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Elementary Methods of Graph Ramsey Theory

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Elementary Methods of Graph Ramsey Theory

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To our families

Preface

Ramsey theory is named after British mathematician Frank P. Ramsey (February 22, 1903–January 19, 1930) who published a paper "On a problem of formal logic" in 1929. Ramsey theory has become a flourishing branch of extremal combinatorics. Just as Theodore S. Motzkin pointed out, the main spirit of Ramsey theory is that

"Complete disorder is impossible!"

Ramsey theory was subsequently developed by Paul Erdős (March 26, 1913–September 20, 1996), a Hungarian mathematician, who was working on many mathematical problems, particularly in combinatorics, graph theory and number theory. Earlier than Frank P. Ramsey, Issai Schur (January 10, 1875–January 10, 1941) and van der Waerden (February 2, 1903–January 2, 1996) obtained similar results in number theory. We refer the reader to the book *Ramsey Theory* by Graham, Rothschild and Spencer (1990) for a systematically introduction and the book *Erdős on Graphs: His Legacy of Unsolved Problems* by Chung and Graham (1999) for many unsolved problems. As an important method on Ramsey theory, we would like to refer the reader to the book *The Probabilistic Method* by Alon and Spencer (2016) for a systematically introduction. For a comprehensive understanding of random graphs which are closely related to Ramsey theory, we refer the reader to three books on this field: The books *Random Graphs* by Bollobás (2001, 2nd ed.), *Random Graphs* by Janson, Łuczak and Ruciński (2000), and *Introduction to Random Graphs* by Frieze and Karoński (2016).

The number of research papers on Ramsey theory before 1970s was not substantial. The Combinatorial Conference at Balatonfüred, Hungary 1973, in honor of Paul Erdős for his 60th birthday, was a milestone in Ramsey theory history. There were more than two dozen talks devoted to what is now called *Ramsey theory*. Many papers have been published after this conference. One striking feature is the invention of many modern methods that involve ideas from various branches of mathematics such as probability, algebra, geometry, and analysis.

Graph Ramsey theory is an important area that serves not only as an abundant source but also as a testing ground of these methods and many other new methods.

Despite substantial advances in graph Ramsey theory, most outstanding problems are far from being solved. The new insights generated by tackling these problems will most likely lead to new tools and techniques. Due to these reasons, graph Ramsey theory is full of vitality and hence deserves much more research efforts.

The book emphasizes making the text easier for the students to learn. To overcome difficulties to access sporadic results in an extensive literature, we set out to describe the material in this elementary book, which aims to provide an introduction to graph Ramsey theory. The prerequisites for this book are minimal: we only require that the reader be familiar with elementary level of graph theory, calculus, probability and linear algebra. To make this book as self-contained as possible, we attempt to introduce the theory from scratch, for instance, some results rely on the properties of finite fields, so we laid down the background beforehand. We believe that this book, intended for beginning graduate students, can serve as an entrance to this beautiful theory. To facilitate better understanding of the material, this book contains some standard exercises in which a large part of the exercises are not difficult since our book serves as a primer on this topic.

We have used the manuscript of the book as lecture notes more than 20 years in many universities including these in mainland of China, Hong Kong and Taiwan, etc. We also used it many times for summer schools supported by Natural Science Foundation of China. The selected topics are almost independent so that beginners may skip some chapters, sections, and proofs, particularly that are marked with asterisks. We are sorry for not being able to incorporate many deep results into this book. As most listeners in the short terms are preferably interested in the specific topics, they can obtain a clearer picture on the topics from the selected chapters instead of the whole book.

There are thirteen chapters in this book, divided mainly according to both the content of the book and methods used for the problems. In Chapter 1, we will introduce some basic definitions and discuss the existences of Ramsey numbers by giving upper bounds. In Chapter 2, we will consider several small Ramsey numbers and a Ramsey number on integers, i.e., Schur number on integers. For algebraic constructions in this chapter, we shall recall some basics of finite fields briefly. In Chapter 3, we will focus on the basic method such as vertices are labeled or picked randomly or semi-randomly, in which we always compute the expectations of random variables. The frequently-used methods to estimate the probability of a variable from expectation including Markov's inequality and Chernoff bound will be introduced in this chapter. In Chapter 4, we will give an overview on random graphs which now has become a flourishing branch. Applications to classic Ramsey numbers due to Erdős (1947) will be given in this chapter, which is always considered as the first conscious application of the probabilistic method, and the graph Ramsey theory is always refereed to as the birthplace of random graphs. This chapter also contains threshold functions for random graphs with certain properties. In Chapter 5, we will introduce Lovász Local Lemma that relaxes the independence of pairwise events to partial independence. We will also give an overview of the Martingales and triangle-free process. In Chapter 6, we shall consider some constructive lower bounds of Ramsey numbers, which tells us that the probabilistic method is more powerful than constructive method for lower bounds of most non-linear Ramsey functions. Also, we introduce a disproof of the conjecture of Borsuk in geometry that is a surprising by-product of graph Ramsey theory. Additionally, this chapter contains basic properties of intersecting hypergraphs. In Chapter 7, Turán numbers will be introduced, in which the Turán numbers of bipartite graphs are tightly related to the corresponding Ramsey numbers in many colors. In Chapter 8, we will introduce communication channel, and the connection between Ramsey theory and communication channel will be revealed. In Chapter 9, we will introduce the method of the dependent random choice, which can be applied to embed a small or sparse graph into a dense graph. Chapter 10 focuses on quasi-random graphs and regular graphs with small second eigenvalues, for which some deep applications especially some graph Ramsey numbers will be included. In Chapter 11, we will introduce an important Ramsey number on integers, i.e. van der Waerden number on arithmetic progression. We will also introduce Szemerédi's regularity lemma which asserts that every large graph can be decomposed into a finite number of parts so that the edges between almost every pair of parts forms a "random-looking" graph. We will give some applications including a classic application on graphs with bounded maximum degree and a Ramsey-Turán number by using the regularity lemma. Several extensions on the regularity lemma will be given. In Chapter 12, we shall discuss some more examples on Ramsey linear functions. The first section of the chapter discusses the linearity of subdivided graphs, and the second is on a special linearity: so called Ramsey goodness, proposed by Burr and Erdős (1983). There are a lot of variants on graph Ramsey theory, some of which will be introduced in Chapter 13, including size Ramsey numbers, induced Ramsey theorem, bipartite Ramsey numbers, and Folkman numbers, etc.

We are deeply indebted to these professors who helped us to learn Ramsey theory, and colleagues who organized seminars and summer schools, as well as students who attended the classes. In particular, we are deeply indebted to Professor Wenan Zang who should be a coauthor if he is not so busy since a large part of the book is chosen from the lecture notes *Introduction to Graph Ramsey Theory* by Y. Li and W. Zang. Finally, we would like to thank the National Science Function of China and the Research Grants Council of Hong Kong for their financial support.

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Jan. 2022

Contents

Chapter 1
Existence

A typical result in Ramsey theory states that if a mathematical object is partitioned into finitely many parts, then one of the parts must contain a sub-object of particular property. The smallest size of the large object such that the sub-object exists is called Ramsey number.

Ramsey theory can be viewed as a generalization of the following well-known *Pigeonhole Principle*. Let k and n be positive integers. When N pigeons are put into k pigeonholes, there exists at least one pigeonhole containing more than n pigeons if N is large enough.

Here the set of pigeons of size N is the large object. The smallest N so that the statement holds is $kn + 1$. For $N \geq kn + 1$, the average size of the substructures is more than n, so there exists a class contains at least $n + 1$ objects. The *averaging technique* used in the argument is one of the oldest "non-constructive" principles: it states only the *existence* of a pigeonhole with more than n pigeons and says nothing about how to find such a pigeonhole.

A cornerstone method in mathematics is *the mathematical induction*, which deduces a general statement for infinitely many parameters from finitely many cases. In contrast to the induction, Ramsey theory does the job to obtain a general statement for all large parameters by excluding finitely many exceptions.

1.1 Terminology

We assume that the readers have learned some standard textbooks in graph theory, a few of which are listed at the end of this book. For terminology and notation that are not defined here, we refer the reader to that such as Bollobás (1994, 2004), Bondy and Murty (2008), Diestel (2010), and West (2001), etc.

Let $G_1, \ldots, G_k$ be graphs. Without specified, all graphs are simple graphs. The *Ramsey number* of $G_1, \ldots, G_k$, denoted by $r_k(G_1, \ldots, G_k)$ or simply $r(G_1, \ldots, G_k)$, is defined to be the smallest integer N such that for any edge-coloring of the complete graph K_N by colors $1, \ldots, k$, there exists some $1 \leq i \leq k$, such that G_i is

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contained in the subgraph spanned by all edges in the color i. The spanned subgraph by edges in the color i may contain isolated vertices which are those (if any) not incident to any edge in the color i. Note that in the definition of Ramsey numbers, there are no restraints on the way of coloring of edges. We call $r(G_1, \ldots, G_k)$ the k-coloring Ramsey number. For simplicity, we write $r_k(G, \ldots, G)$ as $r_k(G)$, write $r(K_\ell, \ldots, K_n)$ as $r(\ell, \ldots, n)$, which is called *classical Ramsey number*. We often use $r_k(n)$ to denote $r_k(K_n)$. If a graph G_i is not complete, then $r(G_1, \ldots, G_k)$ is also called *generalized Ramsey number*.

The case $k = 1$ is trivial as $r_1(G) = |V(G)|$. The most studied case is $k = 2$ in which we often refer the two colors for edges as red and blue. Let $\overline{F}$ be the complementary graph of F. The Ramsey number $r(G, H)$ can be alternatively defined as the smallest positive integer N such that for any graph F of order N, either F contains G as a subgraph or its complement $\overline{F}$ contains H as a subgraph. We always call $r(G, G)$ or simply $r(G)$ the *diagonal* Ramsey number of G, and write $r(n, n)$ or simply $r(n)$ for the diagonal classic Ramsey number $r(K_n, K_n)$ for convenience.

It is a simple fact that $r(K_1, G) = 1$ and $r(K_2, G) = |V(G)|$ for any graph G with $|V(G)| \ge 2$. To see the latter, let $n = |V(G)| \ge 2$, and color all edges of K_{n-1} blue, then there is neither a red K_2 nor a blue G, so $r(K_2, G) > n - 1$. On the other hand, consider a red-blue coloring of edges of K_n. We are done if there exists a red edges. Otherwise, all edges of K_n are colored blue, thus we definitely have a blue G, the upper bound $r(K_2, G) \le n$ follows.

If G is not an empty graph, then $r(\overline{K}_m, G) = m$, where $\overline{K}_m$ is the complementary graph of K_m. It is easy to see that $r(G, H)$ is monotone increasing in the sense that if G_1 is a subgraph of G and H_1 is a subgraph of H, then $r(G_1, H_1) \le r(G, H)$.

Proposition 1.1 *For any graphs G and H, $r(G, H) = r(H, G)$.*

Proof. Let $N = r(G, H)$. From the definition of $r(G, H)$, there exists an edge-coloring of K_{N-1} by red and blue such that there is neither red G nor blue H. For each edge, switch its color to the other. In the new coloring, there is neither red H nor blue G, so $r(H, G) \ge N = r(G, H)$. Similarly, $r(G, H) \ge r(H, G)$. The assertion follows. □

Generally, for multi-color Ramsey number we have the following facts.

(i) If $(H_1, H_2, \ldots, H_k)$ is a permutation of $(G_1, G_2, \ldots, G_k)$, then

$$r(G_1, G_2, \ldots, G_k) = r(H_1, H_2, \ldots, H_k).$$

(ii) $r(G_1, G_2, \ldots, G_{k-1}, K_1) = 1$.
(iii) $r(G_1, G_2, \ldots, G_{k-1}, K_2) = r(G_1, G_2, \ldots, G_{k-1})$.

In order to verify that $r(G, H) \ge N + 1$ for some N, one must have an edge coloring of K_N in red and blue, such that there is neither a monochromatic red G nor a monochromatic blue H. Such coloring is always referred to as a *Ramsey coloring*. Since the two monochromatic graphs, which contain all vertices, can be referred to as F and $\overline{F}$, respectively, verifying $r(G, H) \ge N + 1$ is the same to find a graph F of order N such that F contains no copy of G and its complement $\overline{F}$ contains no copy of H. Such a graph F of order $N = r(G,H) - 1$ is always called a *Ramsey graph* for $r(G,H)$.

Let $\omega(F)$ and $\alpha(F)$ be the clique number and independence number of F, respectively. Then $r(m,n)$ is the smallest N such that for any graph F of order N, we have either $\omega(F) \geq m$ or $\alpha(F) \geq n$. Note that any graph F can yield a lower bound, good or bad, for a Ramsey number. For example, if $\omega(F) = m$, and $\alpha(F) = n$, then we have $r(m+1, n+1) \geq |V(F)| + 1$. However, it is very difficult to find a good lower bound for most Ramsey numbers.

Throughout this book, we use the standard asymptotic notation. For functions $f(n)$ and $g(n)$ that take positive values, we write $f = O(g)$ if $f \leq cg$ for all large n, where $c > 0$ is a constant, $f = \Omega(g)$ if $g = O(f)$ and $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$. Denote $f = o(g)$ if $f/g \to 0$. Finally, $f \sim g$ denotes that $f = (1+o(1))g$, i.e. $f/g \to 1$. We use $\log n$ to denote the natural logarithm based on e.

1.2 General Upper Bounds

In 1929, Frank P. Ramsey, in a fundamental paper on mathematical logic, gave a result whose special case can be stated in graph language as follows.

Theorem 1.1 (Ramsey's theorem) *For $k \geq 2$ and $n_1, n_2, \ldots, n_k \geq 1$, the Ramsey number $r(n_1, n_2, \ldots, n_k)$ exists. If $n_i \geq 2$ for all i with $1 \leq i \leq k$, then*

$$r(n_1, n_2, \ldots, n_k) \leq \sum_{i=1}^{k} r(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_k) - k + 2.$$

Proof. The proof of the upper bound is by induction on $n_1 + n_2 + \cdots + n_k$. Note that $r(2,n) = r(n,2) = n$ and

$$r_k(n_1, \ldots, n_{i-1}, 2, n_{i+1}, \ldots, n_k) = r_{k-1}(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_k),$$

we may assume that $N_i = r(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_k)$ exists for any $i \leq k$, and $n_i \geq 3$. Let

$$N = \sum_{i=1}^{k} N_i - k + 2$$

and consider an edge-coloring of K_N by colors $1, 2, \ldots, k$. We have to show that the subgraph G_i spanned by all edges in some color i contains K_{n_i}. For a vertex v of K_N, let $d_i(v)$ be the degree in G_i. If $d_i(v) \leq N_i - 1$ for each $1 \leq i \leq k$, then

$$N - 1 = \sum_{i=1}^{k} d_i(v) \leq \sum_{i=1}^{k} (N_i - 1) = N - 2,$$

which leads to a contradiction. Thus there is some i such that $d_i(v) \geq N_i$. If there is a $K_{n_i - 1}$ in color i, then we are done since which together with v form a K_{n_1} in color i. Otherwise, by the definition of N_i, the corresponding k edge-coloring of K_{N_i} contains a monochromatic K_{n_j} in some color $j \neq i$. The proof is complete. □

For a graph G of order at least two, denote by G' for a graph obtained by deleting one vertex from G. We can generalize Theorem 1.1 as follows.

Theorem 1.2 *Let $k \geq 2$ be an integer and let $G_1, G_2, \ldots, G_k$ be graphs. Then the Ramsey number $r(G_1, G_2, \ldots, G_k)$ exists. If each G_i has at least two vertices, then*

$$r(G_1, G_2, \ldots, G_k) \leq \sum_{i=1}^{k} r(G_1, \ldots, G_{i-1}, G_i', G_{i+1}, \ldots, G_k) - k + 2.$$

The popularization of Ramsey's Theorem began with its rediscovery in a classic paper of Erdős and Szekeres in 1935.

Theorem 1.3 *For $m \geq 2$ and $n \geq 2$,*

$$r(m, n) \leq r(m-1, n) + r(m, n-1).$$

Moreover,

$$r(m, n) \leq \binom{m+n-2}{m-1}.$$

Proof. The first inequality is a special case of Theorem 1.1. The second can be proved by induction on $m + n$. The case $m = 2$ or $n = 2$ is trivial. For the case $m \geq 3$ and $n \geq 3$, by noting the first inequality and the induction hypothesis, and the fact that

$$\binom{m+n-3}{m-2} + \binom{m+n-3}{m-1} = \binom{m+n-2}{m-1},$$

the second inequality follows. □

Let us consider the first non-trivial Ramsey number $r(3, 3)$, which was a problem for International Mathematical Olympiad.

Theorem 1.4 *We have $r(3, 3) = 6$.*

Proof. By the upper bound in Theorem 1.3, we have $r(3, 3) \leq \binom{4}{2} = 6$. More direct proof is easy. Let F be a graph of order 6 and $v \in V(F)$. Note that v has at least three neighbors in F or in $\overline{F}$, say, three neighbors $S = \{v_1, v_2, v_3\}$ in F. If any two vertices in S are adjacent, then these two vertices and v form a triangle. Otherwise, S induces a triangle in $\overline{F}$ hence $r(3, 3) \leq 6$. Note the fact that $\overline{C_5} = C_5$, which contains no K_3, the lower bound $r(3, 3) > 5$ follows. □

The upper bound by Erdős and Szekeres (1935) stood for a long time, until Yackel (1972), Rödl (unpublished, 1980's), Graham and Rödl (1987) and Thomason (1988) proved that (in different forms)

$$r(m, n) \leq o\left[\binom{m+n-2}{m-1}\right]$$

as m fixed and $n \to \infty$ or $m = n \to \infty$.

A breakthrough of Conlon (2009) further improves the upper bound as

$$r(n,n) \le \frac{1}{n^{c_1 \log n/\log\log n}} \binom{2n-2}{n-1} \le \frac{1}{\exp(c_2 \log^2 n/\log\log n)} 4^n,$$

where c_i and c henceforth are positive constants and the second inequality comes from $\binom{2k}{k} \sim \frac{1}{\sqrt{\pi k}} 4^k$ as $k \to \infty$. Recently, Sah (preprint, 2020^+) improves the upper bound further as

$$r(n,n) \le \frac{1}{\exp(c \log^2 n)} 4^n.$$

However, this does not change the following limit that

$$\overline{\lim_{n\to\infty}}\, r(n,n)^{1/n} \le 4.$$

We shall prove a result emphasizing on the condition for the strict inequality in the above theorem, which is needed for finding exact values of some small Ramsey numbers.

Theorem 1.5 *If G and H are graphs on at least two vertices, then*

$$r(G,H) \le r(G',H) + r(G,H').$$

If both $r(G',H)$ and $r(G,H')$ are even, then

$$r(G,H) \le r(G',H) + r(G,H') - 1.$$

Proof. The first inequality is a special case of Theorem 1.2, and we shall prove the second. On contrary, suppose $N := r(G,H) = r(G',H) + r(G,H')$ which is even. Thus there exists an edge coloring of K_{N-1} such that there is neither red G nor blue H. For any vertex v, we have $d_R(v) \le r(G',H) - 1$ and $d_B(v) \le r(G,H') - 1$. Therefore, we have

$$d_R(v) = r(G',H) - 1 \quad \text{and} \quad d_B(v) = r(G,H') - 1$$

by noting that $d_R(v) + d_B(v) = N - 2 = (r(G',H) - 1) + (r(G,H') - 1)$. Consider the number of edges of R, which is

$$e(R) = \frac{1}{2}(N-1)(r(G',H) - 1).$$

But the right hand side is not an integer since both $N - 1$ and $r(G',H) - 1$ are odd, yielding a contradiction. □

The following is a two-step recursive upper bound, due to Li, Rousseau and Zang (2004), in which the main argument in the proof is to count the number of triangles. Let G'' be any graph obtained by deleting two vertices from G.

Theorem 1.6 *Let G and H be graphs of order at least three, and let $A = r(G'', H)$ and $B = r(G, H'')$. Then*

$$r(G,H) \le A + B + 2 + 2\sqrt{(A^2 + AB + B^2)/3}.$$

In particular, $r(G,G) \le 4r(G,G'') + 2$.

Proof. Let $N = r(G,H) - 1$, $a = A - 1$ and $b = B - 1$. Then there is an edge-coloring of K_N by red and blue in which there is neither red G nor blue H. Denote by V the vertex set of the colored K_N, and let $F_R = (V, E_R)$ and $F_B = (V, E_B)$ denote the red and blue subgraphs, respectively, where (E_R, E_B) is the corresponding partition of the edge set of K_N. Then the number M of monochromatic triangles is

$$M = \frac{1}{3}\left(\sum_{uv\in E_R} |N_R(u)\cap N_R(v)| + \sum_{uv\in E_B} |N_B(u)\cap N_B(v)|\right),$$

where $N_R(u)$ and $N_B(u)$ are neighborhoods of u in F_R and F_B, respectively. Since an edge $uv \in E_R$ and a red copy of G'' in $N_R(u)\cap N_R(v)$ yield a red K_2+G'' hence a red G, we have $|N_R(u)\cap N_R(v)| \le r(G'',H) - 1 = a$. Similarly, $|N_B(u)\cap N_B(v)| \le r(G,H'') - 1 = b$ for an edge $uv \in E_B$. Thus we have

$$M \le \frac{a|E_R| + b|E_B|}{3} = \frac{1}{6}\sum_{v\in V}\Big(a d_R(v) + b d_B(v)\Big).$$

where $d_R(v) = |N_R(v)|$ and $d_B(v) = |N_B(v)|$. As observed by Goodman (1959), the number of non-monochromatic triangles is $\frac{1}{2}\sum_{v\in V} d_R(v)d_B(v)$, it follows that

$$\binom{N}{3} - \frac{1}{2}\sum_{v\in V} d_R(v)d_B(v) \le \frac{1}{6}\sum_{v\in V}\Big(a d_R(v) + b d_B(v)\Big).$$

Or equivalently,

$$\frac{N(N-1)(N-2)}{3} - \sum_{v\in V}\left(d_R(v) + \frac{b}{3}\right)\left(d_B(v) + \frac{a}{3}\right) + \frac{abN}{9} \le 0.$$

Notice that $xy \le (x+y)^2/4$, implying that

$$\left(d_R(v) + \frac{b}{3}\right)\left(d_B(v) + \frac{a}{3}\right) \le \frac{1}{4}\left(N - 1 + \frac{a+b}{3}\right)^2,$$

and thus

$$\frac{N(N-1)(N-2)}{3} - \frac{N}{4}\left(N - 1 + \frac{a+b}{3}\right)^2 + \frac{abN}{9} \le 0.$$

Equivalently, we get

$$(N-1)^2 - 2(A+B)(N-1) - \frac{(A-B)^2}{3} \leq 0,$$

which gives $N - 1 \leq A + B + 2\sqrt{(A^2 + AB + B^2)/3}$ as required. □

The following result due to Walker (1968) is often used in estimating small Ramsey numbers.

Corollary 1.1 *For* $n \geq 3$, $r(n,n) \leq 4r(n-2,n) + 2$.

Denote by $G + H$ the graph obtained from vertex-disjoint G and H by adding edges connecting G and H completely, which is called the join of G and H. Then $B_n = K_2 + \overline{K}_n$ contains n triangles all sharing a common edges. We always call B_n an n-book or simply book graph. We have the following result by Rousseau and Sheehan (1978).

Theorem 1.7 *Let* $m, n \geq 1$ *be integers with* $2(m+n)+1 > (n-m)^2/3$. *Then*

$$r(B_m, B_n) \leq 2(m+n+1).$$

In particular, $r(B_n, B_n) \leq 4n + 2$.

Proof. From the facts that $\overline{K}_m$ can be obtained from B_m by deleting two vertices and $r(\overline{K}_m, B_n) = m$, it follows from Theorem 1.6 that

$$r(B_m, B_n) \leq m + n + 2 + 2\sqrt{(m^2 + mn + n^2)/3}.$$

To get the upper bound as desired, note that the Ramsey number is an integer, we need only to verify that $2\sqrt{(m^2 + mn + n^2)/3} < m + n + 1$, which is equivalent to $2(m+n)+1 > (n-m)^2/3$, as given. □

In the next chapter, we will show the above upper bound for $r(B_n, B_n)$ can be achieved for infinitely many n. Rousseau and Sheehan (1978) also conjectured that there exists a positive constant $c > 0$ such that for all $m, n \geq 1$,

$$r(B_m, B_n) \leq 2(m+n) + c.$$

Recently, Chen, Lin and You (2021+) show that this conjecture holds asymptotically.

1.3 Upper Bounds for $r_k(3)$

Let us write r_k instead of $r_k(3)$ in the proof for convenience. We have known that $r_2 = 6$. The only known exact value for a multicolored Ramsey number is $r_3 = 17$ by Greenwood and Gleason (1955), and we will discuss it in the next chapter. For $k = 4$, we only known that $51 \leq r_4 \leq 62$, where the lower bound is due to Chung (1973) while the upper bound is due to Fettes, Kramer and Radziszowski (2004) which improves that by Sanchez-Flores (1995) and an earlier upper bound due to Folkman (1974). In this section, we focus on the upper bound of r_k when k is large and we will consider further the lower bound of r_k in the next chapter.

Lemma 1.1 *For $k \geq 2$,*

$$r_k(3) \leq 2 + k(r_{k-1}(3) - 1).$$

Proof. The assertion follows from Theorem 1.1. More directive proof is as follows. Let $N = r_k - 1$. There is an edge-coloring of K_N by colors $1, 2, \ldots, k$ such that there is no monochromatic triangle. Denote by $N_i(v)$ for the neighborhood of v in color i. Note that $N_i(v)$ contains no edge in color i since otherwise there is a triangle in color i, so we have $d_i(v) = |N_i(v)| \leq r_{k-1} - 1$. Therefore,

$$r_k - 2 = N - 1 = \sum_{i=1}^{k} d_i(v) \leq k(r_{k-1} - 1).$$

The desired upper bound follows. □

Corollary 1.2 *Let m be a positive integer, and let*

$$c = c(m) = \frac{r_m(3) - 1}{m!} + \sum_{t>m} \frac{1}{t!}.$$

For $k \geq m$,

$$r_k(3) < c \cdot k! + 1.$$

In particular, $r_k(3) < e \cdot k! + 1$ for $k \geq 1$.

Proof. Lemma 1.1 gives that $r_k - 1 \leq 1 + k(r_{k-1} - 1)$. Using this repeatedly, we have

$$\begin{aligned}
r_k - 1 &\leq 1 + k(r_{k-1} - 1) \\
&\leq 1 + k[1 + (k-1)(r_{k-2} - 1)] \\
&\leq 1 + k + k(k-1) + \cdots + k(k-1)\cdots(m+1)(r_m - 1) \\
&= k!\left(\frac{1}{k!} + \frac{1}{(k-1)!} + \cdots + \frac{1}{(m+1)!} + \frac{r_m - 1}{m!}\right),
\end{aligned}$$

and the desired upper bound follows. In particular, $c(1) = e$. □

An improvement can be obtained by noting the following fact.

Lemma 1.2 *Let k and p be even integers. If $r_{k-1}(3) \leq p$, then*

$$r_k(3) \leq k(p - 1) + 1.$$

In particular, if both k and $r_{k-1}(3)$ are even, then

$$r_k(3) \leq k(r_{k-1}(3) - 1) + 1.$$

Proof. If $r_{k-1} \le p-1$, then the assertion follows from Lemma 1.1. So we assume that $r_{k-1} = p$, which is even. Set $N = r_k - 1$, then $N \le 1 + k(r_{k-1} - 1)$ by Lemma 1.1, and we want to show that $N \le k(r_{k-1} - 1)$. On contrary, suppose that $N = 1 + k(r_{k-1} - 1)$. Clearly, N is odd and there is an edge coloring of K_N by colors $1, \ldots, k$ such that there is no monochromatic K_3 in any color. For a fixed color j, let E_j be the set of edges in color j. Since $\sum_x d_j(x) = 2|E_j|$ is even, where the sum is over N vertices and N is odd, we have that there exists a vertex x with $d_j(x)$ is even. Since $d_j(x) \le r_{k-1} - 1$ and r_{k-1} is even, we get $d_j(x) \le r_{k-1} - 2$. Therefore,

$$N - 1 = \sum_{j=1}^{k} d_j(x) \le (r_{k-1} - 2) + (k-1)(r_{k-1} - 1) = k(r_{k-1} - 1) - 1,$$

yielding a contradiction. □

Theorem 1.8 *Let m be even and p odd. If $r_m(3) \le p$, then for any $k \ge m$,*

$$r_k(3) \le k! \left(\frac{p-1}{m!} + \sum_{\substack{m<t\le k,\\ t\ odd}} \frac{1}{t!} \right) + 1.$$

Proof. The idea of the proof is a combination of that in Wan (1997), and Chung and Graham (1999). For fixed even $m \ge 2$ and $k \ge m$, set

$$A_k = A_k(m) = \frac{p-1}{m!} + \sum_{\substack{m<t\le k,\\ t\ odd}} \frac{1}{t!}.$$

Note that for $2n - 1 \ge m$,

$$A_{2n} = A_{2n-1} \quad \text{and} \quad A_{2n+1} = A_{2n} + \frac{1}{(2n+1)!}.$$

We shall prove that $r_k \le k!A_k + 1$ by induction on $k \ge m$. The assertion holds for $k = m$ as $m!A_m + 1 = p$. By the fact that

$$(m+1)!A_{m+1} = (m+1)(p-1) + 1,$$

the assertion for $k = m+1$ follows from Lemma 1.1. We now suppose that $k \ge m+2$. Let $k = 2n \ge m+2$ be an even integer. Then $k - 1 \ge m + 1$, and $k - 1$ is odd. It is easy to verify that

$$(k-1)!A_{k-1} = (k-1)! \left(\frac{p-1}{m!} + \frac{1}{(m+1)!} + \frac{1}{(m+3)!} + \cdots + \frac{1}{(k-1)!} \right)$$

is an odd integer. From the induction assumption $r_{k-1} \le (k-1)!A_{k-1} + 1$, where the right-hand side is even, and Lemma 1.2, we have

$$\begin{aligned} r_k - 1 &\le k(r_{k-1} - 1) = k!A_{k-1} \\ &= k!A_{2n-1} = k!A_{2n} = k!A_k. \end{aligned}$$

So $r_k \le k!A_k + 1$. For r_{2n+1}, from Lemma 1.1 and what just proved,

$$\begin{aligned} r_{2n+1} - 1 &\le 1 + (2n+1)(r_{2n} - 1) \\ &\le 1 + (2n+1)((2n)!A_{2n}) \\ &= 1 + (2n+1)!A_{2n} = (2n+1)!A_{2n+1}, \end{aligned}$$

completing the proof. □

Using a very old result of Folkman (1974) that $r_4(3) \le 65$ and Theorem 1.8, we have $r_k(3) \le ck! + 1$ for $c = (e - e^{-1} + 3)/2$. However, we are more concerned what is the limit of $r_k(3)/k!$ as $k \to \infty$. No matter what $r_m(3)$ we know, we cannot obtain the limit from Theorem 1.8 since it is the limit of $A_\infty(m)$ as $m \to \infty$.

Proposition 1.2 *As $k \to \infty$, the limit of $r_k(3)/k!$ exists.*

Proof. It is clear that the limit of $r_k(3)/k!$ exists if and only if that of $(r_k(3) - 1)/k!$ does, and the limits are the same if they exist. Denote by ℓ for $\underline{\lim}_{k\to\infty} (r_k - 1)/k!$. For any $\epsilon > 0$, there are infinitely many m such that $(r_m - 1)/m! < \ell + \epsilon/2$. Take such large m that $(r_k - 1)/k! > \ell - \epsilon$ for $k \ge m$ and that $\sum_{t>m} 1/t! < \epsilon/2$. From Corollary 1.2, we have

$$\ell - \epsilon < \frac{r_k - 1}{k!} < \frac{r_m - 1}{m!} + \sum_{t>m} \frac{1}{t!} < \ell + \epsilon.$$

Thus $\lim_{k\to\infty} (r_k - 1)/k! = \ell$. □

Problem 1.1 Prove or disprove that the limit of $r_k(3)/k!$ is zero.

1.4 Some Early Ramsey Numbers

This section contains several early generalized Ramsey numbers. The first generalized Ramsey number was due to Gerencsér and Gyárfás in 1967, who computed the Ramsey number of paths. In this section P_{1+n} is a path of length n instead of a Paley graph defined in the next chapter.

Theorem 1.9 *For $n \ge m \ge 1$,*

$$r(P_{1+n}, P_{1+m}) = n + \lceil m/2 \rceil.$$

Proof. We only consider the diagonal case since the proof for general case is similar. In the following, we shall show

$$r(P_{1+n}, P_{1+n}) = n + \lceil n/2 \rceil.$$

The assertion is trivial for $n = 1$ or $n = 2$, so we assume $n \geq 3$ and the assertion holds for $n - 1$. The facts that the graph $G = K_n \cup K_\ell$, where $\ell = \lceil n/2 \rceil - 1$ and its complement contain no P_{1+n} give $r(P_{1+n}, P_{1+n}) \geq N$, where $N = n + \lceil n/2 \rceil$.

In the following, we consider a red-blue edge coloring of K_N and we shall show that there is a monochromatic path of length n. If not, by assumption, we have a monochromatic, say red, path P of length $n-1$. Consider two disjoint blue paths Q_1 and Q_2 consisting of edges between $V(P)$ and $V(K_N) \setminus V(P)$, such that their end-vertices are in $V(K_N)\setminus V(P)$, and they do not contain end-vertices of P. Furthermore, the sum of their lengthes are maximum. We shall prove that the following claim by induction on n. Denote by $u_0, v_0, u_1, v_1, u_2, v_2$ for the end-vertices of P, Q_1, Q_2, respectively.

Claim $V(P) \cup V(Q_1) \cup V(Q_2) = V(K_N)$.

Proof. It is easy to see that $|(V(Q_1) \cup V(Q_2)) \setminus V(P)| \geq 2$ since otherwise there is at most one vertex outside $V(P)$ hence $|V(P)| \geq N - 1 \geq n + 1$. Thus

$$|V(P) \cup V(Q_1) \cup V(Q_2)| \geq n + 2.$$

Suppose the claim is false, i.e., there exists a vertex $x \in V(K_N)$ with $x \notin V(P) \cup V(Q_1) \cup V(Q_2)$. From the structures of Q_1 and Q_2,

$$|V(P) \cap V(Q_i)| = \frac{1}{2}(|V(Q_i)| - 1) = |V(Q_i) \setminus V(P)| - 1.$$

Thus we have

$$\begin{aligned} |V(P) \setminus (V(Q_1) \cup V(Q_2))| &\geq n + 2 - |(V(Q_1) \cup V(Q_2)) \setminus V(P)| \\ &> n + 2 - |V(K_N) \setminus V(P)| \\ &= n + 2 - \lceil n/2 \rceil \\ &= \lfloor n/2 \rfloor + 2. \end{aligned}$$

Thus at least $\lfloor n/2 \rfloor + 3$ vertices of P, implying at least $\lfloor n/2 \rfloor + 1$ internal vertices of P are not covered by $Q_1 \cup Q_2$. Therefore, there exists an internal edge uv of P such that $u, v \notin V(Q_1 \cup Q_2)$.

One of edges xu and xv, say xu, must be blue as otherwise the length of P can be increased by replacing uv by ux and xv. Therefore, the edges u_1u and u_2u are red. Thus the edge u_1v and vu_2 must be blue by the same argument as above. Now

$$Q_1' = Q_1 + u_1v + vu_2 + Q_2,$$

and $Q_2' = \{x\}$ are two paths with the sum of lengths greater than that of Q_1 and Q_2 satisfying the same conditions, yielding a contradiction and proving the claim. □

Consider the four edges $u_0u_1, u_0u_2, v_0v_1, v_0v_2$. All of these edges are blue by the maximality of P. Therefore, the cycle

$$C = Q_1 + u_1u_0 + u_0u_2 + Q_2 + v_2v_0 + v_0v_1$$

is monochromatically blue. Since the vertices of C are in $V(P)$ and out of $V(P)$ alternatively, the length of C is

$$2(N - n) = 2\lceil n/2 \rceil.$$

If n is odd, then the length of C is $n + 1$, which gives a path of length n by removing an edge of C. So we assume that n is even and the length of C is n. If there is a blue edge between C and $V(K_N) \setminus V(C)$, then we can find such blue path again. Thus we assume that all edges between C and $V(K_N) \setminus V(C)$ are red. Therefore, we can find a red path with length $2\lceil n/2 \rceil \geq n$ easily. This contradicts to the maximality of P, completing the proof. □

It takes about 40 years after Gyárfás, Ruszinkó, Sárközy and Szemerédi (2007) showed that

$$r(P_n, P_n, P_n) = \begin{cases} 2n - 1 & \text{if } n \text{ is odd,} \\ 2n - 2 & \text{if } n \text{ is even,} \end{cases}$$

in which the authors used Szemerédi regularity lemma (1976) and the idea of connected matchings which was suggested by Łuczak (1999). We will discuss further on related topics in latter chapters.

The following is due to Chvátal and Harary (1972), and Burr and Roberts (1973), in which $K_{1,n}$ is a star of n edges.

Theorem 1.10 *For positive integers m and n,*

$$r(K_{1,m}, K_{1,n}) = \begin{cases} m + n - 1 & \textit{if } m \textit{ and } n \textit{ are both even,} \\ m + n & \textit{otherwise.} \end{cases}$$

Proof. From the recursive upper bounds in Theorem 1.5, we have

$$r(K_{1,m}, K_{1,n}) \leq r(\overline{K}_m, K_{1,n}) + r(K_{1,m}, \overline{K}_n) = m + n,$$

and if m and n are both even, the inequality is strict. The desired upper bound follows.

For the lower bound, the proof is clear if $m = 1$ or $n = 1$. So we assume that $m \geq 2$ and $n \geq 2$, and separate the remaining proof into two cases.

Case 1 m or n, say m, is odd.

For this case, set

$$Z_{m+n-1} = \{0, 1, 2, \ldots, m + n - 2\}$$

and $A = \{\pm 1, \pm 2, \cdots, \pm (m-1)/2\}$. Define a graph on Z_{m+n-1}, in which two vertices x and y are adjacent if and only if $x - y \in A$. This graph is $(m - 1)$-regular and its complement is $(n - 1)$-regular, which yields that $r(K_{1,m}, K_{1,n}) \geq m + n$.

Case 2 Both m and n are even.

From case 1, we have

$$r(K_{1,m}, K_{1,n}) \geq r(K_{1,m-1}, K_{1,n}) \geq m + n - 1,$$

the desired lower bound follows. □

Let T_n denote a tree of order n. we have the following lower bound.

Lemma 1.3 *For* $n \geq 2$,

$$r(T_n) \geq \frac{4n}{3} - 1.$$

Proof. For a tree T_n, it is unique to separate the vertices of T_n into two color classes. Assume that the sizes of color classes of T_n are m and $n - m$ with $m \leq n - m$. If $G = K_{m-1} \cup K_{n-1}$, or $G = K_{n-m-1} \cup K_{n-m-1}$, then neither G nor $\overline{G}$ contains T_n, which implies $r(T_n) \geq \max\{m + n - 1, 2(n - m) - 1\}$. Minimizing the right side on $1 \leq m \leq n - 1$, we have $r(T_n) \geq \frac{4n}{3} - 1$. □

The lower bound in the above lemma is sharp. A broom $B_{k,\ell}$ is a tree on $n = k + \ell$ vertices obtained by identifying an end-vertex of a path P_ℓ with the central vertex of a star $K_{1,k}$. Erdős et al. (1982) proved that $r(B_{k,\ell}) = k + \lceil \frac{3\ell}{2} \rceil - 1$ for $\ell \geq 2k$ and $k \geq 1$. Yu and Li (2016) determined all other Ramsey numbers of brooms. In particular, for any integer k, ℓ with $k \geq 2$ and $n = k + \ell$,

$$r(B_{k,\ell}) = \begin{cases} n + \lfloor \frac{\ell}{2} \rfloor - 1 & \text{if } \ell \geq 2k - 1, \\ 2n - 2\lfloor \frac{\ell}{2} \rfloor - 1 & \text{if } 4 \leq \ell \leq 2k - 2. \end{cases}$$

Problem 1.2 Find a good expression for $r(T_{1+n}, T_{1+n})$.

The following result of Chvátal (1977) stimulated generalized Ramsey theory greatly.

Theorem 1.11 *For* $k, n \geq 1$,

$$r(T_n, K_k) = (k - 1)(n - 1) + 1.$$

Proof. The complete $(k - 1)$-partite graph $K_{k-1}(n - 1)$ yields the lower bound $r(K_k, T_n) \geq (k - 1)(n - 1) + 1$. To get the reverse inequality, we use induction on n. Let G be a graph on $N = (k - 1)(n - 1) + 1$ vertices. Suppose that $\overline{G}$ contains no K_k, that is to say, $\alpha(G) \leq k - 1$. We shall see that G contains T_n. Let $S \subset V(G)$ be an maximum independent set of size $\ell \leq k - 1$. Outside S, there are $N - \ell \geq (k-1)(n-2)+1$ vertices, which must contain T_{n-1} by induction, where T_{n-1} is a tree obtained from T_n by deleting a vertex v of degree one. Let u be the vertex of T_{n-1} adjacent to v in T_n. Since S is maximum, u has at least one neighborhood in S, yielding T_n as claimed. □

Let C_n be a cycle on n vertices. In the early 1970's, the Ramsey number $r(C_m, C_n)$ was studied by several authors, we refer the reader to Bondy and Erdős (1973), Faudree and Schelp (1974), and Rosta (1973). We conclude this section with the Ramsey number $r(C_3, C_n)$ by Chartrand and Schuster (1971), in which the authors also determined the exact value of $r(C_m, C_n)$ for $m = 4, 5$.

Lemma 1.4 *We have* $r(C_3, C_4) = 7$.

Proof. Consider a red-blue edge coloring of K_7, and denote R and B by graphs induced by the red and blue edges, respectively. It is clear that $\Delta(R) \leq 3$ and $\Delta(B) \leq 3$. This implies that both R and B are 3-regular, which is impossible since the number of vertices is odd. On the other hand, if we color edges of K_6 by red and blue such that the red graph is $K_{3,3}$, then there is neither red K_3 nor blue C_4 hence the lower bound $r(C_3, C_4) > 6$ follows as desired. □

Theorem 1.12 *If $n \geq 4$, then $r(C_3, C_n) = 2n - 1$.*

Proof. We prove the equality by induction on $n \geq 4$, which is just obtained for $n = 4$. Now we assume that $r(C_3, C_n) = 2n - 1$ for $n \geq 4$ and consider the number $r(C_3, C_{n+1})$. Denote by $K_{m,n}$ the complete bipartite graph of order $m + n$ whose vertex set may be partitioned as $V_1 \cup V_2$, where $|V_1| = m$ and $|V_2| = n$. Since $K_{n,n}$ contains no C_3 and its complement contains no C_{n+1}, the lower bound follows.

Let G be a graph of order $2n+1$, and assume G has no C_3. Since $r(C_3, C_n) = 2n-1$, $\overline{G}$ contains a cycle $C_n := u_1u_2 \cdots u_nu_1$. Denote the remaining vertices of $\overline{G}$ (and hence G) by $v_1, v_2, \ldots, v_{n+1}$. If any v_i is adjacent in $\overline{G}$ to two consecutive vertices of C_n, then $\overline{G}$ contains a C_{n+1}, completing the proof. Suppose, then, that no such v_i exists. We consider two cases.

Case 1 There exist two alternate vertices of C_n, say u_j and u_{j+2}, which are respectively joined in $\overline{G}$ to two distinct v_i and $v_{i'}$.

Case 2 No two alternate vertices of C_n are respectively joined in $\overline{G}$ to distinct vertices v_i and $v_{i'}$.

For Case 1, it is easy to check that either G contains C_3 or $\overline{G}$ contains a C_{n+1} by noting $u_{j+1}v_i$ and $u_{j+1}v_{i'}$ are edges in G. For Case 2, note that there is an edge $v_iv_{i'}$ in G since otherwise $\overline{G}$ contains a C_{n+1} as desired and we may assume u_jv_i is an edge in G. It can be shown that either G contains C_3 or $\overline{G}$ contains a C_{n+1}. We leave the details to the reader. The proof is complete. □

1.5 Hypergraph Ramsey Number

We shall conclude this chapter with the existence of the hypergraph Ramsey number.

A *hypergraph* $\mathcal{G}$ on vertex set V is a pair $(V, \mathcal{E})$, where the edge set $\mathcal{E}$ is a family of subsets of V. Let $V^{(r)}$ be the family of all r-subsets of V, which is also denoted by $\binom{V}{r}$. If $\mathcal{E} \subseteq V^{(r)}$, then $\mathcal{G}$ is called r-uniform. Thus a 2-uniform hypergraph is just a graph. When $\mathcal{E} = V^{(r)}$, we say that $\mathcal{G}$ is complete, denoted by $K_n^{(r)}$, where $n = |V|$. If all elements in $V^{(r)}$ are colored in k colors, a subset $X \subseteq V$ is said to be monochromatic when any element in $X^{(r)}$, not element of X itself, has the same color. That is to say, the subset X induces a monochromatic complete sub-hypergraph of $\mathcal{G}$.

Define $r_k^{(r)}(n_1, n_2, \ldots, n_k)$ to be the minimum integer N such that every coloring of $V^{(r)}$ with $|V| = N$ by colors $1, 2, \ldots, k$, then for some i, there exists

$X \subseteq V$ with $|X| = n_i$ that induces a monochromatic $K_{n_i}^{(r)}$ in the color i. Write $r_k^{(r)}(n)$ for $r_k^{(r)}(n, n, \ldots, n)$, $r^{(r)}(m, n)$ for $r_2^{(r)}(m, n)$ and $r_k(n_1, n_2, \ldots, n_k)$ for $r_k^{(2)}(n_1, n_2, \ldots, n_k)$. Note that $r^{(1)}(m, n) = m + n - 1$ and for $r \geq 2$,

$$r^{(r)}(m, n) = \begin{cases} \max\{m, n\} & \text{if } r = \min\{m, n\}, \\ \min\{m, n\} & \text{if } r > \min\{m, n\}. \end{cases}$$

Theorem 1.13 *For $r, m, n \geq 2$, $r^{(r)}(m, n)$ exists and*

$$r^{(r)}(m, n) \leq r^{(r-1)}\left(r^{(r)}(m-1, n),\ r^{(r)}(m, n-1)\right) + 1.$$

Proof. From the fact that $r^{(1)}(m, n) = m+n-1$ and the upper bound that $r^{(2)}(s, t) \leq r^{(2)}(s-1, t) + r^{(2)}(s, t-1)$ in Section 1.2, we see that the assertion holds for $r = 2$. For $r > 2$, we assume that $r^{(r-1)}(s, t)$ exists.

Let V be a set with $N = r^{(r-1)}\left(r^{(r)}(m-1, n),\ r^{(r)}(m, n-1)\right) + 1$ vertices. For a given coloring $c:\ V^{(r)} \to \{\text{red, blue}\}$, a vertex $v \in V$ and $X = V \setminus \{v\}$, define a coloring

$$\bar{c}:\ X^{(r-1)} \to \{\text{red,blue}\},\ \ \bar{c}(S) = c(S \cup \{v\})$$

for $S \in X^{(r-1)}$ as $S \cup \{v\} \in X^{(r)}$. From the definition of $r^{(r-1)}(s, t)$ and

$$|X| = r^{(r-1)}\left(r^{(r)}(m-1, n),\ r^{(r)}(m, n-1)\right),$$

we see that either X contains a subset Y with $|Y| = r^{(r)}(m-1, n)$ such that $Y^{(r-1)}$ is completely red in $\bar{c}$, or X contains a subset Z with $|Z| = r^{(r)}(n, m-1)$ such that $Z^{(r-1)}$ is completely blue in $\bar{c}$. Without loss of generality, we assume that the former is the case. We shall show that Y contains either a subset A with $|A| = m$ such that $A^{(r)}$ is completely red in c or subset B with $|B| = n$ such that $B^{(r)}$ is completely blue in c.

We now have $Y \subseteq X = V \setminus \{v\}$ with $|Y| = r^{(r)}(m-1, n)$. Consider the restriction of the coloring c of $V^{(r)}$ on $Y^{(r)}$. From the definition, we know that either Y contains a subset B with $|B| = n$ such that $B^{(r)}$ is completely blue in c, or Y contains a subset A_0 with $|A_0| = m - 1$ such that $A_0^{(r)}$ is completely red in c. In the former case, we are done. In the latter case, set $A = A_0 \cup \{v\}$, then $|A| = m$ since $v \notin A$. For any $T \in A^{(r)}$, if $v \notin T$, then $T \in A_0^{(r)}$ and hence $c(T)$ is red. Otherwise, $S = T \setminus \{v\} \in A_0^{(r-1)} \subseteq Y^{(r-1)}$, so $\bar{c}(S)$ is red since $Y^{(r-1)}$ is completely red in $\bar{c}$. The definition of $\bar{c}$ implies that $c(T)$ is red since $c(T) = \bar{c}(S)$, thus $A^{(r)}$ is completely red in c. □

The following is the multicolor case for the existence of hypergraph Ramsey number. Denote $r_i = r_k^{(r)}(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_k)$.

Theorem 1.14 *For $r, k, n_1, \ldots, n_k \geq 2$, $r_k^{(r)}(n_1, \ldots, n_k)$ exists and*

$$r_k^{(r)}(n_1, \ldots, n_k) \leq r_k^{(r-1)}(r_1, r_2, \ldots, r_k) + 1.$$

The above results do not give rational upper bounds for the hypergraph Ramsey numbers other than the cases $r = 2$ and small n_i. Erdős, Hajnal, and Rado (1965) showed that there are positive constants c and c' such that

$$2^{cn^2} < r^{(3)}(n,n) < 2^{2^{c'n}}.$$

They also conjectured that $r^{(3)}(n,n) > 2^{2^{cn}}$ for some constant $c > 0$, and Erdős offered a \$500 reward for a proof. Similarly, for $r \geq 4$, there is a difference of one exponential between the known upper and lower bounds for $r^{(r)}(n,n)$, i.e.,

$$t_{r-1}(cn^2) \leq r^{(r)}(n,n) \leq t_r(c'n),$$

where the tower function $t_r(x)$ is defined by $t_1(x) = x$ and $t_{i+1}(x) = 2^{t_i(x)}$. The study of 3-uniform hypergraphs is particularly important for our understanding of hypergraph Ramsey numbers. This is because of an ingenious construction called the stepping-up lemma due to Erdős and Hajnal (see, e.g., Chapter 4.7 in the book by Graham, Rothschild and Spencer (1990)). Their method allows one to construct lower bound colorings for uniformity $r+1$ from colorings for uniformity r, effectively gaining an extra exponential each time it is applied. Unfortunately, the smallest r for which it works is $r = 3$. Therefore, proving that $r^{(3)}(n,n)$ has doubly exponential growth will allow one to close the gap between the upper and lower bounds for $r^{(r)}(n,n)$ for all uniformities r. There is some evidence that the growth rate of $r^{(r)}(n,n)$ is closer to the upper bound, namely, that with four colors instead of two this is known to be true. Erdős and Hajnal (see, e.g., Graham, Rothschild and Spencer (1990)) constructed a 4-coloring of the triples of a set of size $2^{2^{cn}}$ which does not contain a monochromatic subset of size n. This is sharp up to the constant c. It also shows that the number of colors matters a lot in this problem and leads to the question of what happens in the intermediate case when we use three colors. The 3-color Ramsey number $r_3(n,n,n)$ is the minimum N such that every 3-coloring of the triples of an N-element set contains a monochromatic set of size n. In this case, Erdős and Hajnal (1989) have made some improvement on the lower bound 2^{cn^2} (see also in Chung and Graham(1998)), showing that $r_3(n,n,n) \geq 2^{cn^2 \log^2 n}$. Conlon, Fox and Sudakov (2010) substantially improved this bound, extending the above-mentioned stepping-up lemma of these two authors to show that there exists a constant $c > 0$ such that

$$r^{(3)}(n,n,n) \geq 2^{n^{c \log n}}.$$

For off-diagonal Ramsey numbers, a classical argument of Erdős and Rado (1952) demonstrates that

$$r^{(r)}(s,n) \leq 2^{\binom{r^{(r-1)}(s-1,n-1)}{r-1}}.$$

Conlon, Fox and Sudakov (2010) obtained that for fixed $s \geq 4$ and sufficiently large n,

$$2^{cn \log n} \leq r^{(3)}(s,n) \leq 2^{c' n^{s-2} \log n}.$$

where the upper bound improves the exponent of that due to Erdős and Rado (1952) by a factor of $n^{s-2}/\text{poly}\log n$, while the lower bound confirms a conjecture of Erdős and Hajnal (1972). Mubayi and Suk (three papers in 2017-2018) considered further for 4-uniform case, in particular, the authors obtained that

$$r^{(4)}(5,n) \geq 2^{n^{c\log n}}, \quad \text{and} \quad r^{(4)}(6,n) \geq 2^{2^{cn^{1/5}}},$$

where $c > 0$ is a constant.

1.6 Exercises

1. Suppose that $S \subseteq \{1, 2, \ldots, 2n\}$ with $|A| = n + 1$. Show that there exists a pair of numbers in S such that one divides the other. (Hint: Write each $s \in S$ in the form $s = 2^k m$, where m is odd.)

2. An application of the pigeonhole principle is to prove a famous result of Erdős-Szekeres (1935): Let $A = (a_1, \ldots, a_n)$ be a sequence of n different real numbers. Prove that if $n \geq st + 1$, then either A has an increasing subsequence of $s + 1$ terms or a decreasing one of $t + 1$ terms. (Hint: Associate each a_i to a pair (x_i, y_i), where x_i (y_i) is the number of terms in the longest increasing (decreasing) subsequence ending (starting) at a_i. Place a_i in the pigeonhole of a grid of n^2 pigeonholes with coordinates (x_i, y_i).)

3. Give an easier proof for the existence of n in the last exercise, say $n \leq r(s + 1, t + 1)$.

4. Prove Theorem 1.2.

5. Prove that $r(K_m + \overline{K}_n, K_p + \overline{K}_q) \leq \binom{m+p-1}{m} n + \binom{m+p-1}{p} q$. (A. Thomason, 1982)

6. Let G be a graph on N vertices. Prove that the number of triangles contained in G and $\overline{G}$ is

$$\binom{N}{3} - \frac{1}{2} \sum_{v \in V(G)} d(v)(N - 1 - d(v)).$$

(R. Goodman, 1959.)

7. Let $\ell = r(K_n, K_{n-2})$. Show $r(K_n, K_n) \leq r(B_\ell, B_\ell)$ and hence Walker's bound $r(K_n, K_n) \leq 4r(K_n, K_{n-2}) + 2$.

8. Let $k \geq 2$. Prove that

$$r(n_1 + 1, \ldots, n_k + 1) \leq \frac{(n_1 + \cdots + n_k)!}{n_1! \cdots n_k!}.$$

9. Prove that

$$r(n, n) \leq 4r(n - 1, n - 1) - 2.$$

(Hint: In a Ramsey coloring, each of $|N_R(x) \cap N_B(y)|$ and $|N_B(x) \cap N_R(y)|$ is at most $r(n-1, n-1)-1$. The sum over $x \neq y$ counts each non-monochromatic triangle twice.)

10. Color all non-empty subsets (not the points) of $[N]$ by k colors. Prove that, if N is large enough, then there are two disjoint non-empty subsets A, B such that A, B and $A \cup B$ have the same color. (Hint: Let $N = r_k(3)$, and χ a k-coloring of subsets of $[N]$. Color any edge ij with $i < j$ of K_N on $[N]$ by color $\chi([i, j-1])$.)

11. Show that

$$r(T, K_{n_1}, \cdots, K_{n_k}) = (m-1)(r(K_{n_1}, \ldots, K_{n_k}) - 1) + 1,$$

where T is a tree of order m.

12. Assume that $m-1$ divides $n-1$. Show that for every tree T on m vertices, $r(T, K_{1,n}) = m + n - 1$.

13. Complete the proof of Theorem 1.12.

14.* Let $B_{k,\ell}$ be a tree on $n = k + \ell$ vertices obtained by identifying an end-vertex of a path P_ℓ with the central vertex of a star $K_{1,k}$. Prove that $r(B_{k,\ell}) = k + \lceil \frac{3\ell}{2} \rceil - 1$ for $\ell \geq 2k$ and $k \geq 1$. (Erdős et al., 1982)

15. Burr and Erdős (See Chung and Graham, 1998) asked to prove that $r(n+1, n) > (1+c)r(n, n)$ for some fixed $c > 0$. From known results, find the pairs (m, n) with $2 \leq m \leq n$ such that $r(m-1, n+1) \leq r(m, n)$.

Chapter 2
Small Ramsey Numbers

What are exact values of $r(m,n)$? This is more challenging than to show their existence. Let us call the classical Ramsey number $r(m,n)$ to be the *small Ramsey number* if m and n are small. In this chapter, we shall obtain some exact values of small Ramsey numbers in the first section. To get more, it is necessary to have a short introduction on finite fields in the second section. The exact value of $r_3(3)$ is an early application of finite field in the graph Ramsey theory. Relating to $r_k(3)$, a Ramsey function on integers is the Schur function, which will be discussed in the third section. One of the most important graphs constructed by finite field are Paley graphs, which are highly symmetric and yield almost all currently best known lower bounds of small Ramsey numbers, and exact values of infinitely many book graphs B_n. Also, Paley graphs form a family of quasi-random graphs, see Chapter 10. We have some context on graph spectra in the sixth section, particularly that related to the independence numbers of graphs, which gives exact independence numbers of infinitely many Paley graphs, showing such Paley graphs are not good for the lower bounds of classical Ramsey numbers.

2.1 Ramsey Folklore

As mentioned in Chapter 1, if $N = r(G,H)$, then there exists a graph F of order $N-1$ such that F contains no G and its complement $\overline{F}$ contains no H, for which F is called a *Ramsey graph* for $r(G,H)$. To illustrate the idea, we shall find the first four nontrivial classical Ramsey numbers.

Theorem 2.1 *We have four exact values of Ramsey numbers as follows.*

$$r(3,3) = 6,\ r(3,4) = 9,\ r(3,5) = 14,\ r(4,4) = 18.$$

Proof. We have $r(3,3) = 6$ in the last chapter. Since both $r(3,3) = 6$ and $r(2,4) = 4$ are even, by Theorem 1.5, we have $r(3,4) \le 9$. Moreover,

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$$r(3,5) \le r(2,5) + r(3,4) \le 14. \tag{2.1}$$

To obtain a lower bound of $r(3,5)$, let us consider the graph G on vertex set $Z_{13} = \{0, 1, \ldots, 12\}$ (the integers modulo 13). Denote by $A = \{1, 5, 8, 12\} = \{\pm 1, \pm 5\}$, which consists of all non-zero cubic (mod 13) of Z_{13}. Connect vertices i and j by an edge if and only if $i - j \in A$. It is easy to check that G is triangle-free and $\alpha(G) = 4$. Thus $r(3,5) \ge 14$ hence $r(3,5) = 14$. Furthermore the equalities in (2.1) hold, which implies that $r(3,4) = 9$.

From $r(3,4) = 9$, we have $r(4,4) \le 2r(3,4) = 18$. To get a lower bound, let us consider the graph with vertex set $Z_{17} = \{0, 1, \ldots, 16\}$, in which i and j are adjacent if and only if $i - j \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$, the set of non-zero quadratics (mod 17). This graph shows that $r(4,4) \ge 18$. We will see that this graph is indeed a Paley graph later. □

Greenwood and Gleason (1955) computed four exact values as in the above theorem. They also found that the exact value of $r(3,3,3)$ is 17, see the next section, which is the only known exact value among all nontrivial classical Ramsey number in three or more colors. Graver and Yackel (1968) determined that $r(3,6) = 18$ and $r(3,7) = 23$. No other classical Ramsey number is found without aid of computers. Using the computers, Grinstead and Roberts (1982) found that $r(3,8)$ is between 28 and 29, and they obtained $r(3,9) = 36$; Mckay and Zhang (1992) finally determined $r(3,8) = 28$. MaKay and Radziszowki (1995) computed $r(4,5) = 25$. All known non-trivial classical Ramsey numbers $r(m,n)$ and some bounds at present are listed in Table 2.1. The Ramsey graphs for $r(3,5)$ and $r(4,4)$ are illustrated in Fig. 2.1.

$m\backslash n$	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28	36	40/43
4		18	25	35/41	49/61	56/84	69/115	92/149
5			43/49	58/87	80/143	101/216	121/316	141/442

Table 2.1 Some values and bounds of $r(m,n)$.

Radziszowski's dynamic survey (1994) offers up-to-date information on small Ramsey numbers. The paucity of known exact values $r(m,n)$ indicates the difficulty in this area. Also this paucity stimulates our curiosity to find more classical Ramsey numbers. The following story came from Spencer (1994).

Erdős asks us to imagine an alien force, vastly more powerful than us, landing the earth and demanding the value of $r(5,5)$ *or they will destroy our planet. In this case, he claims, "we should marshall all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for* $r(6,6)$. *In that case, we should attempt to destroy the aliens."*

Perhaps as the improvement of computers and algorithms, the value of $r(5,5)$ and even $r(6,6)$ can be obtained in the near future. But for a bit larger n, the exact value of $r(n,n)$ is still far away from being tractable. However, more "exact" results are known on generalized Ramsey numbers, some of which will be discussed in latter chapters.

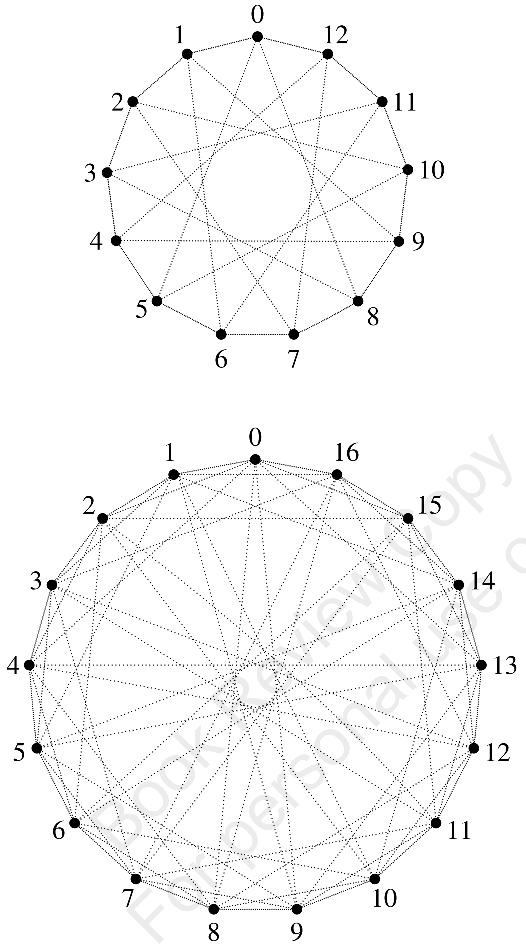

Fig. 2.1 Ramsey graphs for $r(3, 5)$ and $r(4, 4)$

2.2 Finite Field and $r_3(3)$

Greenwood and Gleason (1955) proved that $r_3(3) = 17$ by partitioning a finite field, which is an example for applications of finite fields in graph Ramsey theory. On finite fields, one of elementary facts is that there exists a field $F(q)$ of q elements if and only if q is a prime power. A finite field $F(q)$ is often called Galois field thus denoted by $GF(q)$. For simplicity, we use the notation $F(q)$ or F_q. On $F(q)$ there are two operations, addition and multiplication. When $q = p$ is a prime, the set of elements of $F(p)$ can be viewed as

$$Z_p = \{0, 1, \ldots, p-1\},$$

where the addition and multiplication are more pleasing since they are arithmetic modulo p. The tables for two operations + and $\cdot$ for elements in Z_5 are as follows.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Table 2.2 Operations on Z_5

Recall an easy fact from group theory as follows.

