Fourier Transform and Roth's Theorem

Dr. William Gasarch, Kelin Zhu

University of Marylnad

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Outline

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Theorem (VDW)

For all positive integers r and k, there exists positive integer N such that any r-coloring of $[N] = \{0, ..., N - 1\}$ contains a monochromatic k-AP.

Can we go further?

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Conjecture (Erdos-Turan)

[<.->] Let $r_k(N)$ be the size of the largest subset of [N] containing no k-AP. Then, for all positive integers r,

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ET would imply a mono k-AP of each color in VDW.

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Today: new upper bounds on $r_k(N)$ established in 2024 using high-powered analysis

Definition (DFT)

Let Z_N denote the integers modulo N. Also let $\chi(z) = e^{\frac{-2\pi i z}{N}}$. Then, the DFT of a function $f : Z_N \to \mathbb{C}$, denoted as \hat{f} , is defined as:

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We will work in Z_N . Although not all Z_N APs are \mathbb{Z} APs, we will place restrictions on the Z_N AP we find so that it is also a \mathbb{Z} AP.

Theorem (Plancherel)

$$\sum_{x \in Z_N} |f(x)|^2 = \frac{1}{N} \sum_{m \in Z_N} |\widehat{f}(m)|^2$$

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Theorem (Convolution (unconventional))

Define

$$(f*g)(x) = \sum_{y \in Z_N} f(y)g(x-y)$$

Then for any m,

$$\widehat{(f * g)}(m) = \widehat{f}(m)\widehat{g}(m)$$

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Figure: Two fourier coefficients of the quadratic residues mod 199 - an example of a set with small fourier coefficients

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Figure: Example of a large fourier coefficient

Hence, we can center an AP of "length" *cN* at the "occupied arc" so that it has a high density intersection with *mA*. We will prove that we can choose the common difference so that the AP consists of enough terms and doesn't "roll over" (isn't split into multiple \mathbb{Z} APs) when we undo multiplication by *m*.

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If N is large enough that we can repeat this process infinitely, A will have infinite density in some AP, impossible.

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Let $B = A \cap [\frac{N}{3}, \frac{2N}{3}]$. Note that if x, y, z is a 3-AP in Z_N such that $x + z \equiv 2y \pmod{N}$, with $x, y \in B$ and $z \in A$, then it is also a 3-AP in \mathbb{N} . Let Q be the number of 3-APs in A where $x, y \in B$. Then,

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$$Q = \sum_{\substack{x,y,z,x+y \equiv 2z \\ x,y,z,m}} B(x)B(y)A(z)$$
$$= \frac{1}{N} \sum_{\substack{x,y,z,m \\ x,y,z,m}} B(x)B(y)A(z)\chi(-m(x+z-2y))$$
$$= \frac{1}{N} \sum_{\substack{m \\ x,y,z,m}} \widehat{B}(m)\widehat{B}(-2m)\widehat{A}(m)$$

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At this point, we split the sum into $\frac{1}{N}|B|^2|A| + \sum_{m \neq 0} \widehat{B}(m)\widehat{B}(-2m)\widehat{A}(m).$ The term $\frac{1}{N}\sum_{m \neq 0} \widehat{B}(m)\widehat{B}(-2m)\widehat{A}(m)$ will be denoted by the "error term" *E*.

Small Fourier Coefficients

Apply Cauchy-Schwartz and Plancherel:

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Apply Cauchy-Schwartz and Plancherel:

$$\begin{split} E| &\leq \frac{1}{N} \max_{m \neq 0} |\widehat{A}(m)| \left| \sum_{m \neq 0} \widehat{B}(m) \widehat{B}(-2m) \right| \\ &\leq \frac{1}{N} \max_{m \neq 0} |\widehat{A}(m)| \sum_{m} |\widehat{B}(m) \widehat{B}(-2m)| \\ &\leq \frac{1}{N} \max_{m \neq 0} |\widehat{A}(m)| \left(\sum_{m} |\widehat{B}(m)|^2 \right)^{\frac{1}{2}} \left(\sum_{m} |\widehat{B}(-2m)|^2 \right)^{\frac{1}{2}} \\ &= \max_{m \neq 0} |\widehat{A}(m)| \left(\sum_{m} |B(m)|^2 \right)^{\frac{1}{2}} \left(\sum_{m} |B(-2m)|^2 \right)^{\frac{1}{2}} \\ &= \max_{m \neq 0} |\widehat{A}(m)| |B| \end{split}$$

If |B| is small, then one of $[0, \frac{N}{3})$ and $[\frac{2N}{3}, N)$ will have large intersection with *A*. We can then use the "density increase" argument on the next slides.

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If we suppose that $\max_{m\neq 0} |\widehat{A}(m)| \leq \frac{\delta^2}{10}N$ and $|B| \geq \frac{|A|}{5}$. we obtain that $|E| \leq \frac{1}{2N}|B|^2|A|$, and consequentially

$$Q = rac{1}{N}|B|^2|A| + E \geq rac{1}{N}|B|^2|A| - |E| \geq rac{\delta^3}{50}N^2$$

Hence, there are at least $\frac{\delta^3}{50}N^2 - \delta N$ 3-APs in *A* (accounting for the overcounted x = y = z), which is positive as long as $N > \frac{50}{\delta^2}$.

