$ hits $\geq q/6$ intervals by just looking at the subset $\{f(k), f(2k), \ldots, f(qk)\}$.

$\{f(k), f(2k), \ldots, f(qk)\}$:

$f(k) = f_1(k) + k\alpha$. *Key:* Recall $k\alpha \pmod q \le \frac{1}{q} \le 1$.

$f(2k) = f_1(2k) + 2k\alpha$. *Key:* Recall $2k\alpha \pmod q \le \frac{2}{q} \le 1$.

$\vdots \quad \vdots$

$f(qk) = f_1(2k) + qk\alpha$. *Key:* Recall $qk\alpha \pmod q \le \frac{q}{q} \le 1$.

Recap The set $Y_1 = \{f_1(k), \ldots, f_1(qk)\}$ hits $(q+1)/2$ intervals of length 1.

$Z_1 = \{f(k), \ldots, f(qk)\}$ can be viewed as taking every element in Y_1 and adding or subtracting ≤ 1 to it. It is easy to show that Z_1 hits $\ge q/6$ intervals.

Case 2: $k \equiv 0 \pmod q$.

Recall that $|k\alpha \mod q| \le \frac{1}{q}$ Since $k \equiv 0 \pmod q$, there exists $s \in \mathsf{Z}$ such that $k = sq$. Since $k \le q^2$, $s \le q$.

Hence

$|sq\alpha \bmod q| \le \frac{1}{q}$.

Hence $sq\alpha$ is within $\frac{1}{q}$ of an integer multiple of q. Let $r \in \mathsf{Z}$ and $0 \le \epsilon \le \frac{1}{q}$ be such that

$$sq\alpha = rq + \epsilon$$

$$\alpha = \frac{r}{s} + \frac{\epsilon}{q}$$

Let $\epsilon' = \frac{\epsilon}{q}$ so

$$\alpha = \frac{r}{s} + \epsilon' \text{ where } \epsilon' < \frac{1}{q^2}.$$

We can assume r, s have no common factors.

We consider $f_2(x) = x^2 + \frac{r}{s}x \bmod q$.

Let

$$Y_2 = \{f_2(s), f_2(2s), \ldots, f_2(qs)\}$$

(Note that $qs \le q^2$.)

By Theorem 3.4 Y_2 hits at least $q/2$ intervals. We show that

$|Z_2| = |\{f(s), f(2s), \ldots, f(qs)\}| \le$.

$f(s) = s^2 + \alpha s = s^2 + \frac{r}{s}s + \epsilon' s = f_2(s) + \epsilon' s$. *Key:* $\epsilon' s \le \frac{s}{q^2}$.

$f(2s) = (2s)^2 + 2\alpha k = (2s)^2 + 2\frac{r}{s}s + 2\epsilon' s = f_2(2s) + 2\epsilon' s$. *Key:* $2\epsilon' s \le \frac{2s}{q^2}$.

$\vdots \quad \vdots$

$f(qs) = (qs)^2 + s\alpha k = (qs)^2 + s\frac{r}{s}k + q\epsilon' s = f_2(sk) + q\epsilon' s$. *Key:* $q\epsilon' s \le \frac{qs}{q^2} = \frac{s}{q} < 1$.

By the above Key's, for all i, $|f(is) - f_2(is)| \le < 1$.

Recap The set $Y_2 = \{f_2(s), f_2(2s), \ldots, f_2(qs)\}$ hits $q/2$ intervals of length 1.

$Z_2 = \{f(s), f(2s), \ldots, f(qk)\}$ can be viewed as taking every element in Y_2 and adding or subtracting ≤ 1 to it. It is easy to show that Z_2 hits $\ge q/6$ intervals.

∎

4 Theorems About Sign Patterns

Notation 4.1

1. If $a \in \mathsf{R}$ then

$$\text{sign}(a) = \begin{cases} - & \text{if } a < 0 \\ 0 & \text{if } a = 0 \\ + & \text{if } a > 0 \end{cases} \tag{1}$$

2. If $\eta \in \{-, 0, +\}^*$ then $\eta(i)$ is the ith character in η.

We will soon define sign-changes but first do an example.

Example 4.2 Let
$p_1(x, y) = x + 2y - 3$
$p_2(x, y) = -2x + 3y - 7$
$p_3(x, y) = 4x - y$

We care about $(\text{sign}(p_1(x, y)), \text{sign}(p_2(x, y)), \text{sign}(p_3(x, y)))$. We look at this sequence for some values of (x, y).

(x, y)	$(p_1(x, y), p_2(x, y), p_3(x, y))$	sign pattern
$(5, 0)$	$(2, -17, 20)$	$(+, +, -)$
$(1, 4)$	$(6, 3, 0)$	$(+, +, 0)$
$(5, 6)$	$(14, 1, 14)$	$(+, +, +)$
$(-5, 0)$	$(-8, 3, -20)$	$(-, +, -)$
$(-3, 3)$	$(0, 8, -15)$	$(0, +, -)$
$(0, \frac{7}{3})$	$(\frac{5}{3}, 0, -\frac{7}{3})$	$(+, 0, -)$
$(0, 2)$	$(1, -1, -2)$	$(+, -, -)$
$(\frac{1}{2}, 2)$	$(\frac{3}{2}, -2, 0)$	$(+, -, 0)$
$(4, 5)$	$(11, 0, 11)$	$(+, 0, +)$
$(5, 5)$	$(12, -2, 15)$	$(+, -, +)$
$(-5, -1)$	$(-10, 0, -19)$	$(-, 0, -)$
$(-5, -5)$	$(-18, -12, -15)$	$(-, -, -)$
$(0, \frac{3}{2})$	$(0, -\frac{5}{2}, -\frac{3}{2})$	$(0, -, -)$
$(0, 0)$	$(-3, -7, 0)$	$(-, -, 0)$
$(5, -5)$	$(-8, -32, 25)$	$(-, -, +)$
$(5, -1)$	$(0, -20, 21)$	$(0, -, +)$
$(-\frac{5}{7}, \frac{13}{7})$	$(0, 0, -\frac{33}{7})$	$(0, 0, -)$
$(\frac{7}{10}, \frac{14}{5})$	$(\frac{33}{10}, 0, 0)$	$(+, 0, 0)$
$(\frac{1}{3}, \frac{4}{3})$	$(0, -\frac{11}{3}, 0)$	$(0, -, 0)$