Lemma 2.1 *Let G be a multiplicative group and let S be a finite subset of G. If S is closed for multiplication, then S is a subgroup of G.*

Since $p|ab$ implies $p|a$ or $p|b$, so $Z_p^* = Z_p \setminus \{0\}$ is closed for multiplication modulo p hence it is a multiplicative group of order $p-1$, and thus the index of any element x of Z_p^* is a factor of $p-1$; namely it satisfies that the equation $x^{p-1} \equiv 1 \pmod{p}$, hence any element of $F(p)$ satisfies the equation $x^p \equiv x \pmod{p}$. Also, for any $a \in Z_p^*$, $ax \equiv 1 \pmod{p}$ has unique solution in Z_p^*, so the inverse a^{-1} of a exists. Thus Z_p is a finite field of p elements. Formally, when Z_p is viewed as a field, the congruence $a \equiv b \pmod{p}$ should be written as $a = b$, which is an equality in the field. We sometimes do not distinguish the two notations to signify the operations in Z_p.

The field $F(p)$ is the unique field of order p up to isomorphism. To discuss some structure of finite field $F(p^m)$, let us begin with some basics. Of course, we shall constrain ourselves with what are needed here.

Let F be a (finite or infinite) field such as the field Z_p, Q of rational numbers, or R of real numbers. A polynomial

$$f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_m$$

is called a polynomial on F if any coefficient $a_i \in F$. Denote by $F[x]$ for the set of all polynomials on F. Then $F[x]$ is a ring on the ordinary addition and the multiplication of polynomials.

Fix a polynomial $f(x) \in F[x]$ of degree m. We define an equivalence relation $\equiv$ on $F[x]$ for with $g_1(x) \equiv g_2(x)$ if and only if $f(x)|(g_1(x) - g_2(x))$, that is to say, the reminders of $g_1(x)$ and $g_2(x)$ are the same when $f(x)$ divides them. Let

$$g_i(x) = f(x)h_i(x) + r_i(x),$$

where the degree of $r_i(x)$ is less m. Denote by $\langle g(x)\rangle$ for the equivalence class that contains $g(x)$, then $\langle g_i(x)\rangle = \langle r_i(x)\rangle$, and $g_1(x) \equiv g_2(x)$ if and only if $r_1(x) = r_2(x)$.

Let $F[x]/(f(x))$ be the set of all equivalence classes. Then it is a ring on addition and multiplication in an obvious way.

Theorem 2.2 *Let $f(x) \in F[x]$. Then $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible over F.*

This field is called a *residue field* or *quotient field*. To simplify the notation, we use the unique polynomial of minimum degree, namely, the remainder, in an equivalence class to represent the class. Then each element in the field $F[x]/(f(x))$ can be expressed uniquely as

$$r(x) = b_0x^{m-1} + b_1x^{m-2} + \cdots + b_{m-1}, \quad b_i \in F,\ 0 \le i \le m-1,$$

where m is the degree of $f(x)$. So the field $Z_p[x]/(f(x))$ contains p^m elements, which is thus denoted by $F(p^m)$. As an example, the field $F(3^2)$ can be expressed as $Z_3[x]/(f(x))$, where $f(x) \in Z_3[x]$ is irreducible of degree 2, so

$$F(3^2) = \{0,\ 1,\ 2,\ x,\ x+1,\ x+2,\ 2x,\ 2x+1,\ 2x+2\}.$$

The sum of two elements is the sum of two polynomials, but the product of two elements depends on the form of $f(x)$. For $Z_3[x]/(f(x))$, if $f(x) = x^2 + 1$, then $x(x+1) = x^2+x = -1+x = x+2$. If $f(x) = x^2+x+2$, then $x(x+1) = x^2+x = -2 = 1$.

For an element $r(x) \in F(q) = Z_p[x]/(f(x))$ and $s \in Z_p$, where $q = p^m$, each coefficient of $sr(x)$ is the product of s and the corresponding coefficient. So we see an interesting fact that $pr(x) = 0$, which is trivial if we write it as $0r(x) = 0$.

Since $F^* = F(p^m) \setminus \{0\}$ is closed on multiplication so it is a multiplicative group. It is interesting to see that F^* is a cyclic group, its generators are called *primitive elements* of $F(p^m)$. For example, in $Z_3[x]/(x^2+1)$, the element $x+1$ is a primitive element.

i	1	2	3	4	5	6	7	8
$(x+1)^i$	$x+1$	$2x$	$2x+1$	2	$2x+2$	x	$x+2$	1

Table 2.3 A primitive element $(x+1)$ of $Z_3[x]/(x^2+1)$

Let us define the *period* of an irreducible polynomial $f(x) \in Z_p[x]$, denoted by $p(f)$, as the smallest ℓ such that $f(x)|(x^\ell - 1)$ in $Z_p[x]$. Clearly $p(f) \le p^m - 1$, where m is the degree of $f(x)$. As $f(x)|(x^\ell - 1)$ in $Z_p[x]$ is equivalent to $x^\ell = 1$ in $Z_p[x]/(f(x))$, so if we choose $f(x)$ such that $p(f) = p^m - 1$, then x is a primitive element of the field $Z_p[x]/(f(x))$. For example, $f(x) = x^2 + x + 2$ of $Z_3[x]$ is irreducible with $p(f) = 8$, so x is a primitive element of $F(3^2) = Z_3[x]/(x^2+x+2)$.

i	1	2	3	4	5	6	7	8
x^i	x	$2x+1$	$2x+2$	2	$2x$	$x+2$	$x+1$	1

Table 2.4 A primitive element x of $Z_3[x]/(x^2+x+2)$

In conclusion, $Z_p[x]/(f(x))$ is a field of order p^m, where $f(x) \in Z_p[x]$ is irreducible with degree m. This field is unique up to isomorphism. If $p(f) = p^m - 1$, then we can simply express the elements of $GF(p^m)$ as

$$0, x, x^2, \ldots, x^{q-1} = 1.$$

This expression is convenient for the multiplication but not for the addition.

Theorem 2.3 $r_3(3) = 17.$

Proof. The right upper bound follows from Theorem 1.1. Indeed, if the edges of K_{17} are colored by colors 1, 2, 3, a vertex v has at least six neighbors connected by the edges in the same color, say color 1. Two of such neighbors x and y of v and v itself form a monochromatic triangle if the edge xy is colored by 1, or six neighbors induce a K_6 whose edges are colored by colors 2 and 3, which contains a monochromatic triangle as $r(3,3) = 6$. This proves that $r_3(3) \le 17$.

Greenwood and Gleason (1955) proved that $r_3(3) \ge 17$ by considering $F(2^4)$, which is isomorphic to $Z_2[x]/(x^4 + x + 1)$ since $x^4 + x + 1$ is a irreducible in $Z_2[x]$ with period 15. Then the elements of $F(2^4)$ can be identified as

$$\{0, x, x^2, \ldots, x^{15} = 1\},$$

where

$0 = 0$	$x^8 = 1 + x^2$
$x = x$	$x^9 = x + x^3$
$x^2 = x^2$	$x^{10} = 1 + x + x^2$
$x^3 = x^3$	$x^{11} = x + x^2 + x^3$
$x^4 = 1 + x$	$x^{12} = 1 + x + x^2 + x^3$
$x^5 = x + x^2$	$x^{13} = 1 + x^2 + x^3$
$x^6 = x^2 + x^3$	$x^{14} = 1 + x^3$
$x^7 = 1 + x + x^3$	$x^{15} = 1$

Table 2.5 The elements in a field $F(16)$

Note that $-f(x) = f(x)$ in $Z_2[x]$. Let A_0 be the set of all cubic residues of $F^* = F \setminus \{0\}$, then

$$A_0 = \{x^3,\ x^6,\ x^9,\ x^{12},\ x^{15}\}$$
$$= \{x^3,\ x^2 + x^3,\ x + x^3,\ 1 + x + x^2 + x^3,\ 1\}.$$

Then A_0 is a subgroup of multiplicative group F^*. Set

$$A_1 = xA_0 = \{1 + x,\ 1 + x + x^3,\ 1 + x + x^2,\ 1 + x^2 + x^3,\ x\}$$

and

$$A_2 = x^2 A_0 = \{x + x^2,\ 1 + x^2,\ x + x^2 + x^3,\ 1 + x^3, x^2\}.$$

The sets A_0, A_1, A_2 are cosets of A_0 in multiplicative group F^*. Let us call a subset S of $F(q)$ sum-free if the equation $x + y = z$ has no solutions in S. Observe that A_0 is sum-free from the above table hence A_1 and A_2 are also sum-free.

Define an edge coloring of K_{16} on vertex set $F(2^4)$ as follows. Color an edge uv with color i if $u + v \in A_i$. Then all edges are colored by colors 0, 1 and 2. We claim that there is no monochromatic triangle. Suppose that vertices u, v and w induce a monochromatic triangle in color i. Then $u + v \in A_i$, $v + w \in A_i$ and $u + w \in A_i$, contradicting to the fact that A_i is sum-free as $u + w = (u + v) + (v + w)$. Thus $r_3(3) \geq 17$. □

Let us remark that in most applications, we are not concerned with the precise structure of a finite field. If this is the case, then we can write $x \in F(q)$ to signify that x is an arbitrary element of $F(q)$, not necessarily a polynomial of the single variable x.

2.3 Schur Numbers

It is hard to determine the exact classical Ramsey numbers as stated. The only known multi-color classical Ramsey number is $r_3(3) = 17$ proved in the last section. For $k = 4, 5$, we have $51 \leq r_4(3) \leq 62$, where the upper bound was established by Fettes, Kramer and Radziszowski (2004) while the lower bound was obtained by Chung (1973), and $162 \leq r_5(3) \leq 307$, where the lower bound is due to Exoo (1994) and the upper bound is implied by that of $r_4(3)$ from the upper bound $r_k(3) \leq k(r_{k-1}(3) - 1) + 2$ proved in the last chapter.

We are more interested in the asymptotic behavior of $r_k(3)$ as $k \to \infty$. Let $f(n)$ be a function taking non-negative values. Call the function $f(n)$ *super-multiplicative* if $f(m+n) \geq f(m)f(n)$. The following result is elementary.

Lemma 2.2 *Suppose the function $f(n) > 0$ is super-multiplicative. Then $\lim_{n\to\infty} f(n)^{1/n}$ exists and it is equal to $\sup_{n\geq 1} f(n)^{1/n}$. If m is fixed, then*

$$f(n) \geq c f(m)^{n/m},$$

where $c = c(m) > 0$ is a constant.

Proof. Set $\ell = \sup_{n\geq 1} f(n)^{1/n}$. Then $0 < \ell \leq \infty$ and $\overline{\lim}_{n\to\infty} f(n)^{1/n} \leq \ell$. We shall show that $\underline{\lim}_{n\to\infty} f(n)^{1/n} \geq \ell$.

Case 1 $\ell < \infty$.

For any $\epsilon > 0$, there is some m such that $f(m)^{1/m} > \ell - \epsilon$. For any $n \geq m$, let $n = qm + r$ with $0 \leq r < m$. Thus

$$f(n) \geq f(qm)f(r) \geq f(m)^q f(r).$$

Since $q/n \to 1/m$ and $f(r)^{1/n} \to 1$, we have

$$\underline{\lim}_{n\to\infty} f(n)^{1/n} \geq f(m)^{1/m} > \ell - \epsilon.$$

Thus $\underline{\lim}_{n\to\infty} f(n)^{1/n} \geq \ell$ since $\epsilon > 0$ is arbitrary.

Case 2 $\ell = \infty$.

For any $M > 0$, there is $m \geq 1$ such that $f(m)^{1/m} > M$, we can similarly show that $\underline{\lim}_{n\to\infty} f(n)^{1/n} \geq M$. It follows that $\underline{\lim}_{n\to\infty} f(n)^{1/n} = \infty$ since $M > 0$ is arbitrary. This prove the first assertion.

For fixed m, let $n = qm + r$ with $0 \leq r < m$. Then

$$f(n) \geq f(r)f(m)^{(n-r)/m} = \frac{f(r)}{f(m)^{r/m}} f(m)^{n/m}.$$

Let $c = \min\{f(r)/f(m)^{r/m} : 0 \leq r < m\}$. Then $c = c(m)$ is a positive constant and $f(n) \geq cf(m)^{n/m}$ as desired. □

Proposition 2.1 *The function $r_k(3) - 1$ is super-multiplicative. Hence the following limits exist and*

$$\lim_{k\to\infty} r_k(3)^{1/k} = \lim_{k\to\infty} (r_k(3) - 1)^{1/k} = \sup_k (r_k(3) - 1)^{1/k}.$$

Proof. Let us write $r_n(3)$ as r_n in the proof. Set $N = r_n - 1$ and $M = r_m - 1$. Color the edges of K_N by n colors and color edges of K_M by other m colors so that there is no monochromatic triangles in any color. Then, "blow-up" one with another by replacing each vertex v of K_N with a colored K_M, denoted by H_v for this K_M. For any distinct vertices u and v of K_N, if $u' \in V(H_u)$, and $v' \in V(H_v)$, color edge between u' and v' with the color as the edge uv in K_N. We thus have a complete graph on NM vertices whose edges are colored with $n + m$ colors and there is no monochromatic triangles. Therefore

$$r_{m+n} - 1 \geq (r_m - 1)(r_n - 1),$$

as claimed. □

It seems to be very difficult to determine the exact value of the limit of $r_k(3)^{1/k}$. From $r_3(3) = 17$, we have $\lim_{k\to\infty} r_k(3)^{1/k} \geq (r_3(3) - 1)^{1/3} = 2.5\cdots$. Nevertheless, we shall do it better.

Schur defined an extremal number in 1916 as follows. A set S of integers is said to be sum-free if $a, b \in S$ (a and b not necessarily distinct) implies

$$a + b \notin S.$$

Let $[N] = \{1, 2, \ldots, N\}$. A result of Schur (1916) states that if the integers $[\lfloor k!e \rfloor]$ are partitioned in any manner into k classes, then at least one of the classes is not sum-free. Accordingly, the *Schur number* s_k is defined to be the largest positive integer N such that $[N]$ can be partitioned in some manner into k sum-free classes. No many exact values of Schur numbers have been known. It is not hard to verify that $s_1 = 1$, $s_2 = 4$ and $s_3 = 13$. With the aid of a computer, Baumert showed that $s_4 = 44$, reported in Abbott and Hanson (1972). It was showed that $s_5 \geq 160$ by Exoo (1994) and $s_6 \geq 536$ by Fredricksen and Sweet (2000). A relation between $r_k(3)$ and s_k is as follows.

Theorem 2.4 *For any positive integer k,*

$$r_k(3) \geq s_k + 2.$$

Proof. Set $N = s_k$. Then $[N]$ can be partitioned into k sum-free sets $A_1, A_2, \ldots, A_k$. We now color the edges of K_{N+1} on vertex set $\{0\} \cup [N]$ in the following way: color the edge uv with color i if $|u - v| \in A_i$. Since $1 \leq |u - v| \leq N$ for any distinct u and v, all edges are colored. We then claim that there are no monochromatic triangles. Suppose not, some distinct vertices u, v and w induce a monochromatic triangle in color i, where $0 \leq u < v < w \leq N$. Thus $|u - v| = v - u \in A_i$ and $|v - w| = w - v \in A_i$. It follows that $|u - w| = w - u = (w - v) + (v - u) \in A_i$, contradicting to the fact that A_i is sum-free. Therefore, the k-colored K_{N+1} do not contain monochromatic triangle, implying that $r_k(3) \geq N + 2 = s_k + 2$. □

Corollary 2.1 $r_5(3) \geq 162$*, and* $r_6(3) \geq 538$.

The original paper of Schur was motivated by Fermat's Last Theorem. He actually proved the following result.

Theorem 2.5 *For any fixed integer $m \geq 1$, if p is a prime with $p \geq s_m + 2$, then the equation*

$$x^m + y^m \equiv z^m \pmod{p}$$

has a nonzero solution.

Proof. We shall prove the equation $x^m + y^m \equiv 1 \pmod{p}$ has a nonzero solution for $p \geq s_m + 2$. If $[p-1]$ is partitioned into m subsets, then one of subsets is not sum-free as $p - 1 \geq s_m + 1$, and thus there exist a, b, c in this subset such that $a + b = c$. Set $H = \{x^m : x \in Z_p^*\}$, which is a multiplicative subgroup of Z_p^*. Then the index of H, i.e., the number of cosets of Z_p^* on H, is $k = gcd(m, p - 1) \leq m$. The cosets of Z_p^* on H define a partition of Z_p^* such that s and t are in the same coset if and only if $st^{-1} \in H$. Suppose that $a, b, c \in [p - 1]$ from the same coset satisfy that $a + b = c$. Then

$$ac^{-1} + bc^{-1} \equiv 1 \pmod{p}.$$

Since $ac^{-1}, bc^{-1}, 1 \in H$, we obtain that $ac^{-1} = x^m, bc^{-1} = y^m$ for some nonzero x and y in Z_p, proving the assertion. □

A partition $\{A_1, A_2, \ldots, A_k\}$ of $[N]$ is called *symmetric* if any $x \in A_i$ implies that $N + 1 - x \in A_i$. It is clear that any sum-free partition $\{A_1, A_2, \ldots, A_k\}$ of $[s_k]$ is symmetric since $x + y = s_k + 1$ must have a solution for any $x \in A_i$ and $1 \leq i \leq k$. We will see the partition in the proof of the following lemma is symmetric.

Lemma 2.3 *For any positive integer k,*

$$s_{k+1} \geq 3s_k + 1.$$

Proof. Let $\{A_1, A_2, \ldots, A_k\}$ be a sum-free partition of $[s_k]$. Then $x + y = s_k + 1$ has a solution in any A_i. Set $B_{k+1} = \{s_k + 1, s_k + 2, \ldots, 2s_k + 1\}$ and extend A_i to B_i as follows. For any $x \in A_i$, add $3s_k + 2 - x$ in A_i. Since $1 \leq x \leq s_k$, we have

$$2s_k + 2 \leq 3s_k + 2 - x \leq 3s_k + 1,$$

and thus $B_1, B_2, \cdots, B_k, B_{k+1}$ form a partition of $[3s_k + 1]$. It is easy to see that B_{k+1} is sum-free. For $1 \leq i \leq k$ and $x, y \in B_i$, we claim that $x + y \notin B_i$.

Indeed, if both x and y are in A_i, then the assertion follows from $x + y \notin A_i$ and $x + y \leq 2s_k$. If both x and y are in $B_i \setminus A_i$, then the assertion follows from the fact that $x + y \geq 2(2s_k + 2) > 3s_k + 1$. Now we assume that exactly one of them is in A_i, say, $x \in A_i$, and $y \in B_i \setminus A_i$. Clearly, $y = 3s_k + 2 - y'$ for some $y' \in A_i$. Moreover, we obtain that

$$x + y \geq x + 2s_k + 2 \geq 2s_k + 3.$$

Therefore, if $x + y \in B_i$, then $x + y \in B_i \setminus A_i$. It follows that

$$x + y = 3s_k + 2 - z$$

for some $z \in A_i$, which implies that

$$x + z = 3s_k + 2 - y = y',$$

contradicting to the fact that A_i is sum-free.

Therefore, $B_1, B_2, \cdots, B_{k+1}$ form a sum-free partition of $[3s_k + 1]$, and hence $s_{k+1} \geq 3s_k + 1$. □

The following is a result of Abbott and Hanson (1972), which is a generalization of the above lemma.

Theorem 2.6 *Let m and n be positive integers and* $a_n = 2s_n + 1$. *Then*

$$s_{m+n} \geq (2s_m + 1)s_n + s_m.$$

In particular, $a_{m+n} \geq a_m \cdot a_n$, *namely* a_n *is super-multiplicative.*

Proof. For $b = 0, 1, \ldots, s_n$, set

$$A_b = \{b(2s_m + 1) + c \,|c = 1, 2, \ldots, s_m\},$$

and for $c = 1, 2, \ldots, s_m$, set

$$A^c = \{b(2s_m + 1) + c \,|b = 0, 1, \ldots, s_n\}.$$

Let

$$A = \cup_{b=0}^{s_n} A_b = \cup_{c=1}^{s_m} A^c.$$

Partition $A_0 = [s_m]$ into m sum-free classes $C_1, C_2, \ldots, C_m$, and partition A into m classes $D_1, D_2, \ldots, D_m$ by placing A^c in D_i if $c \in C_i$. We claim that each of these classes is sum-free. Suppose to the contrary, D_i is not sum-free, i.e., there are some elements $b_1M + c_1,\ b_2M + c_2,\ b_3M + c_3$ of D_i, where $M = 2s_m + 1$, such that

$$(b_1M + c_1) + (b_2M + c_2) = b_3M + c_3,$$

where c_1, c_2, c_3 are all in C_i. We thus have

$$c_1 + c_2 \equiv c_3 \pmod{M}.$$

However, $2 \le c_1 + c_2 \le 2s_m < M$, and $1 \le c_3 \le s_m < M$, so $c_1 + c_2 = c_3$, contradicting to the fact that C_i is sum-free.

For $b = 1, 2, \ldots, s_n$, set

$$B_b = \{b(2s_m + 1) - c \,|c = 0, 1, \ldots, s_m\}.$$

Let

$$B = \cup_{b=1}^{s_n} B_b.$$

Partition $[s_n]$ into n sum-free classes $C_{m+1}, C_{m+2}, \cdots, C_{m+n}$, and partition B into n classes $D_{m+1}, D_{m+2}, \cdots, D_{m+n}$ by placing B_b in D_{m+i} if $b \in C_{m+i}$. We claim that each D_{m+i} is sum-free. In fact, for any elements $b_1M - c_1,\ b_2M - c_2,\ b_3M - c_3$ of D_{m+i}, where $M = 2s_m + 1$, and b_1, b_2, b_3 are of C_{m+i}, satisfying $b_1 + b_2 \ne b_3$ since C_{m+i} is sum-free. If $b_1 + b_2 \ge b_3 + 1$, then by noting $M - c_1 - c_2 \ge 1$ we obtain that

$$(b_1M - c_1) + (b_2M - c_2) \ge b_3M + 1 > b_3M - c_3.$$

If $b_1 + b_2 \le b_3 - 1$, then $(b_1M - c_1) + (b_2M - c_2) \le (b_3 - 1)M < b_3M - c_3$. Therefore $(b_1M - c_1) + (b_2M - c_2) \ne b_3M - c_3$, which implies that D_{m+i} is sum-free.

It is easy to verify that $A_0, B_1, A_1, B_2, A_2, \cdots, B_{s_n}, A_{s_n}$ is a partition of the set $A \cup B = [(2s_m + 1)s_n + s_m]$ in the natural order. Thus $A \cup B$ has been partitioned into sum-free classes $D_1, D_2, \ldots, D_{m+n}$. The proof is complete. □

Corollary 2.2 *For any fixed positive integer m,*

$$s_k > c(2s_m + 1)^{k/m}$$

for any integer $k > m$, where $c = c(m) > 0$ is a constant.

Proof. The assertion follows from the super-multiplicity of the function $2s_n + 1$ and Lemma 2.2. □

Since $(2s_6 + 1)^{1/6} \ge 1073^{1/6}$ by $s_6 \ge 536$, we thus have $s_k \ge c\,1073^{k/6}$ and $r_k(3) \ge c1073^{k/6}$, where $1073^{1/6} = 3.199\cdots$. The following is a very old conjecture of Erdős.

Conjecture 2.1 (Erdős) The limit of $r_k(3)^{1/k}$ is infinity as $k \to \infty$.

Erdős and Graham (1973), Bondy and Erdős (1973), and Graham, Rothschild and Spencer (1990, Ramsey Theory) considered multicolor Ramsey numbers $r_k(C_{2m+1})$, where C_{2m+1} is a cycle of length $2m+1$. Generally, we have

$$m2^k + 1 \le r_k(C_{2m+1}) < 2m \cdot (k+2)!.$$

For $m = 2$, Li (2009) showed that $r_k(C_5) \le c\sqrt{18^k k!}$ for all $k \ge 3$, where $0 < c < 1/10$ is a constant. In general, Lin and Chen (2019) showed that $r_k(C_{2m+1}) \le c^k\sqrt{k!}$ for all $k \ge 3$, where c is a positive constant depending only on m. For the lower bound, Day and Johnson proved (2017) that

$$r_k(C_{2m+1}) \ge 2m(2+\epsilon)^{k-1}$$

for large k, where $\epsilon = \epsilon(m) > 0$ and $\epsilon \to 0$ as $m \to \infty$. In particular, $r_k(C_5) > c17^{k/4}$. We will give a simpler proof here.

Let $\mathbb{C}_{2m+1} = \{C_3, C_5, \ldots, C_{2m+1}\}$ be the family of odd cycles of length at most $2m+1$. The *class Ramsey number* $r_k(\mathbb{C}_{2m+1})$ is defined as the smallest integer N such that any k-edge-coloring of K_N contains at least a monochromatic cycle in $\mathbb{C}_{2m+1}$. It is clear that

$$r_k(\mathbb{C}_{2m+1}) \le r_k(C_{2m+1}).$$

For a graph G, denote by $g(G)$ the girth of G that is the smallest length of a cycle in G. Let $g_0(G)$ and $g_1(G)$ be even girth and odd girth of G, which are the smallest length of an even cycle and an odd cycle in G, respectively. Clearly $g(G) = \min\{g_0(G), g_1(G)\}$.

For a k-edge-coloring $\mathcal{X}$ of K_N with colors $\{1, 2, \ldots, k\}$, let G_i be the monochromatic graph induced by all edges in color i. We write

$$g_1(\mathcal{X}) = \min_{1\le i\le k} g_1(G_i),$$

which is called the *odd girth* of $\mathcal{X}$.

We write an edge $\{u, v\}$ as uv simply and the color of edge uv in $\mathcal{X}$ as $\mathcal{X}(uv)$.

Lemma 2.4 *For any integer $m \ge 1$, $r_k(\mathbb{C}_{2m+1}) - 1$ is super-multiplicative, i.e.,*

$$r_{k+n}(\mathbb{C}_{2m+1}) - 1 \ge (r_k(\mathbb{C}_{2m+1}) - 1)(r_n(\mathbb{C}_{2m+1}) - 1).$$

Proof. Let $M = r_k(\mathbb{C}_{2m+1}) - 1$ and $N = r_n(\mathbb{C}_{2m+1}) - 1$. Let U and V be the vertex sets of K_M and K_N, respectively. There exists an edge-coloring $\mathcal{X}$ of K_M with colors $1, 2, \ldots, k$ and an edge-coloring $\mathcal{Y}$ of K_N with colors $k+1, k+2, \ldots, k+n$ such that

$$g_1(\mathcal{X}) > 2m+1, \quad g_1(\mathcal{Y}) > 2m+1.$$

Denote by G and H the edge-colored K_M and K_N, respectively. Replace each vertex v of H with a copy of G, which is denoted by G^v. The obtained graph, denoted by $G \times H$, is called the "blow up" of K_N by K_M. Formally, we define $G \times H$ to be an edge-colored complete graph of order MN on vertex set $U \times V$, and an edge-coloring $\mathcal{Z}$ that assigns an edge $(u, v)(u', v')$ of $G \times H$ to be

$$\mathcal{Z}\big((u,v)(u',v')\big) = \begin{cases} \mathcal{X}(uu') & if \;\; v = v', \\ \mathcal{Y}(vv') & otherwise. \end{cases}$$

Let $U = \{u_1, u_2, \ldots, u_M\}$ and $V = \{v_1, v_2, \ldots, v_N\}$. The vertices of $G \times H$ can be labelled as follows.

$u_i \setminus v_j$	v_1	v_2	$\cdots$	v_N
u_1	(u_1, v_1)	(u_1, v_2)	$\cdots$	(u_1, v_N)
u_2	(u_2, v_1)	(u_2, v_2)	$\cdots$	(u_2, v_N)
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
u_M	(u_M, v_1)	(u_M, v_2)	$\cdots$	(u_M, v_N)

Table 2.6 The vertices of $G \times H$

Call an edge to be vertical if it has form $(u_i, v)(u_j, v)$ with $u_i \neq u_j$ and $v \in V$, and horizontal if it has form $(u, v_i)(u, v_j)$ with $u \in U$ and $v_i \neq v_j$, and skew if it has form $(u, v)(u', v')$ with $u \neq u'$ and $v \neq v'$. The edge-coloring $\mathcal{Z}$ assigns a vertical edge $(u_i, v)(u_j, v)$ with color $\mathcal{X}(u_i u_j)$ as same as the color of the corresponding edge $u_i u_j$ in G, and $\mathcal{Z}$ assigns a non-vertical edge $(u, v)(u', v')$ with color $\mathcal{Y}(vv')$ as same as the color of the corresponding edge vv' in H.

Claim $g_1(\mathcal{Z}) > 2m + 1$.

Proof. Suppose to the contrary, there is a monochromatic odd cycle of length at most $2m + 1$. Let (u_{i_1}, v_{j_1}), (u_{i_2}, v_{j_2}), $\ldots$, $(u_{i_{2\ell+1}}, v_{j_{2\ell+1}})$ be a monochromatic odd cycle $C_{2\ell+1}$ with $1 \leq \ell \leq m$.

If the color of this $C_{2\ell+1}$ is one of $1, 2, \ldots, k$ in $\mathcal{X}$, say color 1, since there are no edge between distinct G^v in the color 1, then $u_{i_1}, u_{i_2}, \ldots, u_{i_{2\ell+1}}$ must form a monochromatic odd cycle of length $2\ell + 1$ in same G^v, which contradicts to $g_1(\mathcal{X}) > 2m + 1$.

Therefore, the color of the $C_{2\ell+1}$ must be one of $k + 1, k + 2, \ldots, k + n$ in $\mathcal{Y}$, say $k + 1$, and $v_{j_s} \neq v_{j_{s+1}}$ for $1 \leq s \leq 2\ell + 1$ as the edge $(u_{i_s}, v_{j_s})(v_{i_{s+1}}, v_{j_{s+1}})$ is not vertical.

Consider the monochromatic closed walk $v_{j_1}, v_{j_2}, \ldots, v_{j_{2\ell+1}}, v_{j_1}$. If $v_{j_{s+1}} = v_{j_1}$ for some $s < 2\ell + 1$, then we consider a cycle in this walk formed by $v_{j_1}, v_{j_2}, \ldots, v_{j_{s+1}} = v_{j_1}$, where $v_{j_1}, v_{j_2}, \ldots, v_{j_s}$ are pairwise distinct that form a monochromatic cycle of length s in H. Since $g_1(\mathcal{Y}) > 2m + 1$, we obtain that s must be an even integer. Then we consider the monochromatic closed walk $v_{j_{s+1}}, v_{j_{s+2}}, \ldots, v_{j_{2\ell+1}}, v_{j_{s+1}}$. Repeat the process, we will obtain a monochromatic odd cycle of length at most $2m + 1$ in H, which leads to a contradiction. □

Now, we have the following result by Lemmas 2.2 and 2.4.

Theorem 2.7 *Let m and s be fixed positive integers. Then there exists a constant $c = c(m, s) > 0$ such that*

$$r_k(\mathbb{C}_{2m+1}) \geq c[r_s(\mathbb{C}_{2m+1}) - 1]^{k/s}$$

for all large k.

Note that $r_2(\mathbb{C}_5) = r_2(\{C_3, C_5\}) > 4$ and the red-blue edge-coloring of K_5 contains monochromatic C_5 if there is no monochromatic C_3. Thus we have $r_2(\mathbb{C}_5) = 5$. Now we shall show the following result.

Theorem 2.8 *We have* $r_4(\mathbb{C}_5) > 17$.

Proof. Let K_{17} be defined on Z_{17}, here we use $Z_{17} = \{0, \pm1, \pm2, \ldots, \pm8\}$ for convenience. For $1 \le \ell \le 8$, define H_ℓ be the subgraph consists of all edges ij if and only if $|i - j| \equiv \ell \pmod{17}$. Then, each H_ℓ is a Hamilton cycle and they form an edge partition of K_{17}. Denote $G_1 = H_1 \cup H_3$, $G_2 = H_2 \cup H_6$, $G_3 = H_4 \cup H_5$ and $G_4 = H_8 \cup H_7$.

Claim 1 G_1, G_2, G_3 and G_4 are isomorphic to each other.

Proof. For $i \in Z_{17}$, $\varphi(i) \equiv 2i \pmod{17}$ is an isomorphism from G_1 to G_2. Indeed, ij is an edge of H_1 if and only if $|i - j| \equiv 1 \pmod{17}$, which is equivalent to $|2i - 2j| \equiv 2 \pmod{17}$, i.e., $\varphi(i)\varphi(j)$ is an edge of H_2. Similarly, ij is an edge of H_3 if and only if $\varphi(i)\varphi(j)$ is an edge of H_6. Moreover, by noticing that $-5 \equiv 12 \pmod{17}$ and $7 \equiv 24 \pmod{17}$, we have for $i \in Z_{17}$, $\varphi(i) \equiv 2^{s-1}i \pmod{17}$ is an isomorphism from G_1 to G_s for $s = 3, 4$. □

Claim 2 G_1 contains neither C_3 nor C_5.

Proof. Suppose there exists an triangle in $G_1 = H_1 \cup H_3$, say $i_1i_2i_3i_1$. Since H_1 and H_3 are triangle-free, we have H_1 contains at least one edge of this triangle, and so dose H_3. Without loss of generality, assume that $i_1i_2 \in H_1$ and $i_1i_3 \in H_3$. Thus

$$|i_2 - i_1| \equiv 1 \pmod{17} \quad \text{and} \quad |i_3 - i_1| \equiv 3 \pmod{17},$$

which implies that $|i_3 - i_2| \equiv 2 \pmod{17}$ or $|i_3 - i_2| \equiv 4 \pmod{17}$, contradicting to the fact that i_2i_3 is an edge of $H_1 \cup H_3$. By a similar argument as above, we obtain that G_1 contains no cycle of length 5. This completes the proof. □

From Claims 1 and 2, we have $r_4(\mathbb{C}_5) > 17$ as desired. □

Recall that

$$r_k(C_{2\ell+1}) \ge r_k(\mathbb{C}_{2m+1})$$

for any $1 \le \ell \le m$, so Theorem 2.8 implies that there exists a constant $c > 0$ such that $r_k(C_5) > c \cdot 17^{k/4}$ for all $k \ge 3$.

We will see in latter chapters that $r_k(C_{2m}) = O(k^{m/(m-1)})$ for fixed $m \ge 2$, while $r_k(C_n)$ is linear in n for fixed k.

2.4 Paley Graphs

We have seen some recursive upper bounds for $r(m, n)$ in last chapter. However they are not effective on estimating larger classical Ramsey numbers. For the lower bounds, the situation is even worse.

Property 2.1 Let m_i and n_i be positive integers. Then

$$r(m_1 m_2 + 1, n_1 n_2 + 1) > (r(m_1 + 1, n_1 + 1) - 1)(r(m_2 + 1, n_2 + 1) - 1).$$

Proof. Let G be a graph of order $N_1 = r(m_1 + 1, n_1 + 1) - 1$ with $\omega(G) \le m_1$ and $\alpha(G) \le n_1$. Let H be a graph of order $N_2 = r(m_2 + 1, n_2 + 1) - 1$ with $\omega(H) \le m_2$ and $\alpha(H) \le n_2$. One can "blow up" the graph G by H as follows. For any vertex v of G, replace v by a copy of H, denoted it by H_v. For any pair of vertices x and y from distinct copies H_u and H_v, x and y are adjacent if and only if u and v are adjacent. Inside the same copy H_v, the adjacency of H is preserved. Clearly, the clique number of the new graph is at most $m_1 m_2$ and its independent number is at most $n_1 n_2$. So the claimed inequality follows. □

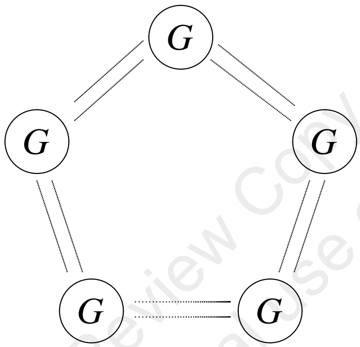

Fig. 2.2 Using G to blow up C_5

On the other hand, we have successfully obtained the right lower bound of $r_3(3)$ by using finite field. Let us return to finite fields for help.

Let us have some discuss on the density of primes. We know that there are infinitely many primes. Let $\pi(n)$ be the number of primes p with $p \le n$, the famous *prime number theorem* states that $\pi(n) \sim n/\log n$, where $\log x$ is the natural logarithmic function. For any integers m and r with $m \ge 2$, $0 < r < m$ and $(r, m) = 1$, Dirichlet's theorem tells us that there are infinitely many primes p of the form $p \equiv r \pmod m$. Let $\pi_m(r, n)$ be the number of these primes with $p \le n$. So Dirichlet's theorem tells us that $\pi_m(r, n) \to \infty$ as $n \to \infty$. Noticing that 2 is the only even prime, we have that for $n > m \ge 2$,

$$\sum_{\substack{r:\, 0<r<m \\ (r,m)=1}} \pi_m(r, n) = \begin{cases} \pi(n) & \text{if } m \text{ is odd,} \\ \pi(n) - 1 & \text{otherwise.} \end{cases}$$

The following result is usually called Siegel-Walfisz Theorem, which says that all summands $\pi_m(r, n)$ in above sum are almost the same for large n. Let $\phi(m)$ be the number of integers $r \in [m]$ with $(r, m) = 1$, which is called the Euler's function.

Theorem 2.9 *If r and m are fixed and $(r, m) = 1$, then, as $n \to \infty$,*

$$\pi_m(r,n) \sim \frac{n}{\phi(m)\log n}.$$

Equivalently, the nth prime p of the form $p \equiv r \pmod m$ *is asymptotically equal to* $\phi(m)n\log n$*. Consequently, if p and p′ are two consecutive primes of this form, then* $p \sim p'$ *as* $p \to \infty$*.*

Theorem 2.10 (Prime number theorem) *As* $n \to \infty$*,* $\pi(n) \sim n/\log n$*, and* $p_n \sim n\log n$*, where* p_n *is the nth prime. Consequently,* $p_n \sim p_{n+1}$*.*

Another way to describe the density of the primes is to estimate the difference $p_{n+1} - p_n$. It has been shown that $p_{n+1} - p_n = O(p_n^a)$, where a is a constant with $0 < a < 1$. The currently known best value of a is $\frac{21}{40} = 0.525$, see Baker, Harman and Pintz (2001).

Our first application of Theorem 2.9 is the case $m = 4$. Since $\phi(4) = 2$, so asymptotically, there are half primes $p \le n$ of the form of $p \equiv 1 \pmod 4$ and half of the form of $p \equiv 3 \pmod 4$.

Let q be a prime power, and F_q the finite field of q elements. Denote by $F_q^* = F_q \setminus \{0\}$. An element $a \in F_q$ is called *quadratic* if $a = b^2$ for some $b \in F_q$. A quadratic element of F_p is usually called a *quadratic residue* $\pmod p$ when p is a prime number.

Let us define a function $\chi(x)$ on F_q as

$$\chi(x) = x^{(q-1)/2}.$$

This function is usually called the *quadratic residue character* of F_q.

Lemma 2.5 *If q is an odd prime power, then*

$$\chi(x) = \begin{cases} 1 & x \text{ is quadratic, } x \neq 0, \\ 0 & x = 0, \\ -1 & x \text{ is non-quadratic.} \end{cases} \tag{2.2}$$

Furthermore, exactly half of elements of F_q^* *are quadratic.*

Proof. Let x be an element of F_q^*. Clearly $\chi(x) = \pm 1$ as

$$(\chi(x) - 1)(\chi(x) + 1) = \chi^2(x) - 1 = x^{q-1} - 1 = 0.$$

Let ν be a primitive element of F_q, i.e.,

$$F_q^* = \{\nu, \nu^2, \ldots, \nu^{q-2}, \nu^{q-1} = 1\}.$$

Note that ν is not quadratic as it is primitive, hence $\chi(\nu) = -1$. Denote

$$S_0 = \{\nu^2, \nu^4, \ldots, \nu^{q-1} = 1\}, \quad \text{and} \quad S_1 = \{\nu, \nu^3, \ldots, \nu^{q-2}\}.$$

Using the facts that $\chi(\nu) = -1$ and $\chi(\nu^k) = \chi^k(\nu)$, we have $\chi(x) = 1$ if and only if $x \in S_0$, as claimed. □

Now, the *Paley graph* P_q is defined as follows. Let $q \equiv 1 \pmod 4$ be a prime power. The *Paley graph* P_q is defined on F_q, and two distinct vertices x and y of F_q are adjacent in P_q if and only if

$$\chi(x-y) = (x-y)^{(q-1)/2} = 1,$$

i.e., $x - y$ is a non-zero quadratic element. Note that $\chi(x-y) = \chi(y-x)$ since $\chi(-1) = 1$ by noticing that $(q-1)/2$ is even as $q \equiv 1 \pmod 4$. As an example, it is easy to verify that the Paley graph P_5 is C_5.

Let A be an additive group S an inverse-closed subset of $A^* = A \setminus \{0\}$. A graph, called the *Cayley graph* with respect to S, is defined as follows. Its vertex set is A, and distinct vertices u and v are adjacent if and only if $u - v \in S$. Clearly, a Paley graph is a special Cayley graph with respect to the subset of non-zero quadratic elements since the inverse of a non-zero quadratic element is also quadratic.

The strongly regular graphs were introduced by Bose (1963). A graph G is said to be a *strongly regular graph* with parameters n, d, λ, μ, denoted by $srg(n, d, \lambda, \mu)$, if it has n vertices, d-regular, and any pair of vertices have λ common neighbors if they are adjacent, and μ common neighbors otherwise. For example, C_5 is an $srg(5, 2, 0, 1)$. The following proposition tells that the complement of a strongly regular graph is also strongly regular.

Proposition 2.2 *If G is a strongly regular graph $srg(n, d, \lambda, \mu)$, then its complement is also an $srg(n, d_1, \lambda_1, \mu_1)$, where*

$$\begin{aligned} d_1 &= n - d - 1, \\ \lambda_1 &= n - 2d + \mu - 2, \\ \mu_1 &= n - 2d + \lambda. \end{aligned}$$

Proof. The value of d_1 can be determined by $d + d_1 = n - 1$. Let u and v be distinct vertices of G. If they are non-adjacent, then $|N(u) \cup N(v)| = 2d - \mu$. The remaining $n-2d+\mu-2$ vertices are the common neighbors of u and v in $\overline{G}$, giving λ_1 as claimed. If u and v are adjacent, then $\{u, v\} \subseteq N(u) \cup N(v)$ and $|N(u) \cup N(v)| = 2d - \lambda$. The remaining $n - 2d + \lambda$ vertices are common neighbors of u and v in $\overline{G}$, yielding μ_1 as claimed. □

For vertex disjoint graphs G and H, let $G \cup H$ be the graph on vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, which is called the union of G and H. Let mG be the union of m copies of G. The union mK_k is an $srg(mk, k-1, k-2, 0)$. On the other hand, if G is an $srg(n, k, \lambda, 0)$, then G is a union of complete graphs of the same order. We sometimes exclude complete and empty graphs as an srg to avoid to define μ and λ, respectively. A relation among the parameters is as follows.

Proposition 2.3 *If G is an $srg(n, d, \lambda, \mu)$, then*

$$d(d - \lambda - 1) = \mu(n - d - 1).$$

Proof. Let v be a vertex and let $M(v)$ be the set of non-neighbors of v. Consider the partition $V(G) = \{v\} \cup N(v) \cup M(v)$. By the definition, $N(v)$ contains d vertices, and $M(v)$ contains $n-d-1$ vertices. Each vertex of $N(v)$ is adjacent to λ vertices in $N(v)$, and hence $d-\lambda-1$ vertices in $M(v)$. Each vertex in $M(v)$ is adjacent to μ vertices in $N(v)$. Counting the edges between $N(v)$ and $M(v)$ in two ways, the required equality follows. □

A graph G is called *vertex-transitive* if for any two vertices a and b of G, there is an automorphism mapping a to b, and it is called *edge-transitive* if for any two edges ab and uv of G, there is an automorphism mapping ab to uv.

Theorem 2.11 *If $q \equiv 1 \ (mod\ 4)$ is a prime power, then the Paley graph P_q is an*

$$srg\left(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4}\right).$$

Furthermore, it is self-complementary, vertex-transitive and edge-transitive.

Proof. Lemma 2.5 implies that P_q is $(q-1)/2$-regular. Since $\sum_x \chi(x) = 0$, we have that the number of common neighbors of two vertices a and b is

$$\sum_{x\in F_q\setminus\{a,b\}} \frac{1+\chi(x-a)}{2}\cdot\frac{1+\chi(x-b)}{2},$$

which equals

$$\frac{q-2}{4} - \frac{\chi(a-b)}{2} + \frac{1}{4}\sum_{x\in F_q\setminus\{a,b\}} \chi(x-a)\chi(x-b).$$

Since $\chi(x-a)\chi(x-b) = \chi(\frac{x-a}{x-b})$ for $x \neq b$, we can write the last term as

$$\frac{1}{4}\sum_{x\in F_q\setminus\{a,b\}} \chi\left(\frac{x-a}{x-b}\right) = \frac{1}{4}\sum_{x\in F_q\setminus\{0,1\}} \chi(x) = \frac{-1}{4}.$$

Thus the number of common neighbors of a and b is $\frac{q-3}{4} - \frac{\chi(a-b)}{2}$, which is $\frac{q-5}{4}$ if a and b are adjacent and $\frac{q-1}{4}$ otherwise.

Fix $a \in F_q^*$ with $\chi(a) = -1$, and define a map ϕ_0 as

$$\phi_0 : V(P_q) \to V(P_q),\ \ \phi_0(x) = ax.$$

Note that the map ϕ_0 is an automorphism between P_q and $\overline{P_q}$ since x and y are adjacent in P_q if and only if $\chi(x-y) = 1$ which is equivalent to $\chi(ax-ay) = -1$, i.e., $\phi_0(x)$ and $\phi_0(y)$ are non-adjacent in P_q. Hence P_q is self-complementary.

Moreover, it is easy to verify that the map $\phi_1(x) = a+b-x$ is an automorphism mapping a to b, and the map $\phi_2(x) = \frac{u-v}{a-b}(x-b)+v$ is an automorphism mapping an edge ab to an edge uv. Therefore, the Paley graph P_q is vertex-transitive and edge-transitive as desired. □

We now can have an equality for $r(B_n, B_n)$ for infinity many n, where $B_n = K_2 + \overline{K}_n$ is an n-book. Note that the Paley graph P_q with $q = 4n + 1$ contains no B_n and P_q is self-complementary. This fact yields that $r(B_n, B_n) \geq 4n + 2$ if $q = 4n + 1$ is a prime power, which together with the upper bound in Theorem 1.7 implies that

$$r(B_n, B_n) = 4n + 2$$

when $4n + 1$ is a prime power.

2.5 Combination of Paley Graphs

It is interesting to see the fact that the Paley graphs P_5 and P_{17} are Ramsey graphs for $r(3)$ and $r(4)$ by Greenwood and Gleason (1955). Unfortunately, no other Paley graphs are found to be the "exact" Ramsey graphs. A result of Shearer (1986) and independently Mathon (1987) was that if the Paley graph P_p contains no K_k, then

$$r(k + 1) \geq 2p + 3. \tag{2.3}$$

This gives the best lower bounds of $r(k)$ for small k (see Table 2.7) until now, except for $k = 4, 5, 6, 8$, see Greenwood and Gleason (1955), Exoo (1989), Kalbfleisch (1965), and Burling and Reyner (1972), respectively. For $k = 5$, we know that $43 \leq r(5) \leq 48$, where the lower bound is due to Exoo (1989) and the upper bound is due to Angeltveit and McKay (2018) respectively.

In the following table, the entries in the first column are values of $k = \omega(P_p) = \alpha(P_p)$; the second p_1 and the third p_2 are the smallest and largest prime p such that $\omega(P_p) = k$, respectively; the fourth n is the number of such primes p; and the last two columns are the lower bounds obtained, in which the better one is listed only.

$\alpha(P_p)$	p_1	p_2	n	$r(\alpha+1) \geq p_2 + 1$	$r(\alpha+2) \geq 2p_2 + 3$
2	5	5	1	$r(3) \geq 6$	
3	13	17	2	$r(4) \geq 18$	
4	29	37	2	$r(5) \geq 38$	
5	41	101	6	$r(6) \geq 102$	$r(7) \geq 205$
6	97	109	2		
7	113	281	10	$r(8) \geq 282$	$r(9) \geq 565$
8	173	373	7		
9	229	797	15	$r(10) \geq 798$	$r(11) \geq 1597$
10	557	709	3		$r(12) \geq 1421$
11	433	1277	32		$r(13) \geq 2557$
12	613	1493	13		$r(14) \geq 2989$
13	853	2741	53		$r(15) \geq 5485$
14	1373	2801	17		$r(16) \geq 5605$

Table 2.7 Independence numbers of small P_p

For vertex disjoint graphs G and H, let $H + G$ be a graph obtained from H and G by adding new edges to connect H and G completely. Since $K_{k+1} = K_1 + K_k$, the following result implies the lower bound (2.3), in which $\delta(G)$ is the minimum degree of G. This is slightly better than that by Lin, Li and Shen (2014).

Theorem 2.12 *Let $q \equiv 1 \pmod 4$ be a prime power. If G is a graph with $\delta(G) \geq 1$ and the Paley graph P_q contains no G, then*

$$r(K_1 + G) \geq 2(q + 1) + 1.$$

We have the following result by considering the largest B_n in P_q when $q = 4n+1$.

Corollary 2.3 *If $4n + 1$ is a prime power, then $r(K_1 + B_n) \geq 8n + 5$.*

In order to give a proof of Theorem 2.12, we need the following construction due to Shearer (1986) and independently Mathon (1987). We will write (u, v) for an edge that connects vertices u and v. Let P_q and P'_q be two disjoint copies of Paley graphs. Let V, V' and E, E' be their corresponding vertex and edge sets, respectively, and let λ, λ' be two additional vertices. We define a new graph H_q with vertex set $\{\lambda, \lambda'\} \cup V \cup V'$ and containing the edges

$$\begin{array}{ll} (\lambda, x),\ (\lambda', x') & x \in V; \\ (x, y),\ (x', y') & (x, y) \in E; \\ (x, y'),\ (x', y) & (x, y) \in \overline{E}. \end{array}$$

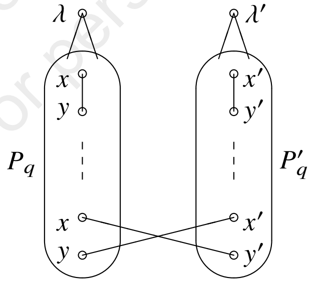

Fig. 2.3 The graph H_q

Let us have the following property of the graph H_q at first.

Lemma 2.6 *Let H_q be defined as above. For any vertex u of H_q, the neighborhood of u induces a subgraph that is isomorphic to the Paley graph P_q.*

Proof. The assertion holds clearly if the vertex u is either λ or λ'. Recall the definition of the Paley graph P_q, $V = F_q$, where $q = 4n+1$ is a prime power. Let β be a primitive element of F_q. Since for any $a, b \in V$, the map $\psi(x) = a + b - x$ is an automorphism mapping a to b, we have that the Paley graph is vertex transitive. Therefore, it suffices to verify the neighborhood of the vertex $0 \in V$ in H_q by symmetry. From the definition of H_q, the neighborhood of 0 is

$$U = \left\{\lambda, 1, \beta^2, \ldots, \beta^{4n-2}; \beta', \beta^{3'}, \ldots, \beta^{4n-1'}\right\}.$$

Denote $H[U]$ by the subgraph induced by the vertices of U in H_q. Define an bijection φ from V to U such that

$$\varphi(0) = \lambda, \quad \varphi(\beta^{2i}) = \frac{1}{\beta^{2i}} \quad \text{and} \quad \varphi(\beta^{2i+1}) = \left(\frac{1}{\beta^{2i+1}}\right)', \quad i = 0, 1, \ldots, 2n-1.$$

P_q	$(0, \beta^{2i})$	(β^{2i}, β^{2j})	$(\beta^{2i+1}, \beta^{2j+1})$	$(\beta^{2i}, \beta^{2j+1})$
$H[U]$	$(\lambda, \frac{1}{\beta^{2i}})$	$(\frac{1}{\beta^{2i}}, \frac{1}{\beta^{2j}})$	$((\frac{1}{\beta^{2i+1}})', (\frac{1}{\beta^{2j+1}})')$	$(\frac{1}{\beta^{2i}}, (\frac{1}{\beta^{2j+1}})')$

Table 2.8 Four types of edges

Clearly, φ is an isomorphism from the Paley graph P_q to $H[U]$ from the definition of H_q. e.g., $\beta^{2i} - \beta^{2j+1}$ is quadratic if and only if $\frac{1}{\beta^{2i}} - \frac{1}{\beta^{2j+1}} = \frac{\beta^{2j+1} - \beta^{2i}}{\beta^{2i}\beta^{2j+1}}$ is non-quadratic by noting $-1 = \beta^{2n}$ is quadratic as $q = 4n+1$. This completes the proof of Lemma 2.6. □

Proof of Theorem 2.12. Let H_q be constructed as above with vertex set $\{\lambda, \lambda'\} \cup V \cup V'$. We aim to show that both H_q and $\overline{H_q}$ contain no copy of $K_1 + G$ as a subgraph. Lemma 2.6 implies that the neighborhood of any vertex of H_q induces a subgraph that is isomorphic to the Paley graph P_q. Hence, H_q contains no copy of $K_1 + G$ as a subgraph from the assumption that the Paley graph P_q contains no copy of G. It remains to verify that $\overline{H_q}$ contains no copy of $K_1 + G$.

Suppose to the contrary that $\overline{H_q}$ contains a copy of $K_1 + G$. Let u be the K_1 of the $K_1 + G$, i.e., the center of $K_1 + G$. We claim $u \neq \lambda$. Otherwise, G is contained in $V' \cup \{\lambda'\}$ completely. Note that λ' has no neighbor in V', G must be contained in V' completely as $\delta(G) \geq 1$. However, this will lead to a contradiction since V' induces the Paley graph P_q containing no copy of G in $\overline{H_q}$. Similarly, $u \neq \lambda'$.

Thus, we assume $u \in V$, say $u = 0$ without loss of generality. From the definition of H_q, the neighborhood of 0 in $\overline{H_q}$ is $W \cup \{0'\}$, where

$$W = \left\{\beta, \beta^3, \ldots, \beta^{4n-1}; \lambda', 1', \beta^{2'}, \ldots, \beta^{4n-2'}\right\}.$$

and denote $H[W]$ by the subgraph induced by the vertices of W in $\overline{H_q}$. Define an bijection φ from V to W such that

$$\varphi(0) = \lambda', \quad \varphi(\beta^{2i}) = \left(\frac{1}{\beta^{2i}}\right)' \quad \text{and} \quad \varphi(\beta^{2i+1}) = \frac{1}{\beta^{2i+1}}, \quad i = 0, 1, \ldots, 2n-1.$$

Similarly, φ is an isomorphism from the Paley graph $\overline{P_q}$ $(= P_q)$ to $H[W]$ from the definition of H_q. e.g., $\beta^{2i} - \beta^{2j+1}$ is non-quadratic if and only if $\frac{1}{\beta^{2i}} - \frac{1}{\beta^{2j+1}} = \frac{\beta^{2j+1} - \beta^{2i}}{\beta^{2i}\beta^{2j+1}}$ is quadratic. i.e., $(\beta^{2i}, \beta^{2j+1})$ is an edge in $\overline{P_q}$ if and only if $(\frac{1}{\beta^{2i}}, \frac{1}{\beta^{2j+1}})$ is an edge in H_q, equivalently, $((\frac{1}{\beta^{2i}})', \frac{1}{\beta^{2j+1}})$ is an edge in $H[W]$.

Now, note that, in $\overline{H_q}$, the neighborhood of the vertex $0'$ is

$$\left\{\lambda, 0, 1, \beta^2, \ldots, \beta^{4n-2}; \beta', \beta^{3'}, \ldots, \beta^{4n-1'}\right\},$$

which is disjoint from W. It follows that G must be contained in W completely as $\delta(G) \geq 1$. However, this is a contradiction since $H[W]$ is isomorphic to the Paley graph P_q which contains no copy of G. The proof of Theorem 2.12 is complete. □

It is time to propose a problem concerning the asymptotic behavior of diagonal Ramsey numbers $r(k, k)$.

Problem 2.1 Prove or disprove that for any $\epsilon > 0$ fixed, if k is large, then

$$r(k+1, k+1) \geq (2 - \epsilon) r(k, k).$$

Does the Paley graph give an exponential lower bound for $r(k, k)$, or equivalently, does $\omega(P_p) \leq C \log p$ hold? Let us define $n(p)$ for the minimum positive integer with $\chi(n(p)) = -1$. Then the set $\{1, 2, \cdots, n(p)\}$ induces a clique in P_p, so we obtain $n(p) \leq \omega(P_p)$. Assuming the Riemann hypothesis for all L-functions of real characters, Ankeny (1952) gave

$$n(p) \leq C \log^2 p.$$

So it is reasonable to believe that the order $\omega(P_p)$ is at most $O(\log^2 p)$ or even smaller. Montgomery (1971) showed that if the above Riemann hypothesis is true, then for some constant $c > 0$, there are infinitely many primes p such that

$$n(p) \geq c \log p \log \log p.$$

Thus it is unlikely that $\omega(P_p)$ can be bounded from above by $C \log p$ for general p, and the situation of using P_q is even worse, where $q = p^m$ with $m \geq 2$, see the next section. However, there is a gap between p_1 and p_2 in Table 2.6, and what we need are bounds for p_2. It seems likely that for sporadic values of p the graphs P_p give good lower bound for $r(k, k)$. Thus it is very interesting to know whether or not $\omega(P_p) \leq C \log p$ infinitely often.

2.6 Spectrum and Independence Number

To obtain more information on Paley graphs, we shall find some spectral bound for independence numbers of regular graphs. This bound will be used also in Chapter

10 to get the right order of the independence numbers of the graphs constructed by algebraic method.

Let us begin with some basics on eigenvalues of adjacency matrix of graph G, which are often called the eigenvalues of G. The list of distinct eigenvalues with their multiplicities is called the *spectrum* of the graph. Clearly, the only eigenvalue of an empty graph is zero. In this section, we admit that the order of each discussed graph is at least two. So if G is a d-regular and connected graph, then $d \geq 1$. The following result is called Perron-Frobenius theorem.

Theorem 2.13 *Let G be a d-regular and connected graph. Then d is an eigenvalue of G of multiplicity one, and $|\lambda| \leq d$ for any eigenvalue λ of G.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $A = (a_{ij})_{n\times n}$ be the adjacency matrix of G. Clearly d is an eigenvalue of A associated with the all-1 eigenvector.

Let λ be an eigenvalue of A associated with eigenvector $X = (x_1, \ldots, x_n)^T$, and let $|x_i| = \max_j |x_j|$. Then $|x_i| > 0$ and

$$|\lambda x_i| = |a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n| = \left|\sum_{v_j \in N(v_i)} x_j\right| \leq d|x_i|.$$

Hence $|\lambda| \leq d$, and the equality holds only if $x_j = x_i$ for all $v_j \in N(v_i)$. We can iterate the argument to reach all coordinates of X as G is connected. Now, let Y be an eigenvector associated with d. Then Y is a constant vector, and thus the dimension of the space of eigenvectors associated with d is one. However, the matrix A is real symmetric, so it has n real eigenvalues (not necessarily distinct) and n orthonormal eigenvectors. Thus the multiplicity of eigenvalue d is one. □

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be all eigenvalues of G. Then

$$\sum_{i=1}^{n} \lambda_i = Tr(A) = 0$$

and $\lambda_n < 0$ for a non-empty graph. An often used spectral bound for independence number can be found in Lovász (1979) as follows.

Theorem 2.14 *Let G be a regular and connected graph of order n and eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then*

$$\alpha(G) \leq \frac{-\lambda_n}{\lambda_1 - \lambda_n} n.$$

Proof. Let A be the adjacency matrix of G, which is real and symmetric. Let $X_1, X_2, \ldots, X_n$ be the ortho-normal basis eigenvectors corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, where $X_1 = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$. Let S be an independent set of G and χ_S its characteristic function, that is the 0-1 vector in which 1 indicates that the corresponding vertex is contained in S. Suppose $\chi_S = \sum_{i=1}^{n} c_i X_i$. Since S is independent, we have

$$\chi_S^T A \chi_S = \sum x_i a_{ij} x_j = 0,$$

as if $x_i = x_j = 1$, then $a_{ij} = 0$. From the ortho-normality of $X_1, X_2, \ldots, X_n$, we have

$$\chi_S^T A \chi_S = \sum_{i=1}^{n} c_i^2 \lambda_i, \qquad \sum_{i=1}^{n} c_i^2 = \chi_S^T \chi_S = |S|$$

and $c_1 = \chi_S^T X_1 = |S|/\sqrt{n}$. It thus follows that

$$\begin{aligned} 0 = \chi_S^T A \chi_S &= \sum_{i=1}^{n} c_i^2 \lambda_i = \lambda_1 c_1^2 + \sum_{i=2}^{n} c_i^2 \lambda_i \\ &\geq \lambda_1 \frac{|S|^2}{n} + \lambda_n \sum_{i=2}^{n} c_i^2 = \lambda_1 \frac{|S|^2}{n} + \lambda_n \left(|S| - \frac{|S|^2}{n} \right), \end{aligned}$$

which implies that

$$|S| \leq \frac{-\lambda_n}{\lambda_1 - \lambda_n} n,$$

as claimed. □

The above equality $|S| = -\lambda_n n/(\lambda_1 - \lambda_n)$ holds if and only if χ_S is a linear combination of the eigenvectors X_1 and X_n or $\lambda_2 = \cdots = \lambda_n$.

