BILL, RECORD LECTURE!!!!

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Small Ramsey Numbers-For HS

Exposition by William Gasarch

April 8, 2025

Lets Party Like Its January of 2019

Recall the first theorem one usually hears in Ramsey Theory and can tell your non-math friends about.

> *If there are 6 people at a party, either 3 know each other or 3 do not know each other.*

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We define graphs and complete graphs and state this theorem in those terms.

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Def A **Graph** $G = (V, E)$ is a set V and a set of unordered pairs from V, called edges. These can easily be drawn.

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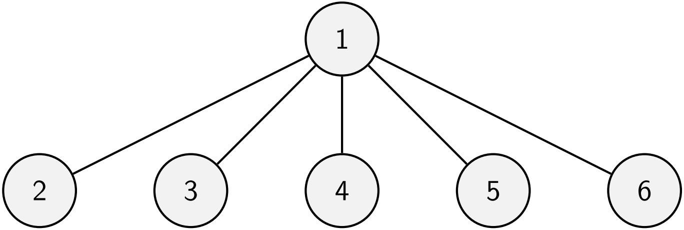

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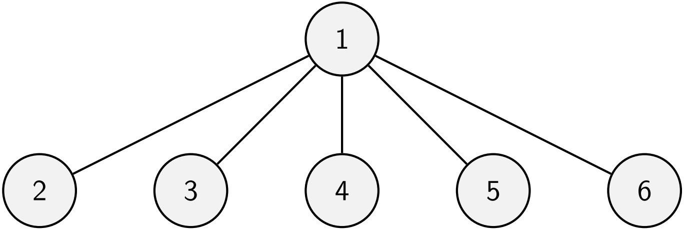

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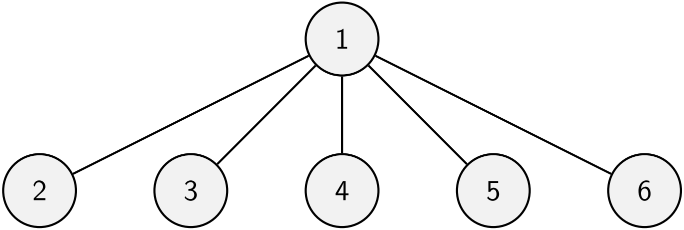

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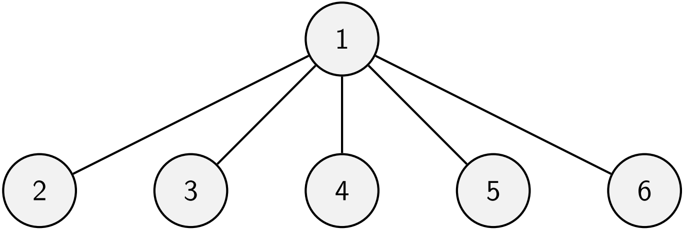

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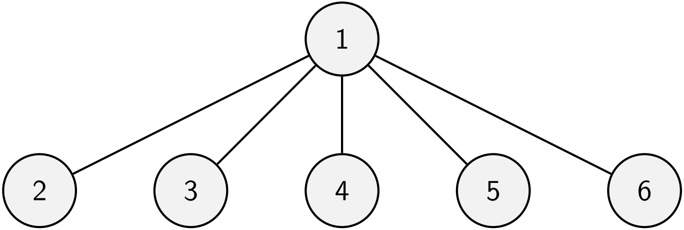

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Def The **degree (deg)** of a vertex is how many edges use it.
In the above graph $\deg(1) = 5$ and
$\deg(2) = \deg(3) = \deg(4) = \deg(5) = \deg(6) = 1$.

Complete Graphs

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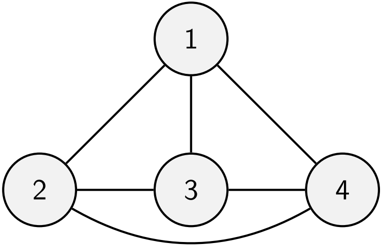

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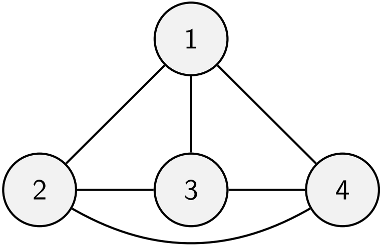

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Note Every vertex of K_n has degree $n - 1$.

