Erdős' (l_6, l_6)

Kelin Zhu

January 25, 2026

1 Needed Lemmas

We will need Stewart's Theorem. For the sake of completeness we state and prove it.

Theorem 1.1. *(Stewart's Theorem) Let T be the triangle in Figure 1. Then $b^2m + c^2n = a(d^2 + mn)$.*

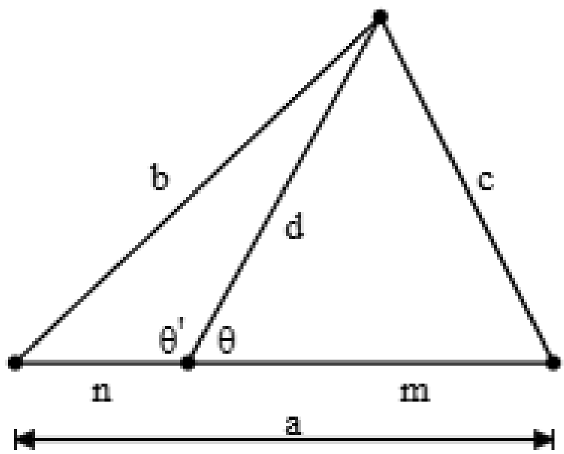

Figure 1: Premise of Stewart's Theorem

Proof. We use the law of cosines twice:
The triangle with sides b-d-n: $b^2 = d^2 + n^2 - 2dn\cos(\theta') = b^2 + d^2 + n^2 + 2dn\cos(\theta)$.
The triangle with sides d-c-m: $c^2 = d^2 + m^2 - 2dm\cos(\theta)$.

Multiply the first equation by n and the second by m to get

$$\begin{aligned} b^2m &= d^2m + n^2m + 2dnm\cos(\theta) \\ c^2n &= d^2n + m^2n - 2dmn\cos(\theta) \end{aligned}$$

Add these two to get

$$b^2m + c^2n = d^2(m+m) + mn(m+n) = d^2a + mna = a(d^2 + mn)$$

□

We will need a lemma about a system of equations over R.

Lemma 1.2. *Let E be the following system of equations*

$$\begin{aligned} a_1 + a_3 &= 2a_2 + \tfrac{1}{3}. \\ a_2 + a_4 &= 2a_3 + \tfrac{1}{3}. \\ a_3 + a_5 &= 2a_4 + \tfrac{1}{3}. \\ a_4 + a_6 &= 2a_5 + \tfrac{1}{3}. \end{aligned}$$

Then there is no real solution for E such that

$$\lfloor a_1 \rfloor \equiv \ldots \equiv \lfloor a_6 \rfloor \pmod 2.$$

Proof. All $\equiv$ are mod 2.
Assume, by way of contradiction, that $a_1, \ldots, a_6 \in \mathsf{R}$ is a solution such that

$$\lfloor a_1 \rfloor \equiv \cdots \equiv \lfloor a_6 \rfloor .$$

For $1 \le i \le 6$ let $b_i = a_i + (i-4)\lfloor a_3 \rfloor + (3-i)\lfloor a_4 \rfloor$. Note that

- $b_3 = a_3 - \lfloor a_3 \rfloor \in [0,1]$. Hence $\lfloor a_3 \rfloor \equiv 0$.
- $b_4 = a_4 - \lfloor a_4 \rfloor \in [0,1]$. Hence $\lfloor a_4 \rfloor \equiv 0$.

Claim 1: $b_1, \ldots, b_6$ is a solution to E.
Proof of Claim 1
We need to show that, for $2 \le i \le 5$

$$\begin{aligned} b_{i-1} + b_{i+1} &= 2b_i + \tfrac{1}{3} \\ a_{i-1} + (i-5)\lfloor a_3 \rfloor + a_{i+1} + (4-i)\lfloor a_4 \rfloor + (i-3)\lfloor a_3 \rfloor + (2-i)\lfloor a_4 \rfloor &= 2(a_i(i-4)\lfloor a_3 \rfloor + (3-i)\lfloor a_4 \rfloor) + \tfrac{1}{3} \\ a_{i-1} + a_{i+1} + (2i-8)\lfloor a_3 \rfloor + (6-2i)\lfloor a_4 \rfloor &= 2a_i + (2i-8)\lfloor a_3 \rfloor + (6-2i)\lfloor a_4 \rfloor + \tfrac{1}{3} \\ a_{i-1} + a_{i+1} &= 2a_i + \tfrac{1}{3} \end{aligned}$$

End of Proof of Claim 1

Claim 2 $\lfloor b_1 \rfloor \equiv \cdots \equiv \lfloor b_6 \rfloor \equiv 0$.
Proof of Claim 2 Since $\lfloor b_3 \rfloor \equiv 0$ it suffices to show that all of the $\lfloor b_i \rfloor$'s have the same parity.

$$\begin{aligned} \lfloor b_i \rfloor &= \lfloor a_i \rfloor + (i-4)\lfloor a_3 \rfloor + (3-i)\lfloor a_4 \rfloor \equiv \lfloor a_i \rfloor + i\lfloor a_3 \rfloor + \lfloor a_4 \rfloor - i\lfloor a_4 \rfloor \\ &\equiv \lfloor a_i \rfloor + \lfloor a_4 \rfloor + i(\lfloor a_3 \rfloor - \lfloor a_4 \rfloor) \end{aligned}$$