The spectrum of a strongly regular graph is determined completely by its four parameters as follows. As λ is used often to signify an eigenvalue, we change the notation $srg(n, d, \lambda, \mu)$ to $srg(n, d, \mu_1, \mu_2)$ to avoid interference.

Theorem 2.15 *Let G be a connected strongly regular graph $srg(n, d, \mu_1, \mu_2)$ with $n \geq 3$. If G is neither complete nor empty, then $\lambda_1 = d$ is an eigenvalue with multiplicity $m_1 = 1$, and any other eigenvalue λ ($\neq \lambda_1$) satisfies*

$$\lambda^2 + (\mu_2 - \mu_1)\lambda + (\mu_2 - d) = 0.$$

The equation has two distinct solutions λ_2 and λ_3 with $\lambda_2 > 0 > \lambda_3$, and λ_3 is an eigenvalue. If $d + (n-1)\lambda_3 \neq 0$, then λ_2 is also an eigenvalue. Their multiplicities m_2 and m_3 can be determined by

$$m_2 + m_3 = n - 1, \quad \textit{and} \quad d + m_2\lambda_2 + m_3\lambda_3 = 0.$$

Proof. Let A be the adjacency matrix of G. Let I and J be the $n \times n$ identity matrix and all-one matrix, respectively. By the definition of A and the fact that A is symmetric with zeros on the main diagonal, the (i,i)-entry of A^2 is $d(v_i) = d$, which can be represented by dI. For $i \neq j$, the (i, j)-entry of A^2 counts common neighbors of vertices v_i and v_j, so it is μ_1 or μ_2, which can be represented by $\mu_1 A$ or $\mu_2(J - I - A)$, respectively. Also the regularity of G can be represented by $AJ = dJ$. So G is an $srg(n, d, \mu_1, \mu_2)$ is equivalent to

$$AJ = dJ, \text{ and } A^2 = (d - \mu_2)I + (\mu_1 - \mu_2)A + \mu_2 J.$$

From Perron-Frobenius Theorem, we have that $\lambda_1 = d$ is an eigenvalue of G with multiplicity 1 with eigenvector $\mathbf{1} = (1, 1, \ldots, 1)^T$. Let $\lambda \neq d$ be another eigenvalue of G and $x \in R^n$ a corresponding eigenvector. Thus $Jx = 0$ by noting $\mathbf{1}^T x = 0$ since eigenvectors corresponding to distinct eigenvalues of real symmetric matrix are orthogonal. As $Ax = \lambda x$ and $A^2 x = \lambda^2 x$, we obtain that

$$\lambda^2 x = (d - \mu_2)x + (\mu_1 - \mu_2)\lambda x,$$

which implies that λ must satisfy that

$$\lambda^2 + (\mu_2 - \mu_1)\lambda + (\mu_2 - d) = 0. \tag{2.4}$$

The equation has two distinct solutions λ_2 and λ_3 ($\lambda_2 > \lambda_3$) can be seen from the fact

$$(\mu_2 - \mu_1)^2 + 4(d - \mu_2) > 0$$

as $\mu_1 = \mu_2 = d$ is impossible. Note that

$$\lambda_2 = \frac{1}{2}\left((\mu_1 - \mu_2) + \sqrt{(\mu_1 - \mu_2)^2 + 4(d - \mu_2)}\right) > 0.$$

Since the smallest eigenvalue of G is negative, it follows that λ_3 is an eigenvalue. Thus, if $d + (n - 1)\lambda_3 \neq 0$, then the multiplicity of λ_3 cannot be $n - 1$, and G has another eigenvalue, which must be λ_2. Also, their multiplicities m_2 and m_3 satisfy

$$1 + m_2 + m_3 = n, \quad \text{and} \quad Tr(A) = d + m_2\lambda_2 + m_3\lambda_3 = 0,$$

which determine m_2 and m_3 completely. □

Lemma 2.7 *If $q \equiv 1\ (mod\ 4)$ is a prime power, then the spectrum of the Paley graph P_q is as follows.*

eigenvalue	$(q-1)/2$	$(\sqrt{q}-1)/2$	$-(\sqrt{q}+1)/2$
multiplicity	1	$(q-1)/2$	$(q-1)/2$

Proof. Perron-Frobenius Theorem yields $\lambda_1 = d = (q - 1)/2$ with multiplicity 1. Using $d = (q - 1)/2$, $\mu_1 = (q - 5)/4$ and $\mu_2 = (q - 1)/4$, the equation (2.4) in the last theorem turns out to be

$$\lambda^2 + \lambda - \frac{q-1}{4} = 0,$$

whose solutions are $(\sqrt{q} - 1)/2$ and $-(\sqrt{q} + 1)/2$. The multiplicities m_2 and m_3 of the two eigenvalues are determined by

$$m_2 + m_3 = q - 1, \quad \frac{q-1}{2} + m_2\frac{\sqrt{q}-1}{2} - m_3\frac{\sqrt{q}+1}{2} = 0,$$

giving $m_2 = m_3 = (q - 1)/2$ as claimed. □

The following result explains why we only use Paley graphs whose order are primes instead of prime powers for classic Ramsey numbers.

Theorem 2.16 *Let* $q = p^{2m} \equiv 1$ (*mod* 4) *and let* P_q *be the Paley graph. Then*

$$\alpha(P_q) = \sqrt{q} = p^m.$$

Proof. Since $q = p^{2m} \equiv 1 \pmod 4$, p is an odd prime. Note that if $m|n$, then there is exactly one subfield of $F(p^n)$ with p^m elements, so $F(p^m)$ can be viewed as a subfield of $F(p^{2m})$. For any distinct x and y of this $F(p^m)$, from the facts that $(p^m+1)/2$ is an integer and $(x-y)^{p^m-1} = 1$ as $x - y \in F^*(p^m)$, we have

$$(x-y)^{(q-1)/2} = \left((x-y)^{p^m-1}\right)^{(p^m+1)/2} = 1.$$

Hence $F(p^m)$ is a clique of the graph P_q implying that $\alpha(P_q) = \omega(P_q) \geq \sqrt{q}$. Also, Theorem 2.14 gives the right upper bound for $\alpha(P_q)$ as $\lambda_1 = (q-1)/2$ and $\lambda_n = -(\sqrt{q}+1)/2$. □

2.7 Exercises

1. Prove that $r(C_4, C_4) = 6$.

2. Give a Ramsey graph for $r(3, 4)$ from the proof of Theorem 2.1.

3. Prove that the Schur number $s_3 = 13$.

4. The proof for $s_4 \geq 44$ can be a partition of $[44] = \{1, 2, \ldots, 44\}$ as

Set 1	1 3 5 15 17 19 26 28 40 42 44
Set 2	2 7 8 18 21 24 27 33 37 38 43
Set 3	4 6 13 20 22 23 25 30 32 39 41
Set 4	9 10 11 12 14 16 29 31 34 35 36

Can we prove the inverse avoiding exhausting method?

5. Show that the line graph of K_n with $n \geq 4$, denoted by $T(n)$, is an $srg(\binom{n}{2}, 2(n-2), n-2, 4)$. Using $T(6)$ and $T(7)$, show that $r(B_2, B_5) = 16$ and $r(B_4, B_6) = 22$. (Rousseau-Sheehan, 1978)

6. For any distinct vertices x and y of the Paley graph P_p, show that there are exactly $(p-1)/4$ vertices $z \notin \{x, y\}$ adjacent with x and not to y.

7. Let P_p be the Paley graph of order p and $k = \omega(P_p)$. Prove that $r(k, k) \geq (p+3)/4$ and $r(k+1, k-1) \geq (p-1)/4$.

8. Prove a general version of Schur's theorem as follows. For every $k \geq 1$ and $m \geq 2$, there exists a positive integer N such that for every partition of $[N]$ into k

classes, one of the classes contains m (not necessarily distinct) numbers $x_1, \ldots, x_m$ such that $x_1 + \cdots + x_{m-1} = x_m$. (Hint: $N = r_k(m)$).

9.* Show that $s_k < ek!$ by the original proof of Schur as follows.

(i) Let $n_0 = s_k$ and let χ be a k-coloring of $[n_0]$ such that there do not exist $x, y \in [n_0]$ such that $\chi(x) = \chi(y) = \chi(x + y)$.

(ii) For some n_1 with $n_0 \le kn_1$, there are $x_0 < x_1 < \cdots < x_{n_1-1}$, which have the same color, say c_0.

(iii) Set $A_0 = \{x_i - x_0 : 1 \le i < n_1\}$. Then $A_0 \cap \chi^{-1}(c_0) = \emptyset$. For some n_2 with $n_1 - 1 \le (k-1)n_2$, there are $y_0 < y_1 < \cdots < y_{n_2-1}$, which have the same color.

(iv) Continue this procedure until $n_k = 1$. Prove $n_0 \le \sum_{i=0}^{k-1} k!/i! < ek!$.

10. Let $B = \{b_1, \ldots, b_n\}$ be a set of nonzero integers. Then there is a sum-free subset A of B with $|A| > n/3$. (Hints: Let $p = 3k+2$ be a prime with $p > 2\max_i |b_i|$, and let $C = [k+1, 2k+1]$. Then C is sum-free in cyclic group Z_p. Randomly choose $x \in [1, p-1]$ and define $d_i = xb_i \pmod{p}$. As x ranges over $[1, p-1]$, d_i does over Z_p^* hence $\Pr(d_i \in C) = |C|/(p-1) > 1/3$. The expected number of b_i such that $d_i \in C$ is more than $n/3$. There is an x, and a subset A of B with $|A| > n/3$, such that $xa \pmod{p} \in C$ for all $a \in A$. Show A is sum-free. (Erdős, 1965))

11. (i) Let H be a finite additive group. Prove that if H^* can be partitioned into k sum-free subsets, then $r_k(3) \ge |H| + 1$.

(ii) Prove $r_3(3) \ge 17$ by partitioning $(Z_2^4)^*$ into three sum-free subsets. (Hint: view the elements of $F(2^4)$ in the proof of Greenwood and Gleason as binary vectors).

12. Let $\widehat{G}$ be the set of distinct characters on a finite abelian group G. Note that if G is cyclic, then $\widehat{G}$ is isomorphic to G.

(i) Let $G = G_1 \times G_2$. Prove that $\widehat{G}$ is isomorphic to $\widehat{G_1} \times \widehat{G_2}$

(ii) Prove that if G is a finite abelian group, then $\widehat{G}$ is isomorphic to G.

(iii) Let G be the Klein group of four elements. Describe $\widehat{G}$.

13. Let G be a d-regular connected graph of order n with eigenvalues $d = \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. If $\lambda = \max_{2 \le i \le n} |\lambda_i|$, then $\alpha(G) \le \lambda n/(\lambda_1 + \lambda)$, hence simply $\alpha(G) < n\lambda/d$. (This is slightly weaker than Theorem 2.14.)

14. Prove that $r(n, n) \le r(B_{n-2,n}, B_{n-2,n})$.

15.* Erdős and Graham (See Chung and Graham, 1998) asked to show that for fixed $m \ge 2$,

$$\lim_{k \to \infty} \frac{r_k(C_{2m+1})}{r_k(3)} = 0.$$

We know that $r_k(3) \le c \cdot k!$ for some constant $c > 0$. Although we do not know whether the above answer to the problem is positive or not, one can prove that $r_k(C_5) \le \sqrt{18^k k!}$ for all $k \ge 3$. (Hint: Li, 2009)

Chapter 3
Basic Probabilistic Method

The probabilistic method is a powerful tool for tackling problems in many areas of mathematics, such as number theory, algebra, analysis, geometry, combinatorics, and computer science, etc. Poineered by Erdős, the probabilistic method has been widely used in combinatorics for more than eight decades. It is an art to design a probability space for a non-random problem. The method works by showing that if one chooses objects randomly from a specified class, the prescribed object has a positive probability to appear. The *basic probabilistic method* means that by calculating the expected value of a random variable. This chapter focus on the basic probabilistic method such as vertices are labeled or picked randomly or semi-randomly. In semi-random method, we shall use average that is the expectation in a uniform probability space. Basic methods are effective in many cases as most random variables are concentrated around expectation. The frequently-used methods estimating such concentration include Markov's inequality and Chernoff bound, which will be introduced in this chapter. We refer the reader to the book *The Probabilistic Method* by Alon and Spencer (2016) for a systematical introduction.

3.1 Some Basic Inequalities

In this subsection, we state some basic inequalities that will be used in the calculations. The reader who is familiar with these inequalities could skip this subsection directly. Throughout this book, we use "log" to denote the natural logarithm based on e.

The following precise formula is the well-known *Stirling formula.*

Lemma 3.1 *For* $n \ge 1$,

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12n+\theta}\right), \quad \textit{where} \;\; 0 < \theta = \theta_n < 1.$$

In particular,

Y. Li, Q. Lin, *Elementary Methods of Graph Ramsey Theory*, Applied Mathematical Sciences 211, https://doi.org/10.1007/978-3-031-12762-5_3

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n},$$

and

$$n! = (1+o(1))\sqrt{2\pi n}\left(\frac{n}{e}\right)^n > \left(\frac{n}{e}\right)^n.$$

Lemma 3.2 *For* $N \geq n \geq 1$*,*

$$\binom{N}{n} \leq \frac{N^n}{n!} \leq \left(\frac{eN}{n}\right)^n.$$

If $n = o(\sqrt{N})$*, then*

$$\binom{N}{n} \sim \frac{N^n}{n!}.$$

Proof. The first two inequalities are immediate from Stirling's formula. If $n = o(\sqrt{N})$, then

$$\binom{N}{n} \Big/ \frac{N^n}{n!} = \frac{N(N-1)\cdots(N-n+1)}{N^n} = \exp\left(\sum_{i=1}^{n-1} \log\left(1 - \frac{i}{N}\right)\right)$$
$$= \exp\left[-\sum_{i=1}^{n-1} \frac{i}{N} + O\left(\frac{(n-1)^3}{N^2}\right)\right],$$

which will tend to 1 as $n \to \infty$, and so the desired asymptotical formula follows. □

Lemma 3.3 *(i) For any* $0 \leq x \leq 1$ *and* $n \geq 0$*,* $(1-x)^n \leq e^{-nx}$*.*

(ii) If $x = x_n \to 0$ *and* $x^2 n \to 0$ *as* $n \to \infty$*, then* $(1-x)^n \sim e^{-nx}$*.*

Proof. The first inequality is clear and for the second inequality, it suffices to note that

$$\log(1-x) = -x + \frac{x^2}{2} + o(x^2),$$

completing the proof. □

Lemma 3.4 *For any* $\lambda > 0$*,*

$$\frac{e^\lambda + e^{-\lambda}}{2} < e^{\lambda^2/2}.$$

Proof. Note that for any x, we have $e^x = 1 + x + \frac{x^2}{2!} + \cdots$, it follows that

$$\frac{e^\lambda + e^{-\lambda}}{2} = \sum_{j=0}^{\infty} \frac{\lambda^{2j}}{(2j)!} < \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{\lambda^2}{2}\right)^j = e^{\lambda^2/2},$$

where the inequality holds since $(2j)! > 2^j j!$ for all $j \geq 2$. □

A real-valued function $f(x)$ is convex if for any $0 \leq \lambda \leq 1$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2).$$

From a geometrical point of view, the convexity of $f(x)$ means that if we draw a line through points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, then the graph of the curve $f(x)$ must lie below that of this line for $x \in [x_1, x_2]$.

The following is known as Jensen's Inequality.

Lemma 3.5 *If* $0 \leq \lambda_i \leq 1$, $\sum_{i=1}^n \lambda_i = 1$ *and* $f(x)$ *is convex, then*

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i).$$

Proof. We use induction on n. For $n = 2$, it follows from the definition. So we assume that the inequality holds for n, and prove it for $n+1$. It suffices to replace the sum of the first two terms in $\sum_{i=1}^{n+1} \lambda_i x_i$ by the term

$$(\lambda_1 + \lambda_2)\left(\frac{\lambda_1}{\lambda_1+\lambda_2}x_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}x_2\right),$$

and then apply the induction hypothesis. □

Jensen's Inequality can be seen as a generalization of the following Cauchy-Schwarz inequality.

Lemma 3.6 *If* $x_1, x_2, \ldots, x_n$ *are non-negative real numbers, then*

$$\sum_{i=1}^n x_i^2 \geq \frac{1}{n}\left(\sum_{i=1}^n x_i\right)^2.$$

Proof. Apply Lemma 3.5 with $f(x) = x^2$ and $\lambda_i = 1/n$ for $1 \leq i \leq n$. □

The following inequality on the arithmetic and geometric means can also be deduced from Jensen's Inequality.

Lemma 3.7 *If* $x_1, x_2, \ldots, x_n$ *are non-negative real numbers, then*

$$\frac{1}{n}\left(\sum_{i=1}^n x_i\right) \geq \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}}.$$

Proof. We apply Lemma 3.5 with $f(t) = 2^t$, $\lambda_i = 1/n$ and $t_i = \log_2 x_i$ for $1 \leq i \leq n$ to obtain that

$$\frac{1}{n}\left(\sum_{i=1}^n x_i\right) = \sum_{i=1}^n \lambda_i f(t_i) \geq f\left(\sum_{i=1}^n \lambda_i t_i\right) = 2^{\frac{1}{n}\sum_{i=1}^n t_i} = \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}},$$

completing the proof. □

3.2 A Lower Bound of $r(n, n)$

The paper of Erdős (1947) is always considered as the first conscious application of the probabilistic method with many remarkable results, although Szele (1943) had applied the probabilistic method to a graph problem. Graph Ramsey theory is always referred to as the birthplace of random graphs. In the original proof of the exponent lower bound for $r(n, n)$ in 1947, Erdős did not use the formal probabilistic language. So his paper has been considered as an informal starting point of random graphs. But in two papers published in 1959 and 1961, Erdős gave a lower bound $c(n/\log n)^2$ for $r(3, n)$ and even wrote probabilities in the titles.

The results in this section are not currently best, but the proofs contain elementary training for asymptotical computing. In some cases, we intentionally give the details by showing how to obtain an optimal constant.

Theorem 3.1 *For* $n \geq 3$,

$$r(n, n) > \frac{n}{e\sqrt{2}} 2^{n/2}.$$

Proof. Let us color each edge of K_N by red and blue randomly and independently, where N is a positive integer to be chosen. Let S be a set of n vertices and A_S be the event that S is monochromatic. It follows that

$$\Pr[A_S] = 2\left(\frac{1}{2}\right)^{\binom{n}{2}} = 2^{1-\binom{n}{2}},$$

as all $\binom{n}{2}$ edges of S must be colored the same. Consider the union of events $\cup A_S$ over all n-sets on $[N]$. We thus have

$$\Pr\left(\bigcup_{S:|S|=n} A_S\right) \leq \sum_{S:|S|=n} \Pr[A_S] = \binom{N}{n} 2^{1-\binom{n}{2}}.$$

If this probability is less than one, then the complement event $\cap_S \overline{A}_S$ has positive probability. Equivalently, there is a point in the probability space for which each event A_S does not appear, i.e., there exists a red/blue edge coloring of K_N such that there is no monochromatic K_n. Hence $r(n, n) > N$.

It remains to find the maximum integer N such that $\Pr[\cup_S A_S] < 1$. From Stirling formula, $n! \geq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, it follows that

$$\binom{N}{n} 2^{1-\binom{n}{2}} \leq \frac{N^n}{n!} 2^{1-\binom{n}{2}} < \frac{2}{\sqrt{2\pi n}} \left(\frac{e\sqrt{2}N}{n 2^{n/2}}\right)^n.$$

Set $N = \lfloor \frac{n}{e\sqrt{2}} 2^{n/2} \rfloor$. We have that the fraction in the parenthesis and hence the probability of $\cup_S A_S$ is less than one. This implies that $r(n, n) \geq N + 1$ as desired. □

The original proof of Erdős (1947) used the counting argument: Let $N = \lfloor 2^{n/2} \rfloor$. Clearly, the number of graphs of N vertices is $2^{\binom{N}{2}}$. (Here the vertices are distinguishable.) Note that the number of different graphs containing a given complete graph of order n is clearly $2^{\binom{N}{2}}/2^{\binom{n}{2}}$ as there are $2^{\binom{n}{2}}$ subgraphs induced by these fixed n vertices. Thus the number of graphs of N vertices containing a complete graph of order n is less than

$$\binom{N}{n}\frac{2^{\binom{N}{2}}}{2^{\binom{n}{2}}} < \frac{N^n}{n!}\cdot\frac{2^{\binom{N}{2}}}{2^{\binom{n}{2}}} < \frac{2^{\binom{N}{2}}}{2},$$

which means that there exists a graph G that contains no complete graph of order n and also no independence set of order n, i.e., there is a coloring without monochromatic K_n. Therefore, we have that $r(n,n) > \lfloor 2^{n/2} \rfloor$.

3.3 Pick Vertices Semi-Randomly

Let us see an interesting example. In 1941, Turán proved a theorem giving a tight bound on the maximum number of edges that a K_r-free graph can have, which has become the cornerstone theorem of extremal graph theory. Consequently, we have a lower bound of the independence number of a graph G that $\alpha(G) \geq \frac{N}{1+d}$, where d is the average degree of G, for which a deterministic proof can be found in Exercises.

Theorem 3.2 *Let $G = (V, E)$ be a graph of order N with degree sequence $\{d(v) : v \in V\}$ and average degree is d. Then*

$$\alpha(G) \geq \sum_{v\in V}\frac{1}{1+d(v)} \geq \frac{N}{1+d}.$$

Proof. Label all vertices in V randomly by $\{1, 2, \ldots, N\}$. Define a set

$$I = \{v \in V : \ell(v) < \ell(w) \text{ for any } w \in N(v)\},$$

where $\ell(v)$ is the label of v. Note that I is a random set determined by ℓ. Let X_v be the indicator random variable for $v \in I$ and let $X = \sum_{v\in V} X_v$. Clearly, $X = |I|$ and its expectation

$$E(X) = \sum_{v\in V} E(X_v) = \sum_{v\in V}\Pr[v \in I] = \sum_{v\in V}\frac{1}{1+d(v)},$$

where the last equality holds since $v \in I$ if and only if v is the least element among v and its neighbors $N(v)$. So there must be a labeling such that $|I| \geq E(X)$. Note that I is an independent set, it follows that $\alpha(G) \geq |I|$ and hence the first inequality holds. For the second inequality, it follows from the fact that the function $f(x) = \frac{1}{1+x}$ is convex. □

In the following, we shall discuss more on the independence number for sparse graphs. In 1980, Ajtai, Komlós and Szemerédi obtained a lower bound for the independence number of triangle-free graphs. The method used by them is now called “semi-random method” or “nibble method” initialized by Rödl (1985), in which they selected objects in many small “nibbles” rather than a big “bite”, and then analyzed how the nibbles change the structure of the remainder.

Let us call a graph G to be H-free if G does not contain H as a subgraph. Recall the Ramsey number $r(H, K_n)$ is the minimum integer N such that any H-free graph G of order N satisfies that $\alpha(G) \geq n$. So it is important to estimate the independence numbers of graphs, in particular, that for H-free graphs.

A greedy algorithm to obtain an independent set is to put a vertex v into the independent set and then delete all neighbors of v, and repeat the process.

In order to produce a larger independent set by this algorithm, we hope to delete less vertices and more edges in each step so that the remaining graph is large and sparse. What a vertex v should be chosen? To obtain some criterion, we define $Q(v)$ to be the number of edges that incident with a neighbor of v, and define

$$Q_0(v) = \sum_{u \in N(v)} d(u).$$

Note that if we delete a vertex v and its neighbors, we delete exactly $Q(v)$ edges.

Lemma 3.8 *For any vertex v in a graph G,*

$$Q(v) \leq Q_0(v),$$

and the equality holds if and only if $N(v)$ contains no edge.

Let us have a property of $Q_0(v)$.

Lemma 3.9 *Let G be a graph with vertex set V. If d is the average degree of G, then the average value of $Q_0(v)$ over $v \in V$ is at least d^2.*

Proof. Let N denote the order of G. Then

$$\frac{1}{N}\sum_{v \in V} Q_0(v) = \frac{1}{N}\sum_{v \in V}\sum_{u \in N(v)} d(u) = \frac{1}{N}\sum_{u \in V} d(u) \sum_{v \in N(u)} 1$$
$$= \frac{1}{N}\sum_{u \in V} d^2(u) \geq \left(\frac{1}{N}\sum_{u \in V} d(u)\right)^2 = d^2,$$

where we have used the convexity of the function $f(x) = x^2$. □

Ajtai, Komlós and Szemerédi (1980) defined a vertex v to be a *groupie* if the average degree of its neighbors is at least the average degree of G. By Lemma 3.9, we know that every graph has a groupie since there is a vertex $v \in V$ so that $Q_0(v) - d \cdot d(v) \geq 0$ as the equality holds on average. By deleting a groupie and its neighbors recursively, they proved that for any triangle-free graph G of order N and average degree d,

$$\alpha(G) \geq cN\frac{\log d}{d}, \tag{3.1}$$

where $c = 1/100$.

Now, let $N = r(3, n) - 1$. Thus there is a triangle-free graph G on N vertices with independence number at most $n-1$. Since each neighborhood of a triangle-free graph is an independent set, we have $\alpha(G) \geq \Delta(G) \geq d$, and hence $n - 1 \geq cN\frac{\log(n-1)}{n-1}$. Therefore,

$$r(3, n) = N + 1 \leq \left(\frac{1}{c} + o(1)\right)\frac{n^2}{\log n}.$$

This bound is much better than the bound $r(3, n) \leq \binom{n+1}{2} \sim \frac{1}{2}n^2$, see Theorem 1.3.

We now look how Shearer (1983) found the vertex for triangle-free graphs, which will give a better lower bound of the independence number for triangle-free graph compared to (3.1).

Theorem 3.3 *For any triangle-free graph G with average degree $d > 1$,*

$$\alpha(G) \geq Nf(d),$$

where $f(x) = \frac{x\log x - x + 1}{(x-1)^2}$.

Proof. In order to find a larger independent set, the key step is to determine a vertex v, which together with $N(v)$, will be deleted. We aim to find a function $f(x)$ such that

$$\alpha(G) \geq Nf(d)$$

for a triangle-free graph G. Naturally, we *assume that $f(x)$ is positive, decreasing*, and more importantly, we hope that $f(x) \geq c\frac{\log x}{x}$ for some constant $c > 1/100$ when x is sufficiently large.

Let $P(v) = d(v) + 1$ and recall $Q(v)$ is the number of edges incident with a neighbor of v. Let H be the graph obtained from G by deleting v and its neighbors. Note that we delete exactly $P(v)$ vertices and $Q(v) = Q_0(v)$ edges since G is triangle-free. Thus H has $N - P(v)$ vertices and $Nd/2 - Q(v)$ edges. So its average degree is

$$d_H = \frac{Nd - 2Q(v)}{N - P(v)}.$$

By induction,

$$\alpha(G) \geq 1 + \alpha(H) \geq 1 + (N - P(v))f(d_H).$$

We do not know which of d and d_H is bigger. However we can swap $f(d_H)$ with $f(d)$, if we *further assume that $f(x)$ is convex* so that we can use the fact $f(x) \geq f(d) + f'(d)(x - d)$. Thus we have

$$\begin{aligned}\alpha(G) &\geq 1 + (N - P(v))f(d_H) \\ &\geq 1 + (N - P(v))\left(f(d) + f'(d)\left(\frac{Nd - 2Q(v)}{N - P(v)} - d\right)\right) \\ &\geq 1 + Nf(d) - P(v)f(d) + f'(d)[(Nd - 2Q(v)) - d(N - P(v))] \\ &= Nf(d) + 1 - P(v)f(d) - (2Q(v) - dP(v))f'(d).\end{aligned}$$

Denote

$$R(v) = 1 - P(v)f(d) - (2Q(v) - dP(v))f'(d).$$

In order to find some vertex v such that $R(v) \geq 0$, let us consider the average of $R(v)$ as follows. Sine $\frac{1}{N}\sum_v Q_0(v) \geq d^2$ by Lemma 3.9 and $f'(x) \leq 0$, it follows that

$$\begin{aligned}\frac{1}{N}\sum_v R(v) &\geq 1 - (d+1)f(d) - (2d^2 - d(d+1))f'(d) \\ &= 1 - (d+1)f(d) - d(d-1)f'(d).\end{aligned}$$

Thus, the function $f(x)$ should satisfy the following differential equation

$$x(x-1)f'(x) + (x+1)f(x) = 1.$$

Solving this differential equation, we obtain that

$$f(x) = \frac{x\log x - x + 1}{(x-1)^2}.$$

Luckily enough, $f(x)$ is positive, decreasing and convex as desired. □

Note that

$$f(x) = \frac{x\log x - x + 1}{(x-1)^2} \sim \frac{\log x}{x}$$

as $x \to \infty$. If $d = 0$, then we can take $f(d) = 1$; and if $0 < d < 1$, then we can take $f(d) = 1/2$ from Turán bound, see Theorem 3.2.

3.4 Independence Number of Sparse Graphs

In 1996, Li and Rousseau generalized Shearer's result from triangle-free graphs to locally sparse graphs.

Lemma 3.10 *For $m \geq 1$ and $x \geq 0$, the function*

$$f_m(x) = \int_0^1 \frac{(1-t)^{1/m}}{m + (x-m)t}\,dt$$

satisfies the differential equation

$$x(x-m)f_m'(x) + (x+1)f_m(x) = 1. \tag{3.2}$$

Moreover, $f_m(x)$ satisfies the following properties:

(1) $f_m(x)$ is positive, decreasing, and convex.

(2) for all $k \geq 0$, $(-1)^k f_m^{(k)}(x) > 0$.

Proof. By differentiating under the integral and then integrating by parts, we have

$$\begin{aligned}
x(x-m)f_m'(x) &= -x(x-m)\int_0^1 \frac{(1-t)^{1/m}t}{(m+(x-m)t)^2}dt \\
&= x\int_0^1 (1-t)^{1/m}t \cdot \frac{d}{dt}\left(\frac{1}{m+(x-m)t}\right)dt \\
&= -x\int_0^1 \left(1-\frac{t}{m(1-t)}\right)\frac{(1-t)^{1/m}}{m+(x-m)t}dt \\
&= -xf_m(x) + \int_0^1 \frac{(1-t)^{1/m}}{m}\left(\frac{1}{1-t}-\frac{m}{m+(x-m)t}\right)dt \\
&= -xf_m(x) + 1 - f_m(x).
\end{aligned}$$

Hence (3.2) follows. The complete monotonicity of $f_m(x)$ can be seen by differentiating under the integral. □

Corollary 3.1 *For $0 \leq x \leq m$, $f_m(x) \leq 1/(1+x)$; and for $x > m \geq 1$,*

$$f_m(x) \geq \frac{\log(x/m)-1}{x}.$$

Proof. The first statement follows from the differential equation (3.2) in Lemma 3.10 immediately since $f_m'(x) < 0$. For the case $x > m \geq 1$, we have that

$$f_m(x) \geq \int_0^1 \frac{(1-t)dt}{m+(x-m)t} = \frac{x\log(x/m)-(x-m)}{(x-m)^2} > \frac{\log(x/m)-1}{x},$$

where the last inequality holds since for $x > m$,

$$(2mx-m^2)\log(x/m) - m(x-m) > 0.$$

This completes the proof. □

It is easy to see that the function $\frac{\log(x/m)-1}{x}$ is decreasing on $m \geq 1$ for any $x > 0$, and it is also decreasing on $x \geq e^2 m$ for any $m \geq 1$.

Theorem 3.4 *Let G be a graph with N vertices and average degree d. Let $a \geq 0$ be an integer. If any subgraph induced by a neighborhood has maximum degree at most a, then*

$$\alpha(G) \geq N f_{a+1}(d).$$

Proof. We prove it by induction on N. If $N \le a+2$, then $d \le a+1$. By Corollary 3.1, we have $1/(d+1) \ge f_{a+1}(d)$. It follows from Turán's theorem that $\alpha(G) \ge \frac{N}{d+1} \ge N f_{a+1}(d)$. So we suppose $N > a+2$ hereafter. By the preceding argument, we may also assume $d > a+1$ since $f_m(x)$ is decreasing on x.

Write G_v for the subgraph induced by the neighborhood of v in G. In case some vertex v of G has degree $N-1$, again by Turán's theorem, we have $\alpha(G_v) \ge \frac{N-1}{a+1}$ as the maximum degree of G_v is at most a. It follows that

$$\alpha(G) \ge \alpha(G_v) \ge \frac{N-1}{a+1} \ge \frac{N}{a+2} = N f_{a+1}(a+1) \ge N f_{a+1}(d),$$

where the equality can be seen from (3.2). So we suppose that the maximum degree of G is at most $N-2$.

Let V be the vertex set of G. For each $v \in V$, let $P(v) = d(v)+1$ and recall $Q(v)$ is the number of edges incident with a neighbor of v. Note that G_v contains at most $\frac{a}{2}d(v)$ edges since the maximum degree of G_v is at most a. So we have

$$Q(v) \ge \sum_{u \in N(v)} d(u) - \frac{a}{2}d(v) = Q_0(v) - \frac{a}{2}d(v).$$

Consequently, by Lemma 3.9, the average value of $Q(v)$ satisfies

$$\frac{1}{N}\sum_{v \in V} Q(v) \ge d^2 - \frac{ad}{2}.$$

Set

$$R(v) = 1 - P(v) f_{a+1}(d) - (2Q(v) - dP(v)) f'_{a+1}(d).$$

Note that the coefficient of $Q(v)$ is positive since $f'_{a+1}(d) < 0$. Thus,

$$\begin{aligned} \frac{1}{N}\sum_{v \in V} R(v) &\ge 1 - (d+1) f_{a+1}(d) + \left((d+1)d - 2d^2 + ad\right) f'_{a+1}(d) \\ &= 1 - (d+1) f_{a+1}(d) - d(d-a-1) f'_{a+1}(d), \end{aligned}$$

which equals 0 by noting (3.2).

Hence there exists a vertex $v_0 \in V$ such that $R(v_0) \ge 0$. Let $R(v_0) = \hat{R}$, $P(v_0) = \hat{P}$ and $Q(v_0) = \hat{Q}$. Thus

$$\hat{R} = 1 - \hat{P} f_{a+1}(d) + (\hat{P}d - 2\hat{Q}) f'_{a+1}(d) \ge 0.$$

Delete v_0 and its neighbors from G, in view of that the maximum degree of G is at most $N-2$, we obtain a nontrivial graph H with $N - \hat{P}$ vertices and $Nd/2 - \hat{Q}$ edges. Note that any subgraph induced by a neighborhood of H has maximum degree at most a, so by induction hypothesis,

$$\alpha(H) \ge (N - \hat{P}) f_{a+1}\left(\frac{Nd - 2\hat{Q}}{N - \hat{P}}\right).$$

Moreover, $f_{a+1}(x) \geq f_{a+1}(d) + f'_{a+1}(d)(x - d)$ for all $x \geq 0$ since $f_{a+1}(x)$ is convex. Consequently, a similar argument as in Theorem 3.3 yields that

$$\alpha(G) \geq 1 + \alpha(H) \geq 1 + (N - \hat{P}) f_{a+1}\left(\frac{Nd - 2\hat{Q}}{N - \hat{P}}\right) \geq N f_{a+1}(d).$$

This completes the proof. □

3.5 Upper Bounds for $r(m, n)$

Now, we are able to improve the constant factor for the upper bounds of $r(m, n)$ due to Ajtai, Komlós and Szemerédi (1980).

Theorem 3.5 *For each fixed $m \geq 2$,*

$$r(m, n) \leq (1 + o(1)) \frac{n^{m-1}}{(\log n)^{m-2}}.$$

Proof. We will prove the assertion by induction on m. For $m = 2$, it is trivial since $r(2, n) = n$.

For $m = 3$, let G be the graph of order $N = r(3, n) - 1$ which contains no triangles and $\alpha(G) \leq n - 1$. Since G is triangle-free, each subgraph induced by the neighborhood of any vertex is empty, and thus its average degree is zero. Let d be the average degree of G. Since each neighborhood of any vertex of G induces an independence set, we have $n - 1 \geq \alpha(G) \geq d$. We apply Theorem 3.4 with $a = 0$ to obtain that

$$n - 1 \geq N f_1(d) \geq N f_1(n - 1) \geq N \frac{\log(n - 1) - 1}{n - 1}.$$

Thus $r(3, n) - 1 < \frac{(n-1)^2}{\log n - 1}$, and it follows by $r(3, n) \leq \frac{n^2}{\log(n/e)}$ for large n.

Suppose the statement holds for $2, 3, \ldots, m$. We proceed to the induction step. Let G be a graph of order $N = r(m + 1, n) - 1$ such that G contains no K_{m+1} and $\alpha(G) \leq n - 1$. Note that for each vertex v of G, we have

- the degree of v is at most $r(m, n) - 1$, and
- the maximum degree of G_v is at most $r(m - 1, n) - 1$, where G_v is the subgraph induced by the neighborhood of v in G.

Denote by $d = r(m, n) - 1$ and $a = r(m - 1, n) - 1$. From the induction hypothesis, we have that for any sufficiently small $\epsilon > 0$, there exists an integer n_0 such that for all $n \geq n_0$,

$$r(m, n) \leq (1 + \epsilon) \frac{n^{m-1}}{(\log n)^{m-2}}, \quad \text{and} \quad r(m - 1, n) \leq (1 + \epsilon) \frac{n^{m-2}}{(\log n)^{m-3}}.$$

Note that $d \geq a + 1$, it follows from Theorem 3.4 and Corollary 3.1 that

$$n > \alpha(G) \geq N f_{a+1}(d) \geq N \frac{\log(d/(a+1)) - 1}{d}.$$

Since the function $f_a(x)$ is decreasing on x, we obtain that

$$n > N \frac{\log(r(m,n)/a) - 1}{r(m,n)} \geq N \frac{\log(n/\log n) - 1}{(1+\epsilon)n^{m-1}/(\log n)^{m-2}},$$

which implies that for large n,

$$r(m+1,n) = N + 1 \leq (1+2\epsilon)\frac{n^m}{(\log n)^{m-1}},$$

completing the proof. □

In the following, we will give another application. Let us list two simple facts at first.

Lemma 3.11 *For any graph G with average degree d, there is a subgraph H of G such that $\delta(H) \geq d/2$.*

Proof. Let G be a graph of order N with average degree $d = d(G)$. As the case $d = 0$ is trivial, we may assume $d > 0$. If $\delta(G) \geq d/2$, then we have nothing to do. Otherwise, deleting a vertex with degree less than $d/2$ from G, then the average degree of the resulting graph, denoted by H, satisfies

$$d(H) \geq \frac{Nd - d}{N - 1} = d(G).$$

Repeat the process, we can obtain a subgraph with minimum degree at least $d/2$ as desired. □

Lemma 3.12 *If a graph G of order N has edge number $e(G) > (m-1)N$, then G contains every tree with m edges.*

Proof. Since the average degree of G is greater than $2(m-1)$, it follows that G contains a subgraph H with minimum degree $\delta(H) > m - 1$. Let T_{m+1} be a tree of m edges. We can embed T_{m+1} into H inductively. Suppose that we have embedded T_k which is the subtree of T_{m+1} into H for $k < m+1$. Note that each vertex of T_k has at least one neighbor outside T_k, thus we can easily find a larger subtree of T_{m+1} as desired. □

Conjecture 3.1 (Erdős-Sós) If G is a graph on N vertices with edge number $e(G) > \frac{m-1}{2}N$, then G contains every tree with m edges.

Ajtai, Komlós, Simonovits and Szemerédi announced (unpublished) that the conjecture is true for sufficiently large m. This conjecture is true for stars and paths, and also many special cases are verified to be true, we refer the reader to Bollobás and Eldridge (1978), Sauer and Spencer (1978), Woźniak (1996), Fan (2013) and other related references.

Now we generalize Theorem 3.4 as follows. Note the condition that $N(v)$ has a maximum degree at most m controls the number of edges in $N(v)$ which are counted twice in summation $\sum_{x \in N(v)} d(x)$. Now, the condition that $N(v)$ contains no T_{m+1} can do the same thing.

Theorem 3.6 *Let G be a graph with N vertices and average degree d. If each neighborhood of G contains no T_{m+1}, then*

$$\alpha(G) \geq N f_{2m-1}(d).$$

If T_{m+1} is a star or a path, then f_{2m-1} can be replaced by f_m.

We conclude this section and hence this chapter with a conjecture of Ajtai, Erdős, Komlós and Szemerédi (1981), which says that the independence numbers of K_m-free graphs have the lower bound similar to that of triangle-free graphs.

Conjecture 3.2 For each fixed integer $m \geq 3$, there exists a constant $c = c(m) > 0$ such that if G is a K_m-free graph with order N and average degree $d > 0$, then

$$\alpha(G) \geq cN \frac{\log d}{d}.$$

For $m = 3$, it has been verified to be true by Ajtai, Komlós and Szemerédi (1980). For general $m \geq 4$, Shearer (1995) proved that $\alpha(G) \geq cN \frac{\log d}{d \log \log d}$ for the graphs described in the above conjecture. To confirm the conjecture, a factor $\log \log d$ in the denominator needs to be taken away.

3.6 Odd Cycle versus Large K_n

We have proved the Turán bound $\alpha(G) \geq \sum_v \frac{1}{1+d(v)}$ in Section 3.3, where we labeled vertices randomly. Let us have another result proven in the similar way. Given a graph G with vertex set V, we set

$$N_i(v) = \{w \in V : d(w, v) = i\},$$

which consists of all vertices of distance i from vertex v in G, and denote $d_i(v) = |N_i(v)|$. Thus $d_0(v) = 1$ and $d_1(v) = d(v)$. We do not distinguish the subset $N_i(v)$ and the subgraph of G induced by $N_i(v)$ when there is no danger of confusion. The graph G is called (m, k)-colorable if $N_i(v)$ is k-colorable for any vertex v and any $i \leq m$, that is, there is an assignment of k colors on vertices of $N_i(v)$ so that no two adjacent vertices receive the same color. The following result was first obtained by Shearer (1995) for graphs that has a small odd girth, where the odd girth is the minimum length of an odd cycle in a graph.

Theorem 3.7 *Let $m \geq 2$ and $k \geq 1$ be integers. If G is an (m, k)-colorable graph with vertex set V, then*

$$\alpha(G) \geq c \left(\sum_{v \in V} d(v)^{1/(m-1)} \right)^{(m-1)/m},$$

where $c = \frac{1}{k2^{(m-1)/m}}$ *is a constant.*

Lemma 3.13 *If* G *is an* $(1, k)$*-colorable graph with vertex set* V*, then*

$$\alpha(G) \geq \frac{1}{k} \sum_{v \in V} \frac{d_1(v)}{1 + d_1(v) + d_2(v)}.$$

Proof. Randomly label the vertices of G with a permutation of integers $1, 2, \ldots, N$, where $N = |V|$. Let X be the set consists of all vertices v such that the minimum label of the vertices in $\{v\} \cup N_1(v) \cup N_2(v)$ is on some vertex in $N_1(v)$. Hence the probability that X contains a vertex v is $\frac{d_1(v)}{1+d_1(v)+d_2(v)}$, which implies that the expected size of X is $\sum_{v \in V} \frac{d_1(v)}{1+d_1(v)+d_2(v)}$. It follows that for certain fixed permutation of integers from 1 to N, we have

$$|X| \geq \sum_{v \in V} \frac{d_1(v)}{1 + d_1(v) + d_2(v)}.$$

We aim to find an independent set in this X of size at least $|X|/k$.

To this end, we define a relation R on X as follows. Let $u, v \in X$. Call u and v satisfy the relation R if the minimum label on $\{u\} \cup N_1(u) \cup N_2(u)$ is precisely the same as that on the vertices in $\{v\} \cup N_1(v) \cup N_2(v)$. Clearly R is an equivalence relation, and thus X can be partitioned into certain equivalence classes $X_1, X_2, \ldots, X_p$ for some positive integer p. For each $1 \leq i \leq p$, by the definition of relation R, all vertices in X_i share a neighbor v_i in common, such that for any $w_i \in X_i$, the label of $v_i \in N_1(w_i)$ is the minimum label on vertices in $\{w_i\} \cup N_1(w_i) \cup N_2(w_i)$. Hence $X_i \subseteq N_1(v_i)$ and $v_i \neq v_j$ for $i \neq j$.

We claim that there is no edge between X_i and X_j whenever $1 \leq i \neq j \leq p$. To justify it, assume to the contrary: some $w_i \in X_i$ is adjacent to some $w_j \in X_j$. Therefore, $v_i w_i w_j v_j$ forms a path of length three. By the definition of X_i, we see that the label on v_i is minimum among $\{w_i\} \cup N_1(w_i) \cup N_2(w_i)$ and hence it is less than that on v_j since $v_j \in N_2(w_i)$. Similarly, by considering w_j, we have that the label on v_j is less than that on v_i, yielding a contradiction.

Since $X_i \subseteq N_1(v_i)$ for each $1 \leq i \leq p$ is k-colorable, there is an independent set Y_i in X_i with $|Y_i| \geq |X_i|/k$. It follows from the above claim that $\cup_{i=1}^{p} Y_i$ is an independent set of size at least $\sum_{i=1}^{p} |X_i|/k = |X|/k$, as desired. □

Lemma 3.14 *If* G *is an* (m, k)*-colorable graph with vertex set* V*, then for any* $1 \leq \ell \leq m + 1$*,*

$$\alpha(G) \geq \frac{1}{2k} \sum_{v \in V} \frac{1 + d_1(v) + \cdots + d_{\ell-1}(v)}{1 + d_1(v) + \cdots + d_\ell(v)}.$$

Proof. The proof goes along the same line as that of the preceding lemma, so we only give a sketch here.

Randomly label the vertices of G with a permutation of the integers $1, 2, \ldots, N$, where $N = |V|$. Let X be the set consists of all vertices v such that the minimum label on the vertices in $\cup_{j=0}^{\ell} N_j(v)$ lies in $\cup_{j=0}^{\ell-1} N_j(v)$. Thus for certain fixed permutation of the integers from 1 to N, we have

$$|X| \geq \sum_{v \in V} \frac{1 + d_1(v) + \cdots + d_{\ell-1}(v)}{1 + d_1(v) + \cdots + d_\ell(v)}.$$

We aim to prove that there is an independent set in this X of size at least $|X|/(2k)$.

To this end, define an equivalence relation R on X such that $u \sim v$ in R if the minimum label on the vertices in $\cup_{j=0}^{\ell} N_j(u)$ is precisely the same as that on the vertices in $\cup_{j=0}^{\ell} N_j(v)$. Hence X can be partitioned into certain equivalence classes $X_1, X_2, \ldots, X_p$ for some integer $p \geq 1$. For each $1 \leq i \leq p$, suppose that v_i possesses the minimum label on the vertices in $\cup_{j=0}^{\ell} N_j(u)$ for any $u \in X_i$. It is clear that the distance between each vertex in X_i and v_i is at most $\ell - 1$ from the definition of X.

Based on these v_i, we can deduce that there is no edge between X_i and X_j whenever $1 \leq i \neq j \leq p$. Now partition each X_i into subsets $X_{i,j}$, $1 \leq j \leq \ell - 1$, such that the distance between every vertex in $X_{i,j}$ and v_i is j. Since each $X_{i,j}$ contains an independent set $Y_{i,j}$ of size at least $|X_{i,j}|/k$, where $1 \leq i \leq p$ and $0 \leq j \leq \ell - 1$, it follows that one of

$$\bigcup_{1 \leq i \leq p} \bigcup_{odd\ j} Y_{i,j} \quad \text{and} \quad \bigcup_{1 \leq i \leq p} \bigcup_{even\ j} Y_{i,j}$$

is an independent set of size at least $\frac{1}{2k} | \cup_{i=1}^{p} \cup_{j=0}^{\ell-1} X_{i,j}| = \frac{|X|}{2k}$. □

Now, we are ready to give a proof for Theorem 3.7.

Proof of Theorem 3.7. Applying Lemma 3.13 and Lemma 3.14 repeatedly,

$$\alpha(G) \geq \frac{1}{k(m-1)} \sum_{v \in V} \left(\frac{d_1(v)}{1 + d_1(v) + d_2(v)} + \frac{1 + d_1(v) + d_2(v)}{2(1 + d_1(v) + d_2(v) + d_3(v))} + \cdots + \frac{1 + d_1(v) + \cdots + d_{m-1}(v)}{2(1 + d_1(v) + \cdots + d_m(v))} \right).$$

Since the arithmetic mean $(x_1 + x_2 + \cdots + x_n)/n$ is no less than the geometric mean $\sqrt[n]{x_1 x_2 \cdots x_n}$ by Lemma 3.7, we obtain that

$$\alpha(G) \geq \frac{1}{k 2^{(m-2)/(m-1)}} \sum_{v \in V} \left(\frac{d_1(v)}{1 + d_1(v) + \cdots + d_m(v)} \right)^{1/(m-1)}.$$

By the condition that $N_i(v)$ is k-colorable, there is an independent set in $N_i(v)$ of size $\alpha_i(v) \geq d_i(v)/k$. Moreover, since there is no edge between $N_i(v)$ and $N_j(v)$ whenever $i - j \equiv 0 \pmod 2$, we can deduce that

$$2\alpha(G) \geq 1 + \alpha_1(v) + \cdots + \alpha_m(v) \geq \frac{1}{k}(1 + d_1(v) + \cdots + d_m(v)).$$

Therefore

$$\alpha(G) \geq \frac{1}{k2^{(m-2)/(m-1)}} \sum_{v \in V} \left(\frac{d_1(v)}{2k\alpha(G)} \right)^{1/(m-1)},$$

the desired statement follows. □

Erdős et al. (1978) proved that $r(C_{2m+1}, K_n) \leq cn^{1+1/m}$ for fixed $m \geq 1$, where $c = c(m) > 0$ is a constant. Li and Zang (2001, 2003), and Sudkov (2002) independently obtained that $r(C_{2m+1}, K_n) \leq c(n^{m+1}/\log n)^{1/m}$. We shall discuss the upper bound of Ramsey number $r(C_{2m}, K_n)$ in Chapter 7.

Lemma 3.15 *Let $m \geq 1$ be an integer. If a graph G contains no C_{2m+1}, then G is $(m, 2m-1)$-colorable.*

Proof. Let G be defined on vertex set V. We assume that G is connected without loss of generality. For a fixed vertex v and any $i \leq m$, we need to verify that $N_i(v)$ is $(2m-1)$-colorable, where $N_i(v)$ is the vertex subset consists of all vertices of distance i from vertex v in G. It is easy to see that there is a spanning tree T of G rooted at v such that $d_T(v, x) = d_G(v, x)$ for any vertex x of G, namely, T preserves the distance from v to any x. Embed T on a plane such that there is no edges of T crossing and label all vertices in a dictionary order.

For a fixed i, $1 \leq i \leq m$, suppose that $N_i(v) = \{y_1, y_2, \dots\}$ as labeled and $d_i(v) = |N_i(v)| \geq 2m-1$. Consider the subgraph H_i of G induced by $N_i(v)$, and assign each edge of H_i a direction from the end vertex of smaller index to the larger one.

Claim For $1 \leq i \leq m$, H_i contains no directed path of length $2m-1$.

Proof. Suppose that for some $1 \leq i \leq m$, H_i contains a directed path of length $2m-1$ on vertices in order as $y_{k_1} y_{k_2} \dots y_{k_{2m}}$ with $k_1 < k_2 < \cdots < k_{2m}$. Let us write $v_j = y_{k_j}$ and let

$$d^* = \max_{1 \leq j \leq 2m-1} d_T(v_j, v_{j+1}) := d_T(v_s, v_{s+1}).$$

From the construction of T and the labeling of $N_1(v), \dots, N_i(v)$, we know that for any pair r and t with $1 \leq r \leq s$ and $s+1 \leq t \leq 2m$,

$$d^* = d_T(v_r, v_t).$$

Moreover, whatever the value of d^*, we would find a cycle C_{2m+1} of G, which would yield a contradiction as desired. For example, if $d^* = 2$, then by noting that $d_T(v_1, v_{2m}) = 2$ and there is a unique path in T, say $v_1 u v_{2m}$, connecting v_1 and v_{2m} of length $d^* = 2$, we obtain that $u v_1 v_2 \dots v_{2m} u$ form a C_{2m+1} in G, which is impossible. Generally, suppose that $d^* = 2h$ for $1 \leq h \leq m$. Then $d_T(v_h, v_{2m-h+1}) = 2h$, and hence the unique path of length $2h$ in T connecting v_h and v_{2m-h+1} and the path $v_h \dots v_{2m-h+1}$ would form a C_{2m+1} in G, which is impossible. □

We may suppose that H is connected. Fix a vertex $y_0 \in N_i(v)$, and we assign color ℓ to a vertex $y \in N_i(v)$ if the maximum length of the directed path between y

and y_0 in $N_i(v)$ is ℓ. Clearly, any two adjacent vertices receive distinct colors. Note that the maximum length of directed path in $N_i(v)$ is at most $2m-2$, so we just use colors of $\{0, 1, \ldots, 2m-2\}$, and thus $N_i(v)$ is $(2m-1)$-colorable. □

Now we give the upper bound for $r(C_{2m+1}, K_n)$ for $m \geq 2$.

Theorem 3.8 *Let $m \geq 2$ be a fixed integer. For all sufficiently large n,*

$$r(C_{2m+1}, K_n) \leq c\frac{n^{1+1/m}}{(\log n)^{1/m}},$$

where $c = c(m) > 0$ is a constant.

Proof. Let $N = r(C_{2m+1}, K_n) - 1$. Let G be a C_{2m+1}-free graph on N vertices with $\alpha(G) \leq n-1$. Suppose that $N > c\frac{n^{1+1/m}}{(\log n)^{1/m}}$ for some suitable constant $c > 0$, we aim to find a contradiction. Let V be the vertex set of G, and let $d = N^{1/(m+1)}(\log N)^{m/(m+1)}$. Denote

$$V_0 = \{v \in V |\ d(v) < d\},$$

and $V_1 = V \setminus V_0$. In the following, $c_i = c_i(m)$ are all positive constants.

Case 1 $|V_0| > N/2$.

Let G_1 be the subgraph of G induced by V_0. Then the average degree of G_1 is at most d. For any vertex $v \in V_0$, the neighborhood $N_{G_1}(v)$ in G_1 does not contain a path of order $2m$, so Theorem 3.6 implies that for large n,

$$\alpha(G) \geq \frac{c_1 N \log d}{d} \geq c_2 \left(N^m \log N\right)^{1/(m+1)} > n.$$

Case 2 $|V_1| \geq N/2$.

By Theorem 3.7, we obtain that for large n,

$$\begin{aligned}\alpha(G) &\geq c_3\left(\sum_{v\in V} d(v)^{1/(m-1)}\right)^{(m-1)/m} \geq c_3\left(\sum_{v\in V_1} d(v)^{1/(m-1)}\right)^{(m-1)/m}\\ &\geq c_4 N^{(m-1)/m} d^{1/m} \geq c_5\Big(N^m \log N\Big)^{1/(m+1)} > n,\end{aligned}$$

completing the proof. □

The following proof is due to Sudakov (2002) that in fact contains a deterministic algorithm.

The Second Proof of Theorem 3.8. Let

$$N = \left\lfloor \frac{an^{1+1/m}}{(\log n)^{1/m}} \right\rfloor, \quad \text{and} \quad d = bn^{1/m}(\log n)^{(m-1)/m}.$$

where $a = a(m)$ and $b = b(m)$ are constants to be chosen. Let G be a graph of order N that contains no C_{2m+1}. We aim to verify that $\alpha(G) \geq n$, which implies that $r(C_{2m+1}, K_n) \leq N$. Starting with $G' = G$ and $I = \emptyset$. If G' has a vertex of degree greater than d we do the following iterative procedure, otherwise, we stop.

Pick a vertex v with $d(v) = d_1(v) > d$. Since

$$d_{m+1}(v) = d_1(v)\frac{d_2(v)}{d_1(v)} \cdots \frac{d_{m+1}(v)}{d_m(v)} \leq N,$$

there exists some i, $1 \leq i \leq m$, such that

$$\frac{d_{i+1}(v)}{d_i(v)} \leq \left(\frac{N}{d}\right)^{1/m} \leq \left(\frac{an}{b\log n}\right)^{1/m}.$$

Take the smallest i with this property. Note that $d_1(v) > (N/d)^{1/m}$ and thus we have $d_i(v)/d_{i-1}(v) > (N/d)^{1/m}$, so

$$\frac{d_{i-1}(v)}{d_i(v)} < \left(\frac{d}{N}\right)^{1/m} < 1$$

for large n. By Lemma 3.15, $N_i(v)$ is $(2m-1)$-colorable hence it contains an independent set I' of size $|I'| \geq d_i(v)/(2m-1)$. Enlarge I to $I \cup I'$ and remove $N_{i-1}(v) \cup N_i(v) \cup N_{i+1}(v)$ from G'. Note that

$$\begin{aligned} d_{i-1}(v) + d_i(v) + d_{i+1}(v) &= \left(\frac{d_{i-1}(v)}{d_i(v)} + 1 + \frac{d_{i+1}(v)}{d_i(v)}\right) d_i(v) \\ &\leq \left(2 + \left(\frac{an}{b\log n}\right)^{1/m}\right) d_i(v) \\ &\leq (2m-1)\left(2 + \left(\frac{an}{b\log n}\right)^{1/m}\right)|I'| \\ &= A|I'| \end{aligned}$$

for any large n, where $A = (2m-1)(2 + (\frac{an}{b\log n})^{1/m})$. Clearly all neighbors of I' have been removed thus I is an independent set after each step, and the ratio between the number of removed vertices and $|I|$ is at most A.

Let $G_1 = (V_1, E_1)$ be a graph in the end of the process, and $G_2 = (V_2, E_2)$ be the graph induced by all removed vertices $V_2 = V \setminus V_1$. We distinguish two cases depending on $|V_2|$.

Case 1 $|V_2| > N/2$.

G_2 has an independent set I of size at least

$$|I| > \frac{N/2}{A} \geq (1-o(1))n\frac{a}{2(2m-1)}\left(\frac{b}{a}\right)^{1/m} > n$$

for large n if a and b satisfy

$$\frac{a}{2(2m-1)}\left(\frac{b}{a}\right)^{1/m} > 1 \tag{3.3}$$

Case 2 $|V_2| \le N/2$.

In this case, $|V_1| > N/2$. Since no vertices in G_1 has degree greater than d, so the average degree of G_1 is at most d, and each subgraph induced by a neighbor of G_1 does not contain a path of length $2m-1$ since G_1 does not contain C_{2m+1}. By Theorem 3.6, we have

$$\alpha(G) \ge \alpha(G_1) \ge \frac{N}{2}\frac{\log(d/(2m)) - 1}{d} \ge \frac{a}{2bm}n > n$$

for large n if a and b satisfy

$$\frac{a}{2bm} > 1. \tag{3.4}$$

To obtain the desired constants a and b, let us look at the case where the equality holds for (3.3) and (3.4). Set $a/b = 2m$ and $a = 2(2m-1)(2m)^{1/m}$. To get what we want, just perturb this solution a little bit, when a and b are slightly larger, such that both (3.3) and (3.4) are satisfied. Thus for all sufficiently large n,

$$r(C_{2m+1}, K_n) \le (1 + o(1))2(2m-1)(2m)^{1/m}\left(\frac{n^{m+1}}{\log n}\right)^{1/m},$$

completing the proof. □

3.7 The First Two Moments

Let X be a random variable, where X takes $\{a_i | i = 1, 2, \ldots\}$. The expected value $E(X)$ of X is defined to be

$$E(X) = \sum_i a_i \Pr(X = a_i).$$

Theorem 3.9 (Markov's Inequality) *Let X be a nonnegative random variable. If $a > 0$, then*

$$\Pr(X \ge a) \le \frac{E(X)}{a}.$$

Proof. Suppose that $\{a_i | i = 1, 2, \ldots\}$ is the set of all values that X takes. We have

$$E(X) = \sum_i a_i \Pr(X = a_i) \geq \sum_{a_i \geq a} a_i \Pr(X = a_i)$$
$$\geq a \sum_{a_i \geq a} \Pr(X = a_i) = a \Pr(X \geq a),$$

as required. □

Corollary 3.2 *If a random variable X only takes nonnegative integer values and $E(X) < 1$, then* $\Pr(X \geq 1) < 1$. *In particular,* $\Pr(X = 0) > 0$.

This is exactly what we used to obtain lower bounds of Ramsey numbers in the last chapter, e.g. Theorem 3.1.

For a positive integer k, the *kth moment* of a real-valued random variable X is defined to be $E(X^k)$, and so the first moment is simply the expected value. Denote by $\mu = E(X)$, and define the *variance* of X as $E((X-\mu)^2)$, which is denoted by σ^2. A basic equality is as follows.

$$\sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2.$$

Also, we call

$$\sigma = \sqrt{E((X-\mu)^2)}$$

as the *standard deviation* of X.

Theorem 3.10 (Chebyshev's Inequality) *Let X be a random variable. For any $a > 0$,*

$$\Pr(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}.$$

Proof. By Markov's inequality,

$$\Pr((X-\mu)^2 \geq a^2) \leq \frac{E((X-\mu)^2))}{a^2}.$$

The assertion follows since $\sigma^2 = E((X-\mu)^2)$. □

By importance, the second moment $E(X^2)$ is second to the first moment $E(X)$. The use of Chebyshev's Inequality is always called the second moment method.

Lemma 3.16 (Second Moment Method) *If X is a random variable, then*

$$\Pr(X = 0) \leq \frac{\sigma^2}{\mu^2} = \frac{E(X^2) - \mu^2}{\mu^2},$$

where $\mu = E(X)$. In particular, $\Pr(X = 0) \to 0$ *if* $E(X^2)/\mu^2 \to 1$.

Proof. By Chebyshev's inequality,

$$\Pr(X = 0) \leq \Pr(|X-\mu| \geq \mu) \leq \frac{\sigma^2}{\mu^2}$$

as desired. □

Intuitively, if σ grows more slowly than μ grows, than $\Pr(X = 0) \to 0$ since σ "pulls" X close to μ thus far away from zero.

3.8 Chernoff Bounds

The Chebshev's inequality is in fact the Markov's inequality on random variable $|X - \mu|$. However, Chebshev's inequality states the probability of a random variable X apart from $E(X)$ is bounded. When this is the case, we say that X is *concentrated*. A concentration bound is used to show that a random variable is very close to its expected value with high probability, so it behaves approximately as one may "expect" it to be.

Chernoff bounds, named after Herman Chernoff, gives exponentially decreasing bounds on tail distributions of sums of independent random variables. They are sharper bounds than the known first or second moment based tail bounds such as Markov's inequality or Chebyshev's inequality, which only yield power-law bounds on tail decay. But Chernoff bounds require the variables to be independent – a condition that neither the Markov's inequality nor the Chebyshev's inequality require.

