Question

Is there a block design with parameters

$$(v, b, r, k, \lambda) = (35, 21, 3, 5, 1),$$

meaning: - $v = 35$ points, - $b = 21$ blocks, - each block has size $k = 5$, - each point occurs in $r = 3$ blocks, - **any two blocks intersect in exactly $\lambda = 1$ point**?

Answer: No, such a design does not exist.

Below is a complete proof.

Step 1: Basic counting checks

First verify that the standard incidence counting identities hold:

- Counting incidences by points and by blocks:

  $$vr = bk.$$

  Here

  $$35 \cdot 3 = 105 \quad \text{and} \quad 21 \cdot 5 = 105,$$

  so this condition is satisfied.

Thus there is no immediate contradiction from incidence counts alone.

Step 2: Count block–block intersections

We now use the assumption that **every pair of distinct blocks intersects in exactly one point**.

There are

$$\binom{21}{2} = 210$$

pairs of blocks. Since each pair intersects in exactly one point, the *total number of block–block intersection points*, counted with multiplicity over points, must be exactly 210.

Step 3: Count intersections via points

Fix a point x. By assumption, x lies in exactly $r = 3$ blocks. These 3 blocks form

$$\binom{3}{2} = 3$$

pairs of blocks that intersect at x.

Thus, each point contributes exactly 3 block–block intersections.

Since there are 35 points, the total number of block–block intersections counted via points is

$$35 \cdot 3 = 105.$$

Step 4: Contradiction

We have now counted the same quantity in two different ways:

- By counting block pairs directly: **210 intersections**
- By summing contributions over points: **105 intersections**

These must be equal, but

$$210\ /\ 105.=$$

This contradiction shows that the assumed design cannot exist.

Conclusion

There is **no block design** with parameters

$$(v, b, r, k, \lambda) = (35, 21, 3, 5, 1)$$

that satisfies the condition that **every pair of blocks intersects in exactly one point**.

The obstruction is purely combinatorial: the number of required block–block intersections is incompatible with the number forced by the replication number $r = 3$.

Remark

This argument is general: whenever every pair of blocks intersects in exactly one point, the parameters must satisfy

$$\binom{b}{2} = v\binom{r}{2},$$

which fails here since

$$\binom{21}{2} = 210 \ / \ 35\binom{3}{2} = 105.$$

Hence the nonexistence follows immediately from this necessary condition.