More Notation

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   Bill Gasarch and the Red Cliques!

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We prove this in the next few slides.

The First Theorem in Ramsey Theory

Thm For all 2-colorings of the edges of K_6 there is a mono K_3.

Focus on Vertex 1

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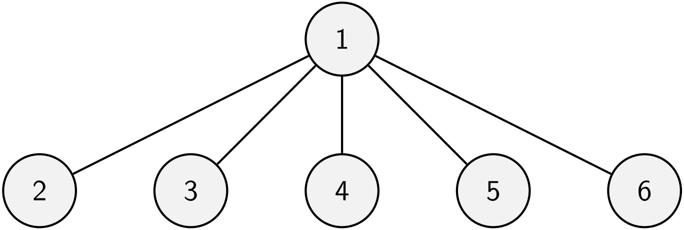

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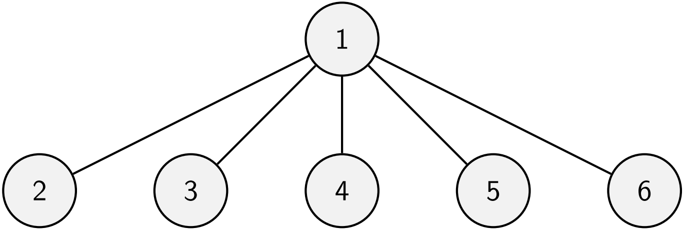

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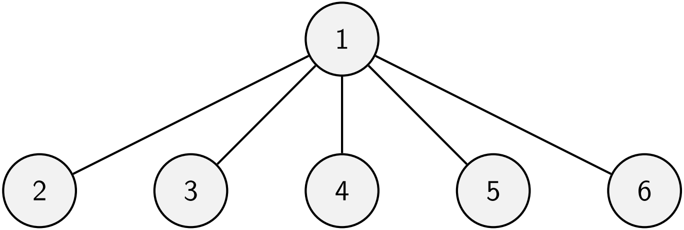

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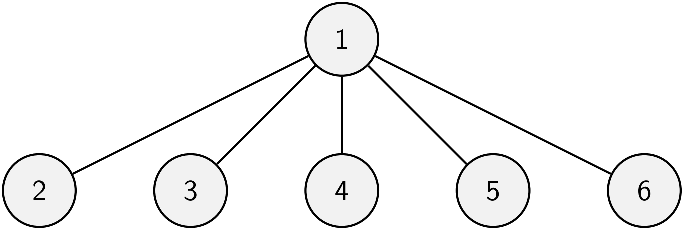

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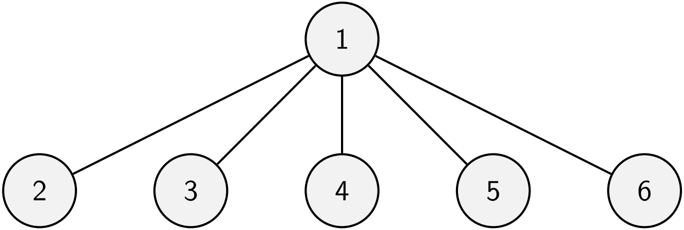

There are 5 edges coming out of vertex 1.
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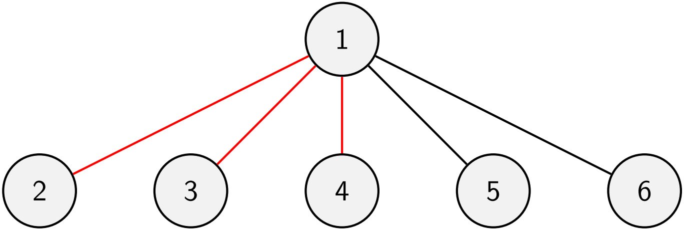
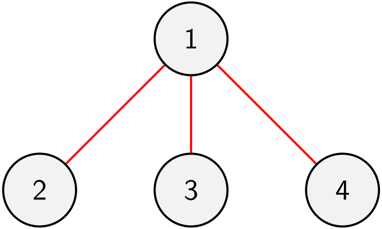
$\exists$ 3 edges from vertex 1 that are the same color.
We can assume $(1, 2)$, $(1, 3)$, $(1, 4)$ are all **RED**.