When S_n is the sum of n independent variables, each variable equals to 1 with probability p and -1 with probability $1 - p$, respectively, the bound can be sharper. Most of the results in this chapter may be found in, or immediately derived from, the seminal paper of Chernoff (1952) while our proofs are self-contained. Recall a set of random variables $X_1, X_2, \ldots, X_n$ are said to be mutually independent if each X_i is independent of any Boolean expression formed from other $(X_j)'s$.

In any form of the Chernoff bounds, we have the following assumption.

Assumption A: Let $X_1, X_2, \ldots, X_n$ be mutually independent variables with the same binomial distribution.

Set

$$S_n = \sum_{i=1}^{n} X_i.$$

All concentration bounds in the remaining part of this section are Chernoff bounds of different forms, which estimate the probability of

$$\Pr(S_n \geq n(\mu + \delta)),$$

where $\mu = E(X_i)$. The symmetric bound on $\Pr(S_n \leq n(\mu - \delta))$ can be obtained similarly.

Theorem 3.11 *Under Assumption A, and suppose*

$$\Pr(X_i = 1) = \Pr(X_i = -1) = \frac{1}{2}$$

for $i = 1, 2, \ldots, n$. *For any* $\delta > 0$,

$$\Pr(S_n \geq n\delta) < \exp\{-n\delta^2/2\}.$$

In particular, $\Pr(S_n \geq a) < \exp\{-a^2/(2n)\}$ *for any* $a > 0$.

Proof. Let $\lambda > 0$ be arbitrary. Clearly,

$$E(e^{\lambda X_i}) = \frac{e^{\lambda} + e^{-\lambda}}{2}.$$

Therefore,

$$\begin{aligned} E(e^{\lambda S_n}) &= E(e^{\lambda X_1})E(e^{\lambda X_2}) \cdots E(e^{\lambda X_n}) \\ &= \left(\frac{e^{\lambda} + e^{-\lambda}}{2}\right)^n < e^{n\lambda^2/2}, \end{aligned}$$

where the inequality follows by noticing Lemma 3.4. Thus, by Markov's inequality, we have that for all $\lambda > 0$,

$$\Pr(S_n \geq n\delta) = \Pr(e^{\lambda S_n} \geq e^{\lambda n\delta}) \leq \frac{E(e^{\lambda S_n})}{e^{\lambda n\delta}} < \exp\{n(\lambda^2/2 - \lambda\delta)\}.$$

Setting $\lambda = \delta$, we obtain the desired result. □

Since X_i is often an indicator variable of some random event, so X_i takes 1 when the event appears and 0 otherwise. The following form of Chernoff bound may be used in more cases.

Theorem 3.12 *Under Assumption A, and suppose*

$$\Pr(X_i = 1) = \Pr(X_i = 0) = \frac{1}{2}$$

for $i = 1, 2, \ldots, n$. *For any* $\delta > 0$,

$$\Pr(S_n \geq n(1 + \delta)/2) < \exp\{-n\delta^2/2\}.$$

Namely, $\Pr(S_n \geq n(1/2 + \delta) < \exp\{-2n\delta^2\}$.

Proof. Set $Y_i = 2X_i - 1$ and $T_n = \sum_{i=1}^{n} Y_i = 2S_n - n$. Then

$$\Pr(Y_i = 1) = \Pr(Y_i = -1) = \frac{1}{2},$$

and $\{Y_i | i = 1, 2, \ldots, n\}$ satisfies Assumption A. Note that $S_n \geq n(1 + \delta)/2$ if and only if $T_n \geq n\delta$. By Theorem 3.11,

$$\Pr(S_n \geq n(1 + \delta)/2) = \Pr(T_n \geq n\delta) < \exp\{-n\delta^2/2\}$$

as claimed. □

Under Assumption A, and suppose

$$\Pr(X_i = 1) = p, \quad \text{and} \quad \Pr(X_i = 0) = 1 - p$$

for $i = 1, 2, \ldots, n$. We say that the sum $S_n = \sum_{i=1}^{n} X_i$ has binomial distribution, denoted by $B(n, p)$. Involved in Theorem 3.12 is the special binomial distribution $B(n, 1/2)$. For the general case, the calculation is slightly more complicated, but the technique is the same. As usual, denote by q for $1 - p$.

Theorem 3.13 *Under Assumption A, and suppose*

$$\Pr(X_i = 1) = p \quad \textit{and} \quad \Pr(X_i = 0) = q$$

for $i = 1, 2, \ldots, n$. There exists $\delta_0 = \delta_0(p) > 0$ so that if $0 < \delta < \delta_0$, then

$$\Pr(S_n \geq n(p + \delta)) < \exp\{-n\delta^2/(3pq)\}.$$

Proof. Denote $a = p + \delta$. By the same argument as in Theorem 3.11,

$$\begin{aligned}
\Pr(S_n \geq na) = \Pr(e^{\lambda S_n} \geq e^{\lambda na}) &\leq \frac{1}{e^{\lambda na}} E(e^{\lambda S_n}) \\
&= \frac{1}{e^{\lambda na}} (pe^{\lambda} + q)^n = (pe^{\lambda(1-a)} + qe^{-\lambda a})^n
\end{aligned}$$

for all $\lambda > 0$. Let $c = 1 - a = q - \delta > 0$. Note that $pe^{\lambda(1-a)} + qe^{-\lambda a}$ is convex on λ, and so it attains the minimum value by taking $\lambda_0 = \log(aq/cp)$, i.e.,

$$\min_{\lambda > 0}(pe^{\lambda c} + qe^{-\lambda a}) = e^{-\lambda_0 a}(pe^{\lambda_0} + q) = \left(\frac{cp}{aq}\right)^a \frac{q}{c} = \left(\frac{p}{a}\right)^a \left(\frac{q}{c}\right)^c.$$

Note that

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4).$$

Recall $a = p + \delta$, it follows that for $0 < \delta < 1 - p$,

$$\begin{aligned}
\log\left(\frac{p}{a}\right)^a &= (p + \delta)\log\left(1 - \frac{\delta}{p + \delta}\right) \\
&= -\delta - \frac{\delta^2}{2(p + \delta)} - \frac{\delta^3}{3(p + \delta)^2} + o(\delta^3),
\end{aligned}$$

and

$$\begin{aligned}
\log\left(\frac{q}{c}\right)^c &= (q - \delta)\log\left(1 + \frac{\delta}{q - \delta}\right) \\
&= \delta - \frac{\delta^2}{2(q - \delta)} + \frac{\delta^3}{3(q - \delta)^2} + o(\delta^3).
\end{aligned}$$

Adding them by terms, the first sum vanishes, and the second is

$$\frac{-\delta^2}{2}\left(\frac{1}{p+\delta}+\frac{1}{q-\delta}\right)=\frac{-\delta^2}{2}\left(\frac{1}{p(1+\delta/p)}+\frac{1}{q(1-\delta/q)}\right)$$
$$=\frac{-\delta^2}{2}\left(\frac{1}{pq}-\frac{(q^2-p^2)\delta}{p^2q^2}+o(\delta)\right)$$
$$=\frac{-\delta^2}{2pq}+\frac{(q-p)\delta^3}{2p^2q^2}+o(\delta^3),$$

and the third is

$$\frac{\delta^3}{3}\left(\frac{1}{(q-\delta)^2}-\frac{1}{(p+\delta)^2}\right)=\frac{\delta^3}{3}\left(\frac{1}{q^2}-\frac{1}{p^2}+O(\delta)\right)$$
$$=\frac{-(q-p)\delta^3}{3p^2q^2}+o(\delta^3).$$

Therefore, for sufficiently small $\delta > 0$,

$$\log\left[\left(\frac{p}{a}\right)^a\left(\frac{q}{c}\right)^c\right]=\frac{-\delta^2}{2pq}+\frac{(q-p)\delta^3}{6p^2q^2}+o(\delta^3)<\frac{-\delta^2}{3pq},$$

it follows that

$$\Pr(S_n \geq n(p+\delta)) < \exp\{-n\delta^2/(3pq)\},$$

completing the proof. □

From the above proof for $p > q$ and Theorem 3.12 for $p = q = 1/2$, we see that if $p \geq 1/2$, the bound can be improved slightly as

$$\Pr(S_n > n(p+\delta)) < \exp\{-n\delta^2/(2pq)\}.$$

We now write out a symmetric form for Theorem 3.13, and omit those for Theorem 3.11 and Theorem 3.12.

Theorem 3.14 *Under Assumption A, and suppose*

$$\Pr(X_i = 1) = p \quad \textit{and} \quad \Pr(X_i = 0) = q$$

for $i = 1, 2, \ldots, n$. *There exists* $\delta_0 = \delta_0(p) > 0$ *such that if* $0 < \delta < \delta_0$, *then*

$$\Pr(S_n \leq n(p-\delta)) < \exp\{-n\delta^2/(3pq)\}.$$

Therefore, $\Pr(|S_n - np| > n\delta)) < 2\exp\{-n\delta^2/(3pq)\}$.

From the proof of Theorem 3.13, we have

$$\Pr(S_n \geq na) \leq \left(\left(\frac{p}{a}\right)^a\left(\frac{q}{c}\right)^c\right)^n=\exp\left\{n\left(a\log\frac{p}{a}+(1-a)\log\frac{q}{1-a}\right)\right\},$$

where $c = 1 - a$. Let $H(x)$ signify the entropy function, i.e.

$$H(x) = x \log \frac{p}{x} + (1-x) \log \frac{q}{1-x}, \quad 0 < x < 1.$$

Thus

$$\Pr(S_n \geq k) \leq \exp\{nH(k/n)\},$$

which is valid also for $k = np$ since $H(p) = 0$.

The following form of Chernoff bound was used by Beck (1983).

Theorem 3.15 *Under Assumption A, and suppose*

$$\Pr(X_i = 1) = p \quad \text{and} \quad \Pr(X_i = 0) = q$$

for $i = 1, 2, \ldots, n$. If $k \geq np$, then

$$\Pr(S_n \geq k) \leq \left(\frac{np}{k}\right)^k \left(\frac{nq}{n-k}\right)^{n-k}.$$

Consequently,

$$\Pr(S_n \geq k) \leq \left(\frac{npe}{k}\right)^k.$$

Proof. The right hand side of the first inequality is just $\exp\{nH(k/n)\}$. For the second inequality, simply note that

$$\left(\frac{nq}{n-k}\right)^{n-k} \leq \left(\frac{n}{n-k}\right)^{n-k} = \left(1 + \frac{k}{n-k}\right)^{n-k} < e^k.$$

Thus the required result follows. □

Recall that $r_k(G)$ is the smallest integer N such that in any $k-$coloring of edges of K_N, there is a monochromatic G. Chung and Graham (1975), and Erdős (1981) proposed a problem to determine $r_k(K_{m,n})$. We now give a lower bound for it as k and m are fixed and $n \to \infty$, in which $\sqrt{n \log n}$ can be replaced by $\sqrt{n}\,\omega(n)$, where $\omega(n) \to \infty$.

Theorem 3.16 *Let k and m be fixed positive integers. There exists a constant $c = c(k,m) > 0$ such that*

$$r_k(K_{m,n}) \geq k^m n - c\sqrt{n \log n}$$

for all large n.

Proof. Set $N = k^m n - c\sqrt{n \log n}$, where c is a constant to be determined. Then

$$n = \left(\frac{1}{k^m} + \frac{c\sqrt{n \log n}}{k^m N}\right) N > (k^{-m} + \delta_n)N = (p + \delta_n)N,$$

where $p = k^{-m}$ and $\delta_n = \frac{c}{2k^{2m}}\sqrt{\frac{\log n}{n}}$. Let us color the edges of K_{N+m} with k colors randomly and independently, such that each edge is assigned in each color with probability $1/k$. Consider a fixed color, say color A, and an arbitrary but fixed set U of m vertices. Let $v_1, v_2, \dots, v_N$ be the N vertices outside U. For each j, define a random variable X_j such that $X_j = 1$ if the edges between v_j and U are all in color A and 0 otherwise. Then $\Pr(X_j = 1) = k^{-m} = p$. Set $S_N = \sum_{i=1}^{N} X_j$. Clearly S_N has the binomial distribution $B(N, p)$ and the event $S_N \geq n$ means that there is a monochromatic $K_{m,n}$ in color A (in which U is the m-vertex part). Hence

$$\Pr(\exists \text{ monochromatic } K_{m,n}) \leq k\binom{N+m}{m}\Pr(S_N \geq n).$$

By virtue of Chernoff bound (Theorem 3.13)

$$\Pr(S_N \geq n) \leq \Pr(S_N \geq (p+\delta_n)N) \leq \exp\{-N\delta_n^2/(3pq)\}.$$

From the facts that

$$-\frac{N\delta_n^2}{3pq} \sim \frac{-c^2 \log n}{12k^m(k^m-1)}$$

and

$$k\binom{N+m}{m} = O(n^m) = O\left(e^{m \log n}\right),$$

we have that the probability that there exists monochromatic $K_{m,n}$ tends to zero as $N \to \infty$ if $c \geq k^m\sqrt{12m}$, which guarantees the existence of an edge-coloring of K_{N+m} with no monochromatic $K_{m,n}$, implying that $r_k(K_{m,n}) > N + m$ for all large n. □

Let $B_n^{(m)}$ be the book graph that consists of n copies of K_{m+1} sharing a common K_m. The above result clearly implies that for fixed k, $m \geq 1$, there is a constant $c = c(k, m) > 0$ such that

$$r_k(B_n^{(m)}) \geq k^m n - c\sqrt{n \log n}$$

for all large n.

Most cases to apply Chernoff bounds are in random graphs, some of which will be discussed in the next chapter.

3.9 Exercises

1. Let $B_1, \dots, B_t$ be a partition of a sample Ω. Adam's Theorem states that for every event A, $\Pr(A) = \sum_i \Pr(A|B_i)\Pr(B_i)$. Prove this theorem. Using it, show that $\Pr(A) \leq \max_i \Pr(A|B_i)$.

2. Prove that in the random graph space $\mathcal{G}(n, p)$, it holds

$$\sum_{G:\, e \in E(G)} \Pr(G) = p,$$

where e is an fixed edge. Explain the equality for cases $p = 0, 1$ and $1/2$.

3. Show that the complement $\overline{G}_p$ of a random graph G_p is exactly G_q, where $q = 1 - p$. When we write $\omega(G_p) = \alpha(G_q)$, what does it mean really?

4. Let v be a fixed vertex of $\mathcal{G}(n, p)$ and let $d(v)$ be the degree of v. Compute the probability of the event $d(v) = k$ and its expectation and variance.

5. For distinct vertices u and v, show

$$\Pr(d(u) \le k_1, d(v) \le k_2) \le \Pr(d(u) \le k_1) \Pr(d(v) \le k_2).$$

6. Define $f(G) = \sum_{v \in V(G)} \frac{1}{1+d(v)}$. Prove that $\alpha(G) \ge f(G)$ by an algorithm as follows. In each step we delete a vertex of maximum degree until no edge left. Let G_1 be the subgraph from G by deleting a vertex of maximum degree of G. Prove $f(G_1) \ge f(G)$.

7. Prove that if the average degree d of a graph G on N vertices satisfies that $0 < d < 1$, then $\alpha(G) \ge N/2$.

8. Erdős and Sós (See Chung and Graham, 1998) asked to prove or disprove that $r(3, n + 1) - r(3, n) = o(n)$. This problem remains unresolved even with the knowledge of Kim's result on $r(3, n)$. Show that $r(3, n) < n^2/\log(n/e)$ for large n.

9. Show that $r(4, n) \le (1 + o(1))r(3, n)n/\log n$. (Hint: Using Kim's result that $r(3, n) \ge cn^2/\log n$.)

10. Prove that there exists some constant $c > 0$ such that if G is a regular graph G on n vertices with girth at least 5, then its independence number is at least $c\sqrt{n} \log n$.

11. Prove that if the conjecture of Erdős-Sós is true, then

$$r(T_{1+m}, T_{1+n}) \le m + n.$$

12.* Let G be a graph with N vertices and average degree d. If each neighborhood of G contains no T_{m+1}, then $\alpha(G) \ge N f_{2m-1}(d)$.

13. In Section 3.6, we have used the fact that if $a_1, \ldots, a_n$ are positive numbers, then $\frac{1}{n} \sum_{i=1}^n a_i \ge \left(\Pi_{i=1}^n a_i\right)^{1/n}$. Prove this by considering the convexity of the function $f(x) = e^x$.

14. Obtain the asymptotically optimal constants in Theorem 3.7.

15. Call a term $1/(1 + d_v)$ in $\alpha(G) \ge \sum_v 1/(1 + d_v)$ as the ratio of independence. Using this concept, explain Lemmas 3.13 and 3.14 intuitively. Furthermore, if $N_\ell(v)$ is k-colorable for each vertex v, then

$$\alpha(G) \ge \frac{1}{k} \sum_{\substack{v \in V \\ d_\ell(v) \ne 0}} \frac{d_\ell(v)}{d_{\ell-1}(v) + d_\ell(v) + d_{\ell+1}(v)}.$$

16. Give a lower bound of the independence number $\alpha(G)$ for triangle free graph. (Hint: Similar to Lemma 3.13)

17. Let $m \geq$ be a fixed integer and let G be a graph of order N and girth at least $2m+2$. Prove that $\alpha(G) \geq \Omega((N \log N)^{m/(m+1)})$. (Hint: $N_m(v)$ is an independent set.)

18.* Use basic method to prove the following result. Let G be a graph of order n that contains at most n triangles. Prove that there is an induced subgraph G_0 of G such that G_0 is triangle-free and its order is at least $0.38n$ for large n. (Hint: Let $c = \sqrt{3}/3$. Consider all subgraphs induced by cn vertices. The expected number of triangles is at most asymptotically $c^3 n$.)

Chapter 4
Random Graph

The study of random graphs should go back to Erdős and Rényi (1947, 1959, 1960, 1961), in which they discovered that the random graphs were often useful in tackling extremal problems in graph theory. Nowadays, random graph has become an active area of research in graph theory and network science. In this chapter, we will give an overview on random graphs. For a comprehensive understanding of random graphs, we refer the reader to books such as *Random Graphs* by Bollobás (2001, 2nd ed.), *Random Graphs* by Janson, Łuczak and Ruciński (2000), and *Introduction to Random Graphs* by Frieze and Karoński (2016) as well as *The Probabilistic Method* by Alon and Spencer (2016), a part of which introduces essentiality of the random graphs.

4.1 Preliminary

For a family of graphs $\mathcal{G} = \{G_1, G_2, \ldots, G_M\}$ with probabilities $\Pr(G_i)$ for $i = 1, 2, \ldots$ such that $0 \le \Pr(G_i) \le 1$ and $\sum_{i\ge 1} \Pr(G_i) = 1$, we have a probability space of random graphs with G_i as basic events. Each G_i is called a *random graph* of $\mathcal{G}$ with probability $\Pr(G_i)$. We shall consider the probability space that consists of graphs on vertex set $V = [N]$, where the vertices are *distinguishable*, and so the edges are distinguishable, too. Note that the complete graph K_N has

$$\binom{N}{1} + \binom{N}{2}2 + \cdots + \binom{N}{k}2^{\binom{k}{2}} + \cdots + \binom{N}{N}2^{\binom{N}{2}}$$

subgraphs. The general term $\binom{N}{k}2^{\binom{k}{2}}$ corresponds to subgraphs that have exactly k vertices, and the last term $\binom{N}{N}2^{\binom{N}{2}}$ corresponds to all spanning subgraphs.

Let us label all edges of K_N on vertex set $V = [N]$ as $e_1, e_2, \ldots, e_m$, where $m = \binom{N}{2}$. Note that the number of graphs on vertex set $[N]$ is 2^m since the edges are distinguishable. The space $\mathcal{G}(N; p_1, \ldots, p_m)$ is defined for $0 \le p_i \le 1$ as follows. It consists of all 2^m spanning graphs on V, in which each edge e_i is selected

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independently with probability p_i. Write $q_j = 1 - p_j$ and $G(p_1, \ldots, p_m)$ for a random element in $\mathcal{G}(N; p_1, \ldots, p_m)$. For a specific graph H in the space with $E(H) = \{e_j : j \in S\}$, where $S \subseteq [m]$ is the index set of edges of H,

$$\Pr(G(p_1, \ldots, p_m) = H)) = \left(\Pi_{j \in S} p_j\right)\left(\Pi_{j \notin S} q_j\right).$$

Note that the event $G(p_1, \ldots, p_m) = H$ is different from that $G(p_1, \ldots, p_m)$ is isomorphic to H since the vertices and edges are distinguishable. We have that $\mathcal{G}(N; p_1, \ldots, p_m)$ is truly a probability space since

$$\sum_H \Pr(G(p_1, \ldots, p_m) = H)) = \sum_{S \subseteq [m]} \left(\Pi_{j \in S} p_j\right)\left(\Pi_{j \notin S} q_j\right) = \Pi_{j=1}^m (p_j + q_j) = 1.$$

When $p_1 = \cdots = p_m = p$, the probability space $\mathcal{G}(N; p_1, \ldots, p_m)$ is written as $\mathcal{G}(N, p)$. In $\mathcal{G}(N, p)$, the probability of a specific graph H with k edges is $p^k (1-p)^{m-k}$: each of the k edges of H has to be selected and none edges of $\overline{H}$ is allowed to be selected. We write $G(N, p)$, or G_p for short, for a random graph in $\mathcal{G}(N, p)$,

$$\Pr(G_p = H) = p^{e(H)} q^{m-e(H)}.$$

Now we have obtained a space of random graphs, and every graph invariant becomes a random variable. For instant, the number of complete graphs of order k in G, denoted by $X_k(G)$, is a random variable on our space of random graphs. The nature of such a random variable depends heavily on p.

In the space $\mathcal{G}(N, 0)$, the probability that the empty graph $\overline{K_N}$ appears is one, and the probability that any other graph appears is zero. Similarly, in the space $\mathcal{G}(N, 1)$, the only graph that appears is K_N. For other but these two extremal cases, i.e. $0 < p < 1$, any graph on vertex set $[N]$ appears with a positive probability. In particular, $\mathcal{G}(N, 1/2)$ could be viewed as the space: it consists of all 2^m graphs on $V = [N]$, and the probability of any graph is equiprobable. This is just a classical probability space. Thus $G_{N,1/2}$ is also obtained by picking any of the 2^m graphs on $V = [N]$ at random with probability 2^{-m}. As p increases from 0 to 1, the random graph G_p *evolves* from empty to full. It is worth remarking that $p = p(N)$ is often a function. No matter p is fixed or not, we tend to be interested in what happens as $N \to \infty$.

In their original paper on random graphs in 1960, Erdős and Rényi used $\mathcal{G}(N, e)$ to denote the random graph with vertex set $V = [N]$ and precisely e edges. For $0 \le e \le m = \binom{N}{2}$ with e fixed, the space $\mathcal{G}(N, e)$ consists of all $\binom{m}{e}$ spanning subgraphs with exactly e edges: which can be turned into a probability space by taking its elements to be equiprobable. Thus, write G_e for a random graph in the space $\mathcal{G}(N, e)$, for a specific graph H in the space, we have that

$$\Pr[G_e = H] = \binom{m}{e}^{-1},$$

where the event $G_e = H$ means that G_e is precisely H, but not only isomorphic to H in general.

It is interesting, as expected, that for $e \sim p\binom{N}{2}$ the spaces $\mathcal{G}(N, e)$ and $\mathcal{G}(N, p)$ are close to each other as $N \to \infty$. In most proofs for existence, the calculations are easier in $\mathcal{G}(N, p)$ than in $\mathcal{G}(N, e)$. So we will work on the probability model $\mathcal{G}(N, p)$ exclusively.

Another point of view may be convenient, in which one colors all edges of the complete graph K_N with probability p, randomly and independently. Thus random graph G_p is viewed as a random coloring of edge set of K_N. The coloring of edge set of K_N is also said a coloring of K_N in short. Recall the definition of Ramsey numbers, we can see why *the relation between random method and Ramsey theory is so natural and tight*.

In many applications, we always need to consider the events that some certain graphs were contained in random graphs. Let F be a given graph on k vertices, and let $S \subseteq [N]$ with $|S| = k$. Let A_S be the event that the subgraph induced by S contains F as a subgraph, then the event $\cup_S A_S$ signifies that F appears in G_p as a subgraph, its probability is hard to calculate since the events A_S have a complex interaction. It is often to bound this probability from above by the expectation of the number of copies of F in the random graph. To get the expectation, let us look the number of copies of F in K_k first. This is closely related to the automorphism group of F.

Recall a permutation (or a bijection) φ of $V(F)$ is an *automorphism* of graph F if $uv \in E(F)$ if and only if $\varphi(u)\varphi(v) \in E(F)$ for any pair of vertices u and v. It is straightforward to verify the set of all automorphisms of F forms a group, called the *automorphism group* of F, and denoted by $\mathcal{A}(F)$. Indeed, it is clear that the identity permutation is an automorphism. If φ is an automorphism of F, then so is its inverse φ^{-1}, and if ψ is a second automorphism of F, then the product $\varphi\psi$ is an automorphism. For example, $\mathcal{A}(K_k)$ is the symmetric group S_k of order $k!$, and $\mathcal{A}(C_k)$ is the dihedral group D_k of order $2k$, one can see Godsil and Royle (2001) for details.

Theorem 4.1 *If F is a graph of order k in which the vertices are labeled, then the number of copies of F such that no two copies are automorphism is $k!/|\mathcal{A}(F)|$.*

Proof. Let $\{v_1, v_2, \ldots, v_k\}$ be the set of labeled vertices. Certainly there are $k!$ labeling of F from this set with some labeled graphs that may be automorphism. Let $F_1, F_2, \ldots, F_{k!}$ be the labeled graphs obtained from F. Note that the relation "F_i is automorphism to F_j" is an equivalence relation, hence each equivalence class contains $|\mathcal{A}(F)|$ elements, implying that there are $k!/|\mathcal{A}(F)|$ equivalent classes in total. This proves the theorem. □

For example, if we label the vertices of a star $K_{1,3}$ as $1, 2, 3, 4$, then any equivalence class is uniquely determined by the label of its center. So there are 4 such classes, and each class contains 6 copies of $K_{1,3}$ with the same label of the center.

In a random graph space $\mathcal{G}(N, p)$, we need to consider the number of copies of F in a labeled complete graph.

Corollary 4.1 *If F is a graph of order k, then the number of copies of F in a labeled complete graph of order k is $k!/|\mathcal{A}(F)|$.*

Let F be a graph of order k. Let $S \subseteq [N]$ with $|S| = k$ and let X_S be the number of copies of F on S. Then $X = \sum_S X_S$ is the number of copies of F in G_p. We have

$$E(X_S) = \frac{k!}{|\mathcal{A}(F)|} p^{e(F)},$$

and

$$E(X) = \binom{N}{k} \frac{k!}{|\mathcal{A}(F)|} p^{e(F)} = \frac{(N)_k}{|\mathcal{A}(F)|} p^{e(F)},$$

where $(N)_k = N(N-1)\cdots(N-k+1)$ is the falling factorial.

Similar formulas hold for the number of *induced* subgraphs. Let Y be the number of induced graph F in G_p. Then

$$E(Y) = \frac{(N)_k}{|\mathcal{A}(F)|} p^{e(F)} q^{\binom{k}{2}-e(F)}.$$

Recall that A_S signifies the event that the subgraph induced by S in G_p contains F as a subgraph, we have

$$\Pr(A_S) \le \frac{k!}{|\mathcal{A}(F)|} p^{e(F)}.$$

Hence

$$\Pr(F \subset G_p) = \Pr(\cup A_S) \le \binom{N}{k} \frac{k!}{|\mathcal{A}(F)|} p^{e(F)} = \frac{(N)_k}{|\mathcal{A}(F)|} p^{e(F)}, \tag{4.1}$$

where the upper bound is exactly $E(X)$.

This can be seen also by the fact that X takes only nonnegative integral values and

$$\begin{aligned}\Pr(\cup A_S) = \Pr(X \ge 1) &= \sum_{i \ge 1} \Pr(X = i) \\ &\le \sum_{i \ge 1} i \Pr(X = i) = E(X).\end{aligned}$$

It seems to be necessary to point out that F is not a random element in $\mathcal{G}(N, p)$ and the above discussion is about appearance of F as a subgraph.

4.2 Lower Bounds for $r(m, n)$

Recall that the lower bound of $r(n, n)$ in Chapter 3 by Erdős, the proof in fact applies the random graphs of $\mathcal{G}(N, 1/2)$, which is a classical probability space as mentioned.

It is interesting to see that this space is the only one that counting argument works since only $\mathcal{G}(N, 1/2)$ is the classic probability space among $\mathcal{G}(N, p)$.

In this section, we will give more lower bounds for classical Ramsey numbers $r(m, n)$. Let us first give a lower bound for $r(m, n)$ by simple applications of random graphs.

Theorem 4.2 *Let m, n and N be positive integers. If for some $0 < p < 1$,*

$$\binom{N}{m} p^{\binom{m}{2}} + \binom{N}{n} (1-p)^{\binom{n}{2}} < 1,$$

then $r(m, n) > N$.

Proof. Consider random graphs G_p in $\mathcal{G}(N, p)$. Let S be a set of m vertices, and A_S the event that S induces a complete graph. Let T be a set of n vertices, and B_T the event that T induces an independent set. Similar to Theorem 3.1, we have $r(m, n) > N$ since $\Pr[(\cup_S A_S) \cup (\cup_T B_T)] < 1$ from the assumption. □

The above result is ineffective in lower bounding of $r(3, n)$. We now examine the lower bound of $r(4, n)$, and we aim to choose a suitable value of p such that N as large as possible for large n. Let us first give an overview. Consider the condition in Theorem 4.2, we roughly estimate $\binom{N}{n}$ as $(eN/n)^n$, and $(1-p)^{\binom{n}{2}}$ as $e^{-p\binom{n}{2}}$, hence $\binom{N}{n}(1-p)^{\binom{n}{2}}$ is roughly

$$\left(\frac{eN}{n}\right)^n \exp\left\{-p\binom{n}{2}\right\} = \left(\frac{eN}{ne^{p(n-1)/2}}\right)^n.$$

To get a better bound, we should balance two terms in the condition such that both terms are less than $1/2$. To this end, we should require that

$$\frac{eN}{ne^{p(n-1)/2}} < 1.$$

So we may take $p = \frac{c_1}{n} \log \frac{N}{n}$ for some constant c_1. On the other hand, we roughly have

$$\binom{N}{4} p^6 \sim \frac{1}{24} N^4 p^6 \sim 1,$$

which implies that $p = c_2 (1/N)^{2/3}$ for some constant c_2. Combining these two expressions, we have $N > n^a$ for some $a > 1$. Thus we can take $p = c_1 \frac{\log n}{n}$ and $N \sim c_2 (n/\log n)^{3/2}$.

Formally, let $p = c_1 \frac{\log n}{n}$ and $N = \lfloor c_2 (n/\log n)^{3/2} \rfloor$, where c_1 and c_2 are positive constants to be chosen satisfying that $c_1^6 c_2^4 < 24$. Then

$$\binom{N}{4} p^6 < \frac{N^4}{24} p^6 \le \frac{c_1^6 c_2^4}{24} \left(\frac{n}{n-1}\right)^6 < 1$$

for large n. For the second term, we have $(1-p)^{\binom{n}{2}} < e^{-pn(n-1)/2} = n^{-c_1 n/2}$ and hence

$$\binom{N}{n}(1-p)^{\binom{n}{2}} < \left(\frac{eN}{n}\right)^n n^{-c_1 n/2} = \left(\frac{eN}{n^{1+c_1/2}}\right)^n,$$

which tends to zero if we take $c_1 \geq 1$. On the other hand, in order to take c_2 as large as possible with $c_1^6 c_2^4 < 24$, we have to take c_1 as small as possible. So we take $c_1 = 1$.

Now, we may hope to optimize the constant c_2. Since we need only $c_2 < 24^{1/4}$, it follows that $c_2 = 24^{1/4} - \epsilon$ will be ok. Thus we have

$$r(4,n) \geq (24^{1/4} - o(1))\left(\frac{n}{\log n}\right)^{3/2}.$$

Hereafter we will choose p with some foresight. For general $m \geq 4$, by taking $p = (m-3)\frac{\log n}{n}$, a similar calculation as above yields that

$$r(m,n) \geq c\left(\frac{n}{\log n}\right)^{(m-1)/2}.$$

We have seen that the property of random graph G_p is sensitive with the value of p. To ensure that G_p contains no K_m (with a positive probability, more precisely, or $\binom{N}{m}p^{\binom{m}{2}}$ is small), it is better to take smaller p. But it is better to take a bigger p to ensure that there is no induced $\overline{K_n}$ (i.e, $\binom{N}{n}(1-p)^{\binom{n}{2}}$ is small). Our task is to balance both sides to obtain a larger N as possible.

We shall improve the lower bounds for $r(n,n)$ and $r(m,n)$ obtained previously by using the so called *deletion method*.

Theorem 4.3 *We have*

$$r(n,n) \geq (1-o(1))\frac{n}{e}2^{n/2}.$$

Proof. Consider the random graphs in $\mathcal{G}(N, 1/2)$. For an n-set S, let X_S be the indicator that S is a clique or an independent set, i.e.,

$$X_S = \begin{cases} 1 & \text{if } S \text{ induces } K_n \text{ or } \overline{K_n}, \\ 0 & \text{otherwise.} \end{cases}$$

Let A_S be the event that S induces K_n or $\overline{K_n}$. We have

$$E[X_S] = \Pr[A_S] = 2\left(\frac{1}{2}\right)^{\binom{n}{2}}.$$

Let $X = \sum_{S:|S|=n} X_S$. Clearly, X is the number of cliques or independent sets of size n. By linearity of expectation,

$$E[X] = \sum_{S:|S|=n} E[X_S] = \binom{N}{n} 2^{1-\binom{n}{2}}.$$

Since there is a point in the probability space for which X does not exceed its expectation, it follows that there exists a graph with at most

$$\binom{N}{n} 2^{1-\binom{n}{2}}$$

n-sets such that every set induces a K_n or a $\overline{K}_n$. By deleting a vertex from each such set, we have the remaining graph contains neither K_n nor $\overline{K}_n$. Thus

$$r(n, n) > N - E(X).$$

The rest of the proof is to find N such that $N - E(X)$ as large as possible. By taking $N = \lfloor \frac{n 2^{n/2}}{e} \rfloor$, from the Stirling formula, we have

$$\binom{N}{n} 2^{1-\binom{n}{2}} < \left(\frac{eN}{n}\right)^n 2^{1-\binom{n}{2}} < 2\left(\frac{e\sqrt{2}N}{n 2^{n/2}}\right)^n \le 2^{n/2+1},$$

which is $o(N)$. Thus $r(n, n) \ge (1 - o(1))N$. □

Theorem 4.4 *For any positive integer m, n and N, and any real number $0 < p < 1$,*

$$r(m, n) > N - \binom{N}{m} p^{\binom{m}{2}} - \binom{N}{n} (1-p)^{\binom{n}{2}}.$$

Consequently, there exists a constant $c > 0$ such that

$$r(m, n) \ge c \left(\frac{n}{\log n}\right)^{m/2}$$

for all large n.

Proof. The first assertion is obvious. For the second, set $N = a(\frac{n}{\log n})^{m/2}$ and $p = (m-2)\frac{\log n}{n}$ such that $a - \frac{(m-2)^{\binom{m}{2}}}{m!} a^m > 0$. Then

$$N_1 = \binom{N}{m} p^{\binom{m}{2}} \sim \frac{(m-2)^{\binom{m}{2}} a^m}{m!} \left(\frac{n}{\log n}\right)^{m/2},$$

and

$$N_2 = \binom{N}{n} (1-p)^{\binom{n}{2}} < \left(\frac{eN}{n}\right)^n e^{-pn(n-1)/2} = \left(\frac{eN}{n^{m/2}}\right)^n \to 0.$$

So if $c < a - \frac{(m-2)^{\binom{m}{2}}}{m!} a^m$, then

$$r(m,n) \geq N - N_1 - N_2 > c\left(\frac{n}{\log n}\right)^{m/2}.$$

This completes the proof. □

In the next chapter, we will see further improvements by using Lovász Local Lemma.

4.3 More Applications of Chernoff Bounds

As a natural application of Chernoff bound, we are concerned with the number of edges in a random graph as follows.

Theorem 4.5 *Let $\mathcal{G}(n,p)$ be a random graph space. If $\delta = \delta(n) > 0$ and $p = p(n) \in (0,1]$ such that $n\delta^2/p \to \infty$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \Pr\left[G_p \in \mathcal{G}(n,p) :\ (p-\delta)\binom{n}{2} \leq e(G_p) \leq (p+\delta)\binom{n}{2}\right] = 1.$$

Proof. For any edge e in K_n, we define a random variable X_e as $X_e = 1$ if e is an edge of G_p and $X_e = 0$ otherwise. Note that $e(G_p) = \sum_e X_e$ and $E(e(G_p)) = p\binom{n}{2}$ as $E(X_e) = p$, and the Chernoff bounds imply the claimed statement. □

An often used measure for sparseness of graphs is K_r-freeness. However, there are K_3-free graphs whose chromatic number can be arbitrarily large, see Mycielski's construction (1955), in which the main idea is as follows. Let $G_1 = K_1$ and $G_2 = K_2$, and generally let G_k be the graph defined on $\{v_1, \ldots, v_n\}$. Now we construct G_{k+1} from G_k by adding $n+1$ new vertices $\{u_1, \ldots, u_n, v\}$ and then for $1 \leq i \leq n$, join u_i to v and all neighbors of v_i. It is not difficult to verify that $\omega(G_k) = 2$ for all $k \geq 2$, and the chromatic number $\chi(G_{k+1}) = \chi(G_k) + 1$ for all $k \geq 1$.

A more general measure for sparseness is to forbid subdivision. A suspended path in graph G is a path $(x_0, x_1, \ldots, x_k)$ in which the inner vertices $x_1, \ldots, x_{k-1}$ have degree two in G. A graph H is a *subdivision* of G if H is obtained from G by replacing edges of G by suspended paths, that is to say, H is obtained by adding vertices on the edges of G.

Hajós conjectured that every graph G with $\chi(G) \geq r$ contains a subdivision of K_r as a subgraph. This conjecture is trivial for $r = 2, 3$, and it is confirmed by Dirac (1952) for $r = 4$, while it remains open for $r = 5, 6$. Catlin (1979) disproved the conjecture for $r \geq 7$ by a constructive proof, but the following disproof for general cases by Erdős and Fajtlowicz (1981) is more powerful. Let $\gamma(G)$ denote the largest r such that G contains a subdivision of K_r as a subgraph. Hajós conjecture is equivalent to that $\gamma(G) \geq \chi(G)$.

Theorem 4.6 *Almost all graphs $G \in \mathcal{G}(n, 1/2)$ satisfy*

$$\chi(G) \geq \frac{n}{2\log_2 n}, \quad \textit{and} \quad \gamma(G) \leq \sqrt{6n}.$$

Proof. Set $k = \lfloor 2\log_2 n \rfloor$. Since the probability that there exists an independent set of size at least k satisfies

$$\Pr(\alpha(G) \geq k) \leq \binom{n}{k} 2^{-\binom{k}{2}} < \left(\frac{e\sqrt{2}n}{k2^{k/2}}\right)^k = o(1)$$

and the fact $\alpha(G)\chi(G) \geq n$ for any graph G, the first statement follows immediately.

In the following, we focus on the second inequality. Set $r = \lceil\sqrt{6n}\rceil$. Clearly $n \leq r^2/6$. Note that there are

$$\binom{n}{r} \leq \left(\frac{en}{r}\right)^r \leq \left(\frac{er}{6}\right)^r$$

potential K_r subdivisions, one for each r-element subset of $V(G)$. Fix such a subset X, we have that each subdivided edge of X must use a vertex of $V(G) \setminus X$ and no two subdivided edge use the same vertex. Since there are $\binom{r}{2}$ suspended paths in a subdivision, and at most $n - r$ of them are of length two or more, which are "really" subdivided edges, it follows that the number of edges induced by X is at least

$$\binom{r}{2} - (n - r) \geq \binom{r}{2} + r - \frac{r^2}{6} \geq \frac{2}{3}\binom{r}{2}.$$

Note that the number of edges induced by X, denoted by $e(X)$, has binomial distribution $B(N, 1/2)$, where $N = \binom{r}{2}$. From Theorem 3.12,

$$\Pr(e(X) \geq N(1+\delta)/2) \leq \exp\{-N\delta^2/2\}.$$

By taking $\delta = 1/3$ hence $\frac{2}{3}\binom{r}{2} = \binom{r}{2}(1+\delta)/2$, we obtain

$$\Pr\left(e(X) \geq \frac{2}{3}\binom{r}{2}\right) \leq \exp\left\{-N\delta^2/2\right\} = \exp\left\{-\frac{1}{18}\binom{r}{2}\right\}.$$

Thus, the probability that the random graph G contains a subdivision of K_r can be upper bounded as follows.

$$\begin{aligned}\Pr(\gamma(G) \geq r) &\leq \sum_X \Pr\left(e(X) \geq \frac{2}{3}\binom{r}{2}\right) \leq \binom{n}{r}\exp\left\{-\frac{1}{18}\binom{r}{2}\right\} \\ &\leq \left(\frac{r^2\exp\{-(r-1)/36\}}{6}\right)^r,\end{aligned}$$

which tends to zero as n tends to infinity. □

From the above result, Hajós conjecture failed badly since *asymptotical almost surely* (a.a.s.) the graph G in $\mathcal{G}(n, 1/2)$ satisfies

$$\chi(G) - \gamma(G) \geq \frac{n}{2\log_2 n} - \sqrt{6n} \to \infty$$

as $n \to \infty$. Furthermore, the gap between the truth and the conjecture is large.

In the following, we give a lower bound for $r(3, n)$ due to Erdős (1961), which is anther application of Chernoff bounds (Theorem 3.15). The following proof by Conlon (Lectures on graph Ramsey theory) was written in a much different way from the original one.

Theorem 4.7 *There exists a constant $c > 0$ such that for all large n,*

$$r(3,n) \geq c\left(\frac{n}{\log n}\right)^2.$$

Proof. Let $N = c(\frac{n}{\log n})^2$. Let $p = \frac{a \log n}{n}$, where a will be chosen later. We color the edges of K_N red with probability p and blue with probability $1 - p$. This graph may have many red triangles. However, let E be a minimal set of red edges which, if recolored blue, would give a triangle-free red graph. It suffices to show that with high probability this recolored graph contains no blue K_n. Let R_E be the subgraph formed by the recolored red edges.

Applying Theorem 3.13 with $p = \frac{a \log n}{n}$ and $\delta = p$, we obtain the probability that there exists a vertex of degree greater than $2pN$ is at most $Ne^{-N\delta^2/(3pq)}$, which tends to zero as n goes to infinity. In the remaining of the proof, all probabilities should be calculated conditional upon this event. However, for convenience, we will ignore this complication by assuming that there are no vertices of degree greater than $2pN$.

Let V denote the vertex set of K_N. For any given n-subset $W \subseteq V$, let A_W be the event that the red subgraph induced by W has an edge xy which is not contained in any red triangle xyz with $z \in V \setminus W$. The critical thing to notice is that if a graph satisfies A_W, then any maximal triangle-free subgraph H of the red graph R_E (formed by recoloring edges) has blue complement which is not monochromatic on W. To see this, suppose that xy is a recolored blue edge in W. Since H is maximal, the graph $H + xy$ must contain a red triangle xyz. But then, by property A_W, z must be in W. So the assertion follows by noting that xz and yz are red. Therefore, we are done if the event $\cap_W A_W$ occurs with positive probability, where the intersection is taken over all W of size n.

We will try and estimate the probability $\Pr(\overline{A}_W)$, where W is a subset of V of size n. If we can show that $\Pr(\overline{A}_W) \leq n^{-n}$, we will be done, since there are only $\binom{N}{n} < (\frac{eN}{n})^n = (\frac{ecn}{\log^2 n})^n$ sets W of size n. We will prove the required inequality in two steps. First, we will show that with high probability, most pairs in W have no common neighbors outside W. Then we shall prove that any given large set of pairs of vertices from W must contain an edge.

Let $d_i = e^{2i}pn/i$ and $N_i = n/e^{2i}$. Let P_i be the probability that at least N_i vertices in $V \setminus W$ have at least d_i neighbors in W.

Claim For all $1 \leq i \leq \log n$, the probability $P_i \leq n^{-2n-1}$.

Proof. Let $d_W(z)$ be the degree of the vertex $z \in V \setminus W$ in W. For each $z \in V \setminus W$, the variable $d_W(z)$ satisfies binomial distribution $B(n, p)$. So Theorem 3.15 implies

$$\Pr\left(d_W(z) \geq e^{2i}pn/i\right) \leq \left(\frac{npe}{e^{2i}pn/i}\right)^{e^{2i}pn/i} = \left(\frac{e^{2i-1}}{i}\right)^{-e^{2i}pn/i} \leq e^{-e^{2i}pn}.$$

Thus

$$P_i \leq \binom{N}{N_i} e^{-(e^{2i}a\log n)N_i} < n^{3ne^{-2i}-an} < n^{-2n-1}$$

as claimed. □

Therefore, adding over all $1 \leq i \leq \log n$, we see that with probability at least $1 - n^{-2n}$, there are at most N_i vertices in $V \setminus W$ which have d_i neighbors in W. Moreover, note that, for $i_0 = (\log n - \log\log n)/2$, $d_{i_0} > 2pN$ provided $c > 0$ is small. Our assumption that all vertices have degree at most $2pN$ therefore implies that there are no vertices with degree d_{i_0} in W. Note that the number of vertices in $V \setminus W$ have at most d_1 neighbors in W is at most $|V \setminus W|$. Hence, the number of pairs of vertices in W which share a neighbor in $V \setminus W$ is at most

$$\begin{aligned} N\binom{d_1}{2} + \sum_{i=2}^{i_0-1} N_{i-1}\binom{d_i}{2} &\leq c\left(\frac{n}{\log n}\right)^2 50a^2\log^2 n + 10a^2 n\log^2 n \sum_{i=2}^{i_0-1}\frac{e^{2i}}{i^2} \\ &\leq 50a^2cn^2 + 20a^2 n\log^2 n\left(\frac{n}{\log n}\right)4(\log n)^{-2} \\ &\leq 50a^2cn^2 + \frac{80a^2n}{\log n} \end{aligned}$$

which may be made as small as any δn^2, for c sufficiently small depending on a and δ. Therefore, for c small, at least $(1-\delta)\binom{n}{2}$ of the edges in W do not have common neighbors in $V \setminus W$.

Note that the event $\overline{A}_W$ appears means that all edges of W share a common neighbor $z \in V \setminus W$. In order to force $\overline{A}_W$, those edges in W having no common neighbor in $V \setminus W$ should not appear. But, for $\delta = 1/2$ and $a = 12$, there are at least $n^2/6$ such edges, and so the probability that all of these edges don't appear is at most

$$(1-p)^{n^2/6} \leq e^{-pn^2/6} = e^{-2n\log n} = n^{-2n}.$$

Note that, since the edges within W and the edges between $V \setminus W$ and W are independent, this latter probability is independent of each of the P_i. Therefore,

$$\Pr(\overline{A}_W) \leq n^{-2n} + n^{-2n} < n^{-n},$$

completing the proof. □

4.4 Properties of Random Graphs

A random graph is obtained by starting with a set of n vertices and adding edges between them at random. Different random graph models produce different probability distributions on graphs, for which the model in this text is classic. Erdős and Rényi (1960) showed that for many monotone-increasing properties of random graphs, graphs of size slightly less than a certain threshold are very unlikely to have the property, whereas graphs with a few more edges are almost certainly to have it. This is known as a phase transition. The second section is devoted to this topic, and the last section covers some deeper discussion. The reader who is just concerned with Ramsey theory could skip this chapter.

4.4.1 Some Behaviors of Almost All Graphs

Given a graph property A, it is often associated with a family Q of graphs as

$$Q = Q(A) = \{G : G \text{ has } A\}.$$

Slightly abusing notation, we do not distinguish the property A and the family Q if there is no danger of confusion. We say that the graphs in $\mathcal{G}(n, p)$ asymptotically almost surely (a.a.s.) have property Q if

$$\lim_{n\to\infty} \Pr[G_p \in Q] = 1.$$

In this case we also say that a.a.s. $G_p \in \mathcal{G}(n, p)$ has property Q. We begin with a classical result of Erdős (1962) which states that almost all graphs seem to behave strangely even though they are sparse. In the following, for a graph G on vertex set V and $S \subseteq V$, we denote $G[S]$ by the subgraph of G induced by S.

Theorem 4.8 *For any $k \geq 1$, there exist positive constants $c = c(k)$ and $\epsilon = \epsilon(k)$ such that the graphs in $\mathcal{G}(n, p)$ with $p = c/n$ a.a.s. satisfy that $\chi(G) \geq k$, and yet $\chi(G[S]) \leq 3$ for any vertex subset S with $|S| \leq \epsilon n$.*

Proof. Let

$$H(x) = -\log\left(x^x(1-x)^{1-x}\right), \qquad 0 < x < 1,$$

and let c and ϵ be positive constants satisfying

$$c > 2k^2 H(1/k) \quad \text{and} \quad c^3 e^5 \epsilon < 3^3. \tag{4.2}$$

Set $p = c/n$ and Consider the random graph $G = G_p$ in $\mathcal{G}(n, p)$. We will show that a.a.s. the graphs in this space satisfy the conditions. If $\alpha(G) \geq n/k$, then $\chi(G) \leq k$. Note that the probability that there exists an independent set of size at least n/k can be upper bounded by

$$\binom{n}{n/k}(1-p)^{\binom{n/k}{2}}.$$

From Stirling formula, we estimate that

$$\binom{n}{n/k} = \frac{n!}{(n/k)!(n-n/k)!} \le \exp\{nH(1/k)\},$$

and

$$(1-p)^{\binom{n/k}{2}} \le \exp\left\{-\frac{pn}{2k}\left(\frac{n}{k}-1\right)\right\} = \exp\left\{-\frac{cn}{2k^2}(1-o(1))\right\}.$$

Therefore,

$$\binom{n}{n/k}(1-p)^{\binom{n/k}{2}} \le \exp\left\{-n\left(\frac{c}{2k^2} - H\left(\frac{1}{k}\right) - o(1)\right)\right\},$$

which tends to zero from (4.2).

Now, suppose that there exists some set S with at most ϵn vertices satisfying that $\chi(G[S]) \ge 4$. Set $t = |S|$, we claim that $G[S]$ would have at least $3t/2$ edges. Suppose that S is a minimal such set. For any $v \in S$, there would be a (proper) 3-coloring of $S \setminus \{v\}$. If v has two or fewer neighbors in $G[S]$ then it would be extended to a 3-coloring of S. Hence the minimum degree of $G[S]$ is at least 3 and the claim follows. The probability that some $t \le \epsilon n$ vertices have at least $3t/2$ edges is less than

$$\sum_{4 \le t \le \epsilon n} \binom{n}{t}\binom{\binom{t}{2}}{3t/2}\left(\frac{c}{n}\right)^{3t/2}.$$

Note that

$$\binom{n}{t} \le \left(\frac{en}{t}\right)^t \quad \text{and} \quad \binom{\binom{t}{2}}{3t/2} \le \left(\frac{et}{3}\right)^{3t/2},$$

so we obtain that each term of the sum is at most

$$\left(\frac{en}{t}\right)^t \left(\frac{et}{3}\right)^{3t/2} \left(\frac{c}{n}\right)^{3t/2} = \left(\frac{c^{3/2}e^{5/2}t^{1/2}}{3^{3/2}n^{1/2}}\right)^t.$$

Hence

$$\sum_{4 \le t \le n^{1/4}} \binom{n}{t}\binom{\binom{t}{2}}{3t/2}\left(\frac{c}{n}\right)^{3t/2} \le n^{1/4}\left(\frac{c^{3/2}e^{5/2}n^{1/4}}{3^{3/2}n^{1/2}}\right)^4 = o(1).$$

Moreover,

$$\sum_{n^{1/4} < t \le \epsilon n} \binom{n}{t}\binom{\binom{t}{2}}{3t/2}\left(\frac{c}{n}\right)^{3t/2} \le \epsilon n \left(\frac{c^{3/2}e^{5/2}}{3^{3/2}}\epsilon^{1/2}\right)^{n^{1/4}} = o(1)$$

by noting (4.2). Thus we have that a.a.s. no such set S exists, which completes the proof. □

From the above theorem, we know that in random graphs, the neighbors of average number vertices distribute evenly in every part of the vertex set. So their clique numbers and independence numbers are relatively small, while their chromatic numbers are large. For a graph G, the girth $g(G)$ is the smallest length of a cycle in G. A historic result of Erdős (1959) states that both of $\chi(G)$ and $g(G)$ can be arbitrarily large.

Theorem 4.9 *For any fixed ℓ and k, there exists a graph G such that $g(G) > \ell$ and $\chi(G) > k$.*

Proof. Fix $0 < \theta < 1/\ell$, and let $p = n^{\theta-1}$. Consider the random graph G in $\mathcal{G}(n, p)$. Let $X = X(G)$ be the number of cycles of length at most ℓ in G. Note that the automorphism group $\mathcal{A}(C_m) = 2m$ (see e.g. Godsil and Royle (2001)). Therefore,

$$E(X) = \sum_{i=3}^{\ell} \frac{(n)_i}{2i} p^i \le \sum_{i=3}^{\ell} \frac{n^{\theta i}}{2i} = o(n)$$

as $\theta\ell < 1$, where $(n)_i$ is the falling factorial $n(n-1)\cdots(n-i+1)$. On the other hand,

$$E(X) = \sum_i i \Pr(X = i) \ge \frac{n}{2} \Pr(X \ge n/2),$$

which implies that $\Pr(X \ge n/2) = o(1)$ since $E(X) = o(n)$.

Set $m = 3n^{1-\theta} \log n$. It is easy to see that

$$\Pr(\alpha(G) \ge m) \le \binom{n}{m}(1-p)^{\binom{m}{2}} < \left(ne^{-p(m-1)/2}\right)^m = o(1).$$

Thus, there exists a graph G of large order n such that $X(G) < n/2$ and $\alpha(G) < m$. By deleting a vertex from each cycle of length at most ℓ, we obtain a graph G^* of order at least $n/2$, which satisfies $g(G^*) > \ell$ and $\alpha(G^*) < m$, which implies that

$$\chi(G^*) \ge \frac{|V(G^*|}{\alpha(G^*)} \ge \frac{n/2}{m} \ge \frac{n^\theta}{6\log n} > k,$$

completing the proof. □

4.4.2 Parameters of Random Graphs★

We are ready to discuss some parameters of random graph G_p for fixed p. It is easy to see some parameters are concentrated around their expectations. The following result was due to Shamir and Spencer (1987).

Theorem 4.10 *If $G_p \in \mathcal{G}(n, p)$, then*

$$\Pr\left(|\chi(G_p) - E(\chi(G_p))| > \lambda\sqrt{n-1}\right) < 2e^{-\lambda^2/2}.$$

Proof. Consider the vertex exposure martingale $X_1, \ldots, X_n$ on $\mathcal{G}(n, p)$ with the parameter $\chi(G)$. A single vertex can always be given a new color so Azuma's Inequality can apply. □

Similarly, we have

$$\Pr\left(|\omega(G_p) - E(\omega(G_p))| > \lambda\sqrt{n-1}\right) < 2e^{-\lambda^2/2},$$

and

$$\Pr\left(|e(G_p) - E(e(G_p))| > \lambda\sqrt{m}\right) < 2e^{-\lambda^2/2},$$

where $m = \binom{n}{2}$. However, the proofs give no clue that what are these expectations.

Lemma 4.1 *Let* $0 < p < 1$, $a = 1/p$ *and* $\epsilon > 0$ *be fixed, and* $f(x) = \binom{n}{x} p^{\binom{x}{2}}$ *for* $0 \le x \le n$. *Define a positive integer* k *such that*

$$f(k-1) > 1 \ge f(k).$$

Then as $n \to \infty$, $\lceil \omega_n - \epsilon \rceil \le k \le \lfloor \omega_n + \epsilon \rfloor + 1$ *where*

$$\omega_n = 2\log_a n - 2\log_a \log_a n + 2\log_a(e/2) + 1,$$

and $f(k-4) > c(\frac{n}{\log_a n})^3 = n^{3-o(1)}$ *where* $c > 0$ *is a constant.*

Proof. It is easy to see that $k \to \infty$ and $k = o(\sqrt{n})$, thus by Stirling formula, we have

$$f(k) = \binom{n}{k} p^{\binom{k}{2}} \sim \frac{n^k}{k!} p^{k(k-1)/2} \sim \frac{1}{\sqrt{2\pi k}} \left(\frac{en}{k} p^{(k-1)/2}\right)^k.$$

So if $\delta > 0$ is fixed, then for all large n,

$$\frac{en}{k} p^{(k-1)/2} \le 1 + \delta$$

as $f(k) \le 1$. This is equivalent to that

$$k \ge 2\log_a n - 2\log_a k + 2\log_a e + 1 - 2\log_a(1+\delta).$$

Let us set $k \sim 2\log_a n$ first. Note that the difference between the right hand side in the above inequality and ω_n is

$$2\log_a \frac{2\log_a n}{k} - 2\log_a(1+\delta) \to -2\log_a(1+\delta),$$

so $k - \omega_n \ge -2\log_a(1+\delta) + o(1) \ge -\epsilon$ if we take δ small enough. Hence $k \ge \omega_n - \epsilon$.

Similarly, from

$$f(k-1) \sim \frac{1}{\sqrt{2\pi(k-1)}} \left(\frac{en}{k-1} p^{(k-2)/2}\right)^{k-1},$$

we have $\frac{en}{k-1}p^{(k-2)/2} \geq 1$, which gives

$$k \leq 2\log_a n - 2\log_a(k-1) + 2\log_a e + 2.$$

Furthermore, by taking $k \sim 2\log_a n$ first, we obtain $k \leq \omega_n + 1 + o(1) \leq \omega_n + \epsilon + 1$, the desired upper bound for k follows.

Finally, note that

$$f(k-2) > \frac{f(k-2)}{f(k-1)} = \frac{k-1}{n-k+2}a^{k-2} \sim p^2\frac{k}{n}a^k > \frac{c\,n}{\log n},$$

the assertion for $f(k-4)$ follows immediately. □

Lemma 4.2 *For fixed* $0 < p < 1$, $a = 1/p$ *and* $\epsilon > 0$, *a.a.s.* $G_p \in \mathcal{G}(n, p)$ *satisfies that*

$$\omega(G_p) < \lfloor \omega_n + \epsilon \rfloor < 2\log_a n,$$

where $\omega_n = 2\log_a n - 2\log_a \log_a n + 2\log_a(e/2) + 1$ *is the same defined as in Lemma 4.1.*

Proof. Let X_r be the number of r-cliques. Then

$$E(X_r) = f(r) = \binom{n}{r}p^{\binom{r}{2}} \leq \frac{n^r}{r!}p^{r(r-1)/2} < \frac{1}{\sqrt{2\pi r}}\left(\frac{en}{r}p^{(r-1)/2}\right)^r.$$

We shall find some $r = r(n) \to \infty$ such that $E(X_r) \to 0$. This is certainly true if $enp^{(r-1)/2}/r \leq 1$ (hence $r \to \infty$). The same argument in the proof of Lemma 4.1 applies that if $r = \lceil \omega_n + \epsilon \rceil$, then $E(X_r) \to 0$, thus $\Pr[\omega(G_p) \geq r] \to 0$ and $\Pr[\omega(G_p) \leq \lfloor \omega_n + \epsilon \rfloor] \to 1$. □

Note that the above result can be stated as

$$\Pr\left(\omega(G_p) \leq \lceil \omega_n + \epsilon \rceil - 1\right) \to 1 \text{ as } n \to \infty.$$

For a property Q, we say that graphs in $\mathcal{G}(n, p)$ *asymptotically almost surely* (a.a.s.) have property Q if

$$\lim_{n\to\infty} \Pr[G_p \in Q] = 1.$$

Matula (1970, 1972, 1976) was the first to notice that for fixed values of p a.a.s. $G_p \in \mathcal{G}(n, p)$ have clique numbers concentrated on (at most) two values,

$$\lfloor \omega_n - \epsilon \rfloor \leq \omega(G_p) \leq \lfloor \omega_n + \epsilon \rfloor.$$

Results asserting this phenomenon were proved by Grimmett and McDiarmid (1975); and these were further strengthened by Bollobás and Erdős (1976).

In order to reduce the difficulty of the proof and preserve the typical flavor, we slightly weaken the above lower bound $\lfloor \omega_n - \epsilon \rfloor$ by having its asymptotical form a little bit later. Let us discuss the chromatic numbers first. A technical lemma is as follows.

Lemma 4.3 *Let k be the integer defined in Lemma 4.1 and let $\ell = k-4$. Let Y be the maximum size of a family of edge-disjoint cliques of size ℓ in $G \in \mathcal{G}(n,p)$. We have*

$$E(Y) \geq \frac{cn^2}{\ell^4},$$

where $c > 0$ is a constant.