By the hypothesis of this Lemma, $\lfloor a_3 \rfloor \equiv \lfloor a_4 \rfloor$, so

$$\lfloor b_i \rfloor \equiv \lfloor a_i \rfloor + \lfloor a_4 \rfloor .$$

Since all of the $\lfloor a_i \rfloor$'s have the same parity, all the $\lfloor b_i \rfloor$'s have the same parity.
End of Proof of Claim 2

We use that $0 \le b_3, b_4 < 1$ and $(\forall i)[\lfloor b_i \rfloor \equiv 0]$.
Since $b_2 = 2b_3 - b_4 + \frac{1}{3}$, $b_2 \in (-\frac{2}{3}, \frac{7}{3})$. Since $\lfloor b_2 \rfloor \equiv 0$, $\lfloor b_2 \rfloor \in \{0, 2\}$. If $b_2 \ge 2$, then

$$4 \le 2b_2 + b_5 = 2(2b_3 - b_4 + \frac{1}{3}) + (2b_4 - b_3 + \frac{1}{3}) = 3b_3 + 1 < 4$$

which is impossible. Hence $\lfloor b_2 \rfloor = 0$.
By an identical argument $\lfloor b_5 \rfloor = 0$. Now that we know $\lfloor b_2 \rfloor = \lfloor b_3 \rfloor = 0$, we may use the same argument to show that $\lfloor b_1 \rfloor = 0$, similarly $\lfloor b_6 \rfloor = 0$. We then conclude that

$$\begin{aligned} 2 > b_1 + b_6 &= (2b_2 - b_3 + \tfrac{1}{3}) + (2b_5 - b_4 + \tfrac{1}{3}) \\ &= (2(2b_3 - b_4 + \tfrac{1}{3}) - b_3 + \tfrac{1}{3}) + (2(2b_4 - b_3 + \tfrac{1}{3}) - b_4 + \tfrac{1}{3}) \\ &= b_3 + b_4 + 2 \ge 2 \end{aligned}$$

which is a contradiction.

□

2 ℓ_6-ℓ_6 Theorem

The following theorem was proven by Erdös et al. [1].

Theorem (Erdős et al.)**.** *For all n, there exists a 2-coloring of $\mathbb{R}^n$ with no monochromatic ℓ_6*

Proof. The rough motivation is to color $\mathbb{R}^n$ in small spherical shells so that it's impossible for all 6 points to lie in a single shell, but not too small so that we have control over which point lies in which cell.
To formalize this intuition, color $x = (x_1, \ldots, x_n)$ by the quantity $\left\lfloor \frac{|x|^2}{6} \right\rfloor$ (mod 2). Hence, the **open** ball of radius $\frac{1}{6}$ around the origin is colored with 0, the shell surrounding it with thickness $\frac{1}{6}$ is colored with 1 (up to this point the open ball with radius $\frac{1}{3}$ is colored), and the next shell of $\frac{1}{6}$ is colored with 0, and so on. See Figure 2.
Assume, by way of contradiction, that there is a monochromatic ℓ_6. Let the points be denoted by $v_1, \ldots, v_6$, where $v_i - v_{i-1} = d$ for a constant unit vector d. See Figure 3.
For $1 \le i \le 6$ let $a_i = \frac{|v_i^2|}{6}$. By Stewart's Theorem, we have that $(\forall 2 \le i \le 5)[a_{i-1} + a_{i+1} = 2a_i + \frac{1}{3}]$. By Lemma 1.2 this set of 4 equations has no solution such that $\lfloor a_i \rfloor$ have the same parity. That is a contradiction.

□

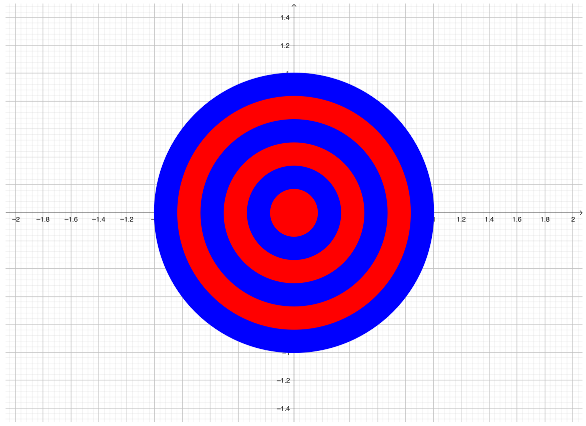

Figure 2: Illustration of our coloring scheme in $\mathbb{R}^2$ in the unit ball

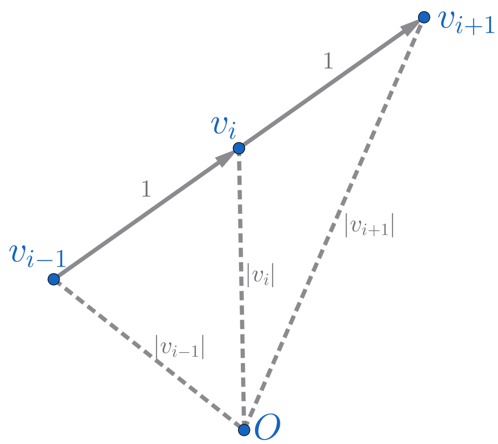

Figure 3: Stewart's theorem concludes that $|v_{i-1}|^2 + |v_{i+1}|^2 = 2|v_i|^2 + 2$.
Divide both sides by 6 to obtain the equations for a_i.

References

[1] P. Erdos, R. Graham, P. Montgomery, B. L. Rothchild, J. Spencer, and E. G. Straus. Euclidean Ramsey theory I. *Journal of Combinatorial Theory B*, 14:341–363, 1973.