(1,2), (1,3), (1,4) are RED



We Look Just at Vertices 1,2,3,4



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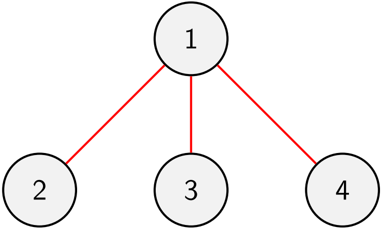

If $(2,3)$ is **RED** then get **RED** Triangle. So assume (2,3) is **BLUE**.

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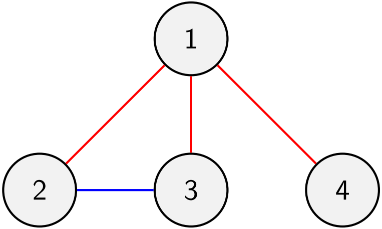

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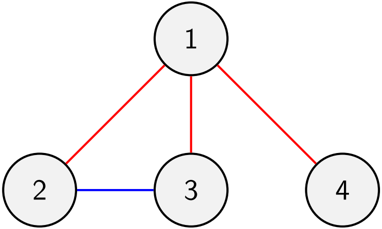

If (3,4) is **RED** then get **RED** triangle. So assume (3,4) is **BLUE**.

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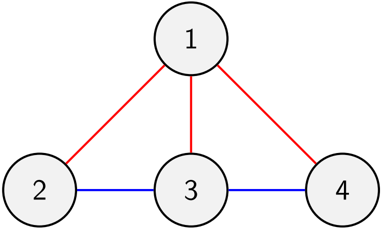

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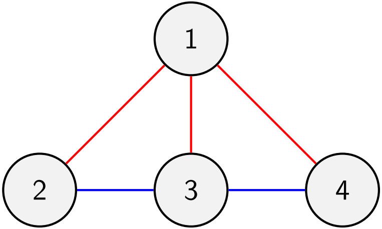

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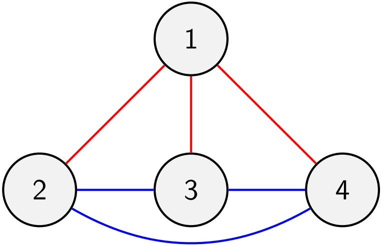

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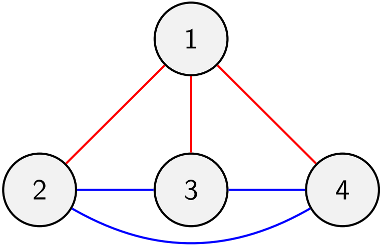

Note that there is a **BLUE** triangle with verts $2, 3, 4$. Done!

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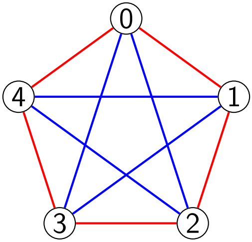

This graph is not arbitrary.
$\mathrm{SQ}_5 = \{x^2 \pmod 5) : 0 \le x \le 4\} = \{0, 1, 4\}$.

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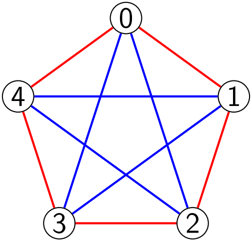

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Two ways to show no mono $\triangle$s on next slide.

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$2(x^2 + y^2 + z^2) \equiv 0$. Divide by 2 (mult by 3) to get
$x^2 + y^2 + z^2 \equiv 0$ which implies $x = y = z$.

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Need to show there are no mono △.

Method 1

The **R** edges are all symmetric so if there is a △ edge can assume one of the edges is $(0,1)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) =$ **R**.
The **B** edges are all symmetric so if there is a △ can assume one of the edges is $(0,2)$. No x with $\mathrm{COL}(0,x) = \mathrm{COL}(0,1) =$ **B**.
PRO Easy Proof. **CON** didn't use SQ_5 so unlikely to generalize.