Proof. Let $\mathcal{L}$ denote the family of ℓ-cliques of G. By Lemma 4.1, we have

$$\mu = E(|\mathcal{L}|) = f(\ell) = \binom{n}{\ell} p^{\binom{\ell}{2}} \geq c_1 \left(\frac{n}{\ell}\right)^3.$$

Let W denote the number of unordered pairs $\{A, B\}$ of ℓ-cliques of G with $A \sim B$, where $A \sim B$ signifies that $2 \leq |A \cap B| < \ell$. Let

$$\Delta = \sum_{A \sim B} \Pr(AB),$$

where the sum is taken over all ordered pairs $\{A, B\}$. Thus $E(W) = \Delta/2$ and

$$\Delta = \binom{n}{\ell} \sum_{i=2}^{\ell-1} \binom{\ell}{i}\binom{n-\ell}{\ell-i} p^{2\binom{\ell}{2}-\binom{i}{2}} = \mu \sum_{i=2}^{\ell-1} \binom{\ell}{i}\binom{n-\ell}{\ell-i} p^{\binom{\ell}{2}-\binom{i}{2}} = \mu \sum_{i=2}^{\ell-1} R_i.$$

Setting $a = 1/p$, we have

$$\frac{R_{i+1}}{R_i} = \frac{(\ell-i)^2}{(i+1)(n-2\ell+i+1)} a^i.$$

If i is small, say bounded, then this ratio is $O((\log_a n)^2/n)$, and if i is large, say $\ell - i = O(1)$, then the ratio is at least $\sqrt{n}$. It is increasing on i, so

$$\Delta = \mu \sum_{i=2}^{\ell-1} R_i \leq 2\mu(R_2 + R_{\ell-1}),$$

where

$$R_2 = \binom{\ell}{2}\binom{n-\ell}{\ell-2} p^{\binom{\ell}{2}-1} = \frac{\ell^2(\ell-1)^2}{2p(n-\ell+2)(n-\ell+1)}\mu \leq \frac{\ell^4}{2pn^2}\mu,$$

and

$$R_{\ell-1} = \ell(n-\ell)p^{\binom{\ell}{2}-\binom{\ell-1}{2}} \leq n\ell p^{\ell-1}.$$

Thus

$$\Delta \leq 2\mu\left(\frac{\ell^4}{2pn^2}\mu + n\ell p^{\ell-1}\right) \leq C\,\frac{\mu^2\ell^4}{n^2}.$$

Let C be a random subfamily of $\mathcal{L}$ defined by setting for each $A \in \mathcal{L}$,

$$\Pr[A \in C] = p_1,$$

where $0 < p_1 < 1$ will be determined. Then $E(|C|) = \mu p_1$. Let W' be the number of unordered pairs $\{A, B\}$ of ℓ-cliques in C with $A \sim B$. Then

$$E(W') = E(W)p_1^2 = \frac{\Delta p_1^2}{2}.$$

Delete from C one set from each such pair $\{A, B\}$. This yields a set C^* of edge-disjoint ℓ-cliques of G and

$$E(Y) \geq E(|C^*|) \geq E(|C|) - E(W') = \mu p_1 - \frac{\Delta p_1^2}{2}.$$

By choosing $p_1 = \frac{\mu}{\Delta} < 1$, we have

$$E(Y) \geq \frac{\mu^2}{2\Delta} \geq \frac{c\,n^2}{\ell^4}$$

as asserted. □

Theorem 4.11 (Bollobás) *Let $0 < p < 1$, $a = 1/p$ be fixed, and let $m = \lceil n/\log_a^2 n \rceil$. Then a.a.s. $G_p \in \mathcal{G}(n, p)$ satisfies that each induced subgraph of order m of G_p has a clique of size at least $r = 2\log_a n - 7\log_a \log_a n$.*

Proof. Let S be an m-set of vertices. We shall bound the probability that S induces no r-clique by $e^{-m^{1+\delta}}$ for all large n (hence all large m), where $\delta > 0$ is a constant. So the probability that there exists an m-set with no r-clique is at most

$$\binom{n}{m} e^{-m^{1+\delta}} < \left(\frac{en}{m}\right)^m e^{-m^{1+\delta}} = \exp\left(m \log_e \frac{en}{m} - m^{1+\delta}\right),$$

which goes to zero, and the assertion follows.

Let X be the maximum number of pairwise edge-disjoint r-cliques sets in this graph (induced by S), where *edge-disjoint* means they share at most one vertex. We shall show that a.a.s. $X \geq 1$ holds. To do this, we invoke Azuma's Inequality. Consider the edge exposure martingale for X that results from revealing G one-edge slot at a time. We have $X_0 = E(X)$ and $X_{\binom{m}{2}} = X$. Clearly the Lipschitz condition $|X_{i+1} - X_i| \leq 1$ is satisfied, so Azuma's Lemma gives

$$\begin{aligned}\Pr(X = 0) &\leq \Pr[X - E(X) \leq -E(X)]\\ &= \Pr\left[X - E(X) \leq -\lambda \binom{m}{2}^{1/2}\right]\\ &\leq e^{-\lambda^2/2} = \exp\left(-\frac{E^2(X)}{m(m-1)}\right),\end{aligned}$$

where $\lambda = E(X)/\binom{m}{2}^{1/2}$. Hence it suffices to find $\delta > 0$ such that $E^2(X) \geq m^{3+\delta}$ for all large n.

To this end, let t_0 be the integer such that $f(t_0 - 1) > 1 \geq f(t_0)$, where $f(x) = \binom{m}{x} p^{\binom{x}{2}}$, and let $t = t_0 - 4$. Then by Lemma 4.1, we have

$$t \geq 2\log_a m - 2\log_a \log_a m - 3 > 2\log_a n - 7\log_a \log_a n,$$

so $t > r$. Let T be the maximum number of edge-disjoint cliques of size t, Then $E(X) \geq E(T)$ and $E(T) \geq cm^2/t^4$ by Lemma 4.3, hence

$$E(X) \geq \frac{cm^2}{t^4} \sim \frac{cn^2}{16(\log_a n)^8},$$

implying that $E^2(X) \geq n^{4-o(1)} \geq n^{3+\delta}$ for any $1 > \delta > 0$ if n is large, which completes the proof. □

Theorem 4.12 (Bollobás) *Let* $0 < p < 1$ *and* $\epsilon > 0$ *be fixed. Denote* $b = 1/q = 1/(1-p)$. *Then a.a.s.* $G_p \in \mathcal{G}(n, p)$ *satisfies that*

$$\frac{n}{2\log_b n} \leq \chi(G_p) \leq (1+\epsilon)\frac{n}{2\log_b n}.$$

Proof. The lower bound holds because a.a.s. $G_p \in \mathcal{G}(n, p)$ satisfies that $\alpha(G_p) \leq 2\log_b n$ and $\chi(G)\alpha(G) \geq n$. The upper bound follows from the above theorem, which is applied for independent sets instead of cliques, because we can almost always select independent set of size $2\log_b n - 7\log_b \log_b n$ until we have only $n/\log_b^2 n < (\epsilon/2)n/(2\log_b n)$ vertices left. We first use at most

$$\frac{n}{2\log_b n - 7\log_b \log_b n} < \left(1 + \frac{\epsilon}{2}\right)\frac{n}{2\log_b n}$$

colors, and then we can complete the coloring by using distinct new colors on each of the remaining vertices. □

Let us remark that Achlioptas and Naor (2005) obtained a result on sparser random graphs as follows. Given $d > 0$, let k_d be the smallest integer k such that $d < 2k\log k$. Then $\chi(G_p)$ for almost all $G_p \in \mathcal{G}(n, d/n)$ is either k_d or $k_d + 1$. This result improves an earlier result of Łuczak (1991) by specifying the form of k_d.

Theorem 4.13 *Let* $0 < p < 1$ *and* $\epsilon > 0$ *be fixed. Then a.a.s.* $G_p \in \mathcal{G}(n, p)$ *satisfies that*

$$(1-\epsilon)2\log_b n \leq \alpha(G_p) < 2\log_b n.$$

Proof. The upper bound follows from Lemma 4.2, and the lower bound follows from Theorem 4.12 and the fact that $\alpha(G) \geq n/\chi(G)$. □

Theorem 4.14 *Let* $0 < p < 1$ *and* $\epsilon > 0$ *be fixed. Then a.a.s.* $G_p \in \mathcal{G}(n, p)$ *satisfies that*

$$(1-\epsilon)2\log_a n \leq \omega(G_p) < 2\log_a n.$$

Proof. This is the complement of Theorem 4.13. □

For some graph parameter $f(G)$, we have seen that there is a function $g(n)$ such that a.a.s. $G_p \in \mathcal{G}(n, p)$ satisfies that

$$(1-\epsilon)g(n) \le f(G_p) \le (1+\epsilon)g(n),$$

hence $f(G)$ concentrate in a small range. We shall call the function $g(n)$ a *threshold* for the parameter f. We will discuss the threshold for probability $p = p(n)$ instead of fixed p, and will consider some other graph parameters in the next chapter.

4.4.3 Threshold Functions

For fixed $0 < p \le 1$, most graphs in $\mathcal{G}(n, p)$ are dense. Bollobás (1988) proved that the chromatic numbers $\chi(G_p)$ for $G_p \in \mathcal{G}(n, p)$ are concentrated at $n/(2\log_{1/q} n)$, where $q = 1 - p$. In this section, we investigate the concentration of edge probability function $p = p(n)$ associated with a property. We will see that random graphs in $\mathcal{G}(n, p)$ behave sensitively on $p = p(n)$. A monumental discovery of Erdős and Rényi (1960) was that many natural graph theoretic properties become true in a very narrow range of $p = p(n)$.

A property Q is said to be *monotone increasing* if G has property Q implies that any graph from G by adding some new edges also has Q. The *monotone decreasing* property can be defined similarly. Thus the property of being connected is monotone increasing and that of being triangle-free is monotone decreasing. Recall that a property Q is associated with a family of graphs. We say that this family of graphs is monotone increasing if so is the property Q. Also we do not distinguish the property and its associated family.

Lemma 4.4 *Let Q be a monotone increasing property. For $G_p \in \mathcal{G}(n, p)$, the function* $\Pr(G_p \in Q)$ *is increasing on p.*

Proof. Let $0 \le p_1(n) < p_2(n) \le 1$. We shall verify

$$\Pr(G_{p_1} \in Q) \le \Pr(G_{p_2} \in Q).$$

Set $p = (p_2 - p_1)/(1 - p_1)$, then $p_2 = p + p_1 - pp_1$. Choose $G \in \mathcal{G}(n, p)$ and $G_1 \in \mathcal{G}(n, p_1)$, independently, and set $G_2 = G \cup G_1$. Namely G_2 is a graph on vertex set $V = [n]$ with edge set $E(G) \cup E(G_1)$, in which each edge e appears with probability

$$\Pr(e) = \Pr(e \in E(G) \cup E(G_1)) = p + p_1 - pp_1 = p_2$$

since the events that e appears in $E(G)$ and in $E(G_1)$ are independent. Thus G_2 is exactly a random graph of $\mathcal{G}(n, p_2)$. As Q is monotone increasing, we have that if G_1 has Q then so does G_2, and thus $\Pr(G_{p_1} \in Q) \le \Pr(G_{p_2} \in Q)$ as claimed. □

Let Q be a monotone increasing property. Erdős and Rényi defined a function $f(n)$ with $0 \le f(n) \le 1$ as a *threshold function* for Q if

$$\lim_{n\to\infty} \Pr(G_p \in Q) = \begin{cases} 0 \text{ if } & p = f(n)/\omega(n), \\ 1 \text{ if } & p = f(n)\omega(n), \end{cases}$$

where $0 < \omega(n) < 1/f(n)$ is a function which tends to infinity, as slowly as desired. For example, if $f(n) = \frac{\log n}{n}$, we may assume that $0 < \omega(n) < \log\log n$. Note that if $f(n)$ is a threshold function for Q, then so is $cf(n)$ for any constant $c > 0$.

Clearly, the definition of $f(n)$ being a threshold function for a monotone increasing property Q is equivalent to

$$\lim_{n\to\infty} \Pr(G_p \in Q) = \begin{cases} 0 \text{ if } & p \ll f(n), \\ 1 \text{ if } & p \gg f(n), \end{cases}$$

where $p \ll f(n)$ means $p = o(f(n))$.

For obvious reason, the above threshold function is in fact a *threshold probability function*. One can certainly define other threshold functions such as the threshold edge function.

The definition of the threshold function for a monotone decreasing property is similar. The definitions mean that whether or not G_p having a property Q changes suddenly even though $p = p(n)$ changes slightly in the moment.

Let $X = X(G)$ be a non-negative integral parameter of G. Since

$$\Pr(X \ge 1) = \sum_{k\ge 1} \Pr(X = k) \le E(X),$$

it follows that $E(X) \to 0$ implies that a.a.s. graphs in $\mathcal{G}(n,p)$ satisfy $X = 0$. And in many cases $E(X) \to \infty$ implies that a.a.s. graphs in $\mathcal{G}(n,p)$ satisfy $X \ge 1$, which can be shown by Chebyshev's inequality often. For example, let X be the number of triangles in $G_p \in \mathcal{G}(n,p)$. Then

$$E(X) = \binom{n}{3} p^3 \sim \frac{1}{6}(np)^3.$$

As we will see in the next theorem that $f(n) = 1/n$ truly is a threshold function for triangle-containedness. Let $p = \gamma/n$ and let $\gamma \to 0$ or $\gamma \to \infty$ signify $\omega(n)$ in the denominator or in the numerator in the definition, respectively. When γ reaches and passes 1, the structure of G_p changes radically. This is called the *double jump* because the structure of G_p changes radically for $\gamma \ll 1$, $\gamma \sim 1$ and $\gamma \gg 1$.

Let us recall the Second Moment Method in the last chapter.

Lemma 4.5 (Second Moment Method) *If X is a random variable, then*

$$\Pr(X = 0) \le \frac{E(X^2) - \mu^2}{\mu^2},$$

where $\mu = E(X)$. *In particular,* $\Pr(X = 0) \to 0$ *if* $E(X^2)/\mu^2 \to 1$.

A graph G with average degree d is called *balanced* if no subgraph of it has average degree greater than d. Complete graphs, cycles and trees are all balanced.

Theorem 4.15 *Let F be a balanced graph with $k \geq 2$ vertices and $\ell \geq 1$ edges. If Q is a property that a graph contains F as a subgraph, then $n^{-k/\ell}$ is a threshold function for Q.*

Proof. To simplify the notation as before, we shall use $p = \frac{\gamma}{n^{k/\ell}}$ with $\gamma \to 0$ and $\gamma \to \infty$ to signify the function $\omega(n)$ in the denominator and numerator, respectively. Let $X = X(G_p)$ be the number of copies of F contained in $G_p \in \mathcal{G}(n, p)$. Denote by a for the number of graphs isomorphic to F on fixed k labeled vertices. As F has ℓ edges, we have

$$\mu = E(X) = \binom{n}{k} a p^{\ell}.$$

By noting the simple facts that $1 \leq a \leq k!$ with k and ℓ fixed, we have $E(X) \leq n^k p^{\ell} = \gamma^{\ell}$ and $E(X) \sim \frac{a}{k!} n^k p^{\ell} = \frac{a}{k!}\gamma^{\ell}$. Thus

$$c_1 \gamma^{\ell} \leq c_2 n^k p^{\ell} \leq \mu = E(X) \leq \gamma^{\ell},$$

where c_1 and c_2 henceforth c_i are positive constants.

When $\gamma \to 0$ as $n \to \infty$, by Markov's inequality,

$$\Pr(G_p \in Q) = \Pr(X \geq 1) \leq E(X) = o(1).$$

It remains to show that $\Pr(G_p \in Q) = \Pr(X \geq 1) \to 1$ when $\gamma \to \infty$. We turn to the Second Moment Method for help since the Markov's inequality does not work in this case.

For any k labeled vertices in $[n]$, we have $a = k!/|\mathcal{A}|$, where $\mathcal{A}$ is the automorphism group of F. Hence there are $a\binom{n}{k}$ potential copies of F on $[n]$. Let

$$\mathcal{F} = \{F_1, F_2, \ldots\}$$

denote the family of these copies. Denote by $F_i \cup F_j$ for the graph with vertex set $V(F_i) \cup V(F_j)$ and edge set $E(F_i) \cup E(F_j)$. The two critical observations are that most pairs F_i and F_j have no vertices in common, and if they have $s \geq 1$ common vertices and these s vertices contains t edges of F_j, then $t/s \leq \ell/k$ since F is balanced.

Let X_i be the indicator function of F_i. Then

$$E(X_i) = \Pr(X_i = 1) = \Pr(G_p \supseteq F_i).$$

Since $X = \sum_i X_i$ and $X_i^2 = X_i$, we have

$$E(X^2) = \mu + \sum_{i \neq j} E(X_i X_j) = \mu + \sum_{i \neq j} \Pr(G_p \supseteq F_i \cup F_j),$$

where the sum is taken over all ordered pairs i and j with $F_i, F_j \in \mathcal{F}$. Set

$$A_0 = \sum_{i \neq j\colon E(F_i) \cap E(F_j) = \emptyset} \Pr(G_p \supseteq F_i \cup F_j),$$

and for $s \geq 1$,

$$A_s = \sum_{i \neq j} \left\{ \Pr(G_p \supseteq F_i \cup F_j) : \ |V(F_i) \cap V(F_j)| = s, E(F_i) \cap E(F_j) \neq \emptyset \right\}.$$

We have $E(X^2) = \mu + \sum_{s=0}^{k} A_s$. Note that if $E(F_i) \cap E(F_j) = \emptyset$, then

$$\Pr(G_p \supseteq F_i \cup F_j) = \Pr(G_p \supseteq F_i) \Pr(G_p \supseteq F_j)$$

from the independency of the events. Thus,

$$\begin{aligned} A_0 &= \sum_{V(F_i) \cap V(F_j) = \emptyset} \Pr(G_p \supseteq F_i \cup F_j) \\ &= \sum_{V(F_i) \cap V(F_j) = \emptyset} \Pr(G_p \supseteq F_i) \Pr(G_p \supseteq F_j) \\ &\leq \Big(\sum_i \Pr(G_p \supseteq F_i) \Big) \Big(\sum_j \Pr(G_p \supseteq F_j) \Big) \\ &= E^2(X) = \mu^2. \end{aligned}$$

For $s \geq 1$, it is expected that A_s is much less than μ^2. Fix F_i, counting F_j that has s common vertices with F_i, in which these s common vertices contain t edges of $E(F_i) \cap E(F_j)$ with $t \leq s\ell/k$ since F is balanced, we have

$$\begin{aligned} \sum_{j\colon |V(F_i) \cap V(F_j)| = s} \Pr(G_p \supseteq F_i \cup F_j) &\leq \sum_{t \leq s\ell/k} \binom{k}{s} \binom{n-k}{k-s} p^{2\ell - t} \\ &\leq c_3 n^{k-s} \sum_{t \leq s\ell/k} p^{2\ell - t} \end{aligned}$$

since k, s, ℓ are fixed and t is bounded. From the fact that there are $a\binom{n}{k}$ elements in $\mathcal{F}$, we obtain

$$\begin{aligned} A_s &\leq a \binom{n}{k} c_3 n^{k-s} \sum_{t \leq s\ell/k} p^{2\ell - t} \leq c_4 n^{2k-s} \sum_{t \leq s\ell/k} p^{2\ell - t} \\ &\leq c_4 (n^k p^\ell)^2 n^{-s} \sum_{t \leq s\ell/k} p^{-t} \leq \frac{c_5 \gamma^{2\ell} n^{-s}}{p^{s\ell/k}} \\ &= \frac{c_5 \gamma^{2\ell}}{(n p^{\ell/k})^s} = \frac{c_5 \gamma^{2\ell}}{\gamma^{s\ell/k}} \leq \frac{c_6 \mu^2}{\gamma^{s\ell/k}}, \end{aligned}$$

where we used the fact that $n^k p^\ell$, γ^ℓ and μ have the same order. So for $s \geq 1$, we have $A_s/\mu^2 \leq c_6/\gamma^{s\ell/k}$, and

$$\frac{E(X^2)}{\mu^2} = \frac{\mu + A_0 + \sum_{s=1}^{k} A_s}{\mu^2} \leq 1 + \frac{1}{\mu} + \sum_{s=1}^{k} \frac{c_6}{\gamma^{s\ell/k}} \leq 1 + o(1) + \frac{c_7}{\gamma^{\ell/k}}.$$

Now the Second Moment Method (see Lemma 3.16) yields that

$$\Pr(X = 0) \leq \Pr(|X - \mu| \geq \mu) \leq \frac{\sigma^2}{\mu^2} = \frac{E(X^2) - \mu^2}{\mu^2} \leq o(1) + \frac{c_7}{\gamma^{\ell/k}},$$

which tends to zero as $\gamma \to \infty$. □

The original definition of threshold function $f(n)$ of Erdős and Rényi comes from $p_1(n) = f(n)/\omega(n)$ and $p_2(n) = f(n)\omega(n)$ hence $\sqrt{p_1(n)p_2(n)}$ is a threshold function, where $p_1/p_2 \to 0$. In many cases, it is just needed that p_1 is slightly less than p_2. A more precise definition of threshold function is as follows.

Let Q be a monotone increasing property of graphs. A function $p_\ell = p_\ell(n)$ is called a *lower threshold function* (ltf) if almost no graphs in $\mathcal{G}(n, p_\ell)$ have Q, and a function $p_u = p_u(n)$ is called an *upper threshold function* (utf) if almost all graphs in $\mathcal{G}(n, p_u)$ have Q.

A realistic situation is very interesting. In a conference, a pair of mathematicians unknown each other can found a common mathematician friend. The distance between two vertices x and y in G is the length of a shortest path between them. The diameter of a graph G, denoted $diam(G)$, is the greatest distance between two vertices of G. The following result called *distance two theorem* gives us a good explanation for this small world phenomenon.

Theorem 4.16 *For any function $\omega(n) \to \infty$ with $\omega(n) < \log n$, set*

$$p_\ell = \sqrt{\frac{2\log n - \omega(n)}{n}}, \quad and \quad p_u = \sqrt{\frac{2\log n + \omega(n)}{n}}.$$

Then p_ℓ and p_u are ltf and utf for the property of graph having diameter two, respectively.

Proof. Enumerate of all pairs of vertices $\{u, v\}$ of $\mathcal{G}(n, p)$ as $e_1, e_2, \ldots, e_m$ with $m = \binom{n}{2}$. For $e_k = \{u, v\}$, let $d(u, v)$ be the distance between u and v. Define

$$X_k = \begin{cases} 0 & d(u, v) \leq 2, \\ 1 & \text{otherwise}, \end{cases}$$

and $X = \sum_{k=1}^{m} X_k$. A non-complete graph G has distance two if and only if $X = 0$. Since the event $d(u, v) \geq 3$ for a pair of non-adjacent vertices is equivalent to that none of other $n - 2$ vertices is adjacent to both u and v, it follows that

$$E(X_k) = \Pr(X_k = 1) = (1 - p)(1 - p^2)^{n-2}.$$

Set $\mu = E(X)$, then

$$\mu = E(X) = \binom{n}{2}(1-p)(1-p^2)^{n-2}.$$

(i) Let $p = p_u = \sqrt{(2\log n + \omega(n))/n}$.

Note that

$$\mu \sim \frac{n^2}{2}(1-p^2)^n \sim \frac{n^2}{2}e^{-np^2} = \frac{1}{2}e^{-\omega(n)} = o(1).$$

Thus $\Pr(X \geq 1) \leq E(X) \to 0$, i.e., a.a.s. graphs in $\mathcal{G}(n,p)$ have diameter at most two since almost no graph in $\mathcal{G}(n,p)$ is complete.

(ii) Let $p = p_\ell = \sqrt{(2\log n - \omega(n))/n}$.

Suppose that $\omega(n) < \log\log n$ without loss of generality. Consider

$$E(X^2) = \sum_{i,j} E(X_iX_j) = A_0 + A_1 + A_2,$$

where $A_s = \sum_{|e_i\cap e_j|=s} E(X_iX_j)$, i.e., the sum in which is taken over all pairs $\{i,j\}$ with e_i and e_j having s vertices in common. Clearly

$$\mu = E(X) \sim \frac{n^2}{2}(1-p^2)^n \sim \frac{n^2}{2}e^{-np^2} = \frac{e^{\omega(n)}}{2},$$

which will tend to infinity since $\omega(n)$ tends to infinity, and

$$A_0 = \sum_{|e_i\cap e_j|=0} E(X_iX_j) \leq \binom{n}{2}\binom{n-2}{2}(1-p)^2(1-p^2)^{2(n-2)} < \mu^2.$$

Moreover,

$$A_2 = \sum_{k=1}^{m} E(X_k) = \mu.$$

We now estimate A_1 that should not be big since A_0 counts most of pairs. For $e_i = \{u,v\}$ and $e_j = \{v,w\}$ with $|e_i \cap e_j| = 1$, we consider the probability of the event $d(u,v) \geq 3$ and $d(v,w) \geq 3$. Applying Lemma 3.14 with $\delta = p/4$, we have that

$$\Pr\left(|N(v)| < \frac{3pn}{4}\right) \leq \exp\left\{-\frac{np}{48q}\right\} < n^{-4}.$$

This means that $|N(v)| \geq \frac{3pn}{4}$ with high probability. Furthermore, the event $d(u,v) \geq 3$ and $d(v,w) \geq 3$ implies that both u and w are not adjacent to any vertex of $N[v]$. Therefore, for fixed u, v and w,

$$\begin{aligned}
&\Pr(d(u,v) \geq 3 \text{ and } d(v,w) \geq 3) \\
&\leq n^{-4} + (1-p)^{3pn/2} \leq n^{-4} + e^{-3np^2/2} \\
&= n^{-4} + e^{-3\log n + 3\omega/2} = n^{-4} + n^{-3}e^{3\omega/2}.
\end{aligned}$$

It follows that

$$A_1 = \sum_{|e_i \cap e_j|=1} E(X_i X_j) \leq 3\binom{n}{3} \cdot \left(n^{-4} + n^{-3}e^{3\omega/2}\right) < 1 + 2\mu^{3/2}.$$

Hence

$$\sigma^2 = E(X^2) - \mu^2 = A_0 + A_1 + A_2 - \mu^2 < 1 + 2\mu^{3/2} + \mu,$$

which and the Second Moment Method yield

$$\Pr(X = 0) \leq \Pr(|X - \mu| \geq \mu) \leq \frac{\sigma^2}{\mu^2} = o(1),$$

proving that a.a.s. no graph in $\mathcal{G}(n, p)$ has diameter two with $p = p_\ell$. □

The further solution for diameter of random graphs is as follows. Let $d \geq 2$ be an integer. If

$$p = n^{1/d-1}(\log(n^2/x))^{1/d},$$

then

$$\Pr(\mathrm{diam}(G_p) = d) \to e^{-x/2},$$

and

$$\Pr(\mathrm{diam}(G_p) = d+1) \to 1 - e^{-x/2}$$

as $n \to \infty$. See Bollobás (2001) for details. The above limit distribution implies that

$$p_{\ell,u} = n^{1/d-1}(2\log n \pm \omega(n))^{1/d}$$

are ltf and utf of graphs being diameter d, respectively.

The diameter two graphs are of interest in graph Ramsey theory. Recall that a graph F is a Ramsey graph for $r(G, H)$ if F is of order $r(G, H) - 1$ such that F contains no copy of G and its complement $\overline{F}$ contains no copy of H. Let G_n be a Ramsey graph of order $n = r(3, k) - 1$. Suppose that G_n is edge maximal for triangle-freeness. Then G_n must be a graph with diameter two. Since the order of n is $k^2/\log k$, the maximum degree of G_n is upper bounded by $k \leq c\sqrt{n \log n}$. It is likely that the minimum degree of G_n has order $\sqrt{n \log n}$ hence the order of its edge density is $\sqrt{\log n/n}$ as that in the above theorem.

Problem 4.1 Let G_n be a Ramsey graph of order $n = r(3, k)-1$ that is edge maximal. Determine the orders of the minimum and maximum degrees of G_n as $k \to \infty$.

The following result gives threshold functions for the property of being connected. A deeper version of the result will be given in the next section.

Theorem 4.17 *Let $\omega(n) \to \infty$ be a function with $\omega(n) < \log n$. Set*

$$p_\ell = \frac{\log n - \omega(n)}{n} \quad \textit{and} \quad p_u = \frac{\log n + \omega(n)}{n}.$$

Then p_ℓ and p_u are ltf and utf for graphs in $\mathcal{G}(n, p)$ with the property of being connected, respectively.

Proof. Let Q be the family of connected graphs. Since Q is monotone increasing, we may assume that $\omega(n) \le \log\log n$ without loss of generality by Lemma 4.4. Let $X_k = X_k(G)$ be the number of components of $G \in \mathcal{G}(n, p)$ that have exactly k vertices.

(i) We first prove that $p = p_\ell$ is a ltf for Q. Set $\mu = E(X_1)$. Note that $(1-p)^n \sim e^{-np}$ since $np^2 \to 0$, it follows that

$$\mu = E(X_1) = n(1-p)^{n-1} \sim ne^{-np} = e^{\omega(n)} \to \infty$$

as n tends to infinity. This may indicate that $\Pr(X_1 = 0) \to 0$, which implies that a.a.s. graphs have isolated vertices and hence they are disconnected. To this end, we will use the Second Moment Method. We need to estimate the variance $\sigma^2 = \sigma^2(X_1)$ hence $E(X_1^2)$. First, note that

$$E[X_1(X_1 - 1)] = n(n-1)(1-p)^{2n-3},$$

which is the expected number of ordered pairs of isolated vertices. Indeed, there are $n(n-1)$ ordered pairs of vertices, and the vertices of each pair are isolated if and only if they are neither adjacent each other nor adjacent to any other $n-2$ vertices, which count $2n-3$ edges. Hence

$$E(X_1^2) = E[X_1(X_1 - 1)] + E(X_1) = \mu + n(n-1)(1-p)^{2n-3}.$$

We thus have

$$\begin{aligned}\sigma^2 = \sigma^2(X_1) &= E[(X_1 - \mu)^2] = E(X_1^2) - \mu^2 \\ &= \mu + n(n-1)(1-p)^{2n-3} - n^2(1-p)^{2n-2} \\ &\le \mu + pn^2(1-p)^{2n-3}.\end{aligned}$$

Since $p = (\log n - \omega(n))/n$ with $\log\log n \ge \omega(n) \to \infty$ and $1 - p \le e^{-p}$, we obtain

$$\begin{aligned}pn^2(1-p)^{2n-3} &\le (1 + o(1))(\log n)ne^{-2np} \\ &= (1 + o(1))(\log n)ne^{-2\log n + 2\omega(n)} \\ &= (1 + o(1))\frac{\log n}{n}e^{2\omega(n)} \to 0.\end{aligned}$$

Thus

$$\sigma^2 = \sigma^2(X_1) \le \mu + 1.$$

This and the Second Moment Method give

$$\Pr(G_p \in Q) \le \Pr(X_1 = 0) \le \Pr(|X_1 - \mu| \ge \mu) \le \frac{\sigma^2}{\mu^2} \to 0,$$

proving that p_ℓ is a ltf for property of being connected.

(ii) Now let $p = p_u = (\log n + \omega(n))/n$. Note that if G is not connected, then it must contains a component of order at most $\lfloor n/2 \rfloor$. So

$$\Pr(G_p \notin Q) = \Pr\left(\sum_{k=1}^{\lfloor n/2 \rfloor} X_k \ge 1\right) \le E\left(\sum_{k=1}^{\lfloor n/2 \rfloor} X_k\right) = \sum_{k=1}^{\lfloor n/2 \rfloor} E(X_k).$$

Note that if a set with k vertices induces a component, then any vertex in it is not adjacent to any vertex out of it. Thus

$$E(X_k) \le \binom{n}{k}(1-p)^{k(n-k)},$$

where we ignore the condition that the set is connected. Therefore,

$$\Pr(G_p \notin Q) \le \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{k}(1-p)^{k(n-k)}.$$

Let us split the sum into two parts S_1 and S_2. Note that $e^{kp} \le 1 + \epsilon$ uniformly for $k \le n^{3/4}$, and $ne^{-np} = e^{-\omega(n)/n}$, we have

$$\begin{aligned} S_1 = \sum_{1 \le k \le n^{3/4}} \binom{n}{k}(1-p)^{k(n-k)} &\le \sum_{1 \le k \le n^{3/4}} \left(\frac{en}{k} e^{-np} e^{kp}\right)^k \\ \le \sum_{1 \le k \le n^{3/4}} \left((1+\epsilon)\frac{e^{1-\omega(n)}}{k}\right)^k &\le \sum_{1 \le k \le n^{3/4}} \left((1+\epsilon)e^{1-\omega(n)}\right)^k \\ \le (1+o(1))(1+\epsilon)e^{1-\omega(n)} \to 0. \end{aligned}$$

Note that for $n^{3/4} < k \le n/2$, we have $\binom{n}{k} \le (en/k)^k \le (en^{1/4})^k$ and

$$(1-p)^{k(n-k)} \le (1-p)^{kn/2} \le e^{-knp/2} < \frac{1}{n^{k/2}},$$

hence

$$S_2 = \sum_{n^{3/4} < k \le \lfloor n/2 \rfloor} \binom{n}{k} (1-p)^{k(n-k)}$$
$$\le \sum_{n^{3/4} < k \le n/2} \left(\frac{e}{n^{1/4}} \right)^k \le (1+o(1)) \left(\frac{e}{n^{1/4}} \right)^{n^{3/4}} \to 0.$$

Thus $S_1 + S_2 \to 0$, proving that a.a.s. graphs in $\mathcal{G}(n,p)$ are connected. □

4.4.4 Poisson Limit

In probability theory, we say that a random variable X has *Poisson distribution* if it takes non-negative integral values and $\Pr(X = k) = \frac{\mu^k}{k!} e^{-\mu}$ for some constant $\mu > 0$, which is the expectation of X (and the variance of X). An elementary fact states that if $X = \sum_{i=1}^n X_i \sim B(n,p)$ and $np \to \mu$ as $n \to \infty$, then $\Pr(X = k) \to \frac{\mu^k}{k!} e^{-\mu}$. This is because for fixed k,

$$\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \sim \binom{n}{k} p^k (1-p)^n$$
$$\sim \frac{n^k}{k!} p^k e^{-np} \to \frac{\mu^k}{k!} e^{-\mu}.$$

In the last section, in order to show that a.a.s. graphs in $\mathcal{G}(n,p)$ are disconnected for $p = (\log n - \omega(n))/n$, we in fact have proved that a.a.s. graphs in $\mathcal{G}(n,p)$ have isolated vertices. Let X be the number of isolated vertices in G_p of $\mathcal{G}(n,p)$, where $p = (\log n + x)/n$. Then $X = \sum_{i=1}^n X_i$, where X_i is the indicator of the ith vertex being isolated. Define $p' = \Pr(X_i = 1)$. Then

$$p' = (1-p)^{n-1} \sim \exp(-np) = \frac{e^{-x}}{n} = \frac{\mu}{n},$$

where $\mu = e^{-x}$, and so $np' \to \mu$. The distribution of X is close to $B(n, p')$ in the sense that $X_1, X_2, \ldots, X_n$ are "almost" mutually independent, so we are expecting that X has limit Poisson distribution.

The approach to the Poisson paradigm introduced in this section is called *Brun's sieve* for its user T. Brun in number theory. Let us begin with a basic identity called *inclusion-exclusion formula*.

In a probability space Ω, let $X_1, X_2, \ldots, X_\ell$ be 0-1 random variables and set

$$X = X_1 + X_2 + \cdots + X_\ell.$$

As usual, denote by $[\ell]$ for $\{1, 2, \ldots, \ell\}$. Define $S_0 = 1$ and

$$S_r = \sum_{\{i_1,\ldots,i_r\} \in \binom{[\ell]}{r}} \Pr(X_{i_1} X_{i_2} \cdots X_{i_r} = 1),$$

where $\binom{[\ell]}{r}$ denotes the set that consists of all r-subsets of $[\ell]$. Note that the elements of Ω satisfy $X_{i_1} X_{i_2} \cdots X_{i_r} = 1$ if and only if $X_{i_1} = 1, X_{i_2} = 1, \ldots, X_{i_r} = 1$, and $S_r = 0$ for $r > \ell$. For general r,

$$S_r = \sum_{\omega \in \Omega} \binom{X(\omega)}{r} \Pr(\omega)$$

as an element ω of the sample space for which $X(\omega) = t$ contributes $\binom{t}{r}$ terms of the defined S_r above. Here and in what follows, we write the formulas appropriate for finite sample space. Following the standard notation, we define the falling factorials by $(X)_0 = 1$ and

$$(X)_r = X(X-1)\cdots(X-r+1).$$

Then

$$S_r = \sum_{\omega \in \Omega} \frac{(X)_r}{r!} \Pr(\omega) = \frac{E((X)_r)}{r!}.$$

The quantity $E((X)_r)$ is called the rth *factorial moment* of X.

Theorem 4.18 (Inclusion-Exclusion Formula) *For each integer $k \geq 0$,*

$$\Pr(X = k) = \sum_{r \geq 0} (-1)^r \binom{k+r}{r} S_{k+r}.$$

Moreover, for each integer $m \geq 0$,

$$\sum_{r=0}^{2m-1} (-1)^r \binom{k+r}{r} S_{k+r} \leq \Pr(X = k) \leq \sum_{r=0}^{2m} (-1)^r \binom{k+r}{r} S_{k+r}.$$

Proof. It is easy to see

$$\begin{aligned} \Pr(X = 0) &= \Pr(X_1 = 0, \ldots, X_\ell = 0) \\ &= 1 - \Pr(\exists\, i, X_i = 1) = S_0 - S_1 + S_2 - \cdots + (-1)^\ell S_\ell. \end{aligned}$$

For general k, using

$$S_{k+r} = \sum_{\omega \in \Omega} \binom{X(\omega)}{k+r} \Pr(\omega),$$

and interchanging orders of the summation, we obtain

$$\sum_{r \geq 0} (-1)^r \binom{k+r}{r} S_{k+r} = \sum_{\omega \in \Omega} \left\{ \sum_{r \geq 0} (-1)^r \binom{k+r}{r} \binom{X(\omega)}{k+r} \right\} \Pr(\omega).$$

For a fixed ω hence fixed $X = X(\omega)$, note that

$$\sum_{r\geq 0}(-1)^r\binom{k+r}{r}\binom{X}{k+r}=\binom{X}{k}\sum_{r\geq 0}(-1)^r\binom{X-k}{r}.$$

If $X<k$, then all terms vanish. If $X=k$, then one term ($r=0$) contributes and the sum is 1. Finally, if $X>k$, the sum vanishes since

$$\sum_{r\geq 0}(-1)^r\binom{X-k}{r}=(1-1)^{X-k}=0.$$

Thus

$$\sum_{r\geq 0}(-1)^r\binom{k+r}{r}\binom{X}{k+r}=\begin{cases}1 \text{ if } X=k,\\ 0 \text{ otherwise,}\end{cases}$$

and

$$\sum_{r\geq 0}(-1)^r\binom{k+r}{r}S_{k+r}=\sum_{\omega\in\Omega}\delta_{k\,X(\omega)}\Pr(\omega)=\Pr(X=k),$$

where δ_{ij} is the Kronecker delta. To verify the second result, note for all $s\geq 0$ and $n\geq 1$,

$$\sum_{r=0}^{s}(-1)^r\binom{n}{r}=(-1)^s\binom{n-1}{s},$$

which can be proved easily by induction on s, hence

$$\begin{aligned}\sum_{r=0}^{s}(-1)^r\binom{k+r}{r}\binom{X}{k+r}&=\binom{X}{k}\sum_{r=0}^{s}(-1)^r\binom{X-k}{r}\\&=\begin{cases}0 & \text{if } X<k,\\ 1 & \text{if } X=k,\\ (-1)^s\binom{X}{k}\binom{X-k-1}{s} & \text{if } X>k.\end{cases}\end{aligned}$$

We thus have

$$\begin{aligned}\sum_{r=0}^{s}(-1)^r\binom{k+r}{r}S_{k+r}&=\sum_{\omega\in\Omega}\left\{\sum_{r=0}^{s}(-1)^r\binom{k+r}{r}\binom{X(\omega)}{k+r}\right\}\Pr(\omega)\\&=\sum_{X(\omega)=k}\Pr(\omega)+\sum_{X(\omega)>k}(-1)^s\binom{X}{k}\binom{X-k-1}{s}\Pr(\omega)\\&=\Pr(X=k)+(-1)^s\sum_{X(\omega)>k}\binom{X}{k}\binom{X-k-1}{s}\Pr(\omega).\end{aligned}$$

In the last line, elements ω with $X(\omega)>k$ make a positive or negative contribution depending on whether s is even or odd. □

Suppose that we have defined a sequence of probability spaces and that in the space $\Omega=\Omega_n$ we have the preceding situation with $\ell=\ell(n)$. If $E((X)_r)\rightarrow\mu^r$

as $n \longrightarrow \infty$, then we can make a precise statement about the limiting distribution of $X = \sum_{i=1}^{\ell} X_i$.

Theorem 4.19 (Poisson Limit) *Suppose that there is a positive number μ such that*

$$\lim_{n\to\infty} S_r = \frac{\mu^r}{r!},$$

equivalently $\lim_{n\to\infty} E((X)_r) = \mu^r$, *for each fixed integer* $r \geq 0$. *Then*

$$\lim_{n\to\infty} \Pr(X = k) = \frac{\mu^k}{k!} e^{-\mu}.$$

Namely, the limiting distribution of X is Poisson with mean μ.

Proof. Refer to the inequalities in the last theorem we have

$$\frac{1}{k!}\sum_{r=0}^{2m-1}(-1)^r \frac{E((X)_{k+r})}{r!} \leq \Pr(X = k) \leq \frac{1}{k!}\sum_{r=0}^{2m}(-1)^r \frac{E((X)_{k+r})}{r!}.$$

Note that for fixed m, we can make the limit (as $n \longrightarrow \infty$) term by term to get

$$\frac{1}{k!}\sum_{r=0}^{2m-1}(-1)^r \frac{\mu^{k+r}}{r!} \leq \lim_{n\to\infty} \Pr(X = k) \leq \frac{1}{k!}\sum_{r=0}^{2m}(-1)^r \frac{\mu^{k+r}}{r!}.$$

Since m is arbitrary, the result follows. □

Theorem 4.20 *For any fixed real number x, let*

$$p = \frac{\log n + x}{n},$$

and let X be the number of isolated vertices in a graph of $\mathcal{G}(n, p)$. We have

$$\lim_{n\to\infty} \Pr[X = k] = \frac{\mu^k}{k!} e^{-\mu},$$

where $\mu = e^{-x}$. In particular, the limiting probability that graph in $\mathcal{G}(n, p)$ has no isolated vertices is $\exp(-e^{-x})$.

Proof. Define X_i as the indicator that the vertex i is an isolated vertex, and define $X = \sum_{i=1}^{n} X_i$. Then X counts the number of isolated vertices in $G_p \in \mathcal{G}(n, p)$ and

$$S_1 = E(X) = n(1-p)^{n-1} \longrightarrow e^{-x}$$

as $n \longrightarrow \infty$. More generally,

$$S_r = \binom{n}{r}(1-p)^{r(n-r)+\binom{r}{2}} \sim \frac{n^r}{r!}(1-p)^{rn} \longrightarrow \frac{\mu^r}{r!},$$

where $\mu = e^{-x}$. The limiting distribution of X follows from Poisson limit theorem as desired. □

Corollary 4.2 *If* $\log n \geq \omega(n) \to \infty$ *as* $n \to \infty$*, then* $p_\ell = \frac{\log n - \omega(n)}{n}$ *and* $p_u = \frac{\log n + \omega(n)}{n}$ *are ltf and utf for graphs in* $\mathcal{G}(n,p)$ *having no isolated vertices, respectively.*

In the following, we shall show that for the same $p = (\log n + x)/n$,

$$\lim_{n\to\infty} \Pr(G_p \text{ has no isolates}) = \lim_{n\to\infty} \Pr(G_p \text{ is connected}) = \exp(-e^{-x}).$$

Thus, in almost every graph, when the last isolated vertex disappears, the graph G_p becomes connected in the evolution of random graph as x increases. Sightly before it becomes connected, a giant component with only a bounded number of vertices outside has formed. In fact, the giant component consists of larger components and the smaller components have great chances to survive.

Theorem 4.21 *For any fixed real number x, let*

$$p = \frac{\log n + x}{n},$$

and let A denote the event that outside of at most one non-trivial component, all vertices are isolated. We have

$$\lim_{n\to\infty} \Pr(A) = 1$$

and

$$\lim_{n\to\infty} \Pr[G_p \textit{ is connected}] = \exp(-e^{-x}).$$

Proof. We begin by identifying the following events in $\mathcal{G}(n,p)$.

A: Outside of at most one non-trivial component, G_p has only isolated vertices.
B: G_p has no isolated vertices.
C: G_p is connected.

Then $C = A \cap B$ and

$$\Pr(B) = \Pr(C) + \Pr(\overline{A} \cap B).$$

To prove that $\Pr(C) \to \exp(-e^{-x})$ as $n \to \infty$, it suffices to show that $\Pr(\overline{A}) \to 0$ since $\Pr(B) \to \exp(-e^{-x})$ from Theorem 4.20.

Let $X \subseteq [n]$ be the vertex set of the largest component of G and let $Y = V \setminus X$. We do not distinguish a vertex set and the subgraph induced by this set if there is no danger of confusion. If $\overline{A}$ holds, then for some $X \subseteq [n]$ with $|X| \geq 2$,

E_1: X is connected;
E_2: Y contains at least one edge;
E_3: There is no edge between X and Y.

Note that these events are independent. Denote P_X, P_Y and P_{XY} by the probabilities of the events E_1, E_2 and E_3, respectively. Let $|X| = k$ and $|Y| = m = n - k$. By distinguishing that $k \le \lfloor n/2 \rfloor$ or $m \le \lfloor n/2 \rfloor$, we have

$$\Pr(\overline{A}) \le \sum_{k=2}^{\lfloor n/2 \rfloor} \binom{n}{k} P_X P_{XY} + \sum_{m=2}^{\lfloor n/2 \rfloor} \binom{n}{m} P_Y P_{XY}. \tag{4.3}$$

To bound $\Pr(\overline{A})$, we use the following facts:

1. $P_X \le k^{k-2} p^{k-1}$;
2. $P_Y = 1 - (1-p)^{\binom{m}{2}}$;
3. $P_{XY} = (1-p)^{mk}$.

The first fact follows since X must contain one of the k^{k-2} possible spanning trees. Consider then the first term on the right hand side of (4.3), we have

$$\sum_{k=2}^{\lfloor n/2 \rfloor} \binom{n}{k} P_X P_{XY} \le \sum_{k=2}^{\lfloor n/2 \rfloor} \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)}.$$

The term corresponding to any fixed $k \ge 2$ can be bounded from above as

$$c_1 n^k p^{k-1} (1-p)^{kn} \le c_1 n^k p^{k-1} e^{-knp} = c_1 e^{-kx} p^{k-1} \to 0,$$

where c_1 and henceforth c_i are positive constants. For any $k \le n/2$,

$$(1-p)^{n-k} \le e^{-(n-k)p} \le e^{-np/2} = \frac{e^{-x/2}}{\sqrt{n}}.$$

Thus

$$\binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)} \le \left(\frac{en}{k}\right)^k k^{k-2} p^{k-1} \left(\frac{e^{-x/2}}{\sqrt{n}}\right)^k$$
$$= \frac{1}{k^2 p} \left(enp \frac{e^{-x/2}}{\sqrt{n}}\right)^k \le \frac{n}{\log n + x} \left(\frac{c_2 \log n}{\sqrt{n}}\right)^k.$$

It follows that

$$\sum_{k=2}^{\lfloor n/2 \rfloor} \binom{n}{k} P_X P_{XY} \le o(1) + \sum_{k \ge 4} \frac{n}{\log n + x} \left(\frac{c_2 \log n}{\sqrt{n}}\right)^k \to 0$$

as $n \to \infty$.

Now set

$$K = \lfloor \sqrt[4]{n} \rfloor, \qquad \text{and} \qquad M = \lceil 2\sqrt{n} \exp(1 - x/2) \rceil,$$

we shall separate the second term on the right hand side of (4.3) into three parts by K and M. Using the facts that

$$P_{XY} = (1-p)^{m(n-m)} \le e^{-m(n-m)p} \le e^{-mnp/2} = \left(\frac{e^{-x/2}}{\sqrt{n}}\right)^m$$

for $m \le \lfloor n/2 \rfloor$ and $\binom{n}{m} \le (en/m)^m$, we have

$$\sum_{m=M}^{\lfloor n/2 \rfloor} \binom{n}{m} P_Y P_{XY} \le \sum_{m=M}^{\lfloor n/2 \rfloor} \binom{n}{m} P_{XY} \le \sum_{m \ge M} \left(\frac{ene^{-x/2}}{m\sqrt{n}}\right)^m$$
$$\le \sum_{m \ge M} \frac{1}{2^m} = \frac{1}{2^{M-1}} \to 0$$

since $M \to \infty$ as $n \to \infty$. On the other hand, if $m < M$, then we have $e^{mp} \le 2$ for all large n since $mp \le Mp \to 0$. Thus

$$P_{XY} = (1-p)^{m(n-m)} \le (e^{-np} e^{mp})^m \le \left(\frac{2e^{-x}}{n}\right)^m.$$

Consequently,

$$\sum_{m=K}^{M-1} \binom{n}{m} P_Y P_{XY} \le \sum_{m=K}^{M-1} \binom{n}{m} P_{XY} \le \sum_{m \ge K} \frac{(2e^{-x})^m}{m!} \to 0$$

as $n \to \infty$ since the sum in the last line is the tail of a convergent series.

Finally, for all $m < K$,

$$P_Y \le 1 - (1-p)^{\binom{K}{2}},$$

which tends to zero uniformly on $m < K$, and it follows that

$$\sum_{m=2}^{K-1} \binom{n}{m} P_Y P_{XY} = o\left(\sum_{m=2}^{K-1} \binom{n}{m} P_{XY}\right) \le o\left(\sum_{m \ge 2} \frac{(2e^{-x})^m}{m!}\right),$$

which tends to zero. Combining these results, we obtain that $\Pr(\overline{A}) \to 0$, completing the proof. □

Corollary 4.3 *If* $\log n \ge \omega(n) \to \infty$ *as* $n \to \infty$, *then* $p_\ell = \frac{\log n - \omega(n)}{n}$ *and* $p_u = \frac{\log n + \omega(n)}{n}$ *are ltf and utf for graphs in* $\mathcal{G}(n, p)$ *of being connected, respectively.*

4.5 Exercises

1. Let $p = (1-\epsilon)\frac{\log n}{n}$. How large can m be such that almost every graph in $\mathcal{G}(n, p)$ has at least m isolated vertices?

2. For $n \ge 2$ and $p \in [0, 1]$, show that if a graph H with n vertices and $e(H) = \lfloor p\binom{n}{2} \rfloor$ or $e(H) = \lceil p\binom{n}{2} \rceil$, then H has the maximum probability to appear in $\mathcal{G}(n, p)$.

3. Show that $\chi(G_{k+1}) = \chi(G_k) + 1$ and $\omega(G_k) = 2$ for $k \geq 2$, where G_k is defined from Mycielski's construction.

4. For events $A_1, \ldots, A_n$, let $a = \sum_{i=1}^{n} \Pr(A_i)$ and $b = \sum_{i<j} \Pr(A_i \cap A_j)$. Prove that $\Pr(\overline{A}_i \cdots \overline{A}_n) \leq (a + 2b)/a^2 - 1$. (Hint: Let X be the number of A_i's that occur. Show $\Pr(X = 0) \leq a^{-2} E[(X - a)^2]$, and expand the right hand expression. (Lovász, 1979))

5. Let $X_1, \ldots, X_n$ be mutually independent 0-1 random variables such that $\Pr(X_i = 1) = p_i$, and let $X = \sum_{i=1}^{n} X_i \pmod 2$. Prove that $\Pr(X = 1) = \frac{1}{2}[1 - \Pi_i(1 - 2p_i)]$. (Hint: Set $Y_i = 1 - 2X_i$ and $Y = Y_1 \cdots Y_n$, and observe that $E(Y) = 1 - 2\Pr(Y = -1)$.

6. (i) Show that the probability that a fixed subset S of $\mathcal{G}(n, 1/2)$ is contained in the neighborhood of a vertex is $(n - |S|)2^{-|S|}$.

(ii) Show that almost every graph G in $\mathcal{G}(n, 1/2)$ has $\Delta(G) \geq n/2 + \sqrt{n}$ and $\delta(G) \leq n/2 - \sqrt{n}$.

7. (i) In $\mathcal{G}(3, 1/2)$, let $X_i(H)$ $(0 \leq i \leq 3)$ be defined as in Section 5.3, in which we reveal $G_p \in \mathcal{G}(3, 1/2)$ one edge at a time. Prove that $X_0(H)$, $X_1(H)$, $X_2(H)$, $X_3(H)$ is a martingale.

(ii) In $\mathcal{G}(3, 1/2)$, let $Y_i(H)$ $(0 \leq i \leq 3)$ be defined as in Section 5.3, in which we reveal $G_p \in \mathcal{G}(3, 1/2)$ one vertex at a time. Prove that $Y_1(H)$, $Y_2(H)$, $Y_3(H)$ is a martingale.

8. Explain that the neighbors of a vertex of a random graph are likely to be "spreading" by Theorem 4.8.

9. Let $k > 0$ be an integer, and let $p \geq (6k \log n)/n$. Prove that almost all graphs in $\mathcal{G}(n, p)$ have independence number less than $n/(2k)$.

10. Prove that if $np \to 0$ as $n \to \infty$, then almost all graphs in $\mathcal{G}(n, p)$ are forests. (Hint: Count the expected number of cycles and apply Markov's inequality.)

11. Let $f(k)$ be the minimum number of vertices in a triangle-free graph G with $\chi(G) = k$. Compare the upper bounds from different assertions as follows.

(i) From the Mycielski construction mentioned in the excises of the last chapter;

(ii) By refining the proof of Theorem 4.9;

12. Prove that $p = n^{-1/\rho(H)}$ is a threshold function for the appearance of H as a subgraph of G_p, where $\rho(H) = \max\{e(F)/|V(F)| : F \subseteq H\}$.

13.* Prove that for $d \geq 2$, if $p = n^{1/d-1}(\log(n^2/x))^{1/d}$, then $\Pr(\text{diam}(G_p) = d) \to e^{-x/2}$. (Hint: Bollobás, 2001).

Chapter 5
Lovász Local Lemma

When applying the probabilistic method, some typical ways are computing expectation, estimating tails of probability and applying Lovász Local Lemma (Lovász, born on 1948, recipient of the 1999 Wolf prize and the 2021 Abel prize), etc. In particular, the Local Lemma allows one to relax the independence condition slightly in applications, and so we can see improvements by Spencer (1977) on the lower bounds of the classic Ramsey numbers given by Erdős (1947). We will also give an overview of the Martingales and triangle-free process.

5.1 Lovász Local Lemma

In probability theory, if a large number of events are all independent of one another and each has probability less than 1, then there is a positive (possibly small) probability that none of the events will occur. Lovász Local Lemma allows one to relax the independence condition slightly, which is often used to give existence proofs. Differing from the "a.a.s." argument, we are concerned with the existence of an event of small positive probability.

Let $A_1, A_2, \ldots, A_n$ be the events in a probability space. In many combinatorial applications such as a coloring of edges of K_N, any A_i is a "bad" event. We wish that no "bad" event happens, namely

$$\Pr\left(\bigcap_{i=1}^{n} \overline{A}_i\right) > 0 \tag{5.1}$$

such that there is a point (a coloring) which is good. e.g., in the proof for the lower bound of $r(n,n)$, A_S is the event that S is monochromatic for an n-set S. The event A_S is "bad" for us. Therefore, $\cap\overline{A}_S$ is the event that none of n-sets is monochromatic, and $\Pr(\cap\overline{A}_S) > 0$ means that there must be a coloring in which no n-set induces a monochromatic K_n.

It is a trivial fact that if

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$$\sum_{i=1}^{n} \Pr(A_i) < 1,$$

then

$$\Pr\left(\bigcap_{i=1}^{n} \overline{A}_i\right) = 1 - \Pr\left(\bigcup_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} \Pr(A_i) > 0.$$

Also, if $A_1, \ldots, A_n$ are mutually independent, i.e., any A_i is independent of any Boolean function of all other A_j, and $\Pr(A_i) = x_i < 1$ for $1 \leq i \leq n$, then

$$\Pr\left(\bigcap_{i=1}^{n} \overline{A}_i\right) = \prod_{i=1}^{n} (1 - x_i) > 0.$$

The Local Lemma may be understood in terms of taking advantage of *partial independence* of the events $A_1, A_2, \ldots, A_n$ so that (5.1) can be ensured with far weaker condition on the probabilities $\Pr(A_i)$ than $\sum_{i=1}^{n} \Pr(A_i) < 1$.

The argument to follow uses *conditional probability*. Recall that for events A and B with $\Pr(B) > 0$, the conditional probability $\Pr(A|B)$ is given by $\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$.

Events A and B are called independent if $\Pr(A|B) = \Pr(A)$. So two events A and B are independent if and only if $\Pr(A \cap B) = \Pr(A)\Pr(B)$, here we admit that an event of zero probability is independent of any other event. Let us introduce a graph to describe the dependency of events as follows.

Definition 5.1 (Dependency graph) Let $A_1, A_2, \ldots, A_n$ be events in a probability space. A graph D defined on vertex set $[n]$ is called **dependency graph** for events $A_1, A_2, \ldots, A_n$ if every event A_i is independent of any Boolean function of these events in $\{A_j : j \notin N[i]\}$, where $N[i]$ is the closed neighborhood of i in D.

This graph must contain edges between the pairs of dependent events, and it contains such edges only in most applications so the term dependency graph is after. The original Local Lemma is as follows, see Erdős and Lovász (1975).

Theorem 5.1 *Suppose that $d \geq 1$ and each of the events $A_1, A_2, \ldots, A_n$ has probability p or less, and each vertex in the dependency graph has degree at most d. If*

$$4dp \leq 1,$$

then $\Pr(\cap_{i=1}^{n} \overline{A}_i) > 0$.

The following form of the Lovász Local Lemma is the general form, see Spencer (1977).

Theorem 5.2 *Let $A_1, A_2, \ldots, A_n$ be events in a probability space. If there exist real numbers $x_1, x_2, \ldots, x_n$ such that $0 < x_i < 1$ and for $i = 1, 2, \ldots, n$,*

$$\Pr(A_i) \leq x_i \Pi_{j:\, ij \in E(D)} (1 - x_j),$$

then $\Pr(\cap_{i=1}^{n} \overline{A}_i) \geq \Pi_{i=1}^{n} (1 - x_i) > 0$.

If i is an isolated vertex in the dependency graph D, i.e., the neighborhood of i in D is empty, then we admit $\Pi_{j\in\emptyset}(1-x_j)=1$.

Proof of Theorem 5.2. For $S\subset[n]$, set

$$C_S=\bigcap_{j\in S}\overline{A}_j.$$

The desired result follows directly from the following claim.

Claim If $i\notin S$, then $\Pr(A_i|C_S)\le x_i$.

Proof. The proof is by induction on $|S|$. For $|S|=0$, we admit $C_S=\Omega$, and so the assertion is immediate from the hypothesis that

$$\Pr(A_i|C_S)=\Pr(A_i|\Omega)=\Pr(A_i)\le x_i\Pi_{j:\,ij\in E(D)}(1-x_j)\le x_i.$$

Now assume that $|S|\ge 1$ and form a partition $S=(S_1,S_2)$, where

$$S_1=\{j\in S: ij\in E(D)\}\quad\text{and}\quad S_2=S\setminus S_1.$$

Let us write $\Pr(A_i|C_S)$ as

$$\frac{\Pr(A_i\cap C_S)}{\Pr(C_S)}=\frac{\Pr(A_i\cap C_{S_1}\cap C_{S_2})}{\Pr(C_{S_1}\cap C_{S_2})}=\frac{\Pr(A_i\cap C_{S_1}|C_{S_2})}{\Pr(C_{S_1}|C_{S_2})}.$$

Since A_i and C_{S_2} are independent, the numerator

$$\Pr(A_i\cap C_{S_1}|C_{S_2})\le\Pr(A_i|C_{S_2})=\Pr(A_i)\le x_i\Pi_{j\in S_1}(1-x_j).\tag{5.2}$$

In the following, we bound the denominator. If $|S_1|=0$, then

$$\Pr(C_{S_1}|C_{S_2})=\Pr(\Omega|C_{S_2})=1$$

and the claim follows. Otherwise, suppose $S_1=\{j_1,j_2,\ldots,j_r\}$, where $r\ge 1$. Let $\mathcal{D}_0,\mathcal{D}_1,\ldots,\mathcal{D}_r$ be the events defined recursively by

$$\mathcal{D}_0=C_{S_2},\quad\text{and}\quad\mathcal{D}_k=\mathcal{D}_{k-1}\cap\overline{A}_{j_k}\quad\text{for}\quad k=1,2,\ldots,r.$$

They start with $\mathcal{D}_0=C_{S_2}$ and end with $\mathcal{D}_r=C_S$. Note that for each $k=0,1,\ldots,r-1$, the event $\mathcal{D}_k$ has a form of C_T for some set $T\subseteq S\setminus\{j_r\}$. Thus $|T|<|S|$. Using the induction hypothesis on C_T repeatedly, we have

$$\begin{aligned}\Pr(C_{S_1}|C_{S_2}) &= \frac{\Pr(C_S)}{\Pr(C_{S_2})} = \frac{\Pr(\mathcal{D}_r)}{\Pr(\mathcal{D}_0)} = \frac{\Pr(\mathcal{D}_r)}{\Pr(\mathcal{D}_{r-1})} \cdots \frac{\Pr(\mathcal{D}_1)}{\Pr(\mathcal{D}_0)} \\ &= \Pr(\overline{A}_{j_r}|\mathcal{D}_{r-1}) \cdots \Pr(\overline{A}_{j_1}|\mathcal{D}_0) \\ &= (1 - \Pr(A_{j_r}|\mathcal{D}_{r-1})) \cdots (1 - \Pr(A_{j_1}|\mathcal{D}_0)) \\ &\geq (1 - x_{j_r}) \cdots (1 - x_{j_1}) \\ &= \Pi_{j\in S_1}(1 - x_j). \qquad (5.3)\end{aligned}$$

Combining (5.2) and (5.3), we have established the claim. □

Note that $\cap_{i=k+1}^n \overline{A}_i$ has a form of C_S with $k \notin S$. In view of the claim just established,

$$\begin{aligned}\Pr(\cap_{i=1}^n \overline{A}_i) &= \Pr(\overline{A}_1| \cap_{i=2}^n \overline{A}_i) \Pr(\cap_{i=2}^n \overline{A}_i) \\ &= \Pr(\overline{A}_1| \cap_{i=2}^n \overline{A}_i) \Pr(\overline{A}_2| \cap_{i=3}^n \overline{A}_i) \cdots \Pr(\overline{A}_n|\Omega) \\ &= (1 - \Pr(A_1| \cap_{i=2}^n \overline{A}_i)) \cdots (1 - \Pr(A_n|\Omega)) \\ &\geq \Pi_{i=1}^n (1 - x_i).\end{aligned}$$

This completes the proof. □

The following is the *symmetric form* of the Local Lemma.

Corollary 5.1 *Suppose that each of the events of $A_1, A_2, \ldots, A_n$ has probability p or less, and each vertex in the dependence graph has degree at most d. If $e(d+1)p \leq 1$, then* $\Pr(\cap_{i=1}^n \overline{A}_i) > 0$.

Proof. By taking $x_i = 1/(d+1)$ for $1 = 1, 2, \ldots, n$, we shall show

$$\Pr(A_i) \leq x_i \Pi_{j:\, ij\in E(D)}(1 - x_j).$$

Since $(1 - \frac{1}{d+1})^d > 1/e$ for $d \geq 1$, it follows that for any i the right side is at least

$$\frac{1}{d+1}\left(1 - \frac{1}{d+1}\right)^d > \frac{1}{e(d+1)} \geq p,$$

completing the proof. □

Note that the original condition $4dp \leq 1$ can be implied by Corollary 5.1 as $4dp \geq e(d+1)p$ for $d \geq 3$, and if $d = 1, 2$, then $\frac{1}{d+1}(1 - \frac{1}{d+1})^d \geq p$.