There are potentially $3^3 = 27$ sign patterns. (p_1, p_2, p_3) has at least 19. We show that there are exactly 19 and then prove a generalization.

Definition 4.3 Let $p_1, \ldots, p_M \in \mathsf{R}[x_1, \ldots, x_N]$. Let

$$X = (p_1, \ldots, p_M).$$

$\eta \in \{-, 0, +\}^M$ is a *sign pattern for* X if there exists $a_1, \ldots, a_N \in \mathsf{R}$ such that for all $1 \le i \le M$

$$\text{sign}(p_i(a_1, \ldots, a_N)) = \eta(i).$$

Theorem 4.4 *Let $p_1(x, y), p_2(x, y), p_3(x, y)$ be 3 linear expressions with 2 variables. Then, there are at most 19 sign patterns.*

Proof: For $i \in \{1, 2, 3\}$ draw the a line $p_i(x, y) = 0$. Then, the 3 expressions will form 3 lines. For any point on the plane, the location of the point with respect to the line determines the sign. If a point is on the line, the sign is 0. If it is on one side of the line, the sign is either positive or negative, and if the point is on the other side of the line, the sign will be flipped. Therefore, each region divided by the lines will represent a sign pattern with no 0, each line segment will represent a sign pattern with one 0, and the point where lines intersect will represent a sign pattern with at least 2 0s.

We want to obtain the maximum number of regions, line segments, and intersections. Now, start with a single line. There are 2 regions, 1 line segment, and 0 intersections. To obtain the maximum number of components, the second line should intersect the existing line, dividing 2 regions and 1 line segment. It also creates 2 new segments by dividing its own and an intersection. In an optimal situation, with 2 lines there are 4 regions, 4 line segments, and 1 intersection. Finally, by adding another line that intersects the 2 existing lines, it divides 3 regions and 2 line segments, and creates 3 line segments and 2 intersections. There are 7 regions, 9 line segments, and 3 intersections that can be created with 3 lines. In total, there are 19 components, each indicating a sign pattern. ∎

We will generalize Theorem 4.4 to n linear functions. We will then further generalize to n polynomials.

We need a lemma about lines in the plane before we can obtain a lemma about sign changes.

Lemma 4.5 *Let $n \ge 1$.*

1. *There is a way to place n lines in the plane so that there are $\frac{n^2+n+2}{2}$ regions, n^2 line segments, and $\frac{n^2-n}{2}$ intersections, which are in total $2n^2 + 1$ components.*

2. *For all sets of n lines in the plane there are $\le \frac{n^2+n+2}{2}$ regions, $\le n^2$ line segments, and $\le \frac{n^2-n}{2}$ intersections, which are in total $\le 2n^2 + 1$ components.*

Proof: We prove part 2. Part 1 is similar. We prove this by induction on n.
Base case: n**=1** With one line, there are $\frac{1^2+1+2}{2} = 2$ regions, $1^2 = 1$ line segment, and $1^2 - 1 = 0$ intersection.
Induction Hypothesis (IH) For any set of n lines in the plane there are $\le \frac{n^2+n+2}{2}$ regions, $\le n^2$ line segments, and $\le \frac{n^2-n}{2}$ intersections.
Induction Step Assume there is a set of $n + 1$ lines in the plane. View these are n lines plus another line L. By the IH the n lines form $\le \frac{n^2+n+2}{2}$ regions, $\le n^2$ line segments, and $\le \frac{n^2-n}{2}$ intersections.

We look at how many new regions, line segments, and intersections can be created.

1. We show that at most $n+1$ new regions are formed, so the total number of regions is at most $\frac{n^2+n+2}{2}+n+1=\frac{(n+1)^2+(n+1)+2}{2}$,

   Let L_1 be the first line that L hits. The L may have already divided an region in two. For every line that is encountered by L a new region is created. The case that maximizes the number of regions is when L hits all of the lines. That creates n regions. Upon leaving the last line it may create another region.

2. By a proof similar to the one for regions, one can show that at most $n+1$ new line segments are formed, so the total number of line segments is at most $n^2+n+n+1=(n+1)^2$.

3. Line L intersects at most n lines, so there are most n new intersections. Hence the number of intersections is at most $\frac{n^2-n}{2}+n=\frac{(n+1)^2-(n+1)}{2}$.

■

Theorem 4.6 *Let $p_1(x,y),\ldots,p_n(x,y)\in \mathsf{R}[x,y]$ be linear. Then there are at most $2n^2+1$ sign patterns.*

Proof: For $i\in\{1,\ldots,n\}$ draw the a line $p_i(x,y)=0$. Then, the n expressions will form n lines. As in the proof of Theorem 4.4 the number of sign patterns is bounded above bu the sum of the number of regions, line segments, and intersections. By Lemma 4.5 this sum is bounded by $2n^2+1$. ■

We will present a known generalize of Theorem 4.6.

