BILL, RECORD LECTURE!!!!

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Roth's Theorem: A Dense Enough Set Has a 3-AP

Exposition by William Gasarch
and
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April 25, 2025

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2) The $k = 3$ case which involves the Discrete Fourier Transform.

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There may be a HW where you are asked to prove theorems like the 0.67-Theorem.

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The **formal proof** will be long and use interesting math.

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Much of what I said here isn't quite right, but thats the intuition.

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We increase δ enough so that the density goes to ∞.

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Hence $\lim_{n\to\infty} \delta_n = \infty$.

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Next slide will deal with this, BUT WE WILL SKIP.

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View A as
$A \cap \{0, \ldots, N/3\} \cup \{N/3 + 1, \ldots, 2N/3\} \cup \{2N/3, \ldots, N - 1\}$.

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Case 1 The density of $A \cap \{N/3 + 1, \ldots, 2N/3\}$ is $\geq \delta/5$. Then do the proof on $A \cap \{N/3 + 1, \ldots, 2N/3\}$ is $\geq \delta/5$. Will get a legit 3-AP

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Case 2 The density of $A \cap \{N/3 + 1, \ldots, 2N/3\}$ is $< \delta/5$. One can show that either $A \cap \{0, \ldots, N/3\}$ or $A \cap \{N/3 + 1, \ldots, 2N/3\}$ is $> \delta$ (by enough so that if we keep doing this get > 0.67).

Detour:
Discrete Fourier Transform

Discrete Fourier Transform

Discrete Fourier Transform (DFT) Let $N \in \mathbb{N}$. Let $\chi(x) = e^{\frac{-2\pi ix}{N}}$. Then, the DFT of a function $f : \mathbb{Z}_N \to \mathbb{C}$, denoted as $\widehat{f}$, is defined as:

$$\widehat{f}(m) = \sum_{x=0}^{N-1} f(x)\chi(-mx)$$

We use this with f being the indicator function of a set $A \subseteq \mathbb{Z}_N$.

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See next few slides for examples.

Small Fourier Coefficients

Let A be the set of quadratic Residues mod 199. This is a random-looking set.

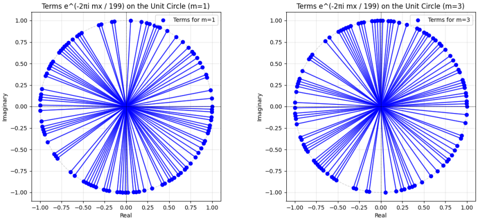

Figure: Left: Summands of $\widehat{A}(1)$. Right: Summands of $\widehat{A}(3)$

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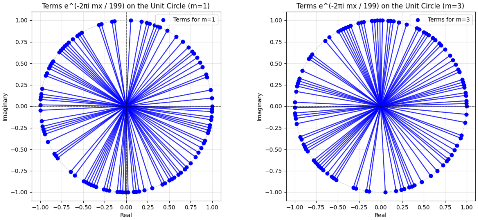

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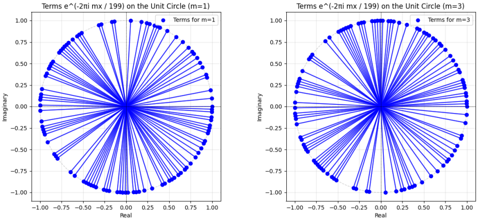

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All of the $\widehat{A}(m)$ for $m \neq 0$ are small.

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$\{17, 18, 59\}$ (Some noise tossed in)

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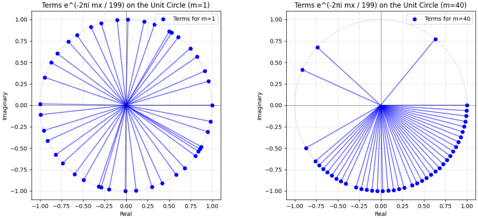

Figure: Left: Summands of $\widehat{A}(1)$. Right: Summands of $\widehat{A}(40)$

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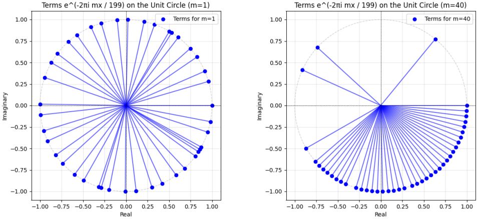

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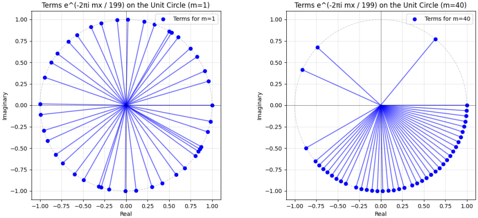

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Non-Rand Since A is non-random, $\exists m \neq 0$, $\widehat{A}(m)$ large.

PROJECT-Write Programs For The Following

Random Sets Given N, Form $A = QR_N$, the set of quad residues mod N. Then find, $\forall m \in A - \{0\}$, $\widehat{A}(m)$. Find the max M Should have $M \ll |A|$.

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$$A = \{a, a+d, \dots, a+Ld\} \text{ The arithmetic is mod } N \text{ .}$$

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We will analyze E very carefully.

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We are done bounding $|E|$.

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After the 4 cases we recap and see why we have the theorem.

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1) We use $|B| \ge \frac{|A|}{5}$ to show that $\frac{1}{N}|B|^2|A|$ is large.

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1) We use $|B| \geq \frac{|A|}{5}$ to show that $\frac{1}{N}|B|^2|A|$ is large.

2) We use $|B| \leq \frac{11|A|}{30}$ and $\max_{m\neq 0} \widehat{A}(m)| \leq \frac{\delta^2 N}{10}$ to show $|E|$ is small.

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Upshot

$$\frac{1}{N}|B|^2|A| \geq \frac{\delta^3 N^2}{25}.$$

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We leave it to the reader to determine N large enough so that this inequality holds.

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One can show from here that $A \cap P$ has high density. We omit this.

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Together get $|Q| \ge 1$.

Case 4 $\max_{m \ne 0} |\widehat{A}(m)| > \frac{\delta^2 N}{10}$. We show there is a long AP P such that the density of A in P is $\ge \delta + \frac{\delta^2}{40}$.

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