We also need the following form of the Local Lemma due to Spencer (1977) who used it to obtain the lower bound of $r(m, n)$. This form is slightly more convenient for some applications.

Corollary 5.2 *Let $A_1, A_2, \ldots, A_n$ be events in a probability space. If there exist numbers $y_1, y_2, \ldots, y_n$ such that for each i,* $0 < y_i \Pr(A_i) < 1$, *and*

$$\log y_i \geq - \sum_{j:\, ij\in E(D)} \log(1 - y_j \Pr(A_j)),$$

then $\Pr(\cap_{i=1}^n \overline{A}_i) > 0$.

Proof. We may suppose that for each i the probability $\Pr(A_i)$ is positive. Let x_i be as in the general form of the Local Lemma and set $y_i = x_i/\Pr(A_i)$ for $i = 1, 2, \ldots, n$. Thus the hypothesis of Lemma 5.2 that

$$\Pr(A_i) \le x_i \Pi_{j:\, ij \in E(D)} (1 - x_j)$$

will take the form

$$y_i \ge \Pi_{j:\, ij \in E(D)} \frac{1}{1 - y_j \Pr(A_j)}.$$

The assertion follows by taking logarithms on both sides of the above inequality. □

Let us have an example to explain that for the Local Lemma, the dependency graph D may contain more edges other than these connecting pairs of dependent events.

Let $\{1, 2, 3\}$ be the vertex set of a K_3, and let the probability space Ω consists of all 2-coloring of the vertices, in which each vertex is colored in red or blue with probability $1/2$ randomly and independently. Clearly $|\Omega| = 8$. For $i < j$, let A_{ij} be the event that the edge $\{i, j\}$ is monochromatic. Note that $\Pr(A_{12}) = \Pr(A_{13}) = \Pr(A_{23}) = 1/2$, and

$$\Pr(A_{12}A_{13}) = \Pr(A_{12}A_{23}) = \Pr(A_{13}A_{23}) = \frac{1}{4}.$$

Thus, events A_{12}, A_{13} and A_{23} are pairwise independent. However, A_{12} is not independent of $A_{13}A_{23}$ since $\Pr(A_{12}A_{13}A_{23}) = \frac{1}{4} \ne \frac{1}{8}$. If we mistakenly use the Local Lemma by letting $E(D) = \emptyset$, then we would set $x_{ij} = 1/2$ with $\Pr(A_{ij}) \le x_{ij} \Pi_{\emptyset}(1 - x_{k\ell})$. Thus we had a wrong conclusion that $\chi(K_3)$ were at most two from that $\Pr(\cap \overline{A}_{ij}) > 0$.

Erdős and Spencer (1991) pointed out that the dependency graph D can be replaced by a graph F on $[n]$ if F satisfies that for each i and each $S \subseteq [n] \setminus N_F[i]$,

$$\Pr(A_i | \cap_{j \in S} \overline{A}_j) \le x_i \Pi_{j:\, ij \in E(F)} (1 - x_j).$$

This condition contains conditional probabilities. To avoid to compute these probabilities in applications and to have a slightly stronger form, we shall specify their idea further. Let us have the following definitions first from Erdős and Spencer (1991).

A graph F defined on $[n]$ is called *lopsidependency graph* (which is called as *negative dependency graph* in Lu and Székely (2007)) for events $A_1, A_2, \ldots, A_n$ if for each $i \in [n]$ and any set $S \subseteq [n] \setminus N_F[i]$,

$$\Pr(A_i | \cap_{j \in S} \overline{A}_j) \le \Pr(A_i). \tag{5.4}$$

It is convenient to say that the event $\cap_{j \in S} \overline{A}_j$ for $S \subseteq [n] \setminus N_F[i]$ is negative to A_i.

Note that a dependency graph is a lopsidependency graph, but the latter may contain less edges, and the dependency graph in the Local Lemma can be replaced by any lopsidependency graph.

Lemma 5.1 (Erdős-Spencer (1991)) *Let $A_1, A_2, \ldots, A_n$ be events with lopsidependency graph F. If there exist $x_1, x_2, \ldots, x_n$ such that for each i, $0 < x_i < 1$ and*

$$\Pr(A_i) \le x_i \Pi_{j:\, ij \in E(F)} (1 - x_j), \tag{5.5}$$

then $\Pr(\cap_{i=1}^n \overline{A}_i) > 0$.

The following form is slightly more convenient for some applications.

Corollary 5.3 *Let $A_1, A_2, \ldots, A_n$ be events with lopsidependency graph F. If there exist numbers $y_1, y_2, \ldots, y_n$ such that for each i, $0 < y_i \Pr(A_i) < 1$, and*

$$\log y_i \ge - \sum_{j:\, ij \in E(F)} \log(1 - y_j \Pr(A_j)),$$

then $\Pr(\cap_{i=1}^n \overline{A}_i) > 0$.

Similar to that of Theorem 5.1, we have that the condition (5.5) can be replaced by $e(d+1)p \le 1$, where d is the maximum degree of the lopsidependency graph, and the original condition is $4dp \le 1$.

Since lopsidependency graphs are bipartite for most applications, the local lemma with lopsidependency graph will be easier to apply.

Let us remark that the dependency of events can be described by a directed graph instead of a graph. A directed graph D on vertices $[n]$ is called *directed dependency graph* for events $A_1, A_2, \ldots, A_n$ if each event A_i is mutually independent of the events in $\{A_j : j \notin N^+[i]\}$, where $N^+[i]$ is the closed out-neighborhood of i. The condition to guarantee $\Pr(\cap \overline{A}_i) > 0$ is that there exist $0 < x_1, x_2, \ldots, x_n < 1$ such that

$$\Pr(A_i) \le x_i \Pi_{j:\, (i,j) \in E(D)} (1 - x_j),$$

where (i, j) is the arc from i to j in directed dependency graph D for the events.

Similarly, the lopsidependency graph in the Local Lemma can be replaced by a directed lopsidependency graph. However, no matter using lopsidependency graph or directed dependency graph in the Local Lemma, the idea is to reduce the edges in the dependency graph.

5.2 Improved Lower Bounds for $r(m, n)$

The Local Lemma has a lot of applications in many fields. The following theorem of Spencer (1975) improves the bound of the classic Ramsey number $r(n, n)$ from deletion method by a factor $\sqrt{2}$.

Theorem 5.3 *We have $r(n, n) \ge (1 - o(1)) \frac{\sqrt{2}}{e} n 2^{n/2}$.*

Proof. Consider the random graph space $\mathcal{G}(N, 1/2)$. For any $S \subseteq V(K_N)$ of size n, let A_S signify the event that "S is monochromatic". Define a graph D with vertex set consisting of all such S and connect vertices S and T in D if and only if $|S \cap T| \geq 2$. Note that the event A_S is independent of any Boolean function of events A_T's with T not adjacent to S, and so D is a dependency graph. Thus for any S, its degree d in D satisfies that

$$d = |\{T : |S \cap T| \geq 2\}| < \binom{n}{2}\binom{N}{n-2}.$$

Applying Corollary 5.1 with $p = \Pr(A_S) = 2^{1-\binom{n}{2}}$, if $ep(d+1) < 1$, then $\Pr(\cap \overline{A}_S) > 0$. Thus $r(n, n) > N$. So it remains to find a positive integer N as large as possible such that

$$e\binom{n}{2}\binom{N}{n-2}2^{1-\binom{n}{2}} < 1.$$

As we did before, the left hand side is less than

$$\frac{en^2}{2}\left(\frac{eN}{n-2}\right)^{n-2}\frac{2}{2^{n(n-1)/2}} = \frac{en^2}{2}\left(\frac{n}{n-2}\right)^{n-2}\left(\frac{eN}{\sqrt{2}n2^{n/2}}\right)^{n-2}.$$

Indeed, for any $\epsilon > 0$, if we take $N = \left\lceil (1-\epsilon)\frac{\sqrt{2}}{e}n2^{n/2} \right\rceil$, then the above will tend to zero as n tends to infinity. □

The above improvement seems negligible in the light of the gap between the upper and lower bounds, but this is the best lower bound we can do until now. Indeed, this is not surprising since the dependencies between events in Theorem 5.3 are not rare compared to the number of events themselves. In the following, we will see the first application of the general form of the Local Lemma by Spencer (1977), which improves that obtained in Chapter 3 greatly. One can see that the dependencies between events in the following result are rare when m is fixed.

Theorem 5.4 *Let $m \geq 3$ be a fixed integer. Then*

$$r(m, n) \geq c\left(\frac{n}{\log n}\right)^{(m+1)/2},$$

where $c = c(m) > 0$ is a constant.

Proof. We give the proof for $m = 3$ and remain the general case as an exercise. Color the edges of K_N in red and blue randomly and independently such that each edge is colored red with probability p and blue with probability $q = 1 - p$, where N and p will be chosen later. For each set of three vertices T, let A_T be the event that T induced a red triangle. Similarly, for each set of n vertices S, let B_S be the event that S induced a blue K_n. It is clear that $\Pr(A_T) = p^3$ and $\Pr(B_S) = q^{\binom{n}{2}}$. Two events are dependent if and only if the corresponding subgraphs have a pair of vertices in common. Hence, each event A_T is independent of any Boolean function but at most $3(N-2) < 3N$ other $A_{T'}$ events and at most $(N)_n < N^n$ B_S events; each event B_S is independent of any Boolean function but at most $\binom{n}{2}(N-2) < n^2N/2$ A_T events and at most N^n other $B_{S'}$ events.

We aim to prove that there exist positive numbers a and b satisfying Corollary 5.3, namely, $ap^3 < 1$ and $bq^{\binom{n}{2}} < 1$ hold with $y_i = a$ for each A_T event and $y_j = b$ for each B_S event. Specifically,

$$\log a \geq -3N \log(1 - ap^3) - N^n \log(1 - bq^{\binom{n}{2}}), \tag{5.6}$$

$$\log b \geq -\frac{n^2 N}{2} \log(1 - ap^3) - N^n \log(1 - bq^{\binom{n}{2}}). \tag{5.7}$$

If such a and b are available, then there exists a red/blue edge-coloring of K_N in which there is neither red triangle nor blue K_n, implying $r(3, n) > N$. To this end, set

$$a = 2, \ b = \exp(n \log n), \quad p = \frac{8 \log n}{n}, \quad \text{and} \quad N = c \left(\frac{n}{\log n} \right)^2,$$

where $c \in (0, 1)$ is a constant to be chosen. Using the basic inequality $q = 1-p < e^{-p}$ for $p > 0$, we have

$$N^n bq^{\binom{n}{2}} \leq N^n be^{-p\binom{n}{2}} \leq \exp\{-n \log n\} = o(1).$$

So $bq^{\binom{n}{2}} = o(1)$ and $\log(1 - x) \sim -x$ for $x = bq^{\binom{n}{2}}$, and the common second term in the right-hand side of (5.6) and (5.7)

$$-N^n \log(1 - bq^{\binom{n}{2}}) \sim N^n bq^{\binom{n}{2}} = o(1).$$

Clearly $ap^3 = o(1)$ and so

$$-3N \log(1 - ap^3) \sim 3aNp^3 = o(1).$$

Thus (5.6) holds for all large n.

Finally, note that the first term of the right hand side of (5.7) is asymptotically

$$\frac{n^2 N}{2} \cdot ap^3 = 8^3 cn \log n.$$

So (5.7) holds for large n if we choose c such that

$$1 > 8^3 c. \tag{5.8}$$

The proof is complete. □

Erdős, Faudree, Rousseau and Schelp (1987), and Krivelevich (1995) generalized Spencer's lower bound from K_m to a fixed graph H. Li and Zang (2003) generalized it further to $r(H, G_n)$, where the order of G_n is n and $e(G_n) = n^{2-o(1)}$. Set

$$\rho(H) = \frac{e(H) - 1}{v(H) - 2},$$

where $v(H)$ and $e(H)$ are the order and the size of H, respectively. Sudakov (2008) proved that for fixed graph H, there exists a constant $c > 0$ depending only on H such that every graph G with m edges,

$$r(H, G) \geq c \left(\frac{m}{\log m} \right)^{\rho(H)/(1+\rho(H))}. \tag{5.9}$$

We also give a generalization as follows, see Dong, Li and Lin (2009).

Theorem 5.5 *Let H be a fixed graph with $v(H) \geq 3$, and let G_n be a graph of order n with average degree d_n. For all large n,*

$$r(H, G_n) \geq c \left(\frac{d_n}{\log d_n} \right)^{\rho(H)},$$

where $c = c(H) > 0$ is a constant.

The following gives lower bounds for $r(K_{t,s}, K_n)$ and $r(C_t, K_n)$.

Corollary 5.4 *For fixed $t, s \geq 2$, there exists a positive constant $c = c(t, s)$ (or $c = c(t)$) such that*

$$r(K_{t,s}, K_n) \geq c \left(\frac{n}{\log n} \right)^{(st-1)/(s+t-2)},$$
$$r(C_t, K_n) \geq c \left(\frac{n}{\log n} \right)^{(t-1)/(t-2)}.$$

We can apply Corollary 5.3 to simplify the calculations of (5.6) and (5.7). Indeed, we can define a lopsidependency graph F by connecting the events of different types, i.e., those A type events and B type events, that have common edges. Any pair of events of the same type are positive each other and a pair of events of different types that do not have common edges are independent. So we only need

$$\log a \geq -N^n \log(1 - bq^{e(G_n)}),$$
$$\log b \geq -e(G_n)N^{m-2} \log(1 - ap^{e(F)})$$

instead of (5.6) and (5.7).

Note that the lower bounds of (5.9) and that in Theorem 5.5 cannot replace each other. To see this, let us assume that G_n is a graph of order n, and $m = e(G_n) = \Theta(n^{1+a})$ for some constant a with $0 < a \leq 1$. Then the lower bounds for $r(F, G_n)$ given by two theorems are

$$c_1 \left(\frac{n}{(\log n)^{1/a}} \right)^{a\rho} \quad \text{and} \quad c_2 \left(\frac{n}{(\log n)^{1/(1+a)}} \right)^{\frac{(1+a)\rho}{1+\rho}},$$

respectively, where $\rho = \rho(F)$. So which bound is stronger depends on whether $a\rho > 1$.

Now, let us see another application. A *hypergraph* $\mathcal{H}$ on vertex set $V \neq \emptyset$ is a pair $(V, \mathcal{E})$, where the edge set $\mathcal{E}$ is a family of subsets of V. We say a coloring of the vertices of a hypergraph $\mathcal{H}$ is *proper* if no edge is monochromatic, and $\mathcal{H}$ is said to be k-colorable if there exists a proper k-coloring for its vertices. Using the original condition $4dp \leq 1$, Erdős and Lovász (1975) proved that an r-uniform hypergraph $\mathcal{H}$ is 2-colorable if each edge of $\mathcal{H}$ intersects at most 2^{r-3} other edges. As the first application of the Local Lemma, this result becomes a specific problem in derandomization of the Local Lemma, see e.g. Beck (1991).

Theorem 5.6 *Let $r \geq k \geq 2$ be integers. If each edge of a hypergraph $\mathcal{H}$ has at least r vertices and every edge intersects at most $k^{r-1}/4(k-1)^r$ other edges, then the vertices of $\mathcal{H}$ can be k-colored such that each color meets each edge.*

Proof. Let $\mathcal{H}$ be the hypergraph with vertex set V and edge set $\mathcal{E}$, where $V = [n]$ and $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$. Let the probability space consist of all k-colorings of V, in which each vertex is colored by one of these k colors with probability $1/k$ randomly and independently. Let A_i be the event that e_i does not receive all colors. Clearly $\Pr(A_i) \leq k(1 - 1/k)^r$. By assumption, the degree of the dependence graph of any event satisfies that $d \leq k^{r-1}/4(k-1)^r$, and thus the original condition of the Local Lemma can be applied as $4dp \leq 1$. So the assertion follows. □

Before the above-mentioned result of Erdős and Lovász, a similar result had appeared shown by basic probabilistic method as follows.

Theorem 5.7 (Erdős-Selfridge) *Let $\mathcal{H} = (V, \mathcal{E})$ be an r-uniform hypergraph. If $|\mathcal{E}| < 2^{r-1}$, then $\mathcal{H}$ is 2-colorable.*

Proof. The basic probabilistic method gives

$$\Pr(\cup A_e) \leq \sum \Pr(A_e) = \frac{|\mathcal{E}|}{2^{r-1}} < 1,$$

where A_e is the event that the edge e is monochromatic defined as that in the proof of the last theorem. □

We can improve this result slightly as follows by using lopsidependency Local Lemma. we leave it as an exercise.

Theorem 5.8 *Let $r \geq k \geq 2$ be integers. If every edge of a hypergraph $\mathcal{H}$ has at least r vertices and every edge intersects at most $k^r/4(k-1)^{r+1} - 1$ other edges, then the vertices of $\mathcal{H}$ can be k-colored such that each color meets each edge.*

For $k = 2$, the bound $k^r/4(k-1)^{r+1} - 1$ becomes $2^{r-2} - 1$ that is asymptotically twice as the bound 2^{r-3}. The bound $k^{r-1}/4(k-1)^r$ in Theorem 5.6 comes from the proof by the original condition $4dp \leq 1$, which can be improved as $k^r/e(k-1)^r - 1$ by using condition $e(d+1)p \leq 1$. Hence the bound $k^r/4(k-1)^{r+1} - 1$ can be improved as $k^r/e(k-1)^{r+1} - k/(k-1)$.

We have seen that the probabilistic method has a lot of applications with much better results than that by elementary combinatorial method. However, we shall see some other methods have much success for some topics. Let us conclude this section with two jokes given by Spencer (1994) to say that for many topics, unlike that for Turán's bound for independence number shown in previously, the probabilistic method cannot provide "exact" results often. The problem that Spencer joked is serious. In order to have verisimilitude, we write the joked results as usual in "academic language" but without indices.

The following results are due to Joker, who used the basic probabilistic method.

Theorem (Joker) *Let S and T be nonempty sets. If $|T| > \binom{|S|}{2}$, then there exists an injection $f : S \to T$.*

Proof. Consider the probability space consisting of all maps from S to T, in which each map appears randomly and independently with the same probability. For any unordered pair of points x and y of S, let A_{xy} signify the event $f(x) = f(y)$. Since for any fixed pair x and y,

$$|\{f : S \to T : f(x) = f(y)\}| = |T|^{|S|-1},$$

we have $\Pr(A_{xy}) = 1/|T|$ and

$$\Pr\left(\cup_{\{x,y\}\subseteq S} A_{xy}\right) \le \sum_{\{x,y\}\subseteq S} \frac{1}{|T|} = \frac{1}{|T|}\binom{|S|}{2} < 1,$$

which implies that $\Pr(\cap_{\{x,y\}\subseteq S}\overline{A}_{xy}) > 0$ and hence the desired injection exists. □

Later Joker amused himself by improving the above result by using the Local Lemma. The new result is tight up to a multiplicative constant.

Theorem (Joker) *Let S and T be nonempty sets. If $|T| \ge 2e|S|$, then there exists an injection $f : S \to T$.*

Proof. The same as that for Joke 1 but apply the Local Lemma. In the dependence graph, the vertex A_{xy} is adjacent to $A_{xy'}$ with $y' \in S\setminus\{y\}$ and $A_{x'y}$ with $x' \in S\setminus\{x\}$. Let $d = 2(|S| - 2)$, then the independence graph is d regular, in which the event A_{xy} is mutually independent to all non-neighbors. As $p = 1/|T|$, the condition ensures $e(d + 1)p < 1$, and so the symmetric form of the Local Lemma gives that $\Pr(\cap_{\{x,y\}\subseteq S}\overline{A}_{xy}) > 0$, implying the existence of the desired injection. □

5.3 Martingales and Triangle-Free Process*

Most parameters of a random graph are concentrated around their expectations. To describe such phenomena, martingale is a powerful tool, which may liberate us from drudgery computations.

Let X and Y be random variables on a probability space Ω. Given $Y = y$ with $\Pr(Y = y) > 0$, we define a conditional expectation $E(X|Y = y)$ as

$$E(X|Y=y) = \sum_x x \Pr(X=x|Y=y),$$

which depends on y. As Y is random, we have a new random variable

$$E(X|Y).$$

For an element $\omega \in \Omega$, if $Y(\omega) = y$, then $E(X|Y)$ takes value $E(X|Y=y)$ at ω.

Lemma 5.2 $E[E(X|Y)] = E[X]$.

Proof. From the definition, we have

$$\begin{aligned}
E[E(X|Y)] &= \sum_y E[X|Y=y] \Pr(Y=y) \\
&= \sum_y \left(\sum_x x \Pr[X=x|Y=y] \right) \Pr(Y=y) \\
&= \sum_x x \left(\sum_y \Pr[X=x|Y=y] \Pr(Y=y) \right) \\
&= \sum_x x \Pr(X=x) = E(X)
\end{aligned}$$

as asserted. □

A *martingale* is a sequence $X_0, X_1, \ldots, X_m$ of random variables such that for $0 \le i < m$,

$$E(X_{i+1}|X_i) = X_i.$$

Namely, $E(X_{i+1}|X_i = x) = x$ for any given $X_i = x$.

Imagine one walks on a line randomly, at each step he moves one unit to the left or right with probability p, or stands still with probability $1 - 2p$. Let X_i be the position of i step. This is a martingale as the expected position after $i+1$ steps equals the actual position after i steps.

Let us look at some martingales used in graph theory. The first is called the *edge exposure martingale* on chromatic numbers, in which we reveal G_p *one edge-slot at a time*. Let the random graph space $\mathcal{G}(n, p)$ be the underlying probability space. Set $m = \binom{n}{2}$, and label the potential edges on vertex set $[n]$ by $e_1, e_2, \ldots, e_m$ in any manner. We define $X_0(H), X_1(H), \ldots, X_m(H)$ for a given graph H on vertex set $[n]$, which are random variables if H is a random graph in $\mathcal{G}(n, p)$. Let $X_0(H) = E(\chi(G_p))$. For general i,

$$X_i(H) = E[\chi(G_p)|e_j \in G_p \text{ if and only if } e_j \in H, 1 \le j \le i].$$

In other words, $X_i(H)$ is the expected value of $E[\chi(G_p)]$ under the condition that the set of the first i edges of G_p equals that of H while the remaining edges are not seen and considered to be random. Note that X_0 is a constant $E(\chi(G_p))$ and $X_m = \chi(H)$.

In Fig. 4.1, the probability space is $\mathcal{G}(3, 0.5)$, so $X_0 = E(\chi(G_p)) = 2$, and $X_1(H) = 2.25$ if $e_1 \in E(H)$, and $X_1(H) = 1.75$ otherwise. Thus $E(X_1|X_0) = 2 = X_0$. The random variables X_2 and X_3 take 4 values and 8 values, respectively, and $E(X_{i+1}|X_i) = X_i$ for $i = 1, 2$. e.g.

$$E(X_2|X_1 = 2.25) = \frac{1}{4}(3 + 3 \times 2), \quad \text{and} \quad E(X_3|X_2 = 2.5) = \frac{1}{2}(3 + 2).$$

Hence this is a martingale on the random space $\mathcal{G}(3, 0.5)$.

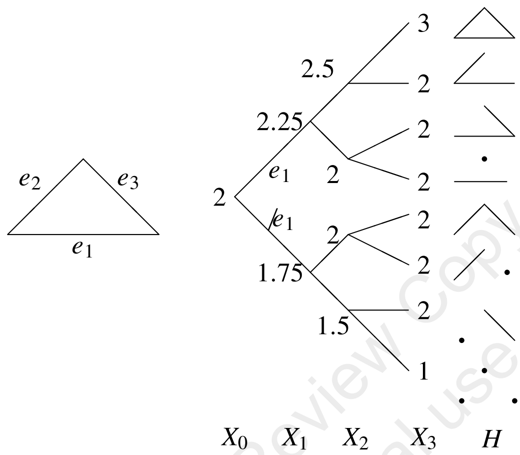

Fig. 4.1 An edge exposure martingale

The second is called the *vertex exposure martingale* on chromatic numbers, in which we reveal G_p *one vertex-slot at a time*. Let the random graph space $\mathcal{G}(n, p)$ be the underlying probability space. We define $Y_1 = E(\chi(G_p))$ and

$$Y_i(H) = E[\chi(G_p)|E_i(G_p) = E_i(H)],$$

where $E_i(H)$ is the edge set induced by the vertex set $\{1, \ldots, i\}$. In other words, $Y_i(H)$ is the expected value of $E[\chi(G_p)]$ under the condition that the set of the edges of G_p induced by the first i vertices equals that of H while the remaining edges are not seen and considered to be random. Note that Y_1 is a constant $E(\chi(G_p))$ and $Y_n = \chi(H)$. Note that the vertex exposure martingale is a subsequence of the edge exposure martingale.

In Fig. 4.2, the probability space is also $\mathcal{G}(3, 0.5)$, and $Y_1 = E(\chi(G_p)) = 2$, and $Y_2(H) = 2.25$ if $e_1 \in E(H)$, and $Y_2(H) = 1.75$ otherwise. Thus $E(Y_2|Y_1) = 2 = X_1$. The random variable Y_3 take 8 values, and similarly $E(Y_3|Y_2) = Y_2$. e.g. $E(Y_3|Y_2 = 2.25) = (3 + 2 + 2 + 2)/4 = 2.25$. Hence this is a martingale on the random space $\mathcal{G}(3, 0.5)$.

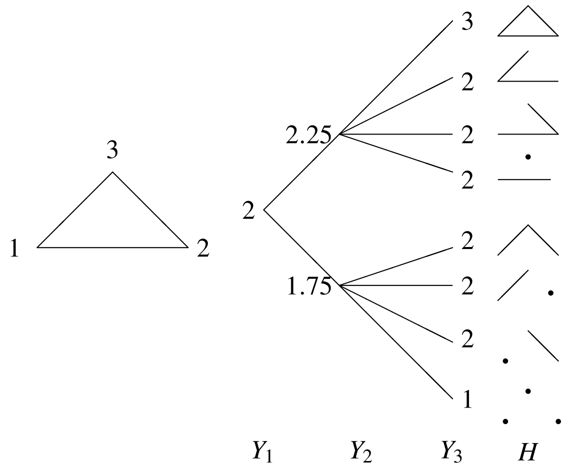

Fig. 4.2 A vertex exposure martingale

Lemma 5.3 *If X is a random variable with $E(X) = 0$ and $|X| \leq 1$, then*

$$E(e^{tX}) < e^{t^2/2}$$

for all $t > 0$.

Proof. For fixed $t \geq 0$, set

$$h(x) = \frac{e^t + e^{-t}}{2} + \frac{e^t - e^{-t}}{2}y, \qquad -1 \leq x \leq 1.$$

Note that the function $f(x) = e^{tx}$ is convex, and $h(x)$ is a line through the point $(-1, f(-1))$ and $(1, f(1))$ as $f(-1) = h(-1)$ and $f(1) = h(1)$, hence $e^{tx} \leq h(x)$, and

$$E(e^{tX}) \leq E(h(X)) = \frac{e^t + e^{-t}}{2}$$

by noting that $E(X) = 0$, and thus the assertion follows by Lemma 3.4. □

Theorem 5.9 (Azuma's Inequality) *Let $X_0, X_1, \ldots, X_m$ be a martingale with*

$$|X_{i+1} - X_i| \leq 1$$

for all $0 \leq i < m$. We have for any $\lambda > 0$,

$$\Pr[X_m - X_0 \geq \lambda\sqrt{m}] < e^{-\lambda^2/2},$$

and similarly, $\Pr[X_m - X_0 \leq -\lambda\sqrt{m}] < e^{-\lambda^2/2}$.

Proof. We may assume that $X_0 = 0$ by translation. Set $Y_i = X_i - X_{i-1}, i = 1, 2, \ldots, m$. Clearly, $|Y_i| \le 1$ and $E(Y_i|X_{i-1}, \ldots, X_0) = 0$ since $X_0, X_1, \ldots, X_m$ is a martingale from the assumption. Thus Lemma 5.3 yields that

$$E(e^{tY_i}|X_{i-1}, \ldots, X_0) < e^{t^2/2}$$

for any $t > 0$. Therefore,

$$\begin{aligned} E(e^{tX_m}) &= E\left(e^{tX_{m-1}}e^{tY_m}\right) \\ &= E\left(e^{tX_{m-1}}E(e^{tY_m}|X_{m-1}, \ldots, X_0)\right) \\ &\le e^{t^2/2}E(e^{tX_{m-1}}). \end{aligned}$$

This and the induction gave $E(e^{tX_m}) \le e^{mt^2/2}$. Using Markov's Inequality, we obtain

$$\Pr(X_m \ge \lambda\sqrt{m}) = \Pr(e^{tX_m} \ge e^{t\lambda\sqrt{m}}) \le \frac{E(e^{tX_m})}{e^{t\lambda\sqrt{m}}} \le \frac{e^{mt^2/2}}{e^{t\lambda\sqrt{m}}}.$$

Now the assertion follows by letting $t = \lambda/\sqrt{m}$. □

A function f of a graph parameter is said to satisfy the *edge Lipschitz condition* if whenever H and H' differ in only one edge then $|f(H) - f(H')| \le 1$. It satisfies the *vertex Lipschitz condition* if whenever H and H' differ in only one vertex then $|f(H) - f(H')| \le 1$. e.g., the chromatic number $\chi(G)$ satisfies both Lipschitz conditions.

Theorem 5.10 *If f satisfies the edge Lipschitz condition, then the corresponding edge exposure martingale satisfies $|X_{i+1} - X_i| \le 1$. If f satisfies the vertex Lipschitz condition, then the vertex exposure martingale satisfies $|X_{i+1} - X_i| \le 1$.*

Now we conclude this section with a simple application on the chromatic number by Shamir and Spencer (1987) by using Azuma's Inequality.

Theorem 5.11 *For the random graph $G = G(n, p)$,*

$$\Pr[|\chi(G) - \mu| > \lambda\sqrt{n-1}] < 2e^{-\lambda^2/2},$$

where $\mu = E(\chi(G))$.

Proof. Consider the vertex exposure martingale $X_1, \ldots, X_n$ on $G(n, p)$ with $f(G) = \chi(G)$. A single vertex can always be given a new color so the vertex Lipschitz condition applies. Now the assertion follows from the Azuma's Inequality of Theorem 5.9 immediately. □

When $\lambda \to \infty$ arbitrarily slowly, then this result shows that the distribution of $\chi(G)$ is "tightly concentrated" around its expectation.

It is often difficult to show the existence of small events. The Local Lemma is a tool for such proof that improved most lower bounds from basic probabilistic method.

The key for the proof of the Local Lemma (see next chapter) itself is the conditional probability. A revolutionary idea for finding the small events is also "conditional". If we know a certain condition in which the event is likely to appear, then the probability for event is large under the condition. In other word, we try to switch a small event to be a large one conditionally. However, we may encounter difficulties to finger the conditional probability out.

Obtaining the right order of magnitude of $r(m,n)$ even $r(3,n)$ was certainly a challenge in decades. A celebrated result of Kim (1995) showed that the order of $r(3,n)$ is $n^2/\log n$, which was obtained again by Bohman (2009). They used different analysis on the same random graph process, called the triangle-free process. For general constrained graph process, see, e.g., Ruciński and Wormald (1992), Erdős, Suen and Winkler (1995), Bollobás and Riordan (2000), and Osthus and Taraz (2001).

The triangle-free process can be described as follows. We begin with the empty graph, denoted by G_0, on N vertices. At step i we form the graph G_i by adding a new edge to G_{i-1} chosen uniformly at random from the collection of pairs of vertices that neither appear as edges in G_{i-1} nor form triangles when added as edges to G_{i-1}. The process terminates at a maximal triangle-free graph G_M, for which the random variable M is the number of edges of G_M. Note that a maximal triangle-free graph is connected and the number of edges in a triangle-free graph of order N is at most $N^2/4$ (see Chapter 7), we have

$$N-1 \le M \le \frac{N^2}{4}.$$

However, Bohman (2009) proved that a.a.s.

$$c_1 N^{3/2}\sqrt{\log N} \le M \le c_2 N^{3/2}\sqrt{\log N}.$$

From a result in Chapter 3, we have that the independence numbers of such graphs are at least $\Omega(\sqrt{N\log N})$. Remarkably, Kim and Bohman showed that a.a.s. the independence numbers of such graphs are at most $O(\sqrt{N\log N})$, which implies that $r(3,n) \ge \Omega(n^2/\log n)$.

Theorem 5.12 *For some constant $c > 0$,*

$$r(3,n) \ge \frac{cn^2}{\log n}.$$

Let us talk a bit more on the process employed by Bohman. For a set V, let $V^{(2)}$ be the set of all pairs u, v of V, which is the edge set of complete graph on V. The vertex set of our complete graph of order N is on $[N] = \{1, 2, \dots, N\}$. In the evolution of the triangle-free process, we shall track some random sets. Recall that G_i is the graph given by the first i edges selected by the process. The graph G_i partitions $[N]^{(2)}$ into three parts: E_i, O_i and C_i. The set E_i is simply the edge set of G_i. A pair of $[N]^{(2)}$ is open, and in the set O_i, if it can still be added as an edge without violating the triangle-free condition. A pair of $[N]^{(2)}$ is closed, and in the set C_i, if it is neither an edge in the graph nor open; that is, a pair $e = \{u, v\}$ is in C_i if $e \notin E_i \cup O_i$ and there exists a vertex w such that $\{u, w\}, \{v, w\} \in E_i$. Note that e_{i+1} is chosen uniformly at random from O_i. That is to say, each edge of O_i has the same probability $1/|O_i|$ to be chosen as e_{i+1}. We do not express this as $\Pr(e_{i+1} \in O_i) = 1/|O_i|$ since only these edges in random set O_i are available. We refer the reader to Kim (1995) and Bohman (2008) for details.

Some improvements on the constant have been obtained. Bohman and Keevash (2021), and independently Fiz Pontiveros, Griffiths and Morris (2020) improved the lower bound to

$$r(3, n) \geq \left(\frac{1}{4} - o(1)\right) \frac{n^2}{\log n}.$$

With more complicated analysis on K_4-free process, Bohman (2009) also improved the known lower bound of $r(4, n)$, and generally, Bohman and Keevash (2010) obtained that

$$r(m, n) \geq c \left(\frac{n}{\log n}\right)^{(m+1)/2} (\log n)^{1/(m-2)}, \tag{5.10}$$

which improves the lower bound of $r(m, n)$ obtained from the Local Lemma by a factor $(\log n)^{1/(m-2)}$.

5.4 Exercises

1. Use the example in the end of Section 5.1 to explain

(i) Pairwise independent events $A_1, \ldots, A_n$ are not necessarily mutually independent.
(ii) Pairwise independent events $A_1, \ldots, A_n$ with $\Pr(A_i) < 1$ may not imply that $\Pr(\cap \overline{A_i}) > 0$.

2. Using Lovász Local Lemma, give lower bounds for $r_k(K_n)$ and $r_k(K_{n,n})$, compare them with that by basic probabilistic method.

3. Prove the lower bound in Theorem 5.4 by taking

$$p = c_1 N^{-2/(m+1)}, \ n = c_2 N^{2/(m+1)} \log N,$$
$$a = c_3, \qquad b = \exp\left\{c_4 N^{2/(m+1)} \log^2 N\right\},$$

where c_i, $i = 1, \ldots, 4$, are constants such that the Local Lemma applies.

4. Let $\mathcal{H} = (V, \mathcal{E})$ be a simple hypergraph. Prove that $\sum_{v \in V} d(v) = \sum_{e \in \mathcal{E}} |e|$, and

$$\sum_{v \in V} d^2(v) = \sum_{e \in \mathcal{E}} \sum_{v \in e} d(v) = \sum_{e \in \mathcal{E}} \sum_{f \in \mathcal{E}} |e \cap f|.$$

5.* Let $r \geq 2$ be integers. Prove that if every edge of a hypergraph $\mathcal{H}$ has at least r vertices and every edge intersects at most $2^{r-2} - 1$ other edges, then the vertices of $\mathcal{H}$ can be two colored such that each color meets each edge. (Hint: Using Lemma 5.1)

Chapter 6
Constructive Lower Bounds

To compare the lower bounds obtained from probabilistic method previously, we will take a look at the lower bounds of the classic Ramsey numbers from the constructive method. In this chapter, we shall introduce a disproof of a conjecture of Borsuk in geometry and related properties of intersecting hypergraphs. The result on the conjecture of Borsuk reveals a general idea in Ramsey theory that exceptions appear only in cases of small sizes. Recently, Conlon and Ferber (2021) made improvements on the lower bounds of multicolor classic Ramsey numbers $r_k(t)$ for $k \geq 3$, which will be introduced in the last section.

6.1 Constructive Lower Bounds for $r(s, t)$

In this section, let us pay attention to non-diagonal constructive lower bounds for $r(s, t)$. The first constructive lower bound

$$r(3, t) \geq \Omega(t^{3/2}) \tag{6.1}$$

was found by Alon (1994). Some years later, Codenotti, Pudlák and Resta (2000) gave the same constructive lower bound by using an algebraic argument. Subsequently, Alon and Pudlák (2001) generalized the construction of Codenotti, Pudlák and Resta (2000) to give polynomial lower bounds for $r(s, t)$ with the exponent of the polynomials increasing with s. i.e.,

$$r(s, t) \geq \exp\left\{\epsilon\sqrt{\log s/\log\log s} \cdot \log t\right\}.$$

For $s = 4, 5, 6$, the constructive lower bounds were improved by Kostochka, Pudlák and Rödl (2010). We will give a combinatorial proof of the lower bound of (6.1) by Kostochka, Pudlák and Rödl (2010).

Let G be a graph. The *superline graph*, denoted by H_G, of G is constructed as follows. The vertices of H_G are the edges of G, and ef is an edge in H_G if e and

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f are disjoint edges of G and there exists an edge g of G that connects an end of e with an end of f (i.e., if the edges e, g and f form a path in G).

Lemma 6.1 *For every triangle-free graph G and its superline graph H_G, $\alpha(H_G) \leq \alpha(G)$.*

Proof. Let A be an independent set in H_G, which is a matching in G. Let B be the set of vertices in G of the edges in A. Then the subgraph $G[B]$ of G induced by B has no triangles and does not contain paths of length 3. So, the components of $G[B]$ are stars, hence $G[B]$ has an independent set of size $|A|$. □

A *projective plane* of order q, denoted by $PG(2, q)$, consists of a set X of q^2+q+1 elements called *points*, and a family $\mathcal{L}$ of subsets of X called *lines*, satisfying the following properties:

(P1) Every line has $q + 1$ points.
(P2) Any pair of distinct points lie on a unique line.

Lemma 6.2 *A projective plane of order q has the properties as follows.*

(P3) Any point lies on $q + 1$ lines.
(P4) There are $q^2 + q + 1$ lines.
(P5) Any two lines meet at a unique point.

Proof. To prove (P3), we fix a point $x \in X$. There are $q(q+1)$ points different from x, each line through x contains q further points, and there are no other overlaps between these lines (apart from x). So $q(q + 1)$ points of $X \setminus \{x\}$ are partitioned equally into parts by these lines. Therefore there must be $q+1$ lines through x with no remaining point.

To show (P4), let us count the number of the pairs (x, L) with $x \in L$ in two ways. Since each line contains $q+1$ points and each point lies on $q+1$ lines from (P3), we obtain $|\mathcal{L}|(q + 1) = (q^2 + q + 1)(q + 1)$. So $|\mathcal{L}| = q^2 + q + 1$.

Finally, we show (P5). From the property (P2), any two lines meet at most one point. Suppose that there are two lines L_1 and L_2 that have no point in common, and we fix a point $x \in L_1$. From the property (P3), there are q lines different from L_1 that contain x. Therefore, by the pigeonhole principle, one of these q lines contains at least two points of L_2 as each line has $q + 1$ points. This leads to a contradiction to (P2). □

We now have the following modification of the superline graph construction. Let G be a bipartite graph with bipartition of vertices (U, V) and let $\prec$ be a linear ordering of the edges of G. We denote by $H_G^{\prec}$ the graph whose vertices are the edges of G and a pair $\{uv, u'v'\}$, with $u \neq u' \in U$ and $v \neq v' \in V$ is an edge in $H_G^{\prec}$ if either $uv \prec u'v'$ and uv' is an edge in G or $u'v' \prec uv$ and $u'v$ is an edge in G. (In particular, $H_G^{\prec}$ is a subgraph of H_G.)

Lemma 6.3 *For every bipartite graph G on n vertices, every bipartition of G and every ordering $\prec$ of its edges, $\alpha(H_G) < n$.*

Proof. Let A be a set of n edges of G. Then A contains a cycle. Let

$$(u_0, v_0, u_1, v_1, \ldots, u_{k-1}, v_{k-1}, u_0)$$

be a cycle formed by some edges in A. Then for some $0 \le i < k$, $u_i v_i \prec u_{i+1} v_{i+1}$ (where we count $i+1$ modulo k). Hence $\{u_i v_i, u_{i+1} v_{i+1}\}$ is an edge in H_G, which proves that A is not an independent set. □

Proof of (6.1). Let G_q be the incidence graph of the classical projective plane $PG(2, q)$ with $q^2 + q + 1$ points and $q^2 + q + 1$ lines (by Lemma 6.2 (P4)), where q is a prime power. Thus G_q is a regular bipartite graph of degree $q + 1$. Let $\prec$ be an arbitrary ordering of the edges of G_q. We use the graphs $H_{G_q}^{\prec}$. The following properties can be easily verified:

1. $H_{G_q}^{\prec}$ has $(q+1)(q^2+q+1)$ vertices.

2. $H_{G_q}^{\prec}$ is triangle-free.

Indeed, if $p_1 l_1 \prec p_2 l_2 \prec p_3 l_3$ would form a triangle, then (p_1, l_3, p_2, l_2) would be a C_4 in G_q.

3. The largest independent set in $H_{G_q}^{\prec}$ has size at most $q^2 + q + 1$, by Lemma 6.3.

Therefore, the lower bound (6.1) follows. □

6.2 Constructive Lower Bounds for $r(t)$

Attempts have been made over the years to construct Ramsey graphs with small cliques and independent sets. Abbott (1972) gave Ramsey graph of order n by a recursive construction with cliques and independence sets of size $cn^{\log 2/\log 5}$. Nagy (1972) gave a construction reducing the size to $cn^{1/3}$. The breakthrough by Frankl (1977) gave the Ramsey graph of order n with cliques and independent sets of size smaller than n^{ϵ} for any $\epsilon > 0$. This result was further improved to $e^{c(\log n)^{3/4}(\log\log n)^{1/4}}$ in Chung (1981) by using different construction. The current best construction by Frankl and Wilson (1981) implies that there exist Ramsey graphs of order n with cliques and independent sets of size at most $e^{c(\log n \log\log n)^{1/2}}$, hence yielding a super-multiplicative lower bound. It would be a challenge to give a lower bound for $r_2(t)$ of the form $(1+\epsilon)^t$ for some $\epsilon > 0$, which is a problem proposed by Erdős.

Recall a *hypergraph* $\mathcal{H}$ on vertex set $V \neq \emptyset$ is a pair $(V, \mathcal{E})$, where the edge set $\mathcal{E}$ is a family of subsets of V. All hypergraphs are *simple*, that is to say, there is no loop and any pair of edges are distinct as subsets of V (no multiedges). We assume that $e \neq \emptyset$ for all $e \in \mathcal{E}$. Let $V^{(r)}$ be the set of all r-subsets of V. If $\mathcal{E} \subseteq V^{(r)}$, then the hypergraph is called r-uniform. So a graph is a 2-uniform hypergraph. The hypergraph $(V, V^{(r)})$ is called *complete*, denoted by $K_n^{(r)}$, where $n = |V|$. We shall write $K_n^{(2)}$ for K_n as usual.

Proposition 6.1 *If vectors $x_1, x_2, \ldots, x_m$ are linearly independent in a linear space, then m is at most the dimension of the space.*

In the following result, the vertex set consists of all residents in a town, named Oddtown, and edge set consists of all clubs in the town. The citizens form these clubs by some rules, from which the name of the town came. These rules are a bit odd, see Berlekamp (1969).

Theorem 6.1 (Odd-town-theorem) *If a hypergraph $\mathcal{H} = (V, \mathcal{E})$ has the following properties:*
(i) $|e|$ is odd for all $e \in \mathcal{E}$,
(ii) $|e \cap f|$ is even for all $e, f \in \mathcal{E}$ with $e \neq f$,
then $|\mathcal{E}| \leq |V|$.

Proof. Assume $|V| = n$ and $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$. Define a 0-1 vector

$$e_i = (e_{i1},\ e_{i2},\ \ldots,\ e_{in}),$$

where $e_{ij} = 1$ if the vertex $v_j \in e_i$ and 0 otherwise. Then all these row vectors form an $m \times n$ matrix M, which is the incidence matrix of $\mathcal{H}$. Since all e_i are elements of the linear space of n-dimensional vectors over the field $F_2 = \{0, 1\}$, the inner product of $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ is

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Then the two conditions can be nicely expressed by

$$e_i \cdot e_j = \begin{cases} 1 \text{ if } & i = j, \\ 0 \text{ if } & i \neq j. \end{cases}$$

We claim that $e_1, e_2, \ldots, e_m$ are linear independent. Indeed, assume

$$\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_m e_m = 0,$$

where $\lambda_i \in F_2$ for $1 \leq i \leq m$. It is clear that $\lambda_i = 0$ by multiplying both sides of the above equation by e_i. Therefore, the conclusion $m \leq n$ follows from Proposition 6.1 immediately. □

We will use the following result by Fisher (1940).

Theorem 6.2 (Fisher Inequality) *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. If there is an integer $\lambda \geq 0$ such that any pair of distinct edges e and f of $\mathcal{H}$ satisfy $|e \cap f| = \lambda$, then*

$$|\mathcal{E}| \leq |V|.$$

Proof. Let $|V| = n$ and $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$. If $\lambda = 0$, then it is trivial to see $m \leq n$. If $\lambda \geq 1$ and there is an edge, say e_m, with $|e_m| = \lambda$, then $e_m \subseteq e_i$ and $e_i \cap e_j = e_m$ for any $i \neq j$. Set $e'_i = e_i \setminus e_m$, and denote $\mathcal{E}' = \{e'_1, e'_2, \ldots, e'_{m-1}\}$, we obtain a hypergraph $\mathcal{H}' = (V \setminus e_m, \mathcal{E}')$ which satisfies the condition in case 1. Thus $m - 1 \leq n - \lambda$. It follows that $m \leq n$.

In the following, we assume that $\lambda \geq 1$ and any edge contains at least $\lambda + 1$ vertices. In the proof of Oddtown-theorem, e_i is viewed as a row vector of the incidence matrix of $\mathcal{H}$. But now we consider e_i as an element of the linear space of n-dimensional vectors over R (real numbers). In this space, the inner product will be

$$e_i \cdot e_j = \begin{cases} \lambda + \mu_i & \text{if } i = j, \\ \lambda & \text{if } i \neq j, \end{cases}$$

where $\mu_i = |e_i| - \lambda \geq 1$ is an integer.

Claim $e_1, e_2, \ldots, e_m$ are linearly independent.

Proof. Assume

$$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_m e_m = 0,$$

where each α_i for $1 \leq i \leq m$ is a real number. Taking the inner product of both sides by e_i, we have

$$\lambda(\alpha_1 + \alpha_2 + \cdots + \alpha_m) + \alpha_i \mu_i = 0.$$

Setting $\beta = \sum_{j=1}^m \alpha_j$, we obtain $\lambda\beta + \alpha_i \mu_i = 0$, which yields

$$\alpha_i = -\frac{\lambda}{\mu_i}\beta.$$

Summing both sides of the above equation over $1 \leq i \leq m$, we get

$$\beta = -\lambda \left(\sum_{j=1}^{m} \frac{1}{\mu_j} \right) \beta,$$

implying that $\beta = 0$ since otherwise the signs of both sides would be different. Therefore $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$ and the proof of the claim is complete. □

Now the desired inequality $m \leq n$ follows from the above claim and Proposition 6.1 immediately. □

It is easy to give a lower bound of form cn^2 for $r(n, n)$, but it is not trivial to give a lower bound of the form cn^3. The following is a constructive lower bound for $r(n, n)$ due to Nagy (1972), which is much weaker than $cn2^{n/2}$ that given by using the probabilistic method. However, the construction itself is interesting.

Corollary 6.1 *For any integer $n \geq 4$, $r(n, n) > \binom{n-1}{3}$.*

Proof. Associate each vertex of complete graph $K_{\binom{n-1}{3}}$ with an edge of $K_{n-1}^{(3)}$. Color an edge xy of $K_{\binom{n-1}{3}}$ by red if the corresponding edges e_x and e_y of $K_{n-1}^{(3)}$ intersect in one element, otherwise (e_x and e_y intersect in zero or two elements) color xy by blue. The Fisher inequality implies that there are at most $n - 1$ such edges e in $K_{n-1}^{(3)}$

that satisfy the above property, which corresponds to a red clique of order at most $n-1$, so there is no red K_n. Similarly, the Oddtown-theorem implies that there is no blue K_n. Thus the claimed lower bound follows. □

Now we shall try to obtain a super-polynomial lower bound for $r(n,n)$. In the above proofs of Theorem 6.1 and Theorem 6.2, the linear spaces are spaces of vectors. For some cases the linear spaces are spaces of functions. For example, the set of functions $f : \Omega \to F$, where Ω is an arbitrary set and F is a field, forms such a space. The dimension of the space is $|\Omega|$ as follows. Another example is that consisting of all homogeneous polynomials of degree k in n variables over a field. The dimension of this space is $\binom{n+k-1}{k}$, which can be seen as the k-repeatable combinations of n-element sets. Specifically, if $n = k = 3$,

$$x^3,\ y^3,\ z^3,\ x^2y,\ xy^2,\ x^2z,\ xz^2,\ y^2z,\ yz^2,\ xyz$$

form a basis, which consists of 10 polynomials of degree 3.

To check the linear independence in spaces of functions, we may need the following propositions.

Proposition 6.2 (Diagonal Criterion) *For $1 \le i \le m$, let $f_i : \Omega \to F$ be functions. If $a_j \in \Omega$ are elements satisfy that*

$$f_i(a_j) \begin{cases} \neq 0 & \text{if} \quad i = j \\ = 0 & \text{if} \quad i \neq j, \end{cases}$$

then $f_1, f_2, \ldots, f_m$ are linearly independent.

Proposition 6.3 (Triangular Criterion) *For $1 \le i \le m$, let $f_i : \Omega \to F$ be functions. If $a_j \in \Omega$ are elements satisfy that*

$$f_i(a_j) \begin{cases} \neq 0 & \text{if} \quad i = j \\ = 0 & \text{if} \quad i < j, \end{cases}$$

then $f_1, f_2, \ldots, f_m$ are linearly independent.

In order to generalize Odd-town Theorem, we shall introduce a definition as follows. For a set L of integers and integers p, s, we shall write

$$s \in L \pmod p$$

if $s = \ell \pmod p$ for some $\ell \in L$. The negation of this statement will be written as $s \notin L \pmod p$.

The following result is due to Deza, Frankl and Singhi (1983), which is a generalization of the corresponding result (Corollary 6.2) of Ray-Chaudhuri and Wilson (1975).

Theorem 6.3 *Let p be a prime and let L be a set of integers. If $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph with $|V| = n$ and $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$ satisfying*

$$|e_i| \notin L \pmod p \qquad \textit{for} \quad 1 \le i \le m,$$
$$|e_i \cap e_j| \in L \pmod p \qquad \textit{for} \quad 1 \le i \ne j \le m,$$

then

$$m \le \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{|L|} = \sum_{i=0}^{|L|} \binom{n}{i}.$$

Proof. The proof is due to Alon, Babai, Suzuki (1991). Define a polynomial $F(x, y)$: $F_p^n \times F_p^n \to F_p$ as

$$F(x, y) = \Pi_{\ell \in L}(x \cdot y - \ell)$$

in $2n$ variables: $x, y \in F_p^n$, which are n-dimentional vectors with coordinates from the p-element field F_p, where $x \cdot y$ is the inner product, and the integers in L are viewed as elements of F_p. Let $y_i \in F_p^n$ be the incidence vector of e_i $(1 \le i \le m)$. Define $f_i : F_p^n \to F_p$ such that

$$f_i(x) = F(x, y_i).$$

The condition of the theorem ensures

$$f_i(y_j) \begin{cases} \ne 0 \text{ if } & i = j, \\ = 0 \text{ if } & i \ne j. \end{cases}$$

This shows that the restricted $f_1, f_2, \ldots, f_m$ are linearly independent from the diagonal criterion. These equations remain true if the domain of f_i is restricted to $\Omega = \{0, 1\}^n \subseteq F_p^n$, where $\{0, 1\}$ is a subset of F_p hence Ω is a subset of F_p^n.

A polynomial in n variables is called *multilinear* if its degree in each variable is at most one. In $\{0, 1\}$, $x_i^2 = x_i$ for each variable (that takes values 0 or 1 only) and thus every polynomial $f_i : \Omega \to F_p$ is multilinear. Note that the degree of such a polynomial is at most $|L|$, so the dimension of the space consisting of all multilinear polynomials is

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{|L|} = \sum_{i=0}^{|L|} \binom{n}{i},$$

which is an upper bound of m. □

Let us remark that if $L = \{0\}$ and $p = 2$, the above theorem implies that the number of clubs in Oddtown is at most $1 + n$, slightly weaker than that obtained from Oddtown-theorem.

Unlike that in Theorem 6.3, the hypergraph in the following corollary of Ray-Chaudhuri and Wilson (1975) should be uniform, and the intersecting size is not considered in modular form.

Corollary 6.2 *Let L be a set of integers. If* $\mathcal{H} = (V, \mathcal{E})$ *is a uniform hypergraph with* $|V| = n$ *and* $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$ *satisfying*

$$|e_i \cap e_j| \in L \quad \textit{for all} \quad 1 \le i \ne j \le m,$$

then

$$m \leq \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{|L|} = \sum_{i=0}^{|L|} \binom{n}{i}.$$

Proof. Assume that $\mathcal{H}$ is t-uniform. Select a prime $p > t$ and set

$$L' = L \setminus \{t\}.$$

If $t \notin L$, then we can apply Theorem 6.3 to L' by noting $|e_i| = t \notin L'$ for each $1 \leq i \leq m$ and $|e_i \cap e_j| \in L = L'$ for all $1 \leq i \neq j \leq m$. If $t \in L$, then since $p > t$, we can also have that for each $1 \leq i \leq m$, $|e_i| = t \notin L' \pmod{p}$ and $|e_i \cap e_j| \in L$ implying $|e_i \cap e_j| \in L' \pmod{p}$ for all $1 \leq i \neq j \leq m$. It follows from Theorem 6.3 that

$$m \leq \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{|L'|}$$

as desired. □

Note that the above corollary can be viewed as a generalization of Fisher Inequality for uniform hypergraphs by taking $L = \{\lambda\}$.

We now have a constructive super-polynomial lower bound for diagonal Ramsey number $r_2(t)$.

Construction Let p be a prime and let $n > 2p^2$, and let V be a set of size n. Define a complete graph K_N of order $N = \binom{n}{p^2-1}$, whose vertex set is $V^{(p^2-1)}$, i.e., the edge set of $K_n^{(p^2-1)}$. Color an edge $\{e, f\}$ of K_N by red if $|e \cap f| \neq p - 1 \pmod{p}$, and blue otherwise. □

If K_m is a monochromatic red clique, then we set $L = \{0, 1, \ldots, p-2\}$. For any vertex e of K_m, as $|e| = p^2 - 1$ we obtain $|e| = p - 1 \pmod{p}$, and so $|e| \notin L \pmod{p}$. Moreover, any pair of distinct vertices e and f satisfy $|e \cap f| \in L \pmod{p}$ as the edge $\{e, f\}$ is red hence $|e \cap f| \neq p - 1$. So Theorem 6.3 yields

$$m \leq \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{p-1} < 2\binom{n}{p-1},$$

where the last inequality follows from induction on p.

If K_m is a monochromatic blue clique, then we set

$$L = \{(p-1), p + (p-1), \ldots, (p-2)p + (p-1)\}.$$

From the construction, any pair of distinct vertices e and f of K_m satisfy $|e \cap f| = p - 1 \pmod{p}$. Thus $|e \cap f| \in L$, and Corollary 6.2 implies $m < 2\binom{n}{p-1}$.

Note that for $p = 2$, this construction is exactly what is given by Nagy in the proof of Corollary 6.1.

Let $t = 2\binom{n}{p-1}$. Then $\binom{n}{p^2-1} > ct^{p+1}$ for all large n, where $c = c_p > 0$ is a constant depending only on p. So for any fixed p, we have

$$r_2(t) > ct^{p+1}.$$

Basing on the above construction, Frankl-Wilson (1981) obtained the following super-polynomial lower bound for $r_2(t)$.

Theorem 6.4 *Let* $\omega(t) = \log^2 t/(4\log\log t)$. *For any* $\epsilon > 0$ *and large* t,

$$r_2(t) > \exp\{(1-\epsilon)\omega(t)\}.$$

Proof. Let $n = p^3$ and let p be the largest prime such that $2\binom{p^3}{p-1} < t$. From Construction 1, there is no monochromatic K_t. For any $\epsilon > 0$ and large t, using the Prime Number Theorem and elementary estimate of $\binom{p^3}{p-1}$ by Stirling formula, we obtain that

$$\frac{(1-\epsilon)\log t}{4\log\log t} \le p \le \frac{(1+\epsilon)\log t}{4\log\log t}.$$

An easy calculation confirms that the number of vertices of the complete graph satisfies

$$\binom{p^3}{p^2-1} \ge \exp\{(1-\epsilon)\omega(t)\}$$

as desired. □

As a complete r-uniform hypergarph on vertex set V is $K_n^{(r)} = (V, V^{(r)})$, we will write a r-uniform hypergraph $\mathcal{H}$ on V as $\mathcal{H} \subseteq K_n^{(r)}$ in the sence that the edge set of $\mathcal{H}$ is a subset of $V^{(r)}$. Note that the number of edges of a r-uniform complete hypergraph $K_{4p-1}^{(2p-1)}$ is

$$\binom{4p-1}{2p-1} \sim \frac{2^{4p-1}}{\sqrt{2\pi p}}$$

as $p \to \infty$. The following result says that forbidding a single intersecting size $p-1$ in this hypergraph, the number of edges must decrease significantly.

Corollary 6.3 (Omitted Intersection Theorem) *Let* p *be a prime. If hypergraph* $\mathcal{H} \subseteq K_{4p-1}^{(2p-1)}$ *satisfies that no pair of edges of* $\mathcal{H}$ *intersect in exactly* $p-1$ *elements, then the number of edges of* $\mathcal{H}$ *is less than*

$$2\binom{4p-1}{p-1} < 1.7548^{4p-1}.$$

Proof. Set $L = \{0, 1, \ldots, p-2\}$. For any $e_i \in \mathcal{E}$, since $|e_i| = 2p-1$, we obtain that $|e_i| \notin L \pmod p$. For $e_i, e_j \in \mathcal{E}$ with $e_i \neq e_j$, we have $|e_i \cap e_j| \neq p-1$, which implies that $|e_i \cap e_j| \in L \pmod p$. Hence $\mathcal{H}$ satisfies the conditions of Theorem 6.3, it follows that

$$|\mathcal{E}| \le \sum_{i=0}^{p-1}\binom{4p-1}{i} < 2\binom{4p-1}{p-1} < (1.7548)^{4p-1},$$

where the second inequality is an exercise. □

Using Omitted Intersection Theorem, one can construct a graph H_p for a prime p by associating its vertices with edges of $K_{4p-1}^{(2p-1)}$, in which two vertices are adjacent if and only if the corresponding edges of $K_{4p-1}^{(2p-1)}$ intersect in exactly $p-1$ vertices. The order of H_p is $\binom{4p-1}{2p-1}$. Omitted Intersection Theorem gives an upper bound for the independence number of H_p as

$$\alpha(H_p) < 2\binom{4p-1}{p-1} < (1.7548)^{4p-1}$$

since any hypergraph with more than $2\binom{4p-1}{p-1}$ edges must contain two edges e and f such that $|e \cap f| = p-1$. Therefore, we obtain that

$$\chi(H_p) \geq \frac{|V(H_p)|}{\alpha(H_p)} = \frac{\binom{4p-1}{2p-1}}{\alpha(H_p)} > (1.1397)^{4p-1}.$$

We will apply this graph to disprove a conjecture in the next section.

6.3 A Conjecture of Borsuk

The essential concept in Ramsey theory is that exceptions occur in small size of the structures. Sometimes the size can be the dimension of a linear space. This is exactly the case we shall discuss in this section. These small size cases may have bigger effect since they are more concrete, which is possible to lead to an error.

In 1933, Borsuk conjectured that every set in real space R^d can be partitioned into $d+1$ sets of *smaller* diameters. Borsuk's paper is famous as it proved an important conjecture of Ulam and contains the Borsuk's conjecture of himself.

The conjecture has been proved for $d = 1, 2, 3$ and also for all d if the set is special, like centrally symmetric, having smooth surface.

Let $f(d)$ be the minimum integer such that every set in R^d can be partitioned into $f(d)$ sets of smaller diameter. Borsuk's conjecture was $f(d) \leq d+1$. An upper bound as $f(d) \leq (\sqrt{3/2} + o(1))^d$ was obtained by Schramm (1988). This bound looks quite weak compared with the Borsuk's conjecture, but it suddently seemed reasonable when Kahn and Kalai (1992) constructed a set giving that $f(d) > 1.2^{\sqrt{d}}$. It is not only dramatic but also very interesting as the set of counterexample contains just only finite points in R^d.

Theorem 6.5 *If p is a prime and $d = \binom{4p-1}{2}$, then*

$$f(d) > (1.1397)^{\sqrt{2d}} > (1.2)^{\sqrt{d}}.$$

Proof. Let p be a prime and $d = d(p) = \binom{4p-1}{2}$. The Kahn-Kalai hypergraph $K(d)$ is defined as follows. Let V be the vertex set of $K_{4p-1}^{(2p-1)}$ with $|V| = 4p - 1$. Let each pair of distinct vertices $x, y \in V$ associate with a vertex a_{xy} of $K(d)$. That is to say, the vertex set of the hypergraph $K(d)$ is

$$V(K(d)) = \{a_{xy} : x, y \in V, x \neq y\},$$

which can be viewed as the edge set of complete graph on V. We admit $a_{xy} = a_{yx}$ for $x \neq y$. The edges of $K(d)$ are associated with the edges of $K_{4p-1}^{(2p-1)}$ in $V^{(2p-1)}$ as follows. If $e \in V^{(2p-1)}$, then an edge A_e of $K(d)$ is defined as

$$A_e = \{a_{xy} : x \in e, y \in V \setminus e\}.$$

That is to say, A_e is defined by the pairs of V "split by e". Note that $V \setminus e$ is not an edge of $K_{4p-1}^{(2p-1)}$ for any edge e, so $A_e = A_f$ if and only if $e = f$ for $e, f \in V^{(2p-1)}$. Note that $K(d)$ has $d = \binom{4p-1}{2}$ vertices and $\binom{4p-1}{2p-1}$ edges, and it is a $2p(2p-1)$-uniform hypergraph.