Let $p_1,\ldots,p_M\in\mathsf{R}[x_1,\ldots,x_N]$. An obvious bound on the number of sign patterns is 3^M The following lemma, due to Oleinik-Petrovsky-Thom-Milnor (see the the book by Basu-Pollack-Roy [1]), shows that, if $N\ll M$, there are far less than 3^M sign patterns. We omit the proof.

CHAEWOON-KELIN-Want WRITE UP THE PROOF OF THIS. LATER.

Lemma 4.7

1. *Let $D,M,N\in\mathsf{N}$. Let $p_1,\ldots,p_M\in\mathsf{R}[x_1,\ldots,x_N]$. Assume that all of the p_i's are of degree $\le D$. The number of sign patterns for $(p_1,\ldots,p_N)$ is at most $\left(\frac{50DM}{N}\right)^N$.*

2. *Let $M\in\mathsf{N}$. Let $p_1,\ldots,p_M\in\mathsf{R}[x,y]$. Assume that all of the p_i's are of degree ≤ 1 (so linear). The number of sign patterns for $(p_1,\ldots,p_N)$ is at most $625M^2$. (This follows from Part 1).*

5 An Interesting 2-Coloring of R

Definition 5.1 Let r,b be such that $0\le r,b\le 1$ and $r+b=1$. We will form two colorings, $\mathrm{COL}'_{r,b}$ and $\mathrm{COL}_{r,b}$ though we will never use the subscripts—they are understood.

$\mathrm{COL}'\colon \mathsf{Z}_q\to[2]$ is defined as follows: For all $x\in\mathsf{Z}_q$:

- $\Pr(\mathrm{COL}'(x)=\mathrm{RED})=r$.
- $\Pr(\mathrm{COL}'(x)=\mathrm{BLUE})=b$.

Let COL: $\mathsf{R} \rightarrow [2]$ be defined as follows:

$$\mathrm{COL}(z) = \mathrm{COL}'(\lfloor z \rfloor \bmod q).$$

Example 5.2 We take $q = 5$. Let COL$'$ be defined as follows:

$\mathrm{COL}'(0) = \mathrm{RED}$
$\mathrm{COL}'(1) = \mathrm{BLUE}$
$\mathrm{COL}'(2) = \mathrm{BLUE}$
$\mathrm{COL}'(3) = \mathrm{RED}$
$\mathrm{COL}'(4) = \mathrm{RED}$
See Figure 1 for what COL looks like

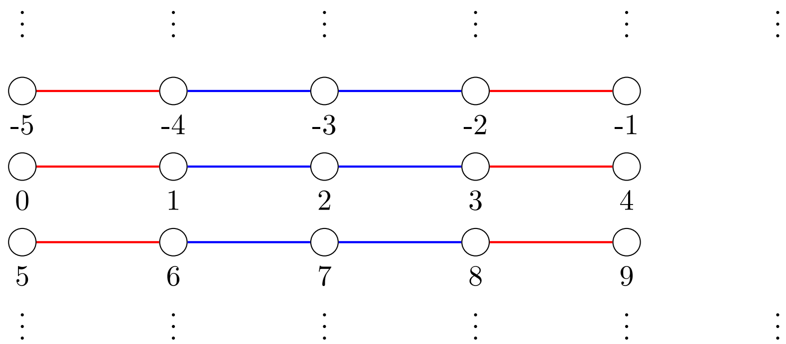

Figure 1: COL from COL′

Lemma 5.3 *Let* $0 \le b, r \le 1$, $b + r = 1$. *Let* q *be a prime. Let* $\mathrm{COL}' \colon \mathsf{Z}_q \to [2]$ *and* $\mathrm{COL} \colon \mathsf{R} \to [2]$ *be as defined in Definition 5.1 (We will not be using how* COL' *was formed. We will only use that* COL *is formed from* COL'*.) Assume there is a* COL*-RED solution to* $y_1 + y_3 = 2y_2 + 2$. *Then there is a* COL'*-RED solution to either*

- $n_1 + n_3 = 2n_2 + 1$, *or*
- $n_1 + n_3 = 2n_2 + 2$, *or*
- $n_1 + n_3 = 2n_2 + 3$.

Proof:

Assume there is a COL-RED solution to $y_1 + y_3 = 2y_2 + 2$. Let $n_1, n_2, n_3 \in \mathsf{Z}$ and $0 \le \epsilon_1, \epsilon_2, \epsilon_3 < 1$ be such that

$y_1 = n_1 + \epsilon_1$,
$y_2 = n_2 + \epsilon_2$,
$y_3 = n_3 + \epsilon_3$.
$\mathrm{COL}(y_i) = \mathrm{COL}'(n_i)$, so the n_i's are all RED.
Then

$$n_1 + \epsilon_1 + n_3 + \epsilon_3 = 2n_2 + 2\epsilon_2 + 2$$

$$n_1 + n_3 = 2n_2 + 2\epsilon_2 - \epsilon_1 - \epsilon_3 + 2$$

Since $n_1, n_2, n_3 \in \mathsf{Z}$, $2\epsilon_2 - \epsilon_1 - \epsilon_3 + 2 \in \mathsf{Z}$.
$2\epsilon_2 - \epsilon_1 - \epsilon_3 + 2 \in \{1, 2, 3\}$. Hence n_1, n_2, n_3 is a COL'-RED solution to

- $n_1 + n_3 = 2n_2 + 1$, or
- $n_1 + n_3 = 2n_2 + 2$, or
- $n_1 + n_3 = 2n_2 + 3$.

