Small Poly VDW Numbers

April 10, 2025

Colorings and Square Differences

The following are all true:

1. There exists a number W_2 such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.

Colorings and Square Differences

The following are all true:

1. There exists a number W_2 such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.
2. There exists a number W_3 such that, for all 3-colorings of $\{1, \ldots, W_3\}$ there exists 2 nums, square-apart, same color.

Colorings and Square Differences

The following are all true:

1. There exists a number W_2 such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.
2. There exists a number W_3 such that, for all 3-colorings of $\{1, \ldots, W_3\}$ there exists 2 nums, square-apart, same color.
3. There exists a number W_4 such that, for all 4-colorings of $\{1, \ldots, W_4\}$ there exists two nums, square-apart, same color.

Colorings and Square Differences

The following are all true:

1. There exists a number W_2 such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.
2. There exists a number W_3 such that, for all 3-colorings of $\{1, \ldots, W_3\}$ there exists 2 nums, square-apart, same color.
3. There exists a number W_4 such that, for all 4-colorings of $\{1, \ldots, W_4\}$ there exists two nums, square-apart, same color.
4. For all c there exists a number W_c ....

Colorings and Square Differences

For all c there exists a number W_c such that for all c-colorings of $\{1, \ldots, W_c\}$ there exists two nums, square-apart, same color.

Colorings and Square Differences

For all c there exists a number W_c such that for all c-colorings of $\{1, \ldots, W_c\}$ there exists two nums, square-apart, same color.

The proofs in the literature of these theorems give EEEEEEEEEENORMOUS bounds on W_2, W_3, W_4, W_c. We look at easier proofs with two **points** in mind:

- Would they be good questions on a HS math competition?
- Which proofs do you prefer?

2-colorings and Square Differences

There exists a number W_2 such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.

2-colorings and Square Differences

There exists a number W_2 such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize W_2.

2-colorings and Square Differences

There exists a number W_2 such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize W_2.

Let COL be a 2-coloring of $\{1, 2, 3, \ldots\}$ with colorings **R** and **B**.
We can assume $\mathrm{COL}(1) = $ **R**.

2-colorings and Square Differences

There exists a number W_2 such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize W_2.

Let COL be a 2-coloring of $\{1, 2, 3, \ldots\}$ with colorings **R** and **B**.
We can assume $\mathrm{COL}(1) = $ **R**.
Since 1 is a square $\mathrm{COL}(2) = $ **B**.

2-colorings and Square Differences

There exists a number W_2 such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize W_2.

Let COL be a 2-coloring of $\{1, 2, 3, \ldots\}$ with colorings **R** and **B**.
We can assume $\mathrm{COL}(1) =$ **R**.
Since 1 is a square $\mathrm{COL}(2) =$ **B**.
Since 1 is a square $\mathrm{COL}(3) =$ **R**.

2-colorings and Square Differences

There exists a number W_2 such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize W_2.

Let COL be a 2-coloring of $\{1, 2, 3, \ldots\}$ with colorings **R** and **B**.
We can assume $\mathrm{COL}(1) =$ **R**.
Since 1 is a square $\mathrm{COL}(2) =$ **B**.
Since 1 is a square $\mathrm{COL}(3) =$ **R**.
Since 1 is a square $\mathrm{COL}(4) =$ **B**.

2-colorings and Square Differences

There exists a number W_2 such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize W_2.

Let COL be a 2-coloring of $\{1, 2, 3, \ldots\}$ with colorings **R** and **B**.
We can assume $\mathrm{COL}(1) =$ **R**.
Since 1 is a square $\mathrm{COL}(2) =$ **B**.
Since 1 is a square $\mathrm{COL}(3) =$ **R**.
Since 1 is a square $\mathrm{COL}(4) =$ **B**.
Since 1 is a square $\mathrm{COL}(5) =$ **R**.

2-colorings and Square Differences

There exists a number W_2 such that, for all 2-colorings of $\{1,\ldots,W_2\}$ there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize W_2.

