BILL, RECORD LECTURE!!!!

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Euclidean Ramsey Theory Chromatic Number of the Plane

Exposition by William Gasarch

July 2, 2025

Ramsey Theory VS Euclidean Ramsey Theory

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B) We do not care about the geometric size. For example, the Square can be any size.
In Euclidean Ramsey Theory we will be seek an object of a certain size, for example the unit square.

For All 2-Colorings of the Plane...

Thm $\forall \, \mathrm{COL}\colon \mathbb{R}^2 \to [2]$ $\exists$ 2 points, same color, 1 inch apart.

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Proof on the next page.

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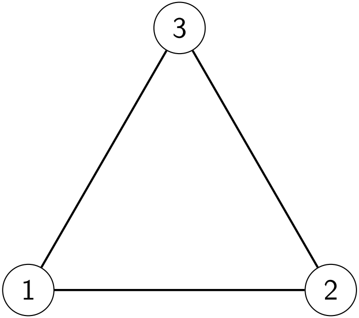

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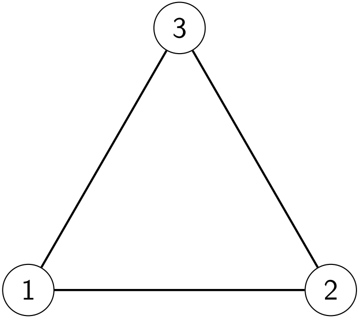

3 vertices and 2 colors. So 2 of the vertices are the same color.

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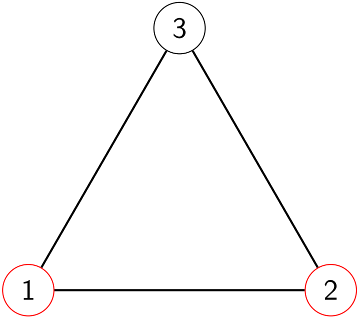

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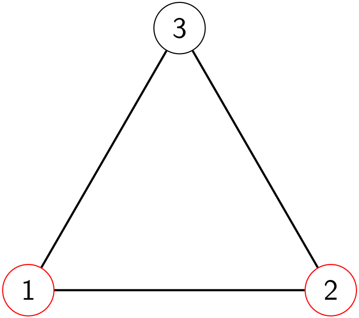

Vertices 1 and 2 are an inch apart.

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We investigate what χ can be.

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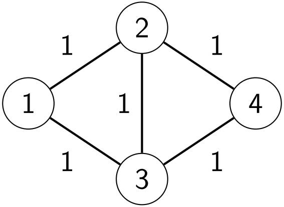

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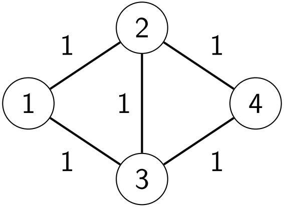

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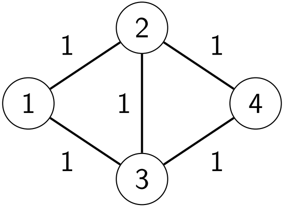

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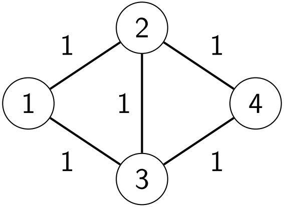

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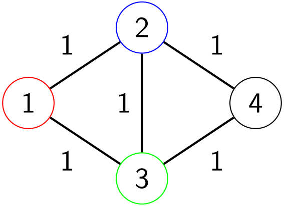

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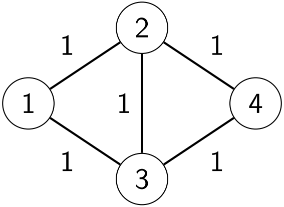

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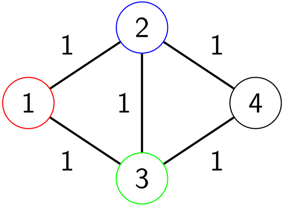

Hence $\mathrm{COL}(4) =$ **R**.

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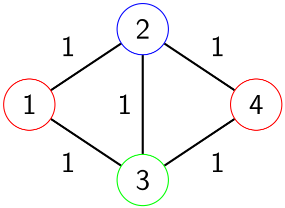

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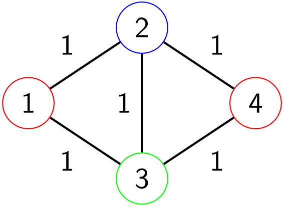

Distance from 1 to 4 is $\sqrt{3}$.

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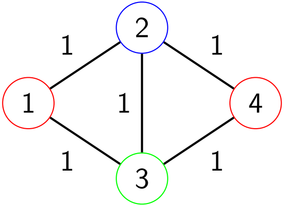

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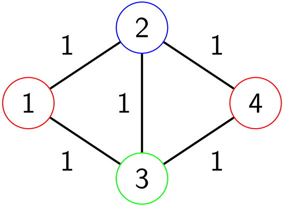

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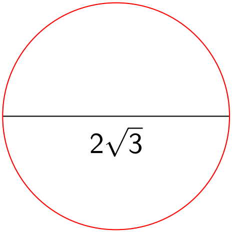

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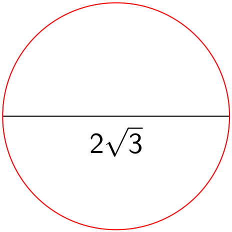

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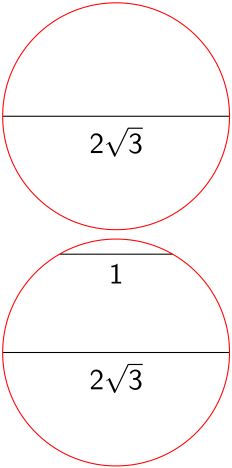

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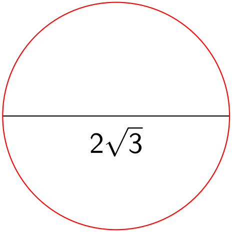

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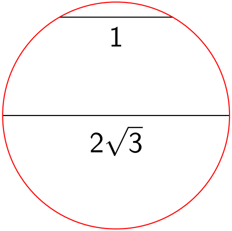

Endpoints of chord are **R** and an inch apart.

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It is a 7-vertex graph drawn in the plane with all sides of length 1.
It is an easy exercise to show that this graph is not 3-colorable.

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So we know that $\chi \geq 5$.

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Here is the 7-coloring:
`https://thatsmaths.com/2022/03/24/the-chromatic-number-of-the-plane/`