∎

Lemma 5.4 *Let $0 \le b, r \le 1$, $b + r = 1$. Let q be a prime. Let $\mathrm{COL}'\colon \mathsf{Z}_q \rightarrow [2]$ as defined in Definition 5.1 (We will not be using how COL' was formed. We will only use using that COL is formed from COL'.) Assume there is no COL'-RED solution to any of*

- $y_1 + y_3 = 2y_2 + a$

where $a \in N^{\ge 2}$
Then there is no COL-RED solution to at least one of

- $n_1 + n_3 = n_2 + b$

where $b \in [a]$

Proof:
Assume there is a COL-RED solution to $y_1+y_3 = 2y_2+a$. Let $n_1, n_2, n_3 \in \mathsf{Z}$ and $0 \le \epsilon_1, \epsilon_2, \epsilon_3 < 1$ be such that
$y_1 = n_1 + \epsilon_1$,
$y_2 = n_2 + \epsilon_2$,
$y_3 = n_3 + \epsilon_3$.
$\mathrm{COL}(y_i) = \mathrm{COL}'(n_i)$, so the n_i's are all RED.
Then

$$n_1 + \epsilon_1 + n_3 + \epsilon_3 = 2n_2 + 2\epsilon_2 + a$$

$$n_1 + n_3 = 2n_2 + 2\epsilon_2 - \epsilon_1 - \epsilon_3 + a$$

Since $n_1, n_2, n_3, a \in \mathsf{Z}$, $2\epsilon_2 - \epsilon_1 - \epsilon_3 + a \in \mathsf{Z}$.
$2\epsilon_2 - \epsilon_1 - \epsilon_3 + 2 \in [a + 1]$.
Since $0 \le \epsilon_1, \epsilon_2, \epsilon_3 < 1$

$$0 \le 2\epsilon_2 - \epsilon_1 - \epsilon_3 + a \le a + 1.$$

Hence n_1, n_2, n_3 is a COL'-RED solution to

- $n_1 + n_3 = n_2 + b$

where $b \in [a + 1]$
▮

Lemma 5.5 *Let $0 \le b, r \le 1$, $b + r = 1$. Let q be a prime. Let $\mathrm{COL}'\colon \mathsf{Z}_q \rightarrow [2]$ and $\mathrm{COL}\colon \mathsf{R} \rightarrow [2]$ be as defined in Definition 5.1 Then the probability that there is a mono solution to $y_1+y_3 = 2y_2+2$ is $\le 3q^2r^3 + 9qr^2$.*

Proof: By Lemma 5.3 the probability that there is a COL'-RED solution to

- $n_1 + n_3 = 2n_2 + 1$, or
- $n_1 + n_3 = 2n_2 + 2$, or

- $n_1 + n_3 = 2n_2 + 3$

bounds the probability that there is a COL-RED solution to $y_1 + y_3 = 2y_2 + 2$.

All $\equiv$ are mod q.

To get the probability that there is a COL$'$-RED solution to $n_1 + n_3 = 2n_2 + 1$ is we need to upper bound the number of solutions to $n_1 + n_3 = 2n_2 + 1$. There are several types of solutions. We list them and the probability that they occur.

- Solutions where n_1, n_2, n_3 are all different. n_1, n_2 determine n_3. Hence there are $\le q^2$ such solutions. The Probably that any of them is RED is $\le q^2r^3$.
- Solutions where $n_1 = n_2$. Then n_1 determine n_3. Hence there are $\le q$ such solutions. The probability that any of them is RED is $\le qr^2$.
- Solutions where $n_1 = n_3$. Then n_1 determine n_2. Hence there are $\le q$ such solutions. The probability that any of them is RED is $\le qr^2$.
- Solutions where $n_2 = n_3$. Then n_2 determine n_1. Hence there are $\le q$ such solutions. The probability that any of them is RED is $\le qr^2$.
- Solutions where n_1, n_2, n_3 are all different. n_1, n_2 determine n_3. Hence there are $\le q^2$ such solutions. The probability that any of them is RED is $\le q^2r^3$.
- There are no solutions where $n_1 = n_2 = n_3$. So this case does not contribute to the probability.

Hence the probability that there is a COL$'$-RED solution to $n_1 + n_3 = 2n_2 + 1$ is

$$\le q^2r^3 + 3qr^2$$

The same reasoning applies to $n_1 + n_3 = n_2 + 2$ and $n_1 + n_3 = n_2 + 3$. Hence the probability that there is a COL$'$-RED solution to any of the three equations is

$$\le 3q^2r^3 + 9qr^2$$

∎

Lemma 5.6 *Let $0 \le a, b, r \le 1$, $b+r = 1$. Let q be a prime. Let $\mathrm{COL}' \colon \mathsf{Z}_q \to [2]$ and $\mathrm{COL} \colon \mathsf{R} \to [2]$ be as defined in Definition 5.1 Then the probability that there is a mono solution to $y_1+y_3 = 2y_2+a$ is*

$$\le (a+1)(q^2r^3 + 3qr^2)$$

Proof: By Lemma 5.4 the probability that there is a COL$'$-RED solution to

- $n_1 + n_3 = 2n_2 + 1$, or
- $n_1 + n_3 = 2n_2 + 2$, or
- $n_1 + n_3 = 2n_2 + 3$

- $\vdots$

- $n_1 + n_3 = 2n_2 + a + 1$

bounds the probability that there is a COL-RED solution to $y_1 + y_3 = 2y_2 + a$.