Method 2 All $\equiv$ are mod 5.
1) Assume a, b, c form a △. Then $a-b, b-c, c-a \in \mathrm{SQ}_5$.
$a-b \equiv x^2$, $b-c \equiv y^2$, $c-a \equiv z^2$.
$x^2 + y^2 + z^2 \equiv 0$. Can show this implies $x = y = z = 0$.
2) Assume a, b, c form a △. Then $a-b, b-c, c-a \notin \mathrm{SQ}_5$. For all p, product of 2 nonsquares mod p is in SQ_p (HW). $2 \notin \mathrm{SQ}_5$ so
$2(a-b) = x^2$, $2(b-c) = y^2$, $2(c-a) = z^2$.
$2(x^2 + y^2 + z^2) \equiv 0$. Divide by 2 (mult by 3) to get
$x^2 + y^2 + z^2 \equiv 0$ which implies $x = y = z$.
UPSHOT $R(3,3) = 6$.

Asymmetric Ramsey Numbers

Definition Let $a, b \geq 2$. $R(a, b)$ is least n such that for all 2-colorings of K_n there is **either** a red K_a or a blue K_b.

1. $R(a, b) = R(b, a)$.
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Proof left to the reader, but its easy.

$R(a,b) \leq R(a-1,b) + R(a,b-1)$

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1. There is a vertex with large **Red** Deg.
2. There is a vertex with large **Blue** Deg.
3. All verts have small **Red** degree and small **Blue** degree.

Some Vertex v Has Large Red Deg

Case 1 $(\exists v)[\deg_R(v) \ge R(a-1, b)]$.

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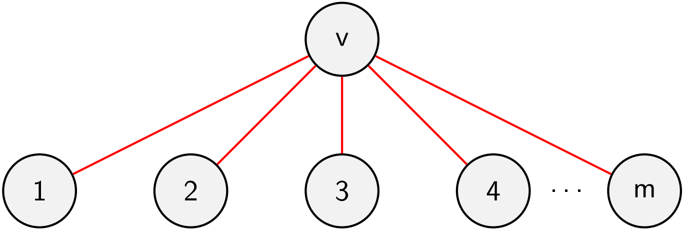

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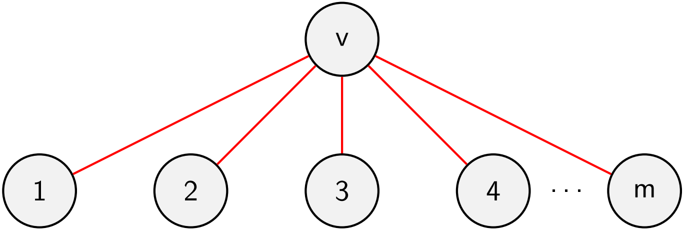

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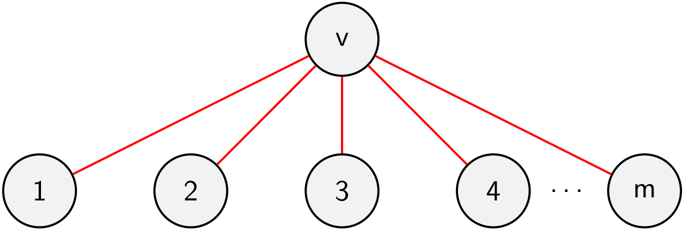

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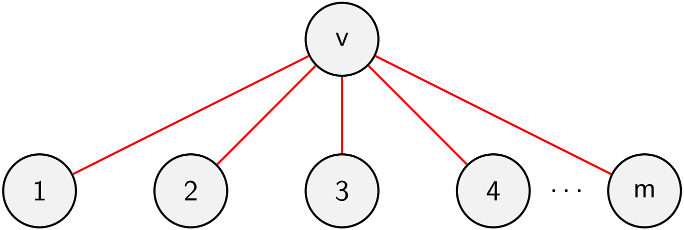

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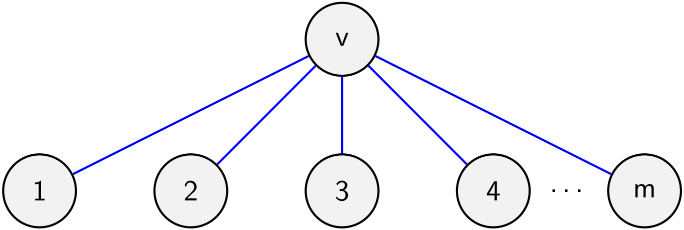