Let COL be a 2-coloring of $\{1,2,3,\ldots\}$ with colorings **R** and **B**.
We can assume $\mathrm{COL}(1) =$ **R**.
Since 1 is a square $\mathrm{COL}(2) =$ **B**.
Since 1 is a square $\mathrm{COL}(3) =$ **R**.
Since 1 is a square $\mathrm{COL}(4) =$ **B**.
Since 1 is a square $\mathrm{COL}(5) =$ **R**.
AH-HA: $\mathrm{COL}(1) = \mathrm{COL}(5)$ and $5 - 1 = 4 = 2^2$. So $W_2 \le 5$.

2-colorings and Square Differences

There exists a number W_2 such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize W_2.

Let COL be a 2-coloring of $\{1, 2, 3, \ldots\}$ with colorings **R** and **B**.
We can assume $\mathrm{COL}(1) =$ **R**.
Since 1 is a square $\mathrm{COL}(2) =$ **B**.
Since 1 is a square $\mathrm{COL}(3) =$ **R**.
Since 1 is a square $\mathrm{COL}(4) =$ **B**.
Since 1 is a square $\mathrm{COL}(5) =$ **R**.
AH-HA: $\mathrm{COL}(1) = \mathrm{COL}(5)$ and $5 - 1 = 4 = 2^2$. So $W_2 \le 5$.
AH-HA: **RBRB** shows that $W_2 \le 5$.

2-colorings and Square Differences

There exists a number W_2 such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize W_2.

Let COL be a 2-coloring of $\{1, 2, 3, \ldots\}$ with colorings **R** and **B**.
We can assume $\mathrm{COL}(1) =$ **R**.
Since 1 is a square $\mathrm{COL}(2) =$ **B**.
Since 1 is a square $\mathrm{COL}(3) =$ **R**.
Since 1 is a square $\mathrm{COL}(4) =$ **B**.
Since 1 is a square $\mathrm{COL}(5) =$ **R**.
AH-HA: $\mathrm{COL}(1) = \mathrm{COL}(5)$ and $5 - 1 = 4 = 2^2$. So $W_2 \le 5$.
AH-HA: **RBRB** shows that $W_2 \le 5$.
So $W_2 = 4$.

2-colorings and Square Differences

There exists a number W_2 such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize W_2.

Let COL be a 2-coloring of $\{1, 2, 3, \ldots\}$ with colorings **R** and **B**.
We can assume $\mathrm{COL}(1) =$ **R**.
Since 1 is a square $\mathrm{COL}(2) =$ **B**.
Since 1 is a square $\mathrm{COL}(3) =$ **R**.
Since 1 is a square $\mathrm{COL}(4) =$ **B**.
Since 1 is a square $\mathrm{COL}(5) =$ **R**.
AH-HA: $\mathrm{COL}(1) = \mathrm{COL}(5)$ and $5 - 1 = 4 = 2^2$. So $W_2 \le 5$.
AH-HA: **RBRB** shows that $W_2 \le 5$.
So $W_2 = 4$.
Upshot Could be easy HS Math Comp Prob. No computer used.

3-colorings and Square Differences

There exists a number W_3 such that, for all 3-colorings of $\{1, \ldots, W_3\}$ there exists 2 nums, square-apart, same color.

3-colorings and Square Differences

There exists a number W_3 such that, for all 3-colorings of $\{1, \ldots, W_3\}$ there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize W_3.

3-colorings and Square Differences

There exists a number W_3 such that, for all 3-colorings of $\{1, \ldots, W_3\}$ there exists 2 nums, square-apart, same color.

Work on in groups and try to minimize W_3.

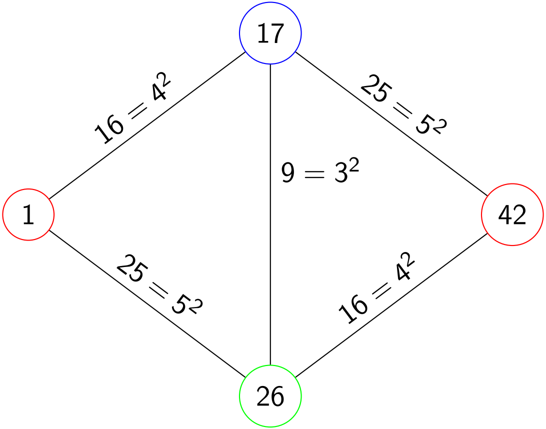


Figure: $\mathrm{COL}(x) = \mathrm{COL}(x + 41)$

Since $\mathrm{COL}(x) = \mathrm{COL}(x+41)\ldots$

Use $\mathrm{COL}(x) = \mathrm{COL}(x+41)$ to finish the proof and find upper bound on W_3.