All $\equiv$ are mod q.

To get the probability that there is a COL$'$-RED solution to $n_1+n_3 = 2n_2+1$ we need to upper bound the number of solutions to $n_1 + n_3 = 2n_2 + 1$. There are several types of solutions. We list them and the probability that they occur.

- Solutions where n_1, n_2, n_3 are all different. n_1, n_2 determine n_3. Hence there are $\le q^2$ such solutions. The Probably that any of them is RED is $\le q^2r^3$.

- Solutions where $n_1 = n_2$. Then n_1 determine n_3. Hence there are $\le q$ such solutions. The probability that any of them is RED is $\le qr^2$.

- Solutions where $n_1 = n_3$. Then n_1 determine n_2. Hence there are $\le q$ such solutions. The probability that any of them is RED is $\le qr^2$.

- Solutions where $n_2 = n_3$. Then n_2 determine n_1. Hence there are $\le q$ such solutions. The probability that any of them is RED is $\le qr^2$.

- Solutions where n_1, n_2, n_3 are all different. n_1, n_2 determine n_3. Hence there are $\le q^2$ such solutions. The probability that any of them is RED is $\le q^2r^3$.

- There are no solutions where $n_1 = n_2 = n_3$. So this case does not contribute to the probability.

Hence the probability that there is a COL$'$-RED solution to $n_1 + n_3 = 2n_2 + 1$ is

$$\le q^2r^3 + 3qr^2$$

The same reasoning applies to $n_1+n_3 = n_2+2$, …, $n_1+n_3 = n_2+a+1$. Hence the probability that there is a COL$'$-RED solution to any of the $a + 1$ equations is

$$\le (a + 1)(q^2r^3 + 3qr^2).$$

∎

Lemma 5.7 *Let q be a prime and let $m \ge q^3$. Let $p_1(x, y), \ldots, p_m(x, y) \in \mathsf{Z}[x, y]$ be such that the following hold:*

1. *For all i, $p_i(x, y)$ is linear in x, y. We intend $p_i(x, y)$ to be a function from R^2 to R.*

2. *The coefficients of $p_i(x, y)$ are quadratic polynomials in i over Z. Formally*

$$p_i(x, y) = a(i)x + b(i)y + c(i)$$

where $a, b, c \in \mathsf{Z}[i]$ and are quadratic.

3. If $a, d \in [0, 2m^2]$ then, for all i, $0 \le p_i(a,d) \le 2m^2$

Let b, r be such that $0 \le b, r \le 1$ and $b + r = 1$. Let COL be as in Definition 5.1. THEN

$$\Pr(\exists a, d \in \mathsf{R}, \mathrm{COL}(p_1(a,d)) = \cdots = \mathrm{COL}(p_m(a,d)) = \mathrm{BLUE}) \le 2500 m^6 b^{m/6}.$$

Proof:

The proof is in two parts

PART ONE: The Set of Intervals Mod q.

Fix $a, d \in \mathsf{R}$. We want to bound

$$\Pr(\mathrm{COL}(p_1(a,d)) = \cdots = \mathrm{COL}(p_m(a,d)) = \mathrm{BLUE}).$$

Recall that $\mathrm{COL}(z) = \mathrm{COL}'(z \bmod q)$. Hence, in order to have

$$\mathrm{COL}(p_1(a,d)) = \cdots = \mathrm{COL}(p_m(a,d)) = \mathrm{BLUE})$$

we need to have the following happen:

- $p_1(a,d) \bmod q$ is in an interval that COL' colors BLUE. This occurs with probability b.
- $p_2(a,d) \bmod q$ is in an interval that COL' colors BLUE. This occurs with probability b.
- Etc until $p_m(a,d) \bmod q$ is in an interval that COL' colors BLUE. This occurs with probability b.

At first glance you might think the probability of this happening is b^m which is small, so good news for us. Alas no. Here is an extreme possibility (that we later show cannot happen): all of the $p_i(a,d) \bmod q$ are in the same interval. This raises the question: how many distinct intervals do we get?

Let $F(x) = p_x(a,d)$. By premise 2, $F(x)$ is quadratic. Also recall that q is prime and $m \ge q^3$. Hence $F(x), m, q$ satisfy the premise of Theorem 3.5. Therefore

$$\{F(1) \bmod q, \ldots, F(m) \bmod q\}$$

will intersect $\ge q/6$ intervals. Hence

$$\{p_1(a,d) \bmod q, \ldots, p_m(a,d) \bmod q\}$$

will intersect $\ge q/6$ intervals.

Hence

$$\Pr(\mathrm{COL}(p_1(a,d)) = \cdots = \mathrm{COL}(p_m(a,d)) = \mathrm{BLUE}) \le b^{q/6}.$$

PART TWO: How Many Sets of Intervals?

The statement

$$\Pr(\mathrm{COL}(p_1(a,d)) = \cdots = \mathrm{COL}(p_m(a,d)) = \mathrm{BLUE}) \le b^{q/6}.$$

is not about a, d: its about interval patterns. That is, we map a, d to the set of intervals mod q that

$$p_1(a,d) \bmod q, \ldots, p_m(a,d) \bmod q$$

are in and then we look at the probability that all of those intervals are the same color.

So the question is: how many sets of intervals can there be?

