0.1 Quadratic VDW Theorem

We prove a lemma from which the Quadratic VDW Theorem will be obvious.

Lemma 0.1.1 *Fix c throughout. For all r there exists $U = U(r)$ such that for all c-colorings χ:$[U] \rightarrow [c]$ one of the following statements holds.*
Statement I: $\exists a, d \in [U]$, $\{a, a+d^2\}$ *same color.* **Statement II:** $\exists a, d_1, \ldots, d_r \in [U]$, $\{a, a+d_1^2, a+d_2^2, \ldots, a+d_r^2\}$ *diff colors.*

Proof:
We define $U(r)$ to be the least number such that this Lemma holds. We will prove $U(r)$ exists by giving an upper bound on it.
Base Case: $r = 1$. $U(1) = 2$.
If 1,2 are the same color take $a = 1$, $d = 1$.
If 1,2 are different colors take $a = 1$, $d_1 = 1$.
Induction Hypothesis Assume $U(r)$ exists, and let
Induction Step: Let $X = W(2U(r), c^{U(r)})$. We show that

$$U(r+1) \leq (X \times U(r))^2 + X \times U(r).$$

Let χ be a c-coloring of

$$[(X \times U(r))^2 + X \times U(r)].$$

View this set as $(X \times U(r))^2$ consecutive elements followed by X blocks of length $U(r)$. Let the last X blocks be

$$B_1, B_2, \ldots, B_X.$$

Restrict χ to the blocks. Let χ^*:$[X] \rightarrow [c^{U(r)}]$ be the coloring viewed as a $c^{U(r)}$-coloring of the blocks. By VDW applied to χ^* and the choice of X there exists $A, D' \in [X]$ such that

$$\{B_A, B_{A+D'}, \ldots, B_{A+(k+2U(r))D'}\} \text{ same color.}$$

Since the blocks viewed as points are D' apart, and each block has $U(r)$ elements in it, corresponding elements in adjacent blocks are $D = D' \times U(r)$ numbers apart.
Consider the coloring of B_A. Since B_A is of size $U(r)$ either
there exists $a, d \in B_A$ such that

- $\exists a, d$ such that $a, a+d^2$ are the same color (we ignore this case since if it happens, we are done), or
- $\exists a, (d'_1)^2, \ldots, (d'_r)^2$ such that

  $a, a+(d'_1)^2, \ldots, a+(d'_r)^2$ are different colors. In this case we also have that, since the B_A+jD are the same color as B_A, that for all $0 \le j \le X$

$$a+jD, a+d_1^2+jD, \ldots, a+d_r^2+jD \text{ are all different colors.}$$

Let the new anchor be $a = a' - D^2$. Let $d_i = D + d'_i$ for all $1 \le i \le r$, and $d_{r+1} = D$. We show that these parameters work.

- $a - D^2$ and $a - D^2 + (D+d_1)^2$ are a square apart.

  If they are the same color, we are done. If they are different colors then let $a - D^2 + (D+d_1)^2$ be colored c_1.
- $a - D^2$ and $a - D^2 + (D+d_2)^2$ are a square apart.

  If they are the same color, we are done. If they are different colors then let $a - D^2 + (D+d_2)^2$ be colored c_2.
- $\vdots \qquad \vdots \qquad \vdots$
- $a - D^2$ and $a - D^2 + (D+d_r)^2$ are a square apart.

  If they are the same color, we are done. If they are different colors then let $a - d^2 + (D+d_r)^2$ be colored c_r.
- $a - D^2$ and a are a square apart.

  If they are the same color, we are done. If they are different colors then let a be colored c_{r+1}.

We need to show that all of the c_i's are different.

Look at $a - D^2 + (D - d_i)^2 = a + 2d_iD$. This is the same color as $a + d_i$. Hence for all $1 \le i < j \le r$ $c_i \ne c_j$.

What about c_{r+1}? This is the color of a which we know is different from the color of any of the $a + d_i^2$ and hence for all $1 \le i \le r$, $c_i \ne c_{r+1}$. ▮