Since $\mathrm{COL}(x) = \mathrm{COL}(x + 41) \ldots$

Use $\mathrm{COL}(x) = \mathrm{COL}(x + 41)$ to finish the proof and find upper bound on W_3.

$$\mathrm{COL}(1) = \mathrm{COL}(1{+}41) = \mathrm{COL}(1{+}2{\times}41) = \cdots = \mathrm{COL}(1{+}41{\times}41)$$

Since $\mathrm{COL}(x) = \mathrm{COL}(x + 41)\ldots$

Use $\mathrm{COL}(x) = \mathrm{COL}(x + 41)$ to finish the proof and find upper bound on W_3.

$$\mathrm{COL}(1) = \mathrm{COL}(1{+}41) = \mathrm{COL}(1{+}2{\times}41) = \cdots = \mathrm{COL}(1{+}41{\times}41)$$

So 1 and 41^2 are a square apart and the same color.
$W_3 \leq 1 + 41^2 = 1682$

Since $\mathrm{COL}(x) = \mathrm{COL}(x + 41)\ldots$

Use $\mathrm{COL}(x) = \mathrm{COL}(x + 41)$ to finish the proof and find upper bound on W_3.

$$\mathrm{COL}(1) = \mathrm{COL}(1{+}41) = \mathrm{COL}(1{+}2{\times}41) = \cdots = \mathrm{COL}(1{+}41{\times}41)$$

So 1 and 41^2 are a square apart and the same color.
$W_3 \leq 1 + 41^2 = 1682$
Can we get better bound on W_3?

Better Bound on W_3

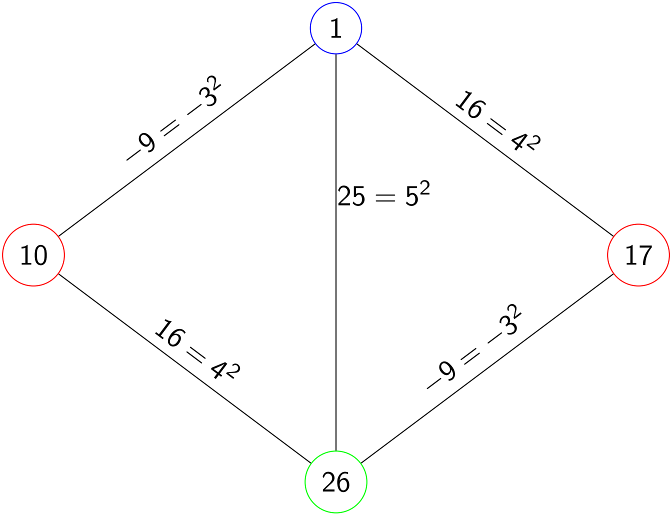


Figure: If $x \geq 10$ then $\mathrm{COL}(x) = \mathrm{COL}(x+7)$, so $W_3 \leq 59$

Reflection on W_3, W_4

Reflection on W_3, W_4

1. Problem 5 (so hard) on UMCP HS Math Comp, 2006:
Show that for all 3-colorings of $\{1, \ldots, 2006\}$ *there exists 2 numbers that are a square apart that are the same color*

Reflection on W_3, W_4

1. Problem 5 (so hard) on UMCP HS Math Comp, 2006:
   Show that for all 3-colorings of $\{1, \ldots, 2006\}$ *there exists 2 numbers that are a square apart that are the same color*
2. 240 took exam, 40 tried this problem, 10 got it right.

Reflection on W_3, W_4

1. Problem 5 (so hard) on UMCP HS Math Comp, 2006:
   Show that for all 3-colorings of $\{1, \ldots, 2006\}$ *there exists 2 numbers that are a square apart that are the same color*
2. 240 took exam, 40 tried this problem, 10 got it right.
3. Bill Gasarch and Matt Jordan proved, by hand, $W_3 = 29$.