Since the coloring is mod q we can assume $a, d \in [0, q)$. By premise 3, $0 \le p_i(a,d) \le 2m^2$. We now ask about the intervals (not mod q).

Consider the following questions:

- Of the intervals $[0,1)$, $[1,2)$, ..., $[2m^2-1, 2m^2)$ which one has $p_1(a,d)$? There are $2m^2$ possibilities. Note that which interval can be determined from the sign changes of the following sequence:

$$p_1(a,d)-1, p_1(a,d)-2), \ldots, p_1(a,d)-2m^2.$$

- Of the intervals $[0,1)$, $[1,2)$, ..., $[2m^2-1, 2m^2)$ which one has $p_2(a,d)$? There are $2m^2$ possibilities.

  Note that which interval can be determined from the sign changes of the following sequence:

$$p_2(a,d)-1, p_2(a,d)-2), \ldots, p_2(a,d)-2m^2.$$

- Etc until Of the intervals $[0,1)$, $[1,2)$, ..., $[2m^2-1, 2m^2)$ which one has $p_m(a,d)$? There are $2m^2$ possibilities.

  Note that which interval can be determined from the sign changes of the following sequence:

$$p_m(a,d)-1, p_m(a,d)-2), \ldots, p_m(a,d)-2m^2.$$

At a first glance it would seem like there are $(2m^2)^m$ possibilities. There are far less.

Consider the sequence of polynomials

$$p_1(x,y)-1, \ldots, p_1(x,y)-2m^2,$$

$$p_2(x,y)-1, \ldots, p_2(x,y)-2m^2,$$

$$\vdots$$

$$p_m(x,y)-1, \ldots, p_m(x,y)-2m^2.$$

The maximum number of sign changes in this sequence is an upper bound on the number of ways $p_1(a,d)$, ..., $p_m(a,d)$ can be in the intervals.

By Lemma 4.7.2 the number of sign changes is $\le 625(2m^3)^2 = 2500m^6$. Hence the number of ways $p_1(a,d), \ldots, p_m(a,d)$ are in the intervals is $\le 2500m^6$.

COMBINE PARTS ONE AND TWO

Map $(a, d) \in \mathsf{R}^2$ to the set of $\geq q/6$ intervals that contain

$$p_1(a, d) \bmod q, \ldots, p_m(a, d) \bmod q.$$

There are at most $2500m^6$ sets of intervals. Each one has probability $\leq b^{q/6}$ of having them all be blue. By the union bound the probability that one of those sets has every interval blue is

$$\leq 2500m^6 b^{q/6}.$$

▮

6 An Anti-Rado Theorem

Theorem 6.1 *Let $m \in \mathsf{N}$ (thing of m as large). Let $p_1(x, y), \ldots, p_m(x, y) \in \mathsf{Z}[x, y]$ be such that the following hold:*

1. *For all i, $p_i(x, y)$ is linear in x, y. We intend $p_i(x, y)$ to be a function from R^2 to R.*
2. *The coefficients of $p_i(x, y)$ are quadratic polynomials in i over Z.*
3. *If $a, d \in [0, 2m^2]$ then, for all i, $0 \leq p_i(a, d) \leq 2m^2$*

Then there exists a 2-coloring of RED *such that the following hold:*

1. *There is no RED solution to $y_1 + y_3 = 2y_2 + 2$.*
2. *There is no BLUE solution to the system of equations $\{p_i(x, y)\}_{i=1}^m$*

Proof: Let $0 \leq b, r \leq 1$ such that $b + r = 1$. We will pick b, r later. Let q a prime such that $m \geq q^3$. Let COL be as in Definition 5.1 with parameters r, s, q.

By Lemma 5.5,

$$\Pr(\text{there is a RED solution to } y_1 + y_3 = 2y_2 + 2) \leq 3q^2r^3 + 9qr^2.$$

By Lemma 5.7,

$$\Pr(\text{there is a BLUE solution to } \{p_i(x, y)\}_{i=1}^m) \leq 2500m^6 b^{q/6}.$$

Hence we want to pick b, r such that

$$3q^2r^3 + 9qr^2 + 2500m^6 b^{q/6} < 1$$

Choose $r = q^{-3/4}$ and $b = 1 - r$
Then,
$3q^2r^3 + 9qr^2 = 3q^{-1/4} + 9q^{-1/2} < 12q^{-1/4} < \frac{1}{2}$
for a sufficiently large q.
Similarly,
$2500m^6 b^{q/6} = 2500m^6(1 - q^{-3/4})^{q/6} \leq 2500m^6(1 - m^{-1/4})^{q/6} \leq 2500m^6(1 - m^{-1/4})^{\frac{m^{1/3}}{6}} < \frac{1}{2}$
for a sufficiently large m.

Therefore, if m and q is sufficiently large, there exists r and b that makes the probability of having a RED solution to $y_1 + y_3 = 2y_2 + 2$ or a BLUE solution to $\{p_i(x, y)\}_{i=1}^m$ is less than 1.

This shows that there exists a 2-coloring of RED such that:

There is no RED solution to $y_1 + y_3 = 2y_2 + 2$ And there is no BLUE solution to $\{p_i(x, y)\}_{i=1}^m$

TO BILL: q gets too big and r gets too small to clearly find out exact number for b using code. Using graphing calculator $q = 10^{11}$ was the first q that makes the equation less than 1, with $r \approx 2.815 \cdot 10^{-8}$

TO CHAEWOON: IF $q = 10^{11}$ THEN $m = q^3 = 10^{33}$. This actually is much better than Conlon-Wu's 10^{33}. DID WE DO ANYTHING TO MAKE IT better (perhaps better version of the signed patterns theorem) OR DID WE JUST TRY HARDER TO MINIMIZE m?