Let $\mathcal{K}(d)$ be the representation of $K(d)$ in R^d, each of which is a column vector for a fixed edge A_e in the incidence matrix. Since $K(d)$ is $2p(2p-1)$-uniform, the distance of the points representing A_{e_1} and A_{e_2} is

$$\sqrt{2\Big(2p(2p-1) - |A_{e_1} \cap A_{e_2}|\Big)}$$

Therefore the maximum distance, which is the diameter of $\mathcal{K}(d)$, is realized between two points of $\mathcal{K}(d)$ if and only if

$$|A_{e_1} \cap A_{e_2}| = \min\left\{|A_e \cap A_{e'}| : e, e' \in V^{(2p-1)}\right\} := \mu.$$

Claim $|A_{e_1} \cap A_{e_2}| = \mu$ if and only if $|e_1 \cap e_2| = p - 1$.

Proof. In fact, $|A_{e_1} \cap A_{e_2}|$ is the number of pairs of V split by *both* e_1 and e_2. It depends only on $|e_1 \cap e_2|$. Assume $|e_1 \cap e_2| = x$ with $0 \leq x \leq 2p - 2$. Then

$$\mu = \min_x \left\{x(x+1) + (2p-1-x)^2\right\}.$$

It is easy to check that this expression has minimum as $x = p - 3/4$ on real numbers hence it does as $x = p - 1$ on integers, which proves the claim. □

Using the claim, the partition of $\mathcal{K}(d)$ into sets of smaller diameter is equivalent to partition the edges of $K(d)$ into classes without intersection of size $p - 1$, and the number of the classes is clearly at least $\chi(H_p)$, where H_p is the graph defined in the last section. From the properties of H_p, we have

$$f(d) \geq \chi(H_p) > (1.1397)^{4p-1} > (1.1397)^{\sqrt{2d}} > (1.2)^{\sqrt{d}},$$

where the third inequality holds as $4p - 1 > \sqrt{2\binom{4p-1}{2}} = \sqrt{2d}$. This completes the proof. □

6.4 Intersecting Hypergraphs*

This section contains more results on the size of edge set of a hypergraph with conditions concerning the intersections of the edges. A hypergraph is called *intersecting* if any pair of edges intersect. An extremal hypergraph $\mathcal{H} = (V, \mathcal{E})$ oppositive to intersecting condition is that the edges are pairwise disjoint, which trivially has $|\mathcal{E}| \le |V|$. Changing the condition as that no pair of edges are comparable under inclusion, Sperner (1928) obtained a totally nontrivial result. We shall mention several classical results in extremal set theory, for which the readers can find more details in some standard textbooks, e.g., Bollobás (1986) or Lovász (1979).

A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is called a *Sperner hypergraph* if no edge of $\mathcal{H}$ is a subset of another. A Sperner hypergraph is also called an *antichain* with respect to the partial order of inclusion.

Theorem 6.6 (Sperner's Theorem) *If $\mathcal{H} = (V, \mathcal{E})$ is a Sperner hypergraph with $|V| = n$, then*

$$|\mathcal{E}| \le \binom{n}{\lfloor n/2 \rfloor}.$$

Proof. The assertion can be seen from the next result by the fact that the function $\binom{n}{x}$ is maximized at $x = \lfloor n/2 \rfloor$. □

A stronger result is as follows, called LYM-inequality after its authors Lubell (1966), Yamamoto (1954), and Meshalkin (1963).

Theorem 6.7 (LYM-inequlity) *If $\mathcal{H} = (V, \mathcal{E})$ is a Sperner hypergraph with $|V| = n$, then*

$$\sum_{e \in \mathcal{E}} \frac{1}{\binom{n}{|e|}} \le 1.$$

Proof. The following proof is due to Lubell (1966), called *Lubell's Permutation Method*. In order to avoid the trivial case, we assume that no edge is empty. Suppose that $V = \{1, 2, \ldots, n\}$. For any subset e of V, let us associate it with a set $P(e)$ of permutations $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ such that

$$e = \{\sigma(1), \ldots, \sigma(|e|)\},$$

where the set equality means the initial segment of σ is a permutation of elements in e. Then $P(e)$ is the set of such permutations σ. The number of such permutations is

$$|P(e)| = |e|!(n - |e|)!.$$

Observe that the Sperner condition is equivalent to that the sets $P(e)$ are pairwise disjoint for $e \in \mathcal{E}$. Hence we have

$$\sum_{e\in\mathcal{E}} |P(e)| = \sum_{e\in\mathcal{E}} |e|!(n-|e|)! \le n!,$$

follows by the inequality as desired. □

Second proof for Theorem 6.7. The second proof can be viewed as a probabilistic version of the first. We still assume that no edge is empty. Choose a permutation σ of $V = \{1, 2, \ldots, n\}$ randomly and uniformly, and associate it with a family $\mathcal{A}_\sigma$ of subsets of V as

$$\mathcal{A}_\sigma = \{\{\sigma(1)\},\ \{\sigma(1), \sigma(2)\},\ \ldots, \{\sigma(1), \sigma(2), \ldots, \sigma(n)\}\},$$

which contains n segments of σ beginning from $\sigma(1)$. Define a random variable

$$X = |\mathcal{E} \cap \mathcal{A}_\sigma|,$$

and X_e as the indicator for $e \in \mathcal{A}_\sigma$. Then $X = \sum_{e\in\mathcal{E}} X_e$ and

$$E(X_e) = \Pr[e \in \mathcal{A}_\sigma] = \frac{1}{\binom{n}{|e|}}$$

as $\mathcal{A}_\sigma$ contains precisely one set of size $|e|$, which is distributed uniformly among all sets of size $|e|$. Thus

$$E(X) = \sum_{e\in\mathcal{E}} \frac{1}{\binom{n}{|e|}}.$$

For each σ, every pair of subsets in $\mathcal{A}_\sigma$ are comparable under the inclusion. The Sperner condition ensures that $X = |\mathcal{E} \cap \mathcal{A}_\sigma| \le 1$, following by $E(X) \le 1$ as desired. □

The second classical result we shall mention concerns t-uniform hypergraphs without disjoint pairs of edges. An easy way to obtain such an edge set is to fix a vertex and take all edges containing it. However, Erdős, Ko, and Rado (1961) proved that we cannot do it better for $t \le n/2$.

Theorem 6.8 (EKR Theorem) *If $\mathcal{H} = (V, \mathcal{E})$ is an t-uniform hypergraph with $|V| = n$ and $t \le n/2$, in which any pair of edges intersect, then*

$$|\mathcal{E}| \le \binom{n-1}{t-1}.$$

Proof. The proof is due to Katona (1972), called *Katona's Cyclic Permutation Method*. Set

$$V = \{0, 1, \ldots, n-1\}.$$

Let σ be a fixed permutation of V and for $0 \le i \le n-1$, set $A(i) = \{\sigma(i), \sigma(i+1), \ldots, \sigma(i+t-1)\}$, where addition is modulo n.

Claim $\mathcal{E}$ contains at most t such sets $A(i)$ where $0 \le i \le n-1$.

Proof. Suppose $A(\ell) \in \mathcal{E}$. There are precisely $2t-2$ of $A(i)$ other than $A(\ell)$ that intersect $A(\ell)$, i.e.,

$$A(\ell - t + 1), \ldots, A(\ell), \ldots, A(\ell + t - 1).$$

Clearly, these sets can be arranged into $t-1$ pairs of nonintersecting sets, and $\mathcal{E}$ can contain at most one member of each of these pairs, proving the claim. □

Put all vertices in a cycle. Image that every vertex being a guest is seated around a big round table with n seats. In each particular seating arrangement there are exactly n contiguous intervals of length t. Let us associate each such interval with a t-set of V, which consists of the vertices in this interval.

It is clear by symmetry that each t-set is associated with the same number of contiguous intervals, i.e., in the same number of seating arrangements. On the other hand, at each seating arrangement, at most a t/n fraction of the intervals can be formed to associate the edges of $\mathcal{E}$ by the claim. Hence the number of edges of $\mathcal{E}$ is at most a t/n fraction of the total t-sets of V, and so

$$|\mathcal{E}| \le \frac{t}{n}\binom{n}{t} = \binom{n-1}{t-1},$$

proving the assertion. □

Second proof for Theorem 6.8. The above proof has a probabilistic version. Choose a permutation σ of V and $i \in V$ randomly, uniformly and independently. Define a random set

$$A = A(\sigma, i) = \{\sigma(i), \sigma(i+1), \ldots, \sigma(i+t-1)\},$$

where addition is modulo n. The above claim can be employed to bound the conditional probability as $\Pr(A \in \mathcal{E}|\sigma) \le t/n$, hence

$$\Pr(A \in \mathcal{E}) = \sum_{\sigma} \Pr(A \in \mathcal{E}|\sigma) \le \frac{t}{n}.$$

However, A is uniformly chosen from all subsets of size t, so

$$\frac{|\mathcal{E}|}{\binom{n}{t}} = \Pr(A \in \mathcal{E}) \le \frac{t}{n},$$

which follows by

$$|\mathcal{E}| \le \frac{t}{n}\binom{n}{t} = \binom{n-1}{t-1}$$

as desired. □

A hypergarph $\mathcal{H} = (V, \mathcal{E})$ is called a *sunflower* with m petals if $|\mathcal{E}| = m$ and

$$e_1 \cap e_2 = \cap_{e \in \mathcal{E}} e$$

for any pair of distinct edges e_1 and e_2 in $\mathcal{E}$. The common intersection is called the *kernel*. Note that a hypergraph with pairwise disjoint edges (with empty kernel) is a sunflower.

The third result we shall mention in this section asserts that if an t-uniform hypergraph has many edges, it must contain a sunflower with specified size, regardless of size of V. The following result is due to Erdős and Rado (1960).

Theorem 6.9 (Sunflower Theorem) *Let $\mathcal{H} = (V, \mathcal{E})$ be a t-uniform hypergraph. If $|\mathcal{E}| > t!(s-1)^t$, then $\mathcal{H}$ contains a sunflower with s petals.*

Proof. Induction on t. For $t = 1$, the hypergraph has more than $s - 1$ edges, which are at least s vertices (1-unform edges). Thus we have a sunflower with s petals and empty kernel.

We then assume that $t \geq 2$ and the assertion is true for $t-1$. Let $\mathcal{T} = \{e_1, \ldots, e_m\}$ be a maximal family of pairwise disjoint edges of $\mathcal{H}$.

Case 1 $m \geq s$.

These m edges form a sunflower with $m \geq s$ edges and empty kernel, and we are done.

Case 2 $m < s$.

Let $A = \cup_{i=1}^m e_i$. Then $|A| = tm \leq t(s-1)$. By the maximality of the family $\mathcal{T}$, every edge of $\mathcal{H}$ intersects some member in $\mathcal{T}$, hence it intersects A. So there is a vertex $x \in A$, which is contained in at least

$$\frac{|\mathcal{E}|}{|A|} > \frac{t!(s-1)^t}{t(s-1)} = (t-1)!(s-1)^{t-1}$$

edges of $\mathcal{H}$. Let us delete x from these edges and consider the hypergraph with vertex set V and edges sets

$$\{e \setminus \{x\} : e \in \mathcal{E}, x \in e\}.$$

This is a $(t-1)$-uniform hypergraph. By the induction hypothesis, this hypergraph contains a sunflower with s petals, say $\{e_1 \setminus \{x\}, \ldots, e_s \setminus \{x\}\}$. Thus we obtain a sunflower in $\mathcal{H}$ with s petals $\{e_1, \ldots, e_s\}$, proving the assertion. □

The conditions in the following theorem are concerning the intersections between two uniform hypergraphs, which have the same vertex set and same number of edges. We call these hypergraphs to be *cross intersecting*.

Theorem 6.10 (Bollobás, 1965) *If hypergraph $(V, \mathcal{E})$ is s-uniform with $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$, and $(V, \mathcal{F})$ is t-uniform with $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$, which satisfy*

(i) $e_i \cap f_i = \emptyset$ for $i = 1, \cdots, m$;
(ii) $e_i \cap f_j \neq \emptyset$ for $1 \leq i \neq j \leq m$,

then

$$m \leq \binom{s+t}{s}.$$

Proof. This is a corollary of the next theorem. □

The proof of the next result is due to Lovász (1977), in which two hypergraphs are called *skew cross intersecting*.

Theorem 6.11 *If hypergraph* $(V, \mathcal{E})$ *is* s*-uniform with* $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$*, and* $(V, \mathcal{F})$ *is* t*-uniform with* $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$*, which satisfy*

(i) $e_i \cap f_i = \emptyset$ *for* $i = 1, \ldots, m$*;*
(ii) $e_i \cap f_j \neq \emptyset$ *for* $1 \le i < j \le m$*,*

then

$$m \le \binom{s+t}{s}.$$

Proof. We need a well known fact from linear algebra as follows. Let $S_1, S_2, \ldots, S_\ell$ be s-dimensional linear subspaces of R^{s+1}. Then

$$\cup_{i=1}^{\ell} S_i \neq R^{s+1}.$$

Associate each vertex $v \in V$ with a vector in $Y(v) \in R^{s+1}$ as

$$Y(v) = (y_1(v), y_2(v), \ldots, y_{s+1}(v)) \in R^{s+1}$$

so that the set of obtained vectors will be in general position, namely, any $s+1$ of them will be linearly independent. This can be done as follows. The first $s+1$ can be placed easily. Suppose that we have placed $n \ge s+1$ vertices. Then any s of them must span a linear subspace of dimension s. Denote by $S_1, S_2, \ldots, S_\ell$ for these subspaces, where $\ell = \binom{n}{s}$. From the mentioned fact, we have $z \in R^{s+1} \setminus \cup_{i=1}^{\ell} S_i$. It is easy to see that any s vectors, which are associated to s vertices that have placed, and z are independent. Then a new vertex in V is associated with the vector z and any $s+1$ of these $n+1$ vectors are independent.

Define polynomials $g_1, g_2, \ldots, g_m$ in $s+1$ variables $x_1, x_2, \ldots, x_{s+1}$ as

$$\begin{aligned} g_i(X) &= \Pi_{v \in f_i} (X \cdot Y(v)) \\ &= \Pi_{v \in f_i} (x_1 y_1(v) + x_2 y_2(v) + \cdots + x_{s+1} y_{s+1}(v)), \end{aligned}$$

where $X = (x_1, x_2, \ldots, x_{s+1})$. Then $g_i(X)$ is a real homogeneous polynomial of degree t, and $g_i(X) = 0$ if and only if there is some $v \in f_i$ such that $Y(v) \perp X$.

The set of vectors $\{Y(v) : v \in e_j\}$ generates a subspace of dimension s since $|e_j| = s$ and the vectors are in general position. Let A_j be the subspace and let a_j be a nonzero vector with $a_j \perp A_j$. From the fact that the vectors $Y(v)$ are in the general position, we know that A_j does not contain any $Y(v)$ with $v \notin e_j$, namely $Y(v) \in A_j$ if and only if $v \in e_j$. This means

$$g_i(a_j) \begin{cases} \neq 0 \text{ if } & i = j, \\ = 0 \text{ if } & i < j. \end{cases}$$

For example, $g_1(a_1) \neq 0$ since otherwise there is some $v \in f_1$ such that $a_1 \perp Y(v)$. From an expression of $Y(v) = \lambda a_1 + a$ for some $\lambda \in R$ and $a \in A_1$ since $a_1 \perp A_1$

implying that a_1 and A_1 spanning R^{s+1}, we get $\lambda = 0$ since $a_1 \perp a$. It follows that $Y(v) = a \in A_1$, implying $v \in e_1$, which contradicts to the condition $e_1 \cap f_1 = \emptyset$. It is clear $g_1(a_i) = 0$ for $i \geq 2$ as a_i is orthogonal to the vectors associated to the elements in $f_1 \cap e_i$.

From the Triangular Criterion we know that $g_1, g_2, \ldots, g_m$ are linearly independent. Therefore m is at most the dimension of the space of homogenous polynomials of degree t in $s + 1$ variables, so

$$m \leq \binom{s+1+t-1}{t} = \binom{s+t}{t} = \binom{s+t}{s},$$

completing the proof. □

It is noteworthy that Erdős and Rado originally called sunflowers Δ-systems, but the term "sunflower" was coined by Deza and Frankl (1981) and is now more widely used. Erdős and Rado (1960) also conjectured that the bound in Theorem 6.9 can be drastically improved.

Conjecture 6.1 Let $s \geq 3$. There exists a constant $c = c(s)$ such that any t-uniform hypergraph $\mathcal{H}$ of size at least c^t contains a sunflower with s petals.

Kostochka (1997) proved that there is a constant $c > 0$ such that any t-uniform hypergraph of size at least $ct! \cdot (\log\log\log t/\log\log t)^t$ must contain a sunflower with 3 petals. Fukuyama (2018+) claimed an improved bound of $t^{(3/4+o(1))t}$ for $s = 3$. Recently, Alweiss, Lovett, Wu and Zhang (2021+) show that for any $s \geq 3$, any t-uniform hypergraph of size at least $(\log t)^{(1+o(1))t}$ must contain a sunflower with s petals. This makes a big step towards the conjecture.

6.5 Lower Bounds of $r_k(t)$ for $k \geq 3$

We know that Erdős (1947) obtained the following lower bound by using probabilistic method.

$$r(t) > 2^{t/2}.$$

The best lower bound by Spencer (1975) is $r(t) > (1 - o(1))\frac{\sqrt{2}}{e}\sqrt{2}^t$, see Theorem 5.3. Similarly, we have that $r_3(t) > 3^{t/2}$. Generally, Lefmann (1987) observed that

$$r_{k_1+k_2}(t) - 1 \geq (r_{k_1}(t) - 1)(r_{k_2}(t) - 1). \tag{6.2}$$

Indeed, we can blow up a k_1-edge-coloring of $K_{r_{k_1}(t)-1}$ with no monochromatic K_t so that each vertex set has order $r_{k_2}(t) - 1$ and then color each of these copies of $r_{k_2}(t) - 1$ separately with the other k_2 colors so that there is again no monochromatic K_t. By using the bounds $r(t) - 1 \geq 2^{t/2}$ and $r_3(t) - 1 \geq 3^{t/2}$, we can repeatedly apply this observation to conclude that

$$r_{3k}(t) > 3^{kt/2}, \quad r_{3k+1}(t) > 2^k 3^{(k-1)t/2}, \quad \text{and} \quad r_{3k+2}(t) > 2^{t/2} 3^{kt/2}.$$

Recently, Conlon and Ferber (2021) improves the above general lower bounds $r_k(t)$ for each fixed $k \geq 3$ via a construction which is partly deterministic and partly random, in which the improvements are exponential in large t. The deterministic part shares some characteristics with a construction of Alon and Krivelevich (1997), in which the authors consider a graph whose vertices are vectors over a finite field where adjacency is determined by the value of their scalar product, while randomness comes in through both random coloring and random sampling.

Theorem 6.12 *For any prime q, $r_{q+1}(t) > 2^{t/2} q^{3t/8+o(t)}$.*

Proof. Let q be a prime. Suppose $t \neq 0 \pmod q$ and let $V \subseteq F_q^t$ be the set consisting of all (row) vectors $v \in F_q^t$ for which $\sum_{i=1}^t v_i^2 = 0 \pmod q$, noting that $q^{t-2} \leq |V| \leq q^t$. Here the lower bound follows from observing that we may pick $v_1, \ldots, v_{t-2}$ arbitrarily and, since every element in F_q can be written as the sum of two squares, there must then exist at least one choice of v_{t-1} and v_t such that $v_{t-1}^2 + v_t^2 = -\sum_{i=1}^t v_i^2$.

We will first color all the pairs $\binom{V}{2}$ and then define a coloring of $E(K_n)$ by restricting our attention to a random sample of n vertices in V. Formally:

Coloring all pairs in $\binom{V}{2}$. For every pair $uv \in \binom{V}{2}$, we define its color $\chi(uv)$ according to the following rules:

(1) If $u \cdot v = i \pmod q$ and $i \neq 0$, then set $\chi(uv) = i$.

(2) If $u \cdot v = 0 \pmod q$, choose $\chi(uv) \in \{q, q+1\}$ uniformly at random, independently of all other pairs.

Mapping $[n]$ into V. Take a random injective map $f : [n] \to V$ and define the color of every edge ij as $\chi(f(i)f(j))$.

Our goal is to upper bound the orders of the cliques in each color class.

Colors $1 \leq i \leq q-1$. There are no i-monochromatic cliques of order larger than t for any $1 \leq i \leq q-1$. Indeed, suppose that $v_1, \ldots, v_s$ form an i-monochromatic clique. We will try to show that they are linearly independent and, therefore, that there are at most t of them. To this end, suppose that

$$u := \sum_{j=1}^{s} \alpha_j v_j = \overline{0}$$

and we wish to show that $\alpha_j = 0 \pmod q$ for all j.

Let $\beta = (v_1, \ldots, v_s)^T$. Observe that since $v_j \cdot v_j = 0 \pmod q$ for all j (our ground set V consists only of such vectors) and $v_k \cdot v_j = i \pmod q$ for each $k \neq j$, by considering all the products $u \cdot v_j$, we obtain that $iJ - iI = \beta\beta^T$, where J is the $s \times s$ all 1 matrix and I is the $s \times s$ identity matrix. Thus the vector $\overline{\alpha} = (\alpha_1, \ldots, \alpha_s)^T$ is a solution to

$$M\overline{\alpha} = \overline{0}$$

with $M = iJ - iI$ since $\beta^T\overline{\alpha} = \overline{0}$. In particular, we obtain that the eigenvalues of M (over Z) are $i(s-1)$ with multiplicity 1 and $-i$ with multiplicity $s-1$. Therefore, if $s \neq 1 \pmod{q}$, the matrix is also non-singular over Z_q, implying that $\overline{\alpha} = 0$, as required. On the other hand, if $s = 1 \pmod{q}$, we can apply the same argument with $v_1, \ldots, v_{s-1}$ to conclude that $s - 1 \leq t$. But, we cannot have $s - 1 = t$, since this would imply that $t = 0 \pmod{q}$, contradicting our assumption. Therefore, we may also conclude that $s \leq t$ in this case.

Colors q and $q + 1$. We call a subset $X \subseteq V$ a *potential clique* if $|X| = t$ and $u \cdot v = 0 \pmod{q}$ for all $u, v \in X$. Given a potential clique X, we let M_X be the $t \times t$ matrix whose rows consist of all the vectors in X. Observe that $M_X \cdot M_X^T = 0$, where we use the fact that each vector is self-orthogonal. First we wish to count the number of potential cliques and later we will calculate the expected number of cliques that survive after we color randomly and restrict to a random subset of order n.

Suppose that X is a potential clique and let $r := \text{rank}(X)$ be the rank of the vectors in this clique. Since $\text{rank}(M_X^T) \leq t - \text{rank}(M_X)$ by noting the vectors in M_X^T satisfy $M_X \cdot x = \overline{0}$, it follows that $r \leq t/2$. By assuming that the first r elements are linearly independent, the number of ways to build a potential clique X of rank r is upper bounded by

$$\left(\prod_{i=0}^{r-1} q^{t-i}\right) \cdot q^{(t-r)r} = q^{tr - \binom{r}{2} + tr - r^2} = q^{2tr - \frac{3r^2}{2} + \frac{r}{2}}.$$

Indeed, suppose that we have already chosen the vectors $v_1, \ldots, v_s \in X$ for some $s < r$. Then, letting M_s be the $s \times t$ matrix with the v_i as its rows, we need to choose v_{s+1} such that $M_s \cdot v_{s+1} = \overline{0}$. Since the rank of M_s is assumed to be s, there are exactly q^{t-s} choices for v_{s+1} in F_q^t and, therefore, at most that many choices for $v_{s+1} \in V$. If, instead, $s \geq r$, then we need to choose a vector $v_{s+1} \in \text{span}\{v_1, \ldots, v_r\}$ and there are at most q^r such choices in V.

Now observe that the function $2tr - \frac{3r^2}{2} + \frac{r}{2}$ appearing in the exponent of the expression above is increasing up to $r = \frac{2t}{3} + \frac{1}{6}$, so the maximum occurs at $t/2$ since $r \leq t/2$. Therefore, by plugging this into our estimate and summing over all possible ranks, we see that the number N_t of potential cliques in V is upper bounded by $q^{5t^2/8 + o(t^2)}$.

The probability that a potential clique becomes monochromatic after the random coloring is $2^{1-\binom{t}{2}}$. Denote by

$$n = 2^{t/2} q^{3t/8 + o(t)}.$$

Suppose now that p is such that $p|V| = 2n$ and observe that $p = nq^{-t+O(1)}$. If we choose a random subset of V by picking each $v \in V$ independently with probability p, the expected number of monochromatic potential cliques in this subset is

$$p^t 2^{1-\binom{t}{2}} N_t \leq q^{-t^2 + o(t^2)} n^t 2^{-\frac{t^2}{2} + o(t^2)} q^{\frac{5t^2}{8} + o(t^2)} = \left(2^{-\frac{t}{2}} q^{-\frac{3t}{8} + o(t)} n\right)^t < \frac{1}{2}.$$

Since our random subset will also contain more than n elements with probability at least $1/2$ from Chernoff bound, there exists a choice of coloring and a choice of subset of order n such that there is no monochromatic potential clique in this subset. This completes the proof. □

Theorem 6.12 implies that

$$r_3(t) > 2^{7t/8+o(t)} \quad \text{and} \quad r_4(t) > 2^{t/2}3^{3t/8+o(t)}.$$

Consequently, we can apply (6.2) to obtain that for each $k \geq 2$,

$$r_k(t) > \left(2^{\frac{7k}{24}+c}\right)^{t-o(t)},$$

where $c > 0$ is a constant. Modifying the construction of that by Conlon and Ferber (2021), Wigderson (2021) further improved the lower bounds $r_k(t)$ for each fixed $k \geq 3$ as follows.

Theorem 6.13 *For each $k \geq 2$, $r_k(t) > (2^{\frac{3k}{8}-\frac{1}{4}})^{t-o(t)}$.*

6.6 Exercises

1. Let $\mathcal{H} = (V, \mathcal{E})$ be a simple hypergraph of order n such that $|e \cap f| \leq t$ for any pair of distinct edges e and f. Prove that $\sum_e |e| \leq n + t\binom{m}{2}$, where m is the number of edges.

2. Let $X = \sum_{i=1^m} A_i$ be a partition of a finite set X into m subsets, and let $a = |X|/m$. Prove that for every $1 \leq b \leq a$, at least $(1 - b/a)|X|$ elements of X belong to subsets of size at least a/b. How many elements of X belong to subsets of size at most ab? (Hint: $m(a/b) \leq |X|/b$.)

3. Prove the dual of Fisher Inequality as follows. Assume that $\mathcal{H} = (V, \mathcal{E})$ is a simple hypergraph such that each pair of vertices is contained in exactly λ edges, then $|\mathcal{E}| \geq |V|$.

4. Generalize Odd-town Theorem for "mod p town" or even "mod p^k town".

5. Prove that the dimension of homogeneous polynomials of degree k in n variables is $\binom{n+k-1}{k}$.

6. Prove the inequality $\sum_{i=0}^{p-1} \binom{4p-1}{i} \leq 2\binom{4p-1}{p-1}$.

7.* Let $n \leq 2r$ and let $A_1, \ldots, A_m$ be a family of r-element subsets of $[n]$ such that $A_i \cup A_j \neq [n]$ for all i, j. Show that $m \leq (1 - r/n)\binom{n}{k}$. (Hint: Apply EKR to $\overline{A}_i$.)

Chapter 7
Turán Number and Related Ramsey Number

Paul Turán (August 18, 1910–September 26, 1976) was a Hungarian mathematician who worked primarily in number theory. As his long collaborator, Paul Erdős wrote of Turán, “In 1940–1941 he created the area of extremal problems in graph theory which is now one of the fastest-growing subjects in combinatorics.” The field is known more briefly today as extremal graph theory. Turán’s best-known result in this area is known as Turán’s theorem, which gives an upper bound on the number of edges in a K_k-free graph. He is also known for the Kövári-Sós-Turán theorem upper bounding the number of edges of bipartite graphs containing no $K_{t,s}$ as a subgraph. For any graph G with chromatic number $\chi(G) \geq 3$, the asymptotic formula of Turán number of G is known from the well-known Erdős-Stone theorem (1946) which is a fundamental theorem in extremal graph theory, see also Erdős and Siminovits (1966), and Siminovits (1968). However, the Turán numbers for most complete bipartite graphs and even cycles are not very well understood. For a survey, we would like to refer the reader to Füredi and Simonovits (2013). In this chapter, we will see that for a bipartite graph B, the Turán number of B is tightly related to the Ramsey numbers involving B.

7.1 Turán Numbers for Non-Bipartite Graphs

Given a graph H, the *Turán number* $ex(n, H)$ is the maximum number of edges of a graph G on n vertices that does not contain H as a subgraph.

Suppose $\chi(H) = k \geq 3$ and let $n = \sum_{i=1}^{k-1} n_i$ such that $|n_i - n_j| \leq 1$ for $1 \leq i < j \leq n$. Thus n_i is either $\lfloor \frac{n}{k-1} \rfloor$ or $\lceil \frac{n}{k-1} \rceil$. Let $K_{n_1,\ldots,n_{k-1}}$ be the balanced complete $(k-1)$-partite graph with the ith part of size n_i for $1 \leq i \leq k-1$. We call such graph $K_{n_1,\ldots,n_{k-1}}$ the Turán graph, denoted by $T_{k-1}(n)$. Let $t_{k-1}(n)$ be the edge number of the Turán graph $T_{k-1}(n)$.

Assume that $n_1 = \lfloor \frac{n}{k-1} \rfloor$ and $n = n_1(k-1) + r$, where $0 \leq r < k-1$. Hence, in the Turán graph $T_{k-1}(n)$, there are r parts of size $n_1 + 1$ and $k - 1 - r$ parts of size n_1. Now we can verify that

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$$t_{k-1}(n) = \binom{n}{2} - \sum_{i=1}^{k-1} \binom{n_i}{2} = \frac{(k-2)n^2}{2(k-1)} - \frac{r(k-1-r)}{2(k-1)}.$$

So we have that for $n \geq k$,

$$\frac{k-2}{k-1}\binom{n}{2} \leq t_{k-1}(n) \leq \frac{(k-2)n^2}{2(k-1)}.$$

Clearly,

$$ex(n, H) \geq t_{k-1}(n)$$

since the Turán graph does not contain H by noting $\chi(H) = k$.

We begin with a slightly less precise theorem with a proof extended from Mantel's ingenious argument for $k = 3$ in 1907.

Theorem 7.1 *Let n and k be positive integers with $k \geq 2$. Then*

$$t_{k-1}(n) \leq ex(n; K_k) \leq \frac{(k-2)n^2}{2(k-1)}.$$

Furthermore both inequalities become equalities if n is a multiple of $k-1$.

Proof. The first inequality follows from the fact that the Turán graph does not contains K_k, and the second is equivalent to show that if a graph G of order n contains no K_k, then $e(G) \leq \frac{(k-2)n^2}{2(k-1)}$. Assign to each vertex $v \in V(G)$ a weight $w(v) \geq 0$ so that $\sum_{v \in V(G)} w(v) = 1$. Let

$$S(w) = \sum_{uv \in E(G)} w(u)w(v),$$

where the sum is taken over unordered pairs of end vertices of all edges.

Suppose that w has been chosen so as to maximize S as $S_{\max}$. Then for $uv \notin E(G)$ we may **claim** that we can make choice such that either $w(u) = 0$ or $w(v) = 0$. To see this, we suppose that

$$\sum_{x \in N(u)} w(x) \geq \sum_{y \in N(v)} w(y). \tag{7.1}$$

Then

$$S(w) = w(u) \sum_{x \in N(u)} w(x) + w(v) \sum_{y \in N(v)} w(y) + S_1,$$

where S_1 is independent of $w(u)$ and $w(v)$. If (7.1) is not an equality, we could increase $w(u)$ by some amount, and decrease $w(v)$ by the same amount, and S would increase, this is impossible. If (7.1) is an equality, we increase $w(u)$ to $w(u)+w(v)$, and decrease $w(v)$ to zero. This claim means that the vertices x with $w(x) > 0$ belong to some clique of G. Label all these positive weights as $w_1, w_2, \ldots, w_t$, where t is the order of the clique. Then

$$S(w) = \sum_{i<j} w_i w_j = \frac{1}{2}\left[\left(\sum_{i=1}^{t} w_i\right)^2 - \sum_{i=1}^{t} w_i^2\right] = \frac{1}{2}\left(1 - \sum_{i=1}^{t} w_i^2\right)$$

with $\sum_{i=1}^{t} w_i = 1$. From Cauchy-Schwarz inequality,

$$1 = \sum_{i=1}^{t} w_i \cdot 1 \leq \sqrt{\sum_{i=1}^{t} w_i^2 \sum_{i=1}^{t} 1} = \sqrt{t \sum_{i=1}^{t} w_i^2},$$

we get the minimum value of $\sum_{i=1}^{t} w_i^2$ to be $1/t$ when all w_i's are equal, and so

$$S_{\max} = \frac{t-1}{2t}.$$

On the other hand, by assigning $w(x) = 1/n$ to each vertex x of G we get $S = e(G)/n^2$, it follows that

$$\frac{e(G)}{n^2} \leq \frac{t-1}{2t} \leq \frac{k-2}{2(k-1)}$$

since $t \leq k-1$. The desired upper bound follows.

If n is a multiple of $k-1$, then $t_{k-1}(n) = (k-2)n^2/(2(k-1))$ hence both inequalities in the theorem become equalities. □

The following is the well known Turán's Theorem, which generalizes Mantel's ingenious argument for $k = 3$ in 1907. When n is not a multiple of $k-1$, the first inequality in Theorem 7.1 becomes an equality, but the second is not.

Theorem 7.2 *Let k and n be integers with $k \geq 2$. If $n = s(k-1) + r$, where $0 \leq r < k-1$, then*

$$ex(n, K_k) = t_{k-1}(n) = \frac{(k-2)n^2}{2(k-1)} - \frac{r(k-1-r)}{2(k-1)}.$$

Furthermore, if a graph with n vertices and $ex(n, K_k)$ edges that contains no K_k, then it is the Turán graph.

Proof. We may assume that $k \geq 3$ since it is trivial for $k = 2$. We shall use induction on $s = \lfloor n/(k-1) \rfloor$. The case $s = 0$ corresponds to $n = r < k-1$, and in this case it is obvious since $ex(n, K_k) = \binom{n}{2}$. The equality holds if and only if the graph is K_n. Now we assume that $s \geq 1$ and G is a graph with $n = s(k-1) + r$ vertices that contains no K_k. It suffices to prove $e(G) \leq t_{k-1}(n)$.

We may assume that G has the maximum possible number of edges subject to this condition. Thus G must contain K_{k-1} as a subgraph, otherwise we could add an edge to G and the resulting graph would still contain no K_k. Pick such a K_{k-1} on vertex set X, let $Y = V(G) \setminus X$. Let $G[Y]$ denote the subgraph of G induced by Y. It is clear that $G[Y]$ has $n - k + 1 = (s-1)(k-1) + r$ vertices, and so by induction, the number of edges of $G[Y]$ satisfies

$$e(G[Y]) \le t_{k-1}(n-k+1) = \frac{(k-2)(n-k+1)^2}{2(k-1)} - \frac{r(k-1-r)}{2(k-1)}$$
$$= t_{k-1}(n) - (k-2)(n-k+1) - \binom{k-1}{2}.$$

Moreover, since G contains no K_k, we have that no vertex in Y is adjacent to all vertices of X. Thus

$$e(G) \le e(G[Y]) + (k-2)(n-k+1) + \binom{k-1}{2} \le t_{k-1}(n),$$

completing the induction step.

Now if G has $t_{k-1}(n)$ edges, then each vertex in Y must be adjacent to $k-2$ vertices of X. Moreover, an inductive argument shows that $G[Y]$ must have $t_{k-1}(n-k+1)$ edges which induces the Turán graph $T_{k-1}(n-k+1)$ with parts

$$Y_1, Y_2, \ldots, Y_{k-1}.$$

If some vertex $y \in Y_i$ does not adjacent to the vertex $x \in X$, then all vertices of Y_i would not adjacent to the vertex $x \in X$. Indeed, if some vertex $y_i \in Y_i$ is adjacent to x, then each vertex in Y_j for $j \neq i$ would not adjacent to all $X \setminus x$ since otherwise for some $y_j \in Y_j$, $\{y_i, y_j\} \cup X \setminus x$ will induce a K_k. It follows that each vertex in Y_j for $j \neq i$ must be adjacent to x, now we can easily get a K_k. Consequently, G is the Turán graph. □

For the case $k = 3$ of the above theorem, we have an amusing proof as follows. If a graph $G = (V, E)$ of order n contains no K_3, then any adjacent vertices u and v do not have neighbor in common, so

$$d(u) + d(v) \le n.$$

From the fact that

$$\sum_{uv \in E} (d(u) + d(v)) = \sum_{x \in V} d^2(x),$$

in which each $d(x)$ is counted $d(x)$ times over $N(x)$, so we have

$$\sum_{x \in V} d^2(x) \le n|E|.$$

Now Cauchy-Schwarz inequality implies that

$$|E| \ge \frac{1}{n} \sum_{x \in V} d^2(x) \ge \left(\frac{1}{n} \sum_{x \in V} d(x)\right)^2 = \left(\frac{2|E|}{n}\right)^2 = \frac{4|E|^2}{n^2}.$$

Therefore, $ex(n, K_3) \le \max_{G:|G|=n} \{|E|\} \le n^2/4$ follows as desired.

For any graph H with $\chi(H) = k$, we have $ex(n, H) \geq t_{k-1}(n)$, and so a graph F with n vertices and $t_{k-1}(n)$ edges may not contain H or K_k as a subgraph. However, adding one more edge to F will force it to contain a K_k. A deep result of Erdős and Stone (1946) states that if $\epsilon > 0$ is fixed, then ϵn^2 more edges ensure that F contains not only a K_k, but a complete k-partite graph $K_k(t)$ with each part of size t for some large t. The Erdős-Stone theorem is a fundamental theorem in extremal graph theory, see also in Erdős and Siminovits (1966) or Siminovits (1968). In particular, we have the following the result.

Theorem 7.3 *For any fixed graph H with $\chi(H) = k \geq 2$,*

$$ex(n, H) = \left(\frac{k-2}{k-1} + o(1)\right)\binom{n}{2}.$$

Sharpening the result of Erdős and Stone (1946), Bollobás and Erdős (1973) proved that the speed of $t \to \infty$ can be at least $\Omega(\log n)$.

Theorem 7.4 *For integer $k \geq 2$ and $\epsilon > 0$, there is an integer $n_0 = n_0(k, \epsilon)$ such that if F is a graph of order $n \geq n_0$ with edge number*

$$e(F) \geq \left(\frac{k-2}{k-1} + \epsilon\right)\binom{n}{2},$$

then F contains a $K_k(t)$ for some $t \geq \frac{\epsilon \log n}{2^k (k-2)!}$.

Lemma 7.1 *For any integer $k \geq 2$ and $0 < \epsilon < 1$, there is an integer $n_0 = n_0(k, \epsilon)$ such that if F is a graph of order $n \geq n_0$ with minimum degree*

$$\delta(F) \geq \left(\frac{k-2}{k-1} + \epsilon\right)n,$$

then F contains a $K_k(t)$ for some $t \geq \frac{\epsilon \log n}{2^{k-2} (k-2)!}$.

Proof. We use the induction on $k \geq 2$. Suppose to the contrary that the assertion is not valid for $k = 2$, then there is a graph F with n vertices and $\delta(F) \geq \epsilon n$ that does not contain a $K_2(t)$ with $t = \lceil \epsilon \log n \rceil$. We say that a set S is covered by a vertex x if x is adjacent to every vertex in S. Every vertex of F covers at least $\binom{\epsilon n}{t}$ sets of t vertices, and no set of t vertices is covered by t vertices. Therefore,

$$n\binom{\epsilon n}{t} \leq (t-1)\binom{n}{t}.$$

This inequality is invalid for $t = \lceil \epsilon \log n \rceil$ with large n, since then

$$\frac{(t-1)\binom{n}{t}}{n\binom{\epsilon n}{t}} < \frac{(t-1)n^t}{n(\epsilon n)^t(1 - 1/(\epsilon n)) \cdots (1 - (t-1)/(\epsilon n))}$$
$$< \frac{t}{n\epsilon^t(1 - t/(\epsilon n))^t} < \frac{2t}{n\epsilon^t} \leq \frac{2t}{n}\left(\frac{1}{\epsilon}\right)^{2\epsilon \log n} \to 0,$$

where we use the fact that $(1-t/(\epsilon n))^t \to 1$. This contradiction proves the assertion for $k = 2$. Now we assume that the assertion is valid for k, and we will show that it also holds for $k+1$.

Let F be a graph on vertex set V with $|V| = n$ and $\delta(F) \geq (\frac{k-1}{k} + \epsilon)n$. Note that the existence of such a graph F implies $0 < \epsilon < 1/k$. Since

$$\delta(F) \geq \left(\frac{k-1}{k} + \epsilon\right) n > \left(\frac{k-2}{k-1} + \frac{1}{k(k-1)}\right) n,$$

by the induction hypothesis, F contains a $K_k(t')$ on vertex set X with $t' = \lceil c(k) \log n \rceil$ vertices in each vertex class, where

$$c(k) = \frac{1/(k(k-1))}{2^{k-2}\,(k-2)!} = \frac{1}{2^{k-2}\,k!}.$$

Let $e(S_1, S_2)$ be the number of edges between the sets S_1 and S_2, and denote

$$Y = \left\{ v \in V \setminus X : \; e(\{v\}, X) \geq \left(\frac{k-1}{k} + \frac{\epsilon}{2}\right) |X| \right\}.$$

Claim $|Y| \geq \epsilon n$.

Proof. To see this, let us consider $e(X, V \setminus X)$. Clearly, each vertex in X is adjacent to at least $\delta(F) - |X| + 1$ vertices in $V \setminus X$, so

$$e(X, V \setminus X) > |X| \left(\left(\frac{k-1}{k} + \epsilon\right) n - |X| \right).$$

Also, for a vertex $v \in V \setminus (X \cup Y)$, it is adjacent to at most $(\frac{k-1}{k} + \frac{\epsilon}{2})|X|$ vertices in X, so we obtain that

$$\begin{aligned} e(X, V \setminus X) &\leq |X||Y| + (n - |X| - |Y|) \left(\frac{k-1}{k} + \frac{\epsilon}{2}\right) |X| \\ &< |X||Y| + (n - |Y|) \left(\frac{k-1}{k} + \frac{\epsilon}{2}\right) |X|. \end{aligned}$$

The above lower and upper bounds of $e(X, V \setminus X)$ give that

$$(2 - k\epsilon)|Y| + 2k|X| > k\epsilon n.$$

Note that $|X| = kt'$. Thus, for large n,

$$|Y| \geq \frac{k\epsilon n - 2k^2 t'}{2 - k\epsilon} > \frac{k\epsilon n}{2} \geq \epsilon n$$

as claimed. □

Now, set

$$t = \left\lceil \frac{\epsilon \log n}{2^{k-1}(k-1)!} \right\rceil \le \left\lceil \frac{k\epsilon t'}{2} \right\rceil.$$

Note that

$$\left\lceil \left(\frac{k-1}{k} + \frac{\epsilon}{2}\right)|X| \right\rceil = \left\lceil (k-1)t' + \frac{k\epsilon}{2}t' \right\rceil \ge (k-1)t' + t.$$

That is to say, each vertex in Y is adjacent to at least $(k-1)t' + t$ vertices in X, which implies that each such vertex covers at least one $K_k(t)$ in existing $K_k(t')$. Since there are $\binom{t'}{t}^k$ such $K_k(t)$ in this $K_k(t')$, it follows on average that there must exist a $K_k(t)$ covered by $|Y|/\binom{t'}{t}^k$ vertices of Y.

Note that $\binom{t'}{t} \le (et'/t)^t$ from Stirling formula, and $t/(et') > \epsilon/e$,

$$\frac{|Y|}{\binom{t'}{t}^k} \ge \epsilon n \left(\frac{t}{et'}\right)^{tk} \ge \epsilon n \left(\frac{\epsilon}{e}\right)^{tk} \ge \epsilon n \exp\left(\frac{k\epsilon \log n \log(\epsilon/e)}{2^{k-1}(k-1)!}\right) > t$$

for large n, hence we can get a subgraph $K_{k+1}(t)$ in F, completing the induction step and hence the proof. □

The condition on the minimum degree of a graph can be weakened to that on the number of edges of a graph.

Lemma 7.2 *Let c, ϵ be positive real numbers and let $n > 3/\epsilon$ be an integer. If F is a graph with n vertices and at least $(c+\epsilon)\binom{n}{2}$ edges, then it contains a subgraph H with $n' \ge \epsilon^{1/2} n$ vertices and $\delta(H) \ge cn'$.*

Proof. The condition $e(F) \ge (c+\epsilon)\binom{n}{2}$ implies that $0 < \epsilon < c + \epsilon \le 1$. If the assertion is not valid, then there is a sequence of graphs $\{G_i : \ell \le j \le n\}$, where $\ell = \lfloor \epsilon^{1/2} n \rfloor$ and the order of G_j is j, such that

$$G = G_n \supseteq G_{n-1} \supseteq \cdots \supseteq G_{\ell+1} \supseteq G_\ell$$

and the only vertex of G_j not in G_{j-1} has degree less than cj in G_j for $n \ge j \ge \ell + 1$. Therefore,

$$e(G_\ell) > (c+\epsilon)\binom{n}{2} - \sum_{j=\ell+1}^{n} cj = \epsilon\binom{n}{2} + c\binom{\ell+1}{2} - cn > \epsilon\binom{n}{2} > \binom{\ell}{2},$$

by noting $\ell = \lfloor \epsilon^{1/2} n \rfloor$, $0 < \epsilon < 1$ and $n \ge 3/\epsilon$. This is a contradiction. □

Proof of Theorem 7.4. If $n > 3/\epsilon$, then Lemma 7.2 implies that F contains a subgraph H of order $n' \ge \epsilon^{1/2} n$ and $\delta(H) \ge (\frac{k-2}{k-1} + \frac{\epsilon}{2})n'$. Now Lemma 7.1 implies that for large n, F contains a $K_k(t)$ with

$$t \ge \frac{(\epsilon/2)\log n'}{2^{k-2}(k-2)!} \ge \frac{\epsilon \log n}{2^k (k-2)!},$$

as desired. □

7.2 Turán Numbers for $K_{t,s}$

The situation of Turán numbers for bipartite graphs is totally different. If the chromatic number $\chi(H) = 2$, then it only gives that $ex(n, H) = o(n^2)$ from Section 7.1. We have also encountered Turán number in Chapter 3, in which a conjecture of Erdős-Sós is equivalent to $ex(n, T_m) \le \frac{m-1}{2}n$ for any tree T with m edges.

The following result of Kövári, Sós and Turán (1954) used the well-known *double counting method.*

Theorem 7.5 *For any positive integers t and s with $t \le s$,*

$$ex(n, K_{t,s}) \le \frac{1}{2}\left((s-1)^{1/t} n^{2-1/t} + (t-1)n\right).$$

Proof. Let G be a graph of order n that contains no $K_{t,s}$. We say that a set is covered by a vertex v if v is joined to every vertex of the set. Since G does not contain $K_{t,s}$, every t-set (i.e., a set with t elements) is covered by at most $s-1$ vertices. Therefore, if G has degree sequence $d_1, d_2, \ldots, d_n$, then

$$\sum_{i=1}^{n} \binom{d_i}{t} \le (s-1)\binom{n}{t}.$$

For fixed $t \ge 1$, define a function on real variable x as

$$\binom{x}{t} = \begin{cases} \frac{x(x-1)\cdots(x-t+1)}{t!} & \text{if } x \ge t-1, \\ 0 & \text{otherwise.} \end{cases}$$

Let d denote the average degree of G. Since $\binom{x}{t}$ is convex, we have

$$n\binom{d}{t} \le \sum_{i=1}^{n} \binom{d_i}{t} \le (s-1)\binom{n}{t}.$$

We may assume that $n \ge d \ge t$, and hence

$$nd(d-1)\cdots(d-t+1) \le (s-1)n(n-1)\cdots(n-t+1),$$

which implies that

$$\left(\frac{d-t+1}{n-t+1}\right)^t \le \frac{d(d-1)\cdots(d-t+1)}{n(n-1)\cdots(n-t+1)} \le \frac{s-1}{n}.$$

It follows that $d-t+1 \le (s-1)^{1/t} n^{-1/t}(n-t+1)$. Thus we have

$$e(G) = \frac{nd}{2} \le \frac{1}{2}\left((s-1)^{1/t}n^{2-1/t} + (t-1)n\right)$$

as claimed. □

We shall improve the upper bound for $ex(n, C_4)$ in the above theorem slightly. The following result due to Kövári, Sós and Turán (1954) (independently Reiman (1958)) is slightly better than that obtained above for the case $s = t = 2$.

Theorem 7.6 *For any positive integer n,*

$$ex(n, C_4) \le \frac{n}{4}(1 + \sqrt{4n-3}) = \frac{1}{2}n^{3/2} + \frac{n}{4} - O(n^{1/2}).$$

Proof. Let G be a graph of order n containing no C_4. Hence

$$\sum_{v \in V(G)} \binom{d(v)}{2} \le \binom{n}{2},$$

that is $\sum_{v\in V(G)} d(v)(d(v)-1) \le n(n-1)$. As the function $x(x-1)$ is convex, we have $nd(d-1) \le n(n-1)$, where d is the average degree of G. Hence $d \le (1+\sqrt{4n-3})/2$, yielding an upper bound for $nd/2$ as required. □

Let us turn to *the problem of Zarankiewicz* which is closely related to Turán number. Denote by $G(m, n)$ for a bipartite graph with m vertices in the first class and n vertices in the second. We shall signify the fact that $K_{s,t}$ is a subgraph of $G(m, n)$ with s vertices in the first class and t in the second by saying that $K_{(s,t)}$ is contained in $G(m, n)$. In other word, we consider the orders of the vertex classes of the bipartite graphs. Note that when $G(m, n)$ does not contain $K_{(s,t)}$, it may contain $K_{(t,s)}$. However, when t is a constant or $t \le s$, we always write the complete $t \times s$ bipartite graph as $K_{t,s}$ instead of $K_{s,t}$. For example, we write a star as $K_{1,s}$.

Define Zarankiewicz number $z(m, n; s, t)$ to be the maximum number of edges in a bipartite graph $G(m, n)$ which does not contain $K_{(s,t)}$. We always assume that $t \le s$ in this chapter and do not assume which is larger between m and n. We shall write $z(n; s)$ for $z(n, n; s, s)$.

The original problem of Zarankiewicz (1951) was asking what is $z(n, n; 3, 3)$ for $3 \le n \le 6$. The argument in the proof of Theorem 7.5 can be applied to obtain that for $t \le s$,

$$z(m, n; s, t) \le (s-1)^{1/t}nm^{1-1/t} + (t-1)m. \tag{7.2}$$

Lemma 7.3 *For any positive integers n, s and t,*

$$ex(n, K_{t,s}) \le \frac{1}{2}z(n, n; s, t).$$

Proof. Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = ex(n, K_{t,s})$ edges that does not contain $K_{t,s}$. Construct a bipartite graph H as follows. Take two disjoint copies of V, say

$$V' = \{v'_1, \ldots, v'_n\} \quad \text{and} \quad V'' = \{v''_1, \ldots v''_n\}.$$

The graph H has bipartition (V', V''), and $v'_i v''_j$ is an edge of H if and only if $v_i v_j$ is an edge of G. Note that v'_i and v''_i are not connected. Clearly $d_G(v_i) = d_H(v'_i) = d_H(v''_i)$, thus $e(H) = 2e(G)$. We know that H does not contain $K_{t,s}$ since G is $K_{t,s}$-free. So $z(n, n; s, t) \geq e(H) = 2e(G)$. □

For fixed $s \geq t \geq 2$ and large n, the asymptotic form of the upper bound of Theorem 7.5 is

$$ex(n, K_{t,s}) \leq \left(\frac{1}{2} + o(1)\right)(s-1)^{1/t} n^{2-1/t}.$$

This result has not been improved for more than a half century. Füredi (1996(a)) did it for fixed $s \geq t \geq 3$.

Theorem 7.7 *If $m \geq s$, $n \geq t$ and $s \geq t$ are positive integers, then*

$$z(m, n; s, t) \leq (s - t + 1)^{1/t} n m^{1-1/t} + tn + tm^{2-2/t},$$

and hence

$$ex(n, K_{t,s}) \leq \frac{1}{2}\left((s - t + 1)^{1/t} n^{2-1/t} + tn + tn^{2-2/t}\right).$$

Proof ⋆**.** The case $t = 1$ is trivial, and $t = 2$ is known from the upper bound (7.2). So we assume $s \geq t \geq 3$. Let $G = G(m, n)$ be a bipartite graph with bipartition

$$V = \{1, 2, \ldots, m\} \quad \text{and} \quad V' = \{1', 2', \ldots, n'\},$$

which does not contain $K_{(s,t)}$. Fix $t - 2$ vertices $1 \leq i_1 < i_2 < \cdots < i_{t-2} \leq m$ in V. Consider all t-subsets of $N(i_1) \cap \cdots \cap N(i_{t-2})$ in V'. Any such set T in V' is covered by at most $s - t + 1$ further more vertices of V as G contains no $K_{(s,t)}$. We thus obtain

$$\sum_{k \neq i_1, \cdots, i_{t-2}} \binom{|N(i_1) \cap \cdots \cap N(i_{t-2}) \cap N(k)|}{t}$$
$$\leq (s - t + 1)\binom{|N(i_1) \cap \cdots \cap N(i_{t-2})|}{t}$$

where the first summation is taken over $k \neq i_1, \ldots, i_{t-2}$. We now need the following lemma in which the function $\binom{x}{t}$ on x is defined in the proof of Theorem 7.5.

Lemma 7.4 *Let $p, t \geq 1$ be integers, and let $c, x_0, x_1, \ldots, x_t \geq 0$ be real numbers. If*

$$\sum_{i=1}^{p} \binom{x_i}{t} \leq c\binom{x_0}{t},$$

then

$$\sum_{i=1}^{p} x_i \le x_0 c^{1/t} p^{1-1/t} + (t-1)p.$$

Proof. Let $\sigma = \sum_{i=1}^{p} x_i$. We suppose $\sigma > (t-1)p$. From the convexity of the function $\binom{x}{t}$, we have $p\binom{\sigma/p}{t} \le \sum_{i=1}^{p} \binom{x_i}{t}$. Hence

$$\frac{p}{c} \le \frac{x_0(x_0-1)\cdots(x_0-t+1)}{(\sigma/p)(\sigma/p-1)\cdots(\sigma/p-t+1)} \le \left(\frac{x_0}{\sigma/p-t+1}\right)^t,$$

the desired inequality follows immediately. □

Now, applying the above lemma with $p = m-t+2$, $c = s-t+1$, and $x_0 = |N(i_1) \cap \cdots \cap N(i_{t-2})|$, we have

$$\begin{aligned}\sum_{k \ne i_1,\ldots,i_{t-2}} &|N(i_1) \cap \cdots \cap N(i_{t-2}) \cap N(k)| \\ &\le (s-t+1)^{1/t}(m-t+2)^{1-1/t}|N(i_1) \cap \cdots \cap N(i_{t-2})| \\ &\quad + (t-1)(m-t+2).\end{aligned}$$

Summing up both sides over $i_1, \ldots, i_{t-2}$, the left hand side is

$$\sum_{i_1<\cdots<i_{t-2}} \sum_{k \ne i_1,\cdots,i_{t-2}} |N(i_1) \cap \cdots \cap N(i_{t-2}) \cap N(k)| = (t-1)\sum_{j=1}^{n}\binom{d(j')}{t-1}.$$

For the right hand side, since

$$\sum_{i_1<\cdots<i_{t-2}} |N(i_1) \cap \cdots \cap N(i_{t-2})| = \sum_{j=1}^{n}\binom{d(j')}{t-2},$$

it follows that

$$\begin{aligned}(t-1)\sum_{j=1}^{n}\binom{d(j')}{t-1} \le (s-t+1)^{1/t}(m-t+2)^{1-1/t}\sum_{j=1}^{n}\binom{d(j')}{t-2} \\ + (t-1)(m-t+2)\binom{m}{t-2}.\end{aligned}$$

We will derive a lower bound for the left-hand side of the above inequlity. Let y_j denote $d(j')$. For $y_i, y_j \ge t-2$, one has

$$\begin{aligned}&[y_i(y_i-1)\cdots(y_i-(t-3)) - y_j(y_j-1)\cdots(y_j-(t-3))] \\ &\quad \times [(y_i-(t-2)) - (y_j-(t-2))] \ge 0,\end{aligned}$$

which yields

$$\left[\binom{y_i}{t-2} - \binom{y_j}{t-2}\right] [(y_i - (t-2)) - (y_j - (t-2))] \geq 0.$$

Therefore

$$\begin{aligned} &\binom{y_i}{t-2}(y_j - (t-2)) + \binom{y_j}{t-2}(y_i - (t-2)) \\ \leq &\binom{y_i}{t-2}(y_i - (t-2)) + \binom{y_j}{t-2}(y_j - (t-2)) \\ = &(t-1)\left[\binom{y_i}{t-1} + \binom{y_j}{t-1}\right]. \end{aligned}$$

Adding up over pairs $\{y_i, y_j\} = \{d(i'), d(j')\}$ for all $1 \leq i, j \leq n$, we have

$$\left(\sum_{j=1}^{n} \binom{d(j')}{t-2}\right)\left(\sum_{j=1}^{n}(d(j') - (t-2))\right) \leq n(t-1)\sum_{j=1}^{n}\binom{d(j')}{t-1}.$$

Combining this with what obtained, we have

$$\begin{aligned} &\frac{1}{n}\left(\sum_{j=1}^{n}\binom{d(j')}{t-2}\right)\left(\sum_{j=1}^{n}(d(j') - (t-2))\right) \\ &\quad \leq (s-t+1)^{1/t}(m-t+2)^{1-1/t}\sum_{j=1}^{n}\binom{d(j')}{t-2} \\ &\qquad + (t-1)(m-t+2)\binom{m}{t-2}. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{j=1}^{n} d(j') - n(t-2) \leq &(s-t+1)^{1/t}(m-t+2)^{1-1/t} n \\ &+ (t-1)(m-t+2)\frac{n\binom{m}{t-2}}{\sum_{j=1}^{n}\binom{d(j')}{t-2}}. \end{aligned}$$

Note that $e(G) = \sum_{j=1}^{n} d(j')$, we have

$$e(G) \leq (s-t+1)^{1/t} n m^{1-1/t} + tn + tm\frac{n\binom{m}{t-2}}{\sum_{j=1}^{n}\binom{d(j')}{t-2}}.$$

If the last fraction is at most $m^{1-2/t}$, we are done. Otherwise, we have

$$\sum_{j=1}^{n}\binom{d(j')}{t-2} < \frac{n}{m^{1-2/t}}\binom{m}{t-2}.$$

Applying Lemma 7.4 with $p = n$, $c = n/m^{1-2/t}$, $x_0 = m$, and $x_i = d(i')$, and replacing t by $t-2$, we obtain that

$$\sum_{j=1}^{n} d(j') < m\left(\frac{n}{m^{1-2/t}}\right)^{1/(t-2)} n^{1-1/(t-2)} + (t-3)n = nm^{1-1/t} + (t-3)n,$$

so $e(G) \le nm^{1-1/t} + (t-3)n$, the desired result follows. □

Corollary 7.1 *For any fixed integers $s \ge t \ge 1$,*

$$ex(n, K_{t,s}) \le \left(\frac{1}{2} + o(1)\right)(s-t+1)^{1/t} n^{2-1/t}.$$

Recall that the k-color Ramsey number $r_k(G)$ is the minimum integer N such that any k-color of edges of K_N contains a monochromatic G. We now discuss a relation between $ex(n, G)$ and $r_k(G)$, where G is a bipartite graph. Note that if $ex(n, G) \le (c + o(1))\frac{1}{2}n^{2-1/t}$ for some constants $c > 0$ and $t \ge 1$, then G must be a bipartite graph.

Theorem 7.8 *If $ex(n, G) \le (c + o(1))\frac{1}{2}n^{2-1/t}$ where $c > 0$ and $t \ge 1$ are fixed, then*

$$r_k(G) \le (1 + o(1))(ck)^t$$

as $k \to \infty$.

Proof. Setting $n = r_k(G) - 1$. Thus there exists a k-coloring of edges of K_n such that there is no monochromatic G. By considering each subgraph induced by edges in a single color and the definition of Turán number, we have

$$\binom{n}{2} \le k\, ex(n, G) \le (c + o(1))\frac{k}{2}n^{2-1/t},$$

which yields $n \le (1 + o(1))(ck)^t$ as required. □

Corollary 7.2 *For any fixed integers s and t with $s \ge t \ge 2$,*

$$r_k(K_{t,s}) \le (1 + o(1))(s-t+1)k^t. \quad (k \to \infty)$$

The idea of the proof of the following result is due to Alon, see Chung and Graham (1999).

Theorem 7.9 *If for large n, $ex(n, G) \ge c_1 n^{2-1/t}$, where $c_1 > 0$ is a constant and $t \ge 1$ is a fixed integer, then there exists a constant $c_2 > 0$ such that*

$$r_k(G) \geq c_2 \left(\frac{k}{\log k}\right)^t$$

as $k \to \infty$.