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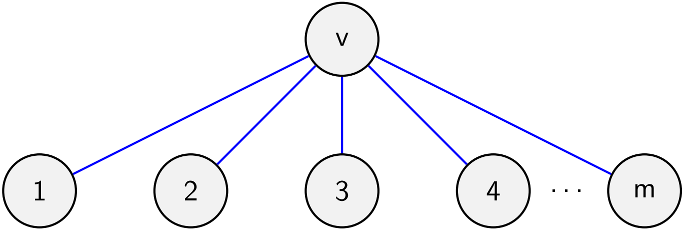

Case 2.1 There is a **Red** K_a in $\{1, \ldots, m\}$. DONE

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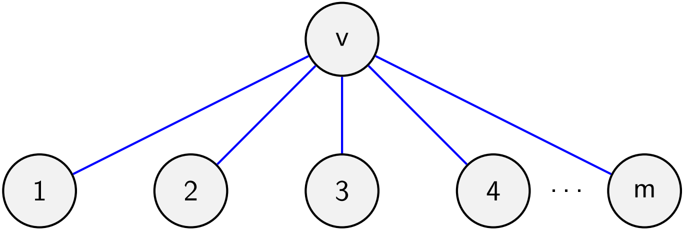

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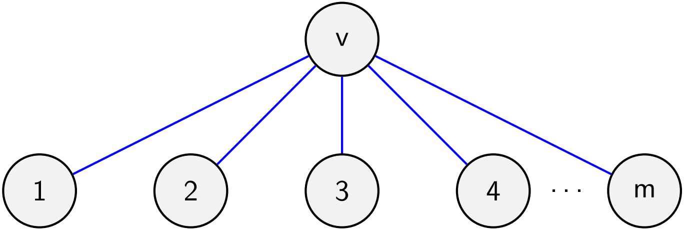

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All Verts: Small Red Deg and Small Blue Deg

Case 3 Negate Case 1 and Case 2:

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Hence

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Hence

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Not possible since every vertex of K_n has degree $n-1$.

Lets Compute Bounds on $R(a, b)$

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Table of Bounds

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Can we make some improvements to this?

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Can we make some improvements to this? YES!
We need a theorem.

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Can we make some improvements to this? YES!
We need a theorem. We first do an example.

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Thm There is NO graph on 9 verts, with every vertex of deg 3.

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So there are $9 \times 3 = 27$ edges.

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We generalize this on the next slide.

Handshake Lemma

Lemma Let $G = (V, E)$ be a graph.

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$$\sum_{v \in V_{\text{odd}}} \deg(v) \equiv 0 \pmod 2.$$

Sum of odds $\equiv 0 \pmod 2$. Must have even numb of them. So $|V_{\text{odd}}| \equiv 0 \pmod 2$.

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Handshake Lemma If all pairs of people in a room shake hands, even number of shakes. (Pre-COVID when people shook hands.)

Corollary of Handshake Lemma

Impossible to have a graph on an odd number of verts where every vertex is of odd degree.

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Impossible to have a graph on an odd number of verts where every vertex is of odd degree.
And NOW to our improvements on small Ramsey numbers.

$R(3,4) \leq 9$ Case 1

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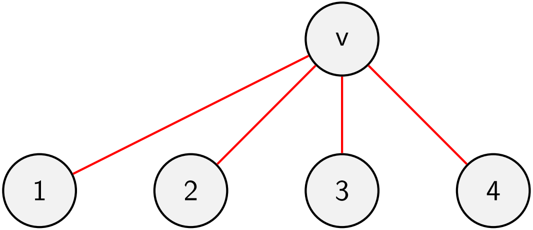

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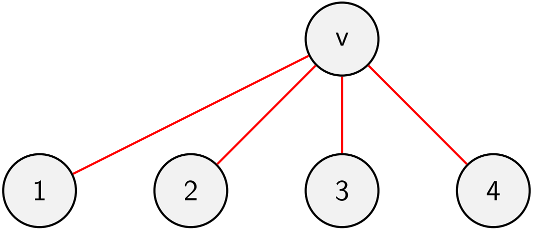
Assume we have a 2-coloring of the edges of K_9.
Case 1 $(\exists v)[\deg_R(v) \ge 4]$.



