BILL, RECORD LECTURE!!!!

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Poly Van Der Warden's (PVDW) Theorem

Exposition by William Gasarch

April 12, 2025

Convention

Whenever I write a, d or a, d_1 or anything of that sort we are assuming $a, d \in \mathbb{N}$ and $a, d \geq 1$.

Recall VDW's Theorem

VDW's Theorem For all k, c there exists $W = W(k, c)$ such that for all $\mathrm{COL}\colon [W] \rightarrow [c]$ there exists a, d such that

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The coloring $RBRBRB \cdots$ has no two naturals 1-apart that have same color.

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Poly VDW Theorem For all $p_1, \ldots, p_k \in \mathbb{Z}[x]$ **with** $(\forall i)[p_i(0) = 0]$, and $c \in \mathbb{N}$, there exists $W = W(p_1, \ldots, p_k; c)$ such that for all $\mathrm{COL}\colon [W] \rightarrow [c]$ there exists a a, d such that

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Credit Where Credit is Due

Poly VDW theorem first proven by Bergelson and Leibman in **Polynomial Extensions of van der Waerden's and Szemeredi's Theorem** Journal of the AMS, Vol 9, 1996. Their paper is here:
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The first Elementary proof was by Walters in **Combinatorial proofs of the Polynomial Van Der Waerden Theorem and the Polynomial Hales-Jewitt Theorem** Journal of the London Math Soc., Vol 61, 2000.
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We present his proof.

Notation

$\mathbf{PVDW}(p_1(x), \ldots, p_k(x); c)$ means
There exists $W = W(p_1, \ldots, p_k; c)$ such that for all
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$\mathbf{PVDW}(p_1(x), \ldots, p_k(x))$ means
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We will discuss them later to contrast with todays proof.

Poly Van Der Warden's (PVDW) Theorem: $\mathrm{PVDW}(x^2)$

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We need to prove $U(r + 1)$ exists.
GOTO WHITE BOARD to prove

$$U(r + 1) \le$$

$$(U(r)W(2U(r), c^{U(r)}))^2 + U(r)W(2U(r), c^{U(r)}).$$

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Poly Van Der Warden's (PVDW) Theorem: $\mathrm{PVDW}(x^2 + x)$

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Keep that in mind.

This Generalizes

Thm Let $A, B \in \mathbb{Z}$. For all $c \in \mathbb{N}$ there exists $W = W(Ax^2 + Bx; c)$ st for all $\mathrm{COL}\colon [W] \rightarrow [c]$, $(\exists a, d)[a, a + Ad^2 + Bd$ same color$]$.

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Poly Van Der Warden's (PVDW) Theorem: $\mathrm{PVDW}(x^2, x^2 + x)$

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We Begin Proof of $\mathrm{PVDW}(x^2, x^2 + x)$

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Poly Van Der Warden's (PVDW) Theorem: $\mathrm{PVDW}(x^2, x)$

Exposition by William Gasarch

April 12, 2025

We Begin Proof of $\mathrm{PVDW}(x^2, x)$

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A Powerful Notation and a General Approach

Exposition by William Gasarch

April 12, 2025

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Proofs used $\mathrm{PVDW}(x^2 - \Box x \ldots, x^2, \ldots, x^2 + \Box x)$ for Base and Ind.

Key There are two lead coefficients and they are for quadratic-degree 2 and linear-degree 1. We will denote this $(1, 1)$: 1 quad lead coeff, 1 linear lead coeffs.

Associate to Each Set of Poly's an Index

Notation Let P be a finite subset of $\mathbb{Z}[x]$ such that $(\forall p \in P)[p(0) = 0]$.
Assume the max degree of a poly is d.
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A Powerful Notation

$\mathrm{PVDW}(1,0)$ means
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But what about $\mathrm{PVDW}(1,0)$? That was proven by VDW.

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Notation Let N^+ be $N \cup \{\omega\}$.
Let $n_d, \ldots, n_1 \in \mathbb{N}^+$ is defined in the obvious way.

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Actual Proof of Poly VDW Theorem

Poly VDW thm proven by ind on the indexes of sets. Ordering:

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2. $\mathrm{PVDW}(\omega, \ldots, \omega) \implies \mathrm{PVDW}(1, 0, \ldots, 0)$.
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3. Are better bounds known? See next slide.

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- Bill- remember to tell them how you learned of Shelah's result.

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So we can obtain a proof of VDW that you can write down nicely.

1. The proof really is the proof I already showed you.
2. While one COULD obtain a clean proof of VDW nobody has bothered writing this up (except me).