Are There Better Bounds on the VDW Numbers?

Exposition by William Gasarch

December 31, 2024

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- Proof is elementary. Can present here but won't.
- Bounds still large. Fifth Level of PR hierarchy.

Deep Math From Search for Better Upper Bounds on VDW Numbers

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It DID succeed! (Oh! Thats a good thing!)

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$$\limsup_{n\to\infty}\frac{|A\cap[n]|}{n}$$

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- 1. For all k, $\{x : x \equiv 0 \pmod{k}\}$ has upper den $\frac{1}{k}$.
- 2. $\{x^2 : x \in \mathbb{N}\}$ has upper den 0.

A Conjecture, 1936

Conjecture If $A \subseteq \mathbb{N}$ has positive upper density then, for all k, A has a k-AP.

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- **Conjecture** If $A \subseteq \mathbb{N}$ has positive upper density then, for all k, A has a k-AP.
- Theorem Conj implies VDW's Theorem. Left to you.
- The hope was that the proof of Conj would require a new proof of VDW's Theorem that would lead to better bounds.

Roth's Theorem, 1952

Theorem If $A \subseteq \mathbb{N}$ has positive upper density then A has a 3-AP.

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- Roth won the Fields Medal in 1958 for his work on Diophantine approximation (so not for this work).

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 - Causes of change: (1) combinatorics using deep math, (2) CS inspired new problems in combinatorics.

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None of these results used mathematics of interest.

Known Lower Bounds

- Easy Use of Prob Method W(k,2) ≥ √k2^{k/2} (Easy extension to 3 colors)
- Very sophisticated use yields W(k,2) ≥ ^{2k}/_{k[€]} (Does not extend to 3 colors.)
- 3. If p is prime then $W(p,2) \ge p(2^p 1)$. Constructive! (Does not extend to 3 colors.)

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- There is also a ConservaMedical Medal- an alternative to the Nobel Prize in Medicine. It went to Donald Trump for his Medical Advice on Covonavirus. I am kidding.