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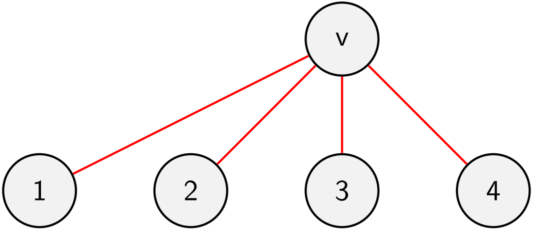

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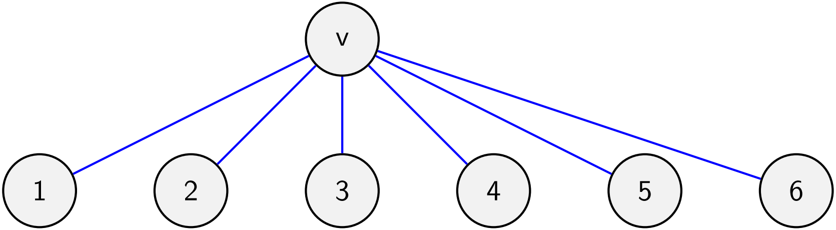
2) If **all** of $\{1,2\}$,$\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$ are **BLUE**, have **BLUE** K_4.

$R(3,4) \leq 9$ Case 2

Case 2 $(\exists v)[\deg_R(v) \leq 2]$, so $\deg_B(v) \geq 6$.



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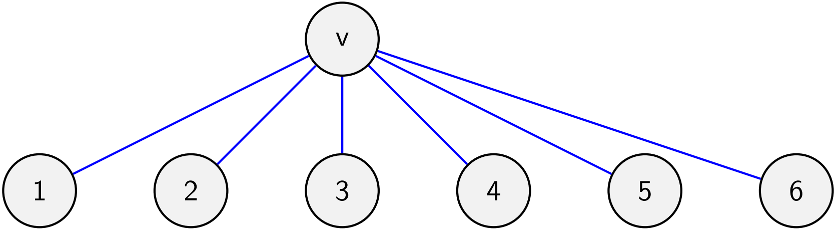

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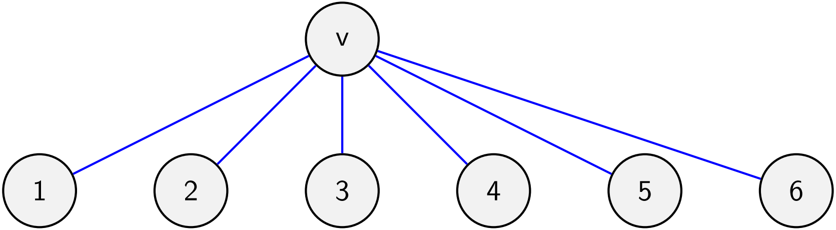

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$R(3,4) \leq 9$ Case 3

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This is impossible!

A Generalization of this Trick

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Proof left to the Reader.

Some Better Upper Bounds

- $R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6.$
- $R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 - 1 = 9.$
- $R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 9 = 14.$
- $R(3,6) \leq R(2,6) + R(3,5) \leq 6 + 14 - 1 = 19.$
- $R(3,7) \leq R(2,7) + R(3,6) \leq 7 + 19 = 26$
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Are these tight? Some yes, some no.

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$R(3,3) = 6$ as shown in prior slide.

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Same idea as above for K_5, but more cases for algebra.
UPSHOT $R(4,4) = 18$ and the coloring used math of interest!

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THATS IT.

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THATS IT.

No other $R(a, b)$ are known using NICE methods.

Summary of Bounds

$R(a,b)$	Old Bound	New Bound	Opt	Int?
$R(3,3)$	6	6	6	Y
$R(3,4)$	10	9	9	Y
$R(3,5)$	15	14	14	Y
$R(3,6)$	21	19	18	Lower-Y
$R(3,7)$	28	27	23	Lower-Y
$R(4,4)$	20	18	18	Y
$R(4,5)$	35	31	25	N
$R(5,5)$	70	62	≤ 46	N

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Applications? Yes, but to other parts of pure math.

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   (Joel Spencer) The Law of Small Numbers: Patterns that persist for small numbers will vanish when the calculations get to hard.

2. Seemed like a nice **Math** problem that would involve interesting and perhaps deep mathematics. No. The work on it is interesting and clever, but (1) the math is not deep, and (2) progress is slow.