Reflection on W_3, W_4

1. Problem 5 (so hard) on UMCP HS Math Comp, 2006: *Show that for all 3-colorings of* $\{1, \ldots, 2006\}$ *there exists 2 numbers that are a square apart that are the same color*
2. 240 took exam, 40 tried this problem, 10 got it right.
3. Bill Gasarch and Matt Jordan proved, by hand, $W_3 = 29$.
4. **Is there a HS-proof that W_4 exists?** Bill wanted to put this problem on the next HS exam to find out. He was (wisely) told **NO**.

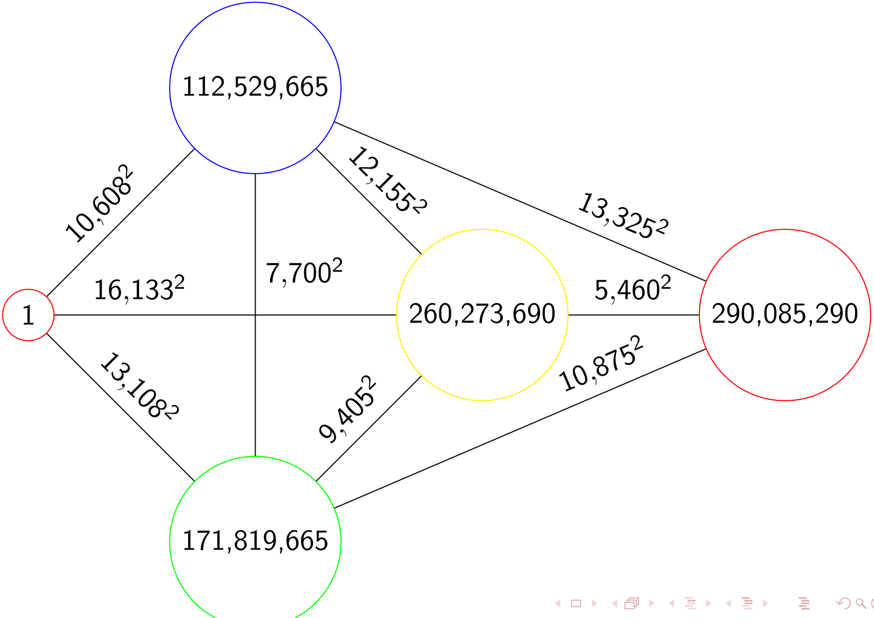
Reflection on W_3, W_4

1. Problem 5 (so hard) on UMCP HS Math Comp, 2006:
   Show that for all 3-colorings of $\{1, \ldots, 2006\}$ *there exists 2 numbers that are a square apart that are the same color*
2. 240 took exam, 40 tried this problem, 10 got it right.
3. Bill Gasarch and Matt Jordan proved, by hand, $W_3 = 29$.
4. **Is there a HS-proof that W_4 exists?** Bill wanted to put this problem on the next HS exam to find out. He was (wisely) told **NO**.
5. The question still remains: Is there a HS proof that W_4 exists?

Reflection on W_3, W_4

1. Problem 5 (so hard) on UMCP HS Math Comp, 2006:
   Show that for all 3-colorings of $\{1,\ldots,2006\}$ *there exists 2 numbers that are a square apart that are the same color*
2. 240 took exam, 40 tried this problem, 10 got it right.
3. Bill Gasarch and Matt Jordan proved, by hand, $W_3 = 29$.
4. **Is there a HS-proof that W_4 exists?** Bill wanted to put this problem on the next HS exam to find out. He was (wisely) told **NO**.
5. The question still remains: Is there a HS proof that W_4 exists? YES. Discovered by Zach Price in 2019 via clever computer search. Next slide.

W_4 Exists: $\mathrm{COL}(x) = \mathrm{COL}(x + 290{,}085{,}290)$



Reflection on W_4

1. Zach's proof shows $W_4 \le 1 + 299,085,290^2$.
   PRO Proof is easy to verify
   CON Number is large, proof does not generalize to W_5.
2. The classical proof.
   PRO Gives bounds for W_c.
   CON Bounds are GINORMOUS, even for W_2.
3. A Computer Search showed that $W_4 = 58$.
   PRO Get exact value.
   CON not human-verifiable. Does not generalize to W_5.

Which do you prefer?