TO CHAEWOON AND KELIN-WE'LL TALK ABOUT HOW/IF WE CAN GET ANY INFO ON THIS POINT. ONE THOUGHT- IF THE INTERVAL THEOREM IS EMPIRICALLY GOOD ENOUGH WITH q^2 OR EVEN q, THAT MIGHT HELP IF WE ONLY CLAIM AN EMPIRICAL RESULT. LATER ∎

A Empirical Results on Intervals

Let $a \in \mathsf{Q}$ and q be a prime. Let $f(x) = x^2 + ax \pmod{q}$. By Theorem 3.4 the set

$$X = \{f(1), \ldots, f(q^2)\}$$

hits at least $q/2$ of the intervals

$$[0, 1), [1, 2), \ldots, [q - 1, q).$$

We suspected that far more intervals are hit, even if X stops at q. We wrote a program to test this.

a	q	Numb of Ints Hit by $f(1), \ldots, f(q)$	Ratio of Ints Hit by $f(1), \ldots, f(q)$	Numb of Ints Hit by $f(1), \ldots, f(q^2)$	Ratio of Ints Hit by $f(1), \ldots, f(q^2)$
0.1	13	8	0.615385	13	1.000000
0.1	17	12	0.705882	17	1.000000
0.1	19	14	0.736842	19	1.000000
0.1	23	15	0.652174	23	1.000000
0.1	29	18	0.620690	29	1.000000
0.1	31	20	0.645161	31	1.000000
0.1	37	24	0.648649	37	1.000000
0.1	41	26	0.634146	41	1.000000
0.1	43	27	0.627907	43	1.000000
0.1	47	31	0.659574	47	1.000000

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
0.2	13	8	0.615385	13	1.000000
0.2	17	10	0.588235	17	1.000000
0.2	19	12	0.631579	19	1.000000
0.2	23	13	0.565217	23	1.000000
0.2	29	19	0.655172	29	1.000000
0.2	31	20	0.645161	31	1.000000
0.2	37	24	0.648649	37	1.000000
0.2	41	27	0.658537	39	0.951220
0.2	43	26	0.604651	42	0.976744
0.2	47	31	0.659574	47	1.000000

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
0.3	13	9	0.692308	13	1.000000
0.3	17	11	0.647059	17	1.000000
0.3	19	12	0.631579	19	1.000000
0.3	23	17	0.739130	23	1.000000
0.3	29	19	0.655172	29	1.000000
0.3	31	20	0.645161	31	1.000000
0.3	37	23	0.621622	37	1.000000
0.3	41	28	0.682927	41	1.000000
0.3	43	29	0.674419	43	1.000000
0.3	47	30	0.638298	47	1.000000

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
0.4	13	9	0.692308	13	1.000000
0.4	17	10	0.588235	17	1.000000
0.4	19	12	0.631579	19	1.000000
0.4	23	12	0.521739	23	1.000000
0.4	29	19	0.655172	29	1.000000
0.4	31	19	0.612903	31	1.000000
0.4	37	23	0.621622	37	1.000000
0.4	41	27	0.658537	39	0.951220
0.4	43	28	0.651163	42	0.976744
0.4	47	27	0.574468	47	1.000000

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
0.5	13	8	0.615385	11	0.846154
0.5	17	11	0.647059	13	0.764706
0.5	19	12	0.631579	15	0.789474
0.5	23	15	0.652174	18	0.782609
0.5	29	18	0.620690	23	0.793103
0.5	31	20	0.645161	24	0.774194
0.5	37	22	0.594595	29	0.783784
0.5	41	26	0.634146	31	0.756098
0.5	43	28	0.651163	33	0.767442
0.5	47	30	0.638298	36	0.765957

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
0.6	13	8	0.615385	13	1.000000
0.6	17	11	0.647059	17	1.000000
0.6	19	10	0.526316	19	1.000000
0.6	23	16	0.695652	23	1.000000
0.6	29	15	0.517241	29	1.000000
0.6	31	20	0.645161	31	1.000000
0.6	37	24	0.648649	37	1.000000
0.6	41	25	0.609756	39	0.951220
0.6	43	27	0.627907	42	0.976744
0.6	47	28	0.595745	47	1.000000

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
0.7	13	7	0.538462	13	1.000000
0.7	17	10	0.588235	17	1.000000
0.7	19	12	0.631579	19	1.000000
0.7	23	13	0.565217	23	1.000000
0.7	29	21	0.724138	29	1.000000
0.7	31	19	0.612903	31	1.000000
0.7	37	22	0.594595	37	1.000000
0.7	41	25	0.609756	41	1.000000
0.7	43	29	0.674419	43	1.000000
0.7	47	31	0.659574	47	1.000000

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
0.8	13	9	0.692308	13	1.000000
0.8	17	12	0.705882	17	1.000000
0.8	19	13	0.684211	19	1.000000
0.8	23	16	0.695652	23	1.000000
0.8	29	17	0.586207	29	1.000000
0.8	31	21	0.677419	31	1.000000
0.8	37	24	0.648649	37	1.000000
0.8	41	29	0.707317	39	0.951220
0.8	43	28	0.651163	42	0.976744
0.8	47	30	0.638298	47	1.000000