Suppose $|B| < \frac{|A|}{5}$, then if we assume WLOG that $|A \cap [0, \frac{N}{3})| \ge A \cap [\frac{2N}{3}, N)|$, we have $\frac{|A \cap [0, \frac{N}{3})|}{|[0, \frac{N}{3})|} \ge \delta + \frac{\delta}{5}$.

Suppose $|B| < \frac{|A|}{5}$, then if we assume WLOG that $|A \cap [0, \frac{N}{3})| \ge A \cap [\frac{2N}{3}, N)|$, we have $\frac{|A \cap [0, \frac{N}{3})|}{|[0, \frac{N}{3})|} \ge \delta + \frac{\delta}{5}$. Now, suppose $\max_{m \ne 0} |\widehat{A}(m)| > \frac{\delta^2}{10}N$. Let the maximum-attaining value of *m* be *r*. Pigeonhole: there exist $0 \le p < q \le \sqrt{N}$ such that $p - q \le \sqrt{N}$ and $r(p - q) \le \sqrt{N} \pmod{N}$. Suppose $|B| < \frac{|A|}{5}$, then if we assume WLOG that $|A \cap [0, \frac{N}{3})| \ge A \cap [\frac{2N}{3}, N)|$, we have $\frac{|A \cap [0, \frac{N}{3})|}{|[0, \frac{N}{3})|} \ge \delta + \frac{\delta}{5}$. Now, suppose $\max_{m \ne 0} |\widehat{A}(m)| > \frac{\delta^2}{10}N$. Let the maximum-attaining value of *m* be *r*. Pigeonhole: there exist $0 \le p < q \le \sqrt{N}$ such that $p - q \le \sqrt{N}$ and $r(p - q) \le \sqrt{N} \pmod{N}$.

Let d = p - q. Then, consider the AP between $-\left\lfloor \frac{\sqrt{N}}{6} \right\rfloor d$ and $\left\lfloor \frac{\sqrt{N}}{6} \right\rfloor d$ with common difference d; let the set of its members be P.

$$\begin{aligned} |\widehat{P}(r)| &= \left| \sum_{x \in P} \chi(-rx) \right| \\ &\geq Re\left(\sum_{x \in P} \chi(-rx) \right) \\ &\geq \frac{|P|}{2} \end{aligned}$$

Because $|-rx| \leq \frac{N}{6}$, so $Re(e^{\frac{-2\pi i r}{N}}) \geq Re(e^{\frac{\pm 2\pi i}{6}}) = \frac{1}{2}. \end{aligned}$

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$$\begin{split} |\widehat{P}(r)| &= \left|\sum_{x \in P} \chi(-rx)\right| \\ &\geq Re\left(\sum_{x \in P} \chi(-rx)\right) \\ &\geq \frac{|P|}{2} \\ \\ Because |-rx| &\leq \frac{N}{6}, \text{ so } Re(e^{\frac{-2\pi i x}{N}}) \geq Re(e^{\frac{\mp 2\pi i}{6}}) = \frac{1}{2}. \\ \text{Let } f(x) &= A(x) - \delta, \text{ and } g(x) = f * P(x). \text{ Note that } \widehat{f}(r) = \widehat{A}(r), \text{ and} \\ &\sum_{m} |g(m)| \geq |\widehat{g}(r)| = |\widehat{f}(r)| |\widehat{P}(r)| \geq \frac{\delta^{2}}{20} N|P| \end{split}$$

Since f has mean value zero, g also does (expand the sum), so

$$\max g(m) \geq \frac{\sum_{m,g(m)\geq 0} g(m)}{N} = \frac{\sum_m |g(m)|}{2N} = \frac{\delta^2}{40} |P|$$

Let the maximum-attaining m be x. By definition,

$$\begin{aligned} \frac{\delta^2}{40} |P| &\leq \sum_y f(y) P(x-y) \\ &= \sum_y A(y) P(x-y) - \sum_y \delta P(x-y) \\ &= |A \cap (x-P)| - \delta |x-P| \end{aligned}$$

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We've shown that the density of A in x - P is at least $\delta + \frac{\delta^2}{40}$. Also, note that the "difference" between the first and last terms of P, and therefore also of x - P, is $2\left\lfloor \frac{\sqrt{N}}{6} \right\rfloor d < N$, so x - P "rolls over" from n - 1 to 0 at most once in \mathbb{Z} .

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We've shown that the density of *A* in x - P is at least $\delta + \frac{\delta^2}{40}$. Also, note that the "difference" between the first and last terms of *P*, and therefore also of x - P, is $2 \lfloor \frac{\sqrt{N}}{6} \rfloor d < N$, so x - P "rolls over" from n - 1 to 0 at most once in \mathbb{Z} .

We can write $x - P = P_1 \cup P_2$ where P_1, P_2 are APs in \mathbb{N} . Easy to show that one of them (*P'*) has size at least $\frac{\delta^2}{80}|P|$ and satisfies $|A \cap P'| \ge \left(\delta + \frac{\delta^2}{80}\right)|P'|$

We've shown:

Proposition

As long as $N > \frac{50}{\delta^2}$, we can either find a 3-AP in A (*) or find an AP P' such that $|P'| = \Omega(\sqrt{N})$ and $|A \cap P'| \ge \left(\delta + \frac{\delta^2}{80}\right)|P'|$

Finish

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We can replace [*N*] with *P'* and reiterate - if *N* is large enough and we never reach \star , we get some AP *P''''''* such that *A* has density > 1 in *P'''''''*, which is impossible.

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Handout - $N = \exp(\exp(c\delta^{-1}))$ suffices.