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
0.9	13	7	0.538462	13	1.000000
0.9	17	11	0.647059	17	1.000000
0.9	19	14	0.736842	19	1.000000
0.9	23	16	0.695652	23	1.000000
0.9	29	20	0.689655	29	1.000000
0.9	31	21	0.677419	31	1.000000
0.9	37	22	0.594595	37	1.000000
0.9	41	31	0.756098	41	1.000000
0.9	43	31	0.720930	43	1.000000
0.9	47	30	0.638298	47	1.000000

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
1.0	13	7	0.538462	7	0.538462
1.0	17	9	0.529412	9	0.529412
1.0	19	10	0.526316	10	0.526316
1.0	23	12	0.521739	12	0.521739
1.0	29	15	0.517241	15	0.517241
1.0	31	16	0.516129	16	0.516129
1.0	37	19	0.513514	19	0.513514
1.0	41	21	0.512195	21	0.512195
1.0	43	22	0.511628	22	0.511628
1.0	47	24	0.510638	24	0.510638

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
1.1	13	7	0.538462	13	1.000000
1.1	17	12	0.705882	17	1.000000
1.1	19	15	0.789474	19	1.000000
1.1	23	15	0.652174	23	1.000000
1.1	29	18	0.620690	29	1.000000
1.1	31	21	0.677419	31	1.000000
1.1	37	26	0.702703	37	1.000000
1.1	41	29	0.707317	41	1.000000
1.1	43	28	0.651163	43	1.000000
1.1	47	31	0.659574	47	1.000000

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
1.2	13	8	0.615385	13	1.000000
1.2	17	10	0.588235	17	1.000000
1.2	19	12	0.631579	19	1.000000
1.2	23	14	0.608696	23	1.000000
1.2	29	20	0.689655	29	1.000000
1.2	31	21	0.677419	31	1.000000
1.2	37	24	0.648649	37	1.000000
1.2	41	29	0.707317	39	0.951220
1.2	43	26	0.604651	42	0.976744
1.2	47	31	0.659574	47	1.000000

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
1.3	13	9	0.692308	13	1.000000
1.3	17	12	0.705882	17	1.000000
1.3	19	11	0.578947	19	1.000000
1.3	23	12	0.521739	23	1.000000
1.3	29	20	0.689655	29	1.000000
1.3	31	20	0.645161	31	1.000000
1.3	37	20	0.540541	37	1.000000
1.3	41	25	0.609756	41	1.000000
1.3	43	24	0.558140	43	1.000000
1.3	47	30	0.638298	47	1.000000

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
1.4	13	7	0.538462	13	1.000000
1.4	17	11	0.647059	17	1.000000
1.4	19	12	0.631579	19	1.000000
1.4	23	15	0.652174	23	1.000000
1.4	29	17	0.586207	29	1.000000
1.4	31	20	0.645161	31	1.000000
1.4	37	21	0.567568	37	1.000000
1.4	41	26	0.634146	39	0.951220
1.4	43	26	0.604651	42	0.976744
1.4	47	30	0.638298	47	1.000000

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
1.5	13	7	0.538462	11	0.846154
1.5	17	9	0.529412	13	0.764706
1.5	19	13	0.684211	15	0.789474
1.5	23	16	0.695652	18	0.782609
1.5	29	17	0.586207	23	0.793103
1.5	31	21	0.677419	24	0.774194
1.5	37	21	0.567568	29	0.783784
1.5	41	24	0.585366	31	0.756098
1.5	43	29	0.674419	33	0.767442
1.5	47	31	0.659574	36	0.765957

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
1.6	13	9	0.692308	13	1.000000
1.6	17	10	0.588235	17	1.000000
1.6	19	13	0.684211	19	1.000000
1.6	23	14	0.608696	23	1.000000
1.6	29	18	0.620690	29	1.000000
1.6	31	18	0.580645	31	1.000000
1.6	37	24	0.648649	37	1.000000
1.6	41	27	0.658537	39	0.951220
1.6	43	28	0.651163	42	0.976744
1.6	47	30	0.638298	47	1.000000

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
1.7	13	8	0.615385	13	1.000000
1.7	17	11	0.647059	17	1.000000
1.7	19	11	0.578947	19	1.000000
1.7	23	16	0.695652	23	1.000000
1.7	29	19	0.655172	29	1.000000
1.7	31	20	0.645161	31	1.000000
1.7	37	24	0.648649	37	1.000000
1.7	41	28	0.682927	41	1.000000
1.7	43	27	0.627907	43	1.000000
1.7	47	30	0.638298	47	1.000000

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
1.8	13	8	0.615385	13	1.000000
1.8	17	11	0.647059	17	1.000000
1.8	19	10	0.526316	19	1.000000
1.8	23	15	0.652174	23	1.000000
1.8	29	17	0.586207	29	1.000000
1.8	31	20	0.645161	31	1.000000
1.8	37	23	0.621622	37	1.000000
1.8	41	26	0.634146	39	0.951220
1.8	43	28	0.651163	42	0.976744
1.8	47	31	0.659574	47	1.000000

a	q	Numb of Ints Hit	Ratio of Ints Hit	Numb of Ints Hit	Ratio of Ints Hit
1.9	13	6	0.461538	13	1.000000
1.9	17	10	0.588235	17	1.000000
1.9	19	14	0.736842	19	1.000000
1.9	23	18	0.782609	23	1.000000
1.9	29	18	0.620690	29	1.000000
1.9	31	20	0.645161	31	1.000000
1.9	37	26	0.702703	37	1.000000
1.9	41	26	0.634146	41	1.000000
1.9	43	27	0.627907	43	1.000000
1.9	47	31	0.659574	47	1.000000

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