Proof. Set $c_2 = (c_1/t)^t$ and $n = \lfloor c_2(k/\log k)^t \rfloor$. Let H denote a graph on n vertices with $e(H) = ex(n, G)$ edges containing no G. Let $H_1, H_2, \ldots, H_k$ be k copies of H placed randomly and independently in the complete graph K_n on n vertices. For each edge of K_n, the probability that the edge is not covered by any graph H_i is precisely

$$p = \left(1 - e(H) \Big/ \binom{n}{2}\right)^k.$$

If $\binom{n}{2}p < 1$, then there is a choice of placing these H_i so that their union covers all edges of K_n. By referring each edge in H_i as one in color i, then any edge of K_n is colored in at least one color. Keep one color for each edge and delete other colors if the edge got more than one colors, then the edges of K_n are colored by k colors, and there is no monochromatic G. Thus $r_k(G) > n$. Now we have

$$\frac{e(H)}{\binom{n}{2}} > \frac{2e(H)}{n^2} \geq \frac{2c_1 n^{2-1/t}}{n^2} = \frac{2c_1}{n^{1/t}} \geq \frac{2c_1 \log k}{c_2^{1/t} k} = 2t \frac{\log k}{k}.$$

This and the fact $1 - x < e^{-x}$ for $x > 0$ imply that

$$\left(1 - e(H) \Big/ \binom{n}{2}\right)^k \leq \left(1 - 2t \frac{\log k}{k}\right)^k \leq \left(\frac{1}{k}\right)^{2t},$$

which yields

$$\binom{n}{2}p \leq \frac{c_2^2}{2} \left(\frac{k}{\log k}\right)^{2t} \left(\frac{1}{k}\right)^{2t} \to 0$$

as $k \to \infty$. This completes the proof. □

It is a widespread belief that the order $n^{2-1/t}$ in the upper bound of $ex(n, K_{t,s})$ is sharp if $s \geq t$ are fixed and $n \to \infty$. If so, then the order of $r_k(K_{t,s})$ is between $(\frac{k}{\log k})^t$ and k^t from the above results. However, when we have a construction to give a lower bound of form $ex(n, K_{t,s}) \geq c_1 n^{2-1/t}$, we often get a lower bound of form $r_k(K_{t,s}) \geq c_2 k^t$, see the forthcoming sections.

7.3 Erdős-Rényi Graph

The starting point of a problem involving complete bipartite graph is usually $C_4 = K_{2,2}$. We begin with a construction of a graph by Erdős and Rényi (1962) (one can see also Erdős, Rényi and Sós (1966) or Brown (1966)), which contains no C_4. This will lead to a tight lower bound for $ex(n, C_4)$.

Let $F = F_q$ be the Galois field with q elements. Define an equivalence relation $\equiv$ on $(F^3)^* = F^3 \setminus \{(0,0,0)\}$ by letting $(a_1, a_2, a_3) \equiv (b_1, b_2, b_3)$ if there is an element $\lambda \in F^* = F \setminus \{0\}$ such that $(a_1, a_2, a_3) = \lambda(b_1, b_2, b_3)$. Let $\langle a_1, a_2, a_3 \rangle$ denote the equivalence class containing (a_1, a_2, a_3), and let V be the set of all equivalence classes.

Now, we define the Erdős-Rényi graph ER_q on vertex set V, in which two distinct vertices $\langle a_1, a_2, a_3 \rangle$ and $\langle x_1, x_2, x_3 \rangle$ are adjacent if and only if

$$a_1x_1 + a_2x_2 + a_3x_3 = 0.$$

This definition is clearly compatible, i.e., it does not depend on the choice of representative elements of the equivalence classes. It is trivial to see that

$$|V| = \frac{q^3 - 1}{q - 1} = q^2 + q + 1.$$

For a vertex $A = \langle a_1, a_2, a_3 \rangle$, since $a_1x_1 + a_2x_2 + a_3x_3 = 0$ has $q^2 - 1$ solutions forming $q + 1$ vertices,

$$d(A) = \begin{cases} q & \text{if } a_1^2 + a_2^2 + a_3^2 = 0, \\ q + 1 & \text{otherwise.} \end{cases}$$

We now come to the point to see the most important fact on ER_q.

Theorem 7.10 *The graph ER_q contains no C_4.*

Proof. Let $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ be distinct vertices. From the definition of ER_q, the vectors (a_1, a_2, a_3) and (b_1, b_2, b_3) are linearly independent. Therefore, the equation system

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 = 0, \\ b_1x_1 + b_2x_2 + b_3x_3 = 0, \end{cases}$$

has exactly $q - 1$ solutions forming only one vertex. So the assertion follows. □

Let $n = q^2 + q + 1$ and let $e(ER_q)$ be the number of edges of ER_q. Then, as $q \to \infty$,

$$ex(n, C_4) \geq e(ER_q) \sim \frac{1}{2}q(q^2 + q + 1) \sim \left(\frac{1}{2} + o(1)\right) n^{3/2}.$$

This together with Theorem 7.5 established by Kővári, Sós and Turán (1954) yield that

$$ex(n, C_4) \sim \left(\frac{1}{2} + o(1)\right) n^{3/2}.$$

Now let us associate the graph ER_q with a more general construction, which is a $(q + 1)$-uniform hypergraph $(X, \mathcal{L})$ called projective plane. However, the members in $\mathcal{L}$ are called lines, and the order of such a plane does not mean the cardinality of X.

Recall a *projective plane* of order q, denoted by $PG(2,q)$, consists of a set X of q^2+q+1 elements called *points*, and a family $\mathcal{L}$ of subsets of X called *lines*, having the following properties:

(P1) Every line has $q+1$ points.
(P2) Any pair of distinct points lie on a unique line.

The only possible projective plane of order $q = 1$ is a triangle. The unique projective plane of order $q = 2$ is the famous *Fano plane*. It contains 7 points, 7 lines, in which each line has 3 points, see Fig 8.1.

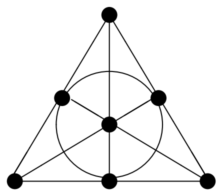

Fig. 8.1 The Fano plane

We restate the additional properties of projective planes as follows, and one can see the proof in Lemma 6.2.

Lemma 7.5 *A projective plane of order q has the properties as follows.*

(P3) Any point lies on $q+1$ *lines.*
(P4) There are q^2+q+1 *lines.*
(P5) Any two lines meet at a unique point.

A nice property of projective planes is their duality. Let $(X,\mathcal{L})$ be a projective plane of order q, and let $M = (m_{x,L})$ be its incidence matrix, in which the rows and columns correspond to points and lines. Each row and column of M has exactly $(q+1)$ $1'$s, and any two rows and any two columns share exactly one 1.

Return to the graph ER_q, whose vertex set is V consisting of q^2+q+1 points (equivalence classes in $(F_q^3)^*$). Let $\langle a_1,a_2,a_3\rangle$ be a point of V which has been defined as above. Define a line $L(a_1,a_2,a_3)$ to be the set of all points $\langle x_1,x_2,x_3\rangle$ in V (not vectors (x_1,x_2,x_3) in $(F_q^3)^*$) for which

$$a_1x_1+a_2x_2+a_3x_3=0.$$

It is easy to see that the definition for lines is compatible, and each line contains exactly $q+1$ points. Note that some lines $L(a_1,a_2,a_3)$ contain point $\langle a_1,a_2,a_3\rangle$ and some do not. Any pair of distinct points $\langle x_1,x_2,x_3\rangle$ and $\langle y_1,y_2,y_3\rangle$ lie on a unique line $L(a_1,a_2,a_3)$ with

$$\begin{cases} a_1x_1+a_2x_2+a_3x_3 = 0, \\ a_1y_1+a_2y_2+a_3y_3 = 0. \end{cases}$$

Therefore, we obtain a projective plane $(V,\mathcal{L})$, where $\mathcal{L}$ consists of all lines defined as above. This projective plane is usually denoted by $PG(2,q)$. Some authors use

$PG(2, q)$ to signify the Erdős-Rényi graph or a bipartite graph, whose bipartition are points and lines, in which a point is adjacent to a line if and only if the point is contained in the line.

No projective plane of order non-prime power is known to exist, and it is conjectured that there is none. It is known that there is no projective plane of order 6, 10 or 14. It is not known whether there is a projective plane of order 12.

The following result gives the exact expression of the edge number $e(ER_q)$ for $q = p^m$, where p is a prime and m is odd. In fact, the assertion holds for any prime power q.

Lemma 7.6 *Let $q = p^m$, where p is a prime and m is odd. There are precisely $q^2 - 1$ non-zero solutions (x_1, x_2, x_3) of the equation*

$$x_1^2 + x_2^2 + x_3^2 = 0$$

in F_q, and hence precisely $q+1$ vertices in ER_q incident with loops. In particular, the eigenvalues of ER_q are $q + 1$, $\pm\sqrt{q}$ with multiplicity 1, and $(q^2 + q)/2$ respectively.

Proof. Label the vertex set of ER_q as

$$V(ER_q) = \{A, B, \cdots, X, \cdots, Y, \cdots, Z\}$$

in some order. We write $X \perp Y$ if and only if $x_1y_1 + x_2y_2 + x_3y_3 = 0$, where $X = \langle x_1, x_2, x_3\rangle$ and $Y = \langle y_1, y_2, y_3\rangle$. Let $n = q^2 + q + 1$ and define an $n \times n$ real matrix $M = (m_{ij})$ by

$$m_{ij} = \begin{cases} 1 \text{ if } X \perp Y, \\ 0 \text{ otherwise,} \end{cases}$$

where X and Y represent the ith vertex and the jth vertex, respectively. We admit $m_{ii} = 1$ if $X \perp X$, that is, X lies on the conic $x_1^2 + x_2^2 + x_3^2 = 0$. All that remains to show is that

$$tr(M) = q + 1,$$

where $tr(M) = \sum_{i=1}^{n} m_{ii}$ is the trace of M. We know that the trace equals the sum of eigenvalues.

Fact 1 Any row of M contains precisely $q + 1$ ones hence $q + 1$ is an eigenvalue of M.

Fact 2 For $i \neq j$, there is exactly one column with 1 in both the ith row and the jth row. Namely, $M_i \cdot M_j = 1$, where M_i and M_j are the ith row and the jth row of M, respectively.

Proof. Suppose that X and Y represent (the vertices) the ith row and the jth row, respectively. Then there is a unique (vertex) row, say the kth row, corresponding to the solution (w_1, w_2, w_3) to the equation system

$$\begin{cases} x_1w_1 + x_2w_2 + x_3w_3 = 0, \\ y_1w_1 + y_2w_2 + y_3w_3 = 0. \end{cases}$$

That is to say, $m_{ik} = m_{jk} = 1$. Note that M is symmetric, so we see that only in the kth column, the elements in both the ith row and the jth row are 1. □

Using these two facts and the symmetry of M, we have

$$M^2 = \begin{pmatrix} q+1 & 1 & \cdots & 1 & 1 \\ 1 & q+1 & \cdots & 1 & 1 \\ \vdots & & & & \vdots \\ 1 & 1 & \cdots & 1 & q+1 \end{pmatrix} = qI + J,$$

where I is the identity matrix and J is the all-ones-matrix. It is easy to see J has the eigenvalues $n = q^2 + q + 1$ with multiplicity 1 and 0 with multiplicity $n - 1 = q^2 + q$. It follows that M^2 has the eigenvalues $q + n = (q+1)^2$ with multiplicity 1 and q with multiplicity $n - 1 = q^2 + q$.

Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of M. Therefore, $\lambda_1 = q+1$ as $q+1$ is an eigenvalue of M with multiplicity 1 from Perron-Frobenius Theorem, and $\lambda_i = \pm\sqrt{q}$ for $i = 2, \ldots, n$. Let s and t be the numbers of eigenvalues of M equal to $\sqrt{q}$ and $-\sqrt{q}$, respectively. Thus $s + t = n - 1$ and

$$tr(M) = (q+1) + (s-t)\sqrt{q}.$$

Since the trace is an integer, we must have $s = t = (n-1)/2 = (q^2+q)/2$ and hence $tr(M) = q + 1$. □

The following result follows easily.

Theorem 7.11 *For any odd prime power q, $e(ER_q) = \frac{1}{2}q(q+1)^2$.*

Proof. By Lemma 7.6, we obtain that

$$e(ER_q) = \frac{1}{2}\Big((q+1)(q^2+q+1) - (q+1)\Big) = \frac{1}{2}q(q+1)^2$$

as required. □

Let $n = q^2 + q + 1$. Hence $q = (\sqrt{4n-3} - 1)/2$ and

$$ex(n, C_4) \geq \frac{1}{4}(n-1)(1 + \sqrt{4n-3}).$$

This is very close to the upper bound $ex(n, C_4) \leq \frac{1}{4}n(1 + \sqrt{4n-3})$ obtained in Theorem 7.6. We will show the lower bound gives the quality for infinite many $n = q^2 + q + 1$ in the next section.

Let us have one more property of graph ER_q.

Lemma 7.7 *All vertices of degree q in graph ER_q are independent.*

Proof. Suppose that $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ are distinct vertices of degree q in ER_q. Then the vectors (a_1, a_2, a_3) and (b_1, b_2, b_3) are linearly independent in

F^3. Therefore, the dimension of the subspace S of F^3 consisting of all solutions (x_1, x_2, x_3) to the equation system

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 = 0, \\ b_1x_1 + b_2x_2 + b_3x_3 = 0 \end{cases}$$

is one. If $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ are adjacent, then both (a_1, a_2, a_3) and (b_1, b_2, b_3) would be in S, a contradiction. □

Among all graphs of the same order that contain no C_4, the extremal graphs for $ex(n; C_4)$ have the largest average degree. So they are expected to give good lower bounds for $r(C_4, K_{1,n})$ if they are near regular. The following results are due to Parsons (1975).

Lemma 7.8 *Let q be a prime power. Then*

$$r(C_4, K_{1,q^2}) \geq q^2 + q + 1,$$
$$r(C_4, K_{1,q^2+1}) \geq q^2 + q + 2.$$

Proof. The graph ER_q has $q^2 + q + 1$ vertices. It contains no C_4 and its minimum degree is q, so the maximum degree of $\overline{ER_q}$ is q^2. This proves the second lower bound in the lemma.

For the first lower bound, let v be a vertex of degree q in the graph ER_q, by Lemma 7.7, each neighbor of v has degree $q + 1$. Deleting the vertex v, we obtain a graph with q^2+q vertices and minimum degree q, so its complement has a maximum degree $q^2 - 1$. This proves the first lower bound. □

Lemma 7.9 *Let $n \geq 2$ be an integer. Then*

$$r(C_4, K_{1,n}) \leq n + \sqrt{n-1} + 2.$$

Proof. Let $N = r(C_4, K_{1,n}) - 1$ and let G be a graph of order N that contains no C_4 and its complement $\overline{G}$ has the maximum degree at most $n - 1$. Note that $r(C_4, K_{1,2}) = 4$ and $r(C_4, K_{1,3}) = 6$, so we suppose that $n \geq 3$ and $N \geq 5$. The fact $\Delta(\overline{G}) \leq n - 1$ implies that for any vertex v of G, $d(v) \geq N - n$. Since the fact $G \not\supseteq C_4$ implies that for any distinct vertices u and v of G, $|N(u) \cap N(v)| \leq 1$, we thus have

$$\sum_{w\in V(G)} \binom{d(w)}{2} \leq \binom{N}{2}. \tag{7.3}$$

This and $d(w) \geq N - n$ give

$$N^2 - 2(n+1)N + n^2 + n + 1 \leq 0. \tag{7.4}$$

The above inequality becomes an equality if and only if G is regular with degree $N - n$ and any pair of vertices have exactly one common neighbor. We now need the following result known as "the Friendship Theorem" of Erdős, Rényi and Sós

(1966). Call the graph $F_k = K_1 + kK_2$ a *Friendship graph* or a *k-fan*, which consists of k triangles with a vertex in common.

Lemma 7.10 (Friendship Theorem) *Let G be a graph with N vertices such that any pair of vertices is joined by exactly one path of length two in G. Then $N = 2k+1$ and $G = F_k$.*

Return to our current proof. If the inequality (7.3) is an equality, then by Lemma 7.10, $G = F_n$, but G is regular, contradicting to $n \geq 3$. So this is not the case and hence the inequality (7.4) is strict, which implies that

$$N^2 - 2(n+1)N + n^2 + n + 2 \leq 0,$$

yielding $r(C_4, K_{1,n}) = N + 1 \leq n + \sqrt{n-1} + 2$. □

Theorem 7.12 *Let q be a prime power. Then*

$$r(C_4, K_{1,q^2}) = q^2 + q + 1,$$
$$r(C_4, K_{1,q^2+1}) = q^2 + q + 2.$$

Proof. The first assertion follows from Lemma 7.8 and Lemma 7.9. However, the second assertion needs more careful analysis, we refer the reader to Parsons (1975) for details for this case. □

Recently, there are more exact values on Ramsey numbers of C_4 versus stars that were obtained, see two papers of Zhang, Chen and Cheng (2017).

7.4 Exact Values of $ex(n, C_4)$ and $z(n; 2)$

In the last section, we have obtained that $ex(n, C_4) \sim n^{3/2}/2$ and hence $z(n; 2) \sim n^{3/2}$. It is hopeless to find all exact values of $ex(n, C_4)$ and $z(n; 2)$ because of the difficulty of constructions for the lower bounds and estimating for exact upper bounds. However, it is possible to be lucky to find such values for infinitely many n.

Let us begin with $z(n; 2)$. The following result is due to Alon and Spencer (1992), in which the definition of the projective plane was introduced in the last section.

Theorem 7.13 *Let q be an integer. If there exists a projective plane $(X, \mathcal{L})$ of order q, then*

$$z(n; 2) = (q^2 + q + 1)(q + 1).$$

Proof. Denote $n = q^2 + q + 1$, which is the number of points in X. Define a bipartite graph G_P with bipartition $(X, \mathcal{L})$ by letting $x \in X$ be adjacent to $L \in \mathcal{L}$ if and only if the point x is on the line L. As two points cannot lie on two lines, we have G_P contains no C_4, and so $z(n; 2) \geq e(G_P)$. Hence the lower bound follows by noting $e(G_P) = (q^2 + q + 1)(q + 1)$.

Now, let G be a bipartite graph on partition (T, B) containing no C_4, where $|T| = |B| = n = q^2 + q + 1$. We shall show that $e(G) \leq (q^2 + q + 1)(q + 1)$. Let b_1, b_2 be a uniformly selected pair of distinct vertices of B. For $t \in T$, let $N(t)$ be the neighborhood of t and $d(t) = |N(t)|$. For a fixed $t \in T$, let I_t be the indicator random variable for t being adjacent to both b_1 and b_2, and let $X = \sum_{t \in T} I_t$. Since t is adjacent to both b_1 and b_2 if and only if both b_1 and b_2 are chosen from $N(t)$, we obtain that

$$E[X] = \sum_{t \in T} E[I_t] = \sum_{t \in T} \binom{d(t)}{2} \Big/ \binom{n}{2}.$$

Note that X is the number of vertices $t \in T$ adjacent to both b_1 and b_2, thus $X \leq 1$ as G contains no C_4. Let $\overline{d} = \frac{1}{n} \sum_{t \in T} d(t)$ be the average degree. Convexity of the function $\binom{y}{2}$ gives

$$\sum_{t \in T} \binom{d(t)}{2} \Big/ \binom{n}{2} \geq n \binom{\overline{d}}{2} \Big/ \binom{n}{2}$$

where the equality holds if and only if all vertices of T have the same degree. Now

$$1 \geq \max X \geq E[X] \geq n \binom{\overline{d}}{2} \Big/ \binom{n}{2},$$

hence

$$\overline{d}(\overline{d} - 1) \leq n - 1,$$

yielding

$$e(G) = n\overline{d} \leq n \frac{1 + \sqrt{1 + 4(n - 1)}}{2} = (q^2 + q + 1)(q + 1)$$

as desired. □

We then turn to find exact values of $ex(n, C_4)$ for $n = q^2 + q + 1$, which will be $\frac{1}{2}q(q + 1)^2$ slightly smaller than $\frac{1}{2}z(n; 2)$ that we just obtained. Recall that the Erdős-Rényi graph ER_q has $n = q^2 + q + 1$ vertices and $q(q + 1)^2/2$ edges, which together with the upper bound obtained in Theorem 7.6 give

$$\frac{1}{4}(n - 1)(1 + \sqrt{4n - 3}) \leq ex(n, C_4) \leq \frac{1}{4}n(1 + \sqrt{4n - 3})$$

when $n = q^2 + q + 1$, i.e.,

$$\frac{1}{2}q(q + 1)^2 \leq ex(n, C_4) \leq \frac{1}{2}(q^2 + q + 1)(q + 1).$$

Erdős, Rényi and Sós (1966) proved that the second inequality is strict, and Erdős (1966) conjectured that the first inequality is an equality. This conjecture was confirmed by Füredi (1996(c)), who obtained a partial answer for $q = 2^k$ in 1983.

Theorem 7.14 *Let $q > 13$ be a prime power. If $n = q^2 + q + 1$, then*

$$ex(n, C_4) = \frac{1}{2}q(q+1)^2.$$

To avoid some burden with extra background for the proof, we only introduce the proof of Füredi for the special case where q is a power of 2.

Lemma 7.11 *Let $q \geq 2$ be an even integer. If G is a C_4-free graph of order q^2+q+1 and $\Delta(G) \leq q+1$, then any vertex of degree $q+1$ is adjacent to at least one vertex of degree of q or less. Consequently, G has at least $q+1$ vertices of degree q or less.*

Proof. Let $v \in V(G)$ be a vertex with $d(v) = q+1$. In the subgraph of G induced by $N(v)$, there is no vertex with degree two or more, thus

$$|E(N(v))| \leq \left\lfloor \frac{q+1}{2} \right\rfloor = \frac{q}{2}$$

as q is even. Noticing that any pair of vertices in $N(v)$ have the unique neighbor v in common in G, and $V(G) \setminus N[v]$ contains q^2-1 vertices, where $N[v] = N(v) \cup \{v\}$, we have

$$\begin{aligned}\sum_{x \in N(v)} d(x) &\leq d(v) + 2|E(N(v))| + |V(G) \setminus N[v]| \\ &\leq (q+1) + q + (q^2-1) = (q+1)^2 - 1.\end{aligned}$$

Thus there exists at least one vertex $x \in N(v)$ with $d(x) \leq q$ as claimed. Set $S = \{x \in V(G) : d(x) \leq q\}$. Note that any vertex of degree $q+1$ is adjacent to at least one vertex of S from what just proved, we get

$$\bigcup_{x \in S} N[x] = V(G).$$

Thus $(q+1)|S| \geq q^2+q+1$, impling that $|S| \geq q+1$ as desired. □

Lemma 7.12 *Let q be an integer. If G is a C_4-free graph of order q^2+q+1 and $\Delta(G) \geq q+2$, then $e(G) \leq q(q+1)^2/2$.*

Proof. Set

$$V = V(G) = \{v_1, v_2, \ldots, v_n\},$$

where $n = q^2+q+1$, and $d(v_1) = \Delta \geq q+2$. Since G contains no C_4, we obtain that

$$|N(v_i) \setminus N(v_1)| \geq |N(v_i)| - 1 = d(v_i) - 1$$

for $2 \leq i \leq n$. Then

$$\begin{aligned}\binom{n-\Delta}{2} &= (\text{number of pairs of } V \setminus N(v_1))\\ &\geq \sum_{2\leq i\leq n} (\text{number of pairs of } N(v_i) \cap (V \setminus N(v_1))\\ &\geq \sum_{2\leq i\leq n} \binom{d(v_i)-1}{2}.\end{aligned}$$

Suppose to the contrary that $e(G) > q(q+1)^2/2$. Since $q(q+1)^2/2$ is an integer, we must have $e(G) \geq q(q+1)^2/2+1$, i.e., $2e(G) \geq (n-1)(q+1)+2$. By Jensen's inequality, we have

$$\begin{aligned}\binom{n-\Delta}{2} &\geq (n-1)\binom{\sum_{2\leq i\leq n}(d(v_i)-1)/(n-1)}{2}\\ &= (n-1)\binom{[2e(G)-(n-1)-\Delta]/(n-1)}{2}\\ &\geq (n-1)\binom{[(n-1)q+2-\Delta]/(n-1)}{2}.\end{aligned}$$

This yields

$$\begin{aligned}&(n-1)(n-\Delta)(n-\Delta-1)\\ \geq&[(n-1)q+2-\Delta][(n-1)(q-1)+2-\Delta].\end{aligned} \tag{7.5}$$

Note that the fact $\Delta q > (q+2)q-1 = n+q-2$ implies

$$(q+1)(n-\Delta) < (n-1)q+2-\Delta,$$

and the fact $\Delta(q-1) \geq (q+2)(q-1) = n-3$ does

$$q(n-\Delta-1) \leq (n-1)(q-1)+2-\Delta.$$

Multiplying the left-hand sides and the right-hand sides of both above inequalities, respectively, we have an inequality contradicting to (7.5). □

Proof of Theorem 7.14 for $q = 2^k$. The Erdős-Rényi graph ER_q gives that $ex(n, C_4) \geq q(q+1)^2/2$. On the other hand, if G is a C_4-free graph of order $n = q^2+q+1$ with q a power of 2, by Lemma 7.12, we may assume that $\Delta(G) \leq q+1$. Thus, by Lemma 7.11, there are at least $q+1$ vertices of degree q or less. Hence

$$2e(G) \leq (q+1)n-(q+1) = q(q+1)^2,$$

the desired equality follows immediately. □

For any positive integer $n = q^2+q+1$ where q is a prime power, Theorem 7.6 and Theorem 7.14 imply that

$$ex(n, C_4) = \frac{1}{2}n^{3/2} + \frac{n}{4} - O(n^{1/2}).$$

A conjecture of Erdős states that for all large n,

$$ex(n, C_4) = \frac{1}{2}n^{3/2} + \frac{n}{4} + o(n^{1/2}).$$

However, this conjecture does not hold in general. Indeed, Ma and Yang (2021) have proved that there exist some real $\epsilon > 0$ and a positive density of integers n such that

$$ex(n, C_4) \le \frac{1}{2}n^{3/2} + \left(\frac{1}{4} - \epsilon\right) n.$$

Here ϵ would be taken as any positive real less than 0.0375.

Let us return to the graphs forbidding C_4 by considering the Ramsey numbers of C_4 in many colors. The following construction of Lazebnik and Woldar (2000) yields a lower bound for $r_k(C_4)$, in which the equality holds for $k = 2$ and $k = 3$. Any monochromatic graph in the construction yields a lower bound for $ex(n, C_4)$.

Theorem 7.15 *If k is a prime power, then*

$$r_k(C_4) \ge k^2 + 2.$$

Proof. Let F_k be the field of k elements and let $V = F_k \times F_k$. Any vertex $v \in V$ can be written as a vector (v_1, v_2), which is distinct to the set $\{v_1, v_2\}$. Let u be an additional vertex out of V. We shall color the edges of K_{k^2+1} on vertex set $V \cup \{u\}$ with k colors so that there is no monochromatic C_4. We will do so for all edges in V first. Let $e = \{(a_1, a_2), (b_1, b_2)\}$ be an edge with both end vertices in V. We assign e with the color α, where

$$\alpha = a_1 b_1 + a_2 + b_2.$$

We claim that there is no monochromatic C_4. Since otherwise, suppose that (a_1, a_2), (b_1, b_2), (c_1, c_2), and (d_1, d_2) are four consecutive (distinct) vertices of a C_4 in some color α. Thus

$$\begin{aligned}\alpha &= a_1 b_1 + a_2 + b_2 = b_1 c_1 + b_2 + c_2 \\ &= c_1 d_1 + c_2 + d_2 = d_1 a_1 + d_2 + a_2,\end{aligned}$$

yielding $(a_1 - c_1)(b_1 - d_1) = 0$. So either $a_1 = c_1$ or $b_1 = d_1$, which imply either $(a_1, a_2) = (c_1, c_2)$ or $(b_1, b_2) = (d_1, d_2)$, a contradiction.

It remains to color the edges of form $\{u, v\}$ with $v = (v_1, v_2) \in V$. For such an edge, we assign it with color v_1, the first coordinate of v. Suppose that there is a monochromatic C_4 in color α, which must contain the vertex u. Let (α, a_2), (b_1, b_2) and (α, c_2) be the other three vertices in the C_4. We have that

$$\alpha = \alpha b_1 + a_2 + b_2 = \alpha b_1 + b_2 + c_2.$$

Thus $a_2 = c_2$, which implies that the vertices (α, a_2) and (α, c_2) are identical, a contradiction. This completes the proof. □

Combining the above Theorem and the upper bound obtained in Corollary 7.2, we obtain the asymptotic formula of $r_k(C_4)$ as follows.

Theorem 7.16 *As* $k \to \infty$, $r_k(C_4) \sim k^2$.

7.5 Constructions with Forbidden $K_{2,s}$

The following construction is due to Füredi (1996(b)), and see also Axenovich, Füredi and Mubayi (2000) with slightly different, which gives lower bounds for $r_k(K_{2,\,s+1})$ and $ex(n, K_{2,\,s+1})$.

Theorem 7.17 *For any fixed integer* $s \geq 1$, $r_k(K_{2,\,s+1}) \sim s\,k^2$ *as* $k \to \infty$.

Proof. The desired upper bound is in the last Chapter. We need to show that

$$r_k(K_{2,\,s+1}) \geq (1 - o(1))s\,k^2$$

as $k \to \infty$. Let q be a prime power such that $k = (q-1)/s$ is an integer. Set $n = (q-1)^2/s = sk^2$. We will color all edges of K_n with slightly more than k colors such that there is no monochromatic $K_{2,\,s+1}$.

Let $F = F_q$ be the q-element finite field, and let $h \in F$ be an element of order s, and

$$H = \{1, h, \ldots, h^{s-1}\}.$$

Denote the cosets of H by

$$H_1, H_2, \ldots, H_k,$$

which partition $F^* = F \setminus \{0\}$. We introduce an equivalence relation "$\equiv$" in $F^* \times F^*$ as $(a_1, a_2) \equiv (x_1, x_2)$ if $(a_1, a_2) = h^t(x_1, x_2)$ for some $h^t \in H$. The equivalence class represented by (a_1, a_2) is denoted by $\langle a_1, a_2\rangle$. Let V be the set of all equivalence classes and $n = |V| = (q-1)^2/s = sk^2$. Consider the complete graph K_n with vertex set V. Color the edge joining two vertices $\langle a_1, a_2\rangle$ and $\langle x_1, x_2\rangle$ with color i if $a_1x_1 + a_2x_2 \neq 0$ and $a_1x_1 + a_2x_2 \in H_i$. Clearly the definition for the coloring is compatible with the equivalence class, that is to say, $a_1x_1 + a_2x_2 \in H_i$, $(a_1, a_2) \equiv (b_1, b_2)$ and $(x_1, x_2) \equiv (y_1, y_2)$ imply $b_1y_1 + b_2y_2 \in H_i$. Note that the edges of form $\{\langle a_1, a_2\rangle, \langle x_1, x_2\rangle\}$ with $a_1x_1 + a_2x_2 = 0$ are still uncolored.

Let G_i denote the graph induced by all edges in color i. We shall show that G_i contains no $K_{2,\,s+1}$. Let $\langle a_1, a_2\rangle$ and $\langle b_1, b_2\rangle$ be a pair of distinct vertices, and consider the equation system

$$\begin{cases} a_1x_1 + a_2x_2 = u, \\ b_1x_1 + b_2x_2 = v. \end{cases} \tag{7.6}$$

We claim the system has at most one solution (x_1, x_2) for every $u, v \in H_i$. Indeed, the solution is unique if the determinant of the system is not 0. Otherwise, there exists $\lambda \in F^*$ such that $(b_1, b_2) = \lambda(a_1, a_2)$. If the system (7.6) has solution (x_1, x_2), then $\lambda u = v$ hence $\lambda = v/u \in H$, contradicting to the fact that (a_1, a_2) and (b_1, b_2) are not equivalent. Finally, there are s^2 possibilities for $u, v \in H_i$ in (7.6). The set of solutions form s equivalence classes hence s vertices. So there are at most s vertices $\langle x_1, x_2\rangle$ joined simultaneously to $\langle a_1, a_2\rangle$ and $\langle b_1, b_2\rangle$.

Now turn to the uncolored edges $\{\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\}$ with $a_1b_1 + a_2b_2 = 0$. Let G_0 be the graph induced by these edges. We are going to color the edges of G_0 by an additional $O(\sqrt{k})$ colors. We first show that G_0 is a union of $K_{k,k}$ and K_k.

Note that the equation

$$a_1x_1 + a_2x_2 = 0$$

has $q-1$ solutions forming $k = (q-1)/s$ equivalence classes hence k vertices, thus vertex $\langle a_1, a_2\rangle$ of G_0 has degree k if $a_1^2 + a_2^2 \neq 0$ and degree $k-1$ if $a_1^2 + a_2^2 = 0$. Let $\{\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\}$ be an edge of G_0, then $a_1b_1 + a_2b_2 = 0$. Set

$$V_1 = \{\langle x_1, x_2\rangle : a_1x_1 + a_2x_2 = 0\} \text{ and } V_2 = \{\langle y_1, y_2\rangle : b_1y_1 + b_2y_2 = 0\}.$$

Then $\langle a_1, a_2\rangle \in V_2$ and $\langle b_1, b_2\rangle \in V_1$, and $|V_1| = |V_2| = k$. For any $\langle x_1, x_2\rangle \in V_1$ and $\langle y_1, y_2\rangle \in V_2$, we have

$$\begin{aligned} x_1y_1 + x_2y_2 &= x_1y_2\left(\frac{y_1}{y_2} + \frac{x_2}{x_1}\right) = x_1y_2\left(-\frac{b_2}{b_1} - \frac{a_1}{a_2}\right) \\ &= -\frac{x_1y_2}{a_2b_2}(a_1b_1 + a_2b_2) = 0. \end{aligned}$$

So V_1 and V_2 are completely connected in G_0. If $V_1 \cap V_2 = \emptyset$, then they induce a complete bipartite graph $K_{k,k}$ that is not connecting to any other vertex of G_0 since the maximum degree of G_0 is k. If $V_1 \cap V_2 \neq \emptyset$, $\langle z_1, z_2\rangle \in V_1 \cap V_2$, then we see that $z_1^2 + z_2^2 = 0$ and the degree of $\langle z_1, z_2\rangle$ in G_0 is $k-1$. Since it is adjacent to all other vertices in V_1 and V_2, we have $V_1 = V_2$ and it induces a complete graph K_k. As any vertex in $V_1 = V_2$ has degree $k-1$, we have that any vertex in this complete graph is not connecting to any other vertex of G_0. Therefore, G_0 is a union of $K_{k,k}$ and K_k as desired.

Note that $r_k(C_4) \sim k^2$ from Lazebnik and Woldar (2000), so we can color the edges of K_{2k} with at most $(1 + o(1))\sqrt{2k} < \sqrt{3k}$ colors such that there is no monochromatic C_4 hence no monochromatic $K_{2,s+1}$. In view of known results about the density of primes (Siegel-Walfisz Theorem, see Walfisz (1936) and Prachar (1957, pp. 144)), let p_j be the jth prime such that $(p_j - 1)/s$ is an integer, and let $n_j = (p_j - 1)/s + \lfloor 3\sqrt{(p_j - 1)/s}\rfloor$, then by what has been proved, $r_{n_j}(K_{2,s+1}) \geq (1 - o(1))s\,n_j^2$. For any n with $n_j \leq n < n_{j+1}$, we have $n \sim n_j$ as $n \to \infty$, thus

$$r_n(K_{2,s+1}) \geq r_{n_j}(K_{2,s+1}) \geq (1 - o(1))s\,n_j^2 = (1 - o(1))s\,n^2,$$

as desired. □

By counting the edges in G_1, we obtain the asymptotic formula of $ex(n, K_{2,s+1})$ as follows.

Corollary 7.3 *For any fixed integer $s \geq 1$,*

$$ex(n, K_{2,\,s+1}) \sim \frac{1}{2}\sqrt{s}n^{3/2}$$

as $n \to \infty$.

Proof. Note that any two vertices $\langle a_1, a_2\rangle$ and $\langle x_1, x_2\rangle$ (all elements are non-zero) are adjacent in G_1 if and only if $a_1x_1 + a_2x_2 \in H_1$ and $a_1x_1 + a_2x_2 \neq 0$. For each fixed vertex $\langle a_1, a_2\rangle$ and $h^t \in H_1$, the solutions of the equation correspond to exactly one vertex. Thus each vertex $\langle a_1, a_2\rangle$ of G_1 is adjacent to $(q-1)$ vertices, one of these might coincide with $\langle a_1, a_2\rangle$ so the degree of the vertex $\langle a_1, a_2\rangle$ is either $q-1$ or $q-2$. Note that G_1 has $(q-1)^2/s$ vertices, so the assertion follows. □

7.6 Constructions with Forbidden $K_{t,s}$

In this section, we first discuss the lower bound of Turán number $ex(n, K_{3,3})$. Brown did not know that his construction is in fact asymptotically sharp when the paper appeared in 1966. Combining his construction and Füredi's upper bound in 1996 (Theorem 7.7), we have an asymptotic formula of $ex(n, K_{3,\,3})$.

Theorem 7.18 *As $n \to \infty$, $ex(n, K_{3,\,3}) \sim \frac{1}{2}n^{5/3}$.*

Let us point out that Brown's construction did not give a good lower bound for $r_k(K_{3,\,3})$. The following construction is due to Kollár, Rónyai and Szabó (1996), and Alon, Rónyai and Szabó (1999), which yields both asymptotic formulas of $ex(n, K_{3,\,3})$ and $r_k(K_{3,\,3})$.

Let q be a prime power and let F_{q^2} be the field of order q^2. For any $X \in F_{q^2}$, set $N(X) = X^{1+q}$, called the norm of X. Note that the zeros of the polynomial $x^q - x$ are precisely the elements of F_q. Since $X^{q^2} = X$ for any $X \in F_{q^2}$ and

$$[N(X)]^q = (X^{1+q})^q = X^q X^{q^2} = X^q X = N(X),$$

we have that $N(X) \in F_q$ for any $X \in F_{q^2}$. Clearly, N is multiplicative as $N(AB) = N(A)N(B)$. We then define a graph H as follows. The vertex set $V(H)$ is $F_{q^2} \times F_q^*$. Two distinct vertices (A, a) and (B, b) in $V(H)$ are connected if and only if

$$N(A + B) = ab.$$

The order of H is $q^2(q-1)$. Note that $B \neq -A$ since otherwise either $a = 0$ or $b = 0$ which is impossible. If (A, a) and (B, b) are adjacent, then (A, a) and $B(\neq -A)$ determine b. Thus H is regular of degree $q^2 - 1$. In particular, the vertex (A, a) has a loop if $N(2A) = a^2$.

Lemma 7.13 *The graph H does not contain $K_{3,3}$ as a subgraph.*

Proof. The lemma is a direct consequence of the following statement: if (D_1, d_1), (D_2, d_2), and (D_3, d_3) are distinct vertices in $V(H)$, then the system of equations

$$\begin{cases} N(X + D_1) = xd_1, \\ N(X + D_2) = xd_2, \\ N(X + D_3) = xd_3 \end{cases} \tag{7.7}$$

has at most two solutions $(X, x) \in F(q^2) \times F^*(q)$.

If (X, x) is a solution of the system (7.7), then

- $X \neq -D_i$ for any $i = 1, 2, 3$,
- $D_i \neq D_j$ for $i \neq j$.

The former is true since $xd_i \neq 0$. For the latter, if $D_i = D_j$, then we have $d_i = d_j$ and hence $(D_i, d_i) = (D_j, d_j)$.

From the system (7.7) and the property that N is multiplicative, we have

$$\begin{cases} N\left(\frac{X+D_1}{X+D_3}\right) = \frac{d_1}{d_3}, \\ N\left(\frac{X+D_2}{X+D_3}\right) = \frac{d_2}{d_3}. \end{cases}$$

Note that a solution (X, x) of (7.7) is uniquely determined by X, so it suffices to show that the last system has at most two solutions on X. This system yields

$$\begin{cases} N\left(\frac{X+D_1}{(X+D_3)(D_1-D_3)}\right) = \frac{d_1}{d_3 N(D_1-D_3)}, \\ N\left(\frac{X+D_2}{(X+D_3)(D_2-D_3)}\right) = \frac{d_2}{d_3 N(D_2-D_3)}. \end{cases}$$

For $i = 1, 2$, if we denote $b_i = d_i/(d_3 N(D_i - D_3))$, $A_i = 1/(D_i - D_3)$, and $Y = 1/(X + D_3)$, then the above equations become

$$\begin{cases} N(Y + A_1) = b_1, \\ N(Y + A_2) = b_2. \end{cases}$$

Note that $(A + B)^q = A^q + B^q$, so $N(Y + A_i) = (Y + A_i)(Y^q + A_i^q)$ and hence the above system equivalents to

$$\begin{cases} (Y + A_1)(Y^q + A_1^q) = b_1, \\ (Y + A_2)(Y^q + A_2^q) = b_2. \end{cases} \tag{7.8}$$

We now refer unknown Y, A_i and b_i as elements of F_{q^2}. Consider the system of equations

$$\begin{cases} (x_1 - a_{11})(x_2 - a_{12}) = b_1, \\ (x_1 - a_{21})(x_2 - a_{22}) = b_2 \end{cases} \tag{7.9}$$

with $a_{11} \neq a_{21}$ and $a_{12} \neq a_{22}$, where a_{ij}, $b_i \in F_{q^2}$. We claim that the system has at most two solutions $(x_1, x_2) \in F_{q^2} \times F_{q^2}$. In fact, from the system we get

$$(a_{11} - a_{21})x_2 + (a_{12} - a_{22})x_1 + a_{21}a_{22} - a_{11}a_{12} = b_2 - b_1.$$

By expressing x_1 in terms of x_2 and substituting it into one equation in system (7.9), we obtain a quadratic equation in x_2 with a nonzero leading coefficient. This has at most two solutions in x_2 and each determines a unique x_1. The claim follows.

Setting $x_1 = Y$, $x_2 = Y^q$, we see the system (7.8) has at most two solutions in unknown Y. These solutions are corresponding with the solutions (X, x) of system (7.7), so the proof is finished. □

It is just one more step to obtain an asymptotic formula of $r_k(K_{3,3})$.

Theorem 7.19 *As* $k \to \infty$, $r_k(K_{3,3}) \sim k^3$.

Proof. The upper bound $r_k(K_{3,3}) \le (1 + o(1))k^3$ comes from Section 7.2. For a lower bound, let q be a prime power. For a complete graph on vertex set $V = F_{q^2} \times F_q^*$, color the edge connecting (A, a) and (B, b) by color $N(A + B)/ab$ if $A + B \ne 0$. For any three points (A_1, a_1), (A_2, a_2) and (A_3, a_3), if the edges connecting (A_i, a_i) and (X, x) receive the same color, then

$$\begin{cases} N(X + A_1)/a_1x = N(X + A_3)/a_3x, \\ N(X + A_2)/a_2x = N(X + A_3)/a_3x, \end{cases}$$

or equivalently,

$$\begin{cases} N\left(\frac{X+A_1}{X+A_3}\right) = \frac{a_1}{a_3}, \\ N\left(\frac{X+A_2}{X+A_3}\right) = \frac{a_2}{a_3}. \end{cases}$$

It can be verified as the proof for Lemma 7.13 that there are at most two solutions of the above system and hence there is no monochromatic $K_{3,3}$.

Now we consider the uncolored edges connecting (A, a) and (B, b) with $A+B = 0$. For any fixed $A \in F_{q^2}$, set

$$V_1 = \{(A, x) : x \in F_q^*\} \quad \text{and} \quad V_2 = \{(-A, y) : y \in F_q^*\}.$$

If $A = 0$, then V_1 $(= V_2)$ induces a complete graph of order $q - 1$, otherwise V_1 and V_2 are disjoint and they form a complete bipartite graph on $2(q - 1)$ vertices. Using the fact that $r_k(C_4) \sim k^2$, we can color the edges of each such graph (in fact the complete graph the same vertex set) with at most $(1 + o(1))(2q)^{1/2}$ additional colors such that there is no monochromatic C_4 hence no monochromatic $K_{3,3}$. The total number of colors is $(1 + o(1))q$, implying the required lower bound. □

The above construction can be generalized to a graph $G_{q,t}$ as follows, which is called *projective norm graph*. Let $V(G_{q,t}) = F_{q^{t-1}} \times F_q^*$ for $t \ge 3$. Two distinct vertices (A, a) and (B, b) are adjacent if and only if $N(A + B) = ab$, where

$$N(X) = X^{1+q+\cdots+q^{t-2}},$$

called the *norm* of X. In this graph, each vertex has degree $q^{t-1} - 1$ (some vertices may have a loop). In Kollár, Rónyai and Szabó (1996), and Alon, Rónyai and Szabó

(1999), the authors obtained that for any fixed $t \ge 3$ and $s \ge (t-1)!+1$, the order of $ex(n, K_{t,s})$ is $n^{2-1/t}$, and the order of $r_k(K_{t,s})$ is k^t. In particular, for $s \ge t = 3$, the generalization and the upper bounds obtained by Füredi's (Theorem 7.7) yield that

$$\left(\frac{1}{2} - o(1)\right)\left(\frac{s-1}{2}\right)^{1/3} n^{5/3} \le ex(n, K_{3,s}) \le \left(\frac{1}{2} + o(1)\right)(s-2)^{1/3} n^{5/3},$$

and

$$(1 - o(1))\frac{s-1}{2}k^3 \le r_k(K_{3,s}) \le (1 + o(1))(s-2)k^3.$$

Erdős and Spencer (1974) proved that $ex(n, K_{t,t}) \ge \Omega(n^{2-1/(t+1)})$ for $t \ge 5$ via an application of the probabilistic method, which was improved by Wolfovitz (2009) to that $ex(n, K_{t,t}) \ge \Omega(n^{2-1/(t+1)}(\log\log n)^{1/(t^2-1)})$. By analyzing of the H-free process, this was further improved by Bohman and Keevash (2010) as

$$ex(n, K_{t,t}) \ge \Omega\left(n^{2-1/(t+1)}(\log n)^{1/(t^2-1)}\right).$$

A natural problem is as follows.

Problem 7.1 Determine the orders of $ex(n, K_{t,t})$ and $r_k(K_{t,t})$ for fixed $t \ge 4$. Is the former $n^{2-1/t}$? Is the latter k^t?

7.7 Turán Numbers for Even Cycles

In this section, we focus on Turán numbers $ex(n, C_{2m})$. For the upper bound, Erdős (1965) claimed an upper bound without proof, which was proved by Bondy and Simonovits (1974). Indeed, Bondy and Simonovits (1974) proved a more general result that if a graph G of order n with edge number $e(G) \ge 100mn^{1+1/m}$, then G contains all even cycles $C_{2\ell}$ for $m \le \ell \in mn^{1/m}$. For $m = 2, 3, 5$, the bounds are tight, see Klein (Erdős 1938), Benson (1966) and Singleton (1966) and later by Wenger (1991), Lazebnik and Ustimenko (1995) and Mellinger and Mubayi (2005). For general m, the best known lower bound on $ex(n, C_{2m})$ is due to Lazebnik, Ustimenko and Woldar (1995), but does not match the upper bound.

The following proof is due to Bondy and Simonovits (1974).

Theorem 7.20 *Let $m \ge 2$ be a fixed integer. We have*

$$ex(n, C_{2m}) \le 10mn^{1+1/m}$$

for $n \ge 10^{m^2}$.

Now we shall prove the bipartite version of Theorem 7.20 first. To do so, we shall have more preparative lemmas. Let $t \ge 1$ be an integer. A coloring of vertices of a graph G, not necessarily proper, is called *t-periodic* if the pair of end vertices of any path of length t have the same color. So all vertices in a connected graph have the same color in any 1-periodic coloring. For $t \geq 2$, if G has enough edges, then the number of colors used in a t-periodic coloring is small.

Proposition 7.1 *If t is the smallest integer such that the cycle C_m is t-periodic, then $t|m$. Moreover, if the cycle C_m is t'-periodic, then $t|t'$.*

Lemma 7.14 *Let $t \geq 1$ be an integer, and let G be a connected graph of order n. If $e(G) \geq 2tn$, then the number of colors in any t-periodic coloring of G is at most two.*

Proof. We separate the proof into several short steps. We first prove that G contains two adjacent vertices joined by two internal vertex-disjoint paths, each of length at least t. We shall call this subgraph as a θ-graph intuitively.

Step 1. $\delta(G) \geq 2t$. We can find such a θ-graph in the following way. Let $x_1x_2\cdots x_m$ be a longest path. Then x_1 is adjacent only to vertices of this path, $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$, say, where

$$2 = i_1 < i_2 < \cdots < i_r,\ r \geq 2t.$$

The cycle $x_1x_2\cdots x_{i_{2t}}x_1$ and the edge $x_1x_{i_t}$ form the desired θ-graph.

Step 2. $\delta(G) < 2t$. Since $e(G) \geq 2tn$, we have that the average degree of G is at least $4t$. From Lemma 3.11, G contains a subgraph H with $\delta(H) \geq 2t$. Thus from step 1, H contains a θ-graph as desired.

Step 3. Any θ-graph has three cycles. Let us denote these cycles C_1, C_2, C_3 of lengths ℓ_1, ℓ_2 and ℓ_3, respectively, where $t + 1 \leq \ell_1 \leq \ell_2 < \ell_3$. Thus

$$\ell_1 + \ell_2 - \ell_3 = 2.$$

The restrictions of the coloring of G to the θ-graph and to each cycle C_i are also t-periodic. Let $t_i \geq 2$ be the smallest integer such that the coloring of C_i is t_i-periodic. Clearly $t_i|t$ and $t_i|\ell_i$ by Proposition 7.1. Also any period on one cycle induces the same period on the other two cycles and hence $t_1 = t_2 = t_3$. Let t^* be the common value of t_i. Then $t^*|\ell_i$ hence $t^*|2$ so $t^* = 1$ or $t^* = 2$, implying that the number of colors in the θ-graph is at most two.

Step 4. Since G is connected, any vertex of G is jointed to some vertex in the θ-graph by a path of length kt, probably using some vertices in the θ-graph. Thus both of the end vertices of the path have the same color. Hence the number of colors in G is at most two from step 3. □

Lemma 7.15 *Let G be a bipartite graph of order $n \geq 10^m$. If the minimum degree $\delta(G) \geq 5mn^{1/m}$, then G contains an even cycle C_{2m}.*

Proof. Fix a vertex x of G and let

$$V_i = \{v \in V(G) : d(v, x) = i\},$$

which is the set of vertices with distance i from x. Since G is bipartite, each V_i is an independent set.

Suppose that G contains no C_{2m}. We claim that

$$|V_i| \geq n^{1/m}|V_{i-1}| \tag{7.10}$$

for $1 \leq i \leq m$, which will lead to a contradiction since (7.10) implies that $|V(G)| > |V_m| \geq n$.

In the following, we aim to prove (7.10). The proof is by induction on i. This is trivial for $i = 1$ since $\delta(G) \geq 5mn^{1/m}$. Suppose (7.10) holds for smaller value of i.

Let H be the subset of G induced by $V_{i-1} \cup V_i$ and let $H_1, H_2, \ldots, H_q$ be the components of H. Write $W_j = V(H_j) \cap V_{i-1}$.

A path $x_1x_2 \cdots x_k$ in G is called *monotonic* if $d(x_i, x)$ is monotonic. This means that a monotonic path passes through each of some consecutive sets $V_j, V_{j+1}, \ldots, V_{j+k}$ exactly once.

We shall show that $e(H_1) < 4m|V(H_1)|$. This is trivial if W_1 contains only one vertex which implies H_1 is a star. We thus assume that W_1 has at least two vertices. Let $p < i - 1$ be the smallest index such that there is a vertex $a \in V_p$ and there are two monotonic paths P_1, P_2 joining a to W_1 which only contain the vertex a in common.

We then show that each vertex of W_1 is jointed to a by a monotonic path. This is clear if $a = x$. Otherwise, for $y \in W_1$, there is a monotonic path P_3 joining y to x. By the minimality of p, P_3 must intersect P_1 or P_2, say P_1, at some vertex z. The path consisting of the section of P_3 between y and z and the section of P_1 between z and a is a monotonic path from y to a.

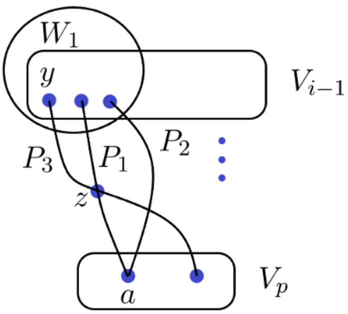

Fig. 1

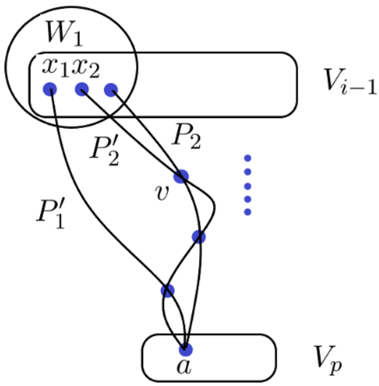

Fig. 2

We now assign colors red and blue to the vertices of W_1 in such a way that if two vertices have different colors, then they are joined to a by internal disjoint monotonic paths. This can be done as follows. Each vertex of W_1 that can be joined to a by a monotonic path disjoint from P_2 is colored red; all other vertices of W_1 are colored blue. To see this is the required coloring, let x_1 and x_2 be vertices of W_1 colored red and blue, respectively. We will show that they are joined to a by internal disjoint monotonic paths.

Let P'_1 be a monotonic path from x_1 to a disjoint from P_2, and let P'_2 be a monotonic path from x_2 to a. Moving along P'_2 from x_2 towards a, let v be the first vertex of $(P'_1 \cup P_2) \setminus \{a\}$ encountered. Such a vertex v exists since x_2 is colored blue. Also we see that v cannot belong to P'_1 for otherwise the section of P'_2 between x_2 and v together with the section of P'_1 between v and a would constitute a monotonic path from x_2 to a disjoint from P_2, contradicting the assumption that x_2 is colored blue. But then $v \in P_2$ and we have a monotonic path $x_2 P'_2 v P_2 a$ disjoint from P'_1 as desired.

We now color the vertices of H_1 in V_i green and show that this coloring of H_1 is t-periodic with $t = 2(m - i + p + 1)$. For, since t is even, if one end vertex of a path of length t in H_1 is green, then so is the other. Also, there can be no path of length t joining a red and a blue vertex, because, if a red x_1 were joined to a blue x_2 by such a path, this path together with vertex-disjoint monotonic paths from x_1 to a and x_2 to a would form a C_{2m}. Therefore, the coloring of H_1 is indeed t-periodic. Since three colors are used in the coloring, Lemma 7.14 implies that

$$e(H_1) < 2t|V(H_1)| < 4m|V(H_1)|.$$

Similarly we have that $e(H_j) < 4m|V(H_j)|$ for $j = 1, 2, \ldots, q$ hence

$$e(H) < 4m|V(H)|.$$

Let H' be the subgraph of G induced by $V_{i-2} \cup V_{i-1}$. The same argument gives

$$e(H') < 4m|V(H')|.$$

Clearly, since $\delta(G) \geq 5mn^{1/m}$,

$$e(H) + e(H') \geq 5mn^{1/m}|V_{i-1}|.$$

Combining these inequalities, we get

$$\begin{aligned} &4m(|V_{i-1}| + |V_i| + |V_{i-1}| + |V_{i-2}|) \\ =&4m(|V(H)| + |V(H')|) > e(H) + e(H') \geq 5mn^{1/m}|V_{i-1}|, \end{aligned}$$

which implies that

$$|V_i| > \frac{1}{4m}\left((5mn^{1/m} - 8m)|V_{i-1}| - 4m|V_{i-2}|\right).$$

Using the induction hypothesis,

$$|V_{i-1}| \geq n^{1/m}|V_{i-2}|.$$

Therefore,

$$\begin{aligned}|V_i| &> \frac{1}{4m}\left(5mn^{1/m} - 8m - \frac{20m^2}{5mn^{1/m}}\right)|V_{i-1}| \\ &> \frac{1}{4m}(5mn^{1/m} - 9m)|V_{i-1}| \\ &> n^{1/m}|V_{i-1}|\end{aligned}$$

as desired. □

We also need the following result.

Lemma 7.16 *For any graph G, there is a subgraph H of G such that H is bipartite and $e(H) \geq e(G)/2$.*

Proof. We may assume that $e(G) > 0$ as the case $e(G) = 0$ is trivial. For a subset S of $V(G)$, write $\overline{S} = V(G) \setminus S$. Let $e(S, \overline{S})$ be the number of edges between S and $\overline{S}$, in which $e(V, \emptyset) = 0$. Maximizing $e(S, \overline{S})$ over all subsets S of $V(G)$, we obtain a spanning bipartite subgraph H on vertex classes S and $\overline{S}$ with $e(H) = e(S, \overline{S})$, in which none of S and $\overline{S}$ is empty as $e(G) > 0$. Then for any vertex v, say $v \in S$, at least half of neighbors of v in G are in $\overline{S}$ since otherwise removing v from S to $\overline{S}$ would increase $e(S, \overline{S})$, contradicting to the maximality of $e(S, \overline{S})$. This follows by $e(H) \geq e(G)/2$.

The Lemma has a simple proof by probabilistic method. Let S be a random set of $V(G)$ defined by $\Pr(v \in S) = 1/2$, independently. Then it is easy to know that the probability of any edge is an edge between S and $\overline{S}$ is $1/2$. Thus the expectation of $e(S, \overline{S})$ is $e(G)/2$, implying that there is some set S such that $e(S, \overline{S}) \geq e(G)/2$. □

Edwards (1972, 1975) proved the essentially best possible result that for every graph G with m edges, there exists a bipartite graph H satisfying

$$e(H) \geq \frac{m}{2} + \sqrt{\frac{m}{8} + \frac{1}{64}} - \frac{1}{8}.$$

This result is tight if G is a complete graph on an odd number of vertices, i.e. whenever $m = \binom{n}{2}$ for some odd integer n.

Proof of Theorem 7.20. Let G be a graph of order $n \geq 10^{m^2}$ with $e(G) \geq 10mn^{1+1/m}$. Lemma 7.16 implies that there is a spanning subgraph H of G such that H is bipartite and $e(H) \geq 5mn^{1+1/m}$. The average degree of H is at least $10mn^{1/m}$. By Lemma 3.11, there is a subgraph F of H satisfying that $\delta(F) \geq 5mn^{1/m}$. Let n_0 be the order of F, then

$$n_0 > \delta(F) \geq 5mn^{1/m} > n^{1/m} \geq 10^m,$$

and $\delta(F) \geq 5mn_0^{1/m}$. Thus by Lemma 7.15, the bipartite graph F contains an even cycle C_{2m}, so does G. This proves the theorem. □

Theorem 7.20 yields that for each fixed $m \geq 2$,

$$ex(n, C_{2m}) \leq 10mn^{1+1/m}.$$

Corollary 7.4 *For each fixed $m \geq 2$,*

$$r_k(C_{2m}) \leq c\, k^{m/(m-1)},$$

where $c = c(m) > 0$ is a constant.

For $m = 2$, we have established the right order for $ex(n, C_4)$. In the following, we shall prove that its order is also right for $m = 3$ and $m = 5$.

Theorem 7.21 *There exists some constant $c = c(m) > 0$ such that*

$$ex(n, C_{2m}) \geq cn^{1+1/m}$$

for $m = 2, 3, 5$.

The constructions for the desired lower bounds are due to Wenger (1991). The same order of lower bound for $m = 3$ has been obtained by Benson (1966), and for $m = 2, 3, 5$ by Lazebnik, Ustimenko and Woldar (1995).

Let q be a prime power. Construct a bipartite graph $H_m(q)$ as follows on vertex classes X and Y, where both X and Y are copies of F_q^m. Thus $|X| = |Y| = q^m$. For two vertices $A \in X$ and $B \in Y$ with

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

they are adjacent if

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{m-1} \\ b_m \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{m-1} \\ 0 \end{pmatrix} + b_m \begin{pmatrix} a_2 \\ a_3 \\ \vdots \\ a_m \\ 1 \end{pmatrix}.$$

For each vertex $A \in X$, the value b_m uniquely determines a neighbor B of A, so each vertex in X has degree q. Hence $H_m(q)$ has $2q^m$ vertices and q^{m+1} edges.

Lemma 7.17 *The bipartite graph $H_m(q)$ is q-regular. The last coordinates of neighbors of any vertex are pairwise distinct, hence they form F_q.*

Proof. For a vertex $A \in X$, any neighbor $B \in Y$ of A is uniquely determined by its last coordinate b_m from the adjacency. Thus A have q neighbors, of which the last coordinates form F_q. Given a vertex $B \in Y$, if $A \in X$ is a neighbor of B, then

$$\begin{pmatrix} b_1 \\ \vdots \\ b_{m-1} \\ a_m \end{pmatrix} = \begin{pmatrix} 1 & b_m & & \\ & \ddots & \ddots & \\ & & 1 & b_m \\ & & & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_{m-1} \\ a_m \end{pmatrix},$$

in which A is uniquely determined by a_m. Thus B has q neighbors, of which the last coordinates form F_q. □

Before giving more properties of $H_m(q)$, recall *Vandermonde matrix* on F_q as follows. For $m \geq 2$ and $a_i \in F_q$, set

$$M_m = M(a_1, a_2, \ldots, a_m) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_m \\ \vdots & \vdots & & \vdots \\ a_1^{m-1} & a_2^{m-1} & \cdots & a_m^{m-1} \end{pmatrix}$$

as an $m \times m$ matrix on F_q. Note that

$$\det(M_m) = \Pi_{1 \leq i < j \leq m}(a_j - a_i).$$

So M_m is singular if and only if there are some $i \neq j$ such that $a_i = a_j$. However, we need a more specific property of the Vandermonde matrix.

Lemma 7.18 *If the ith column of the Vandermonde matrix M_m is a linear combination of the others, then there exists $j \neq i$ such that $a_i = a_j$.*

Proof. Note that $\det(M_m) = \Pi_{1 \leq i < j \leq m}(a_j - a_i) = 0$ by the assumption, so the assertion follows immediately. □

Lemma 7.19 *If $H_m(q)$ contains a cycle of length $2m$, denoted by*

$$C_{2m} = (A_1, B_1, A_2, B_2, \cdots, A_m, B_m),$$

where $A_i \in X$ and $B_i \in Y$, then for each B_i, there exists a B_j with $j \neq i$ such that $b_{im} = b_{jm}$, where b_{im} and b_{jm} are the mth (last) coordinates of B_i and B_j, respectively.

Proof. Let A, B and A' be three consecutive vertices in the cycle C_{2m} with $B \in Y$. By the definition of adjacency we have

$$\begin{pmatrix} a_1 - a_1' \\ \vdots \\ a_{m-1} - a_{m-1}' \\ 0 \end{pmatrix} = -b_m \begin{pmatrix} a_2 - a_2' \\ \vdots \\ a_m - a_m' \\ 0 \end{pmatrix},$$

which gives $a_i - a_i' = -b_m(a_{i+1} - a_{i+1}') = (-b_m)^{m-i}(a_m - a_m')$ hence

$$\begin{pmatrix} a_1 - a'_1 \\ a_2 - a'_2 \\ \vdots \\ a_{m-1} - a'_{m-1} \\ a_m - a'_m \end{pmatrix} = (a_m - a'_m) \begin{pmatrix} (-b_m)^{m-1} \\ (-b_m)^{m-2} \\ \vdots \\ -b_m \\ 1 \end{pmatrix}.$$

Clearly $a_m \neq a'_m$ since otherwise A and A' are the same vertex. By taking A, B and A' as A_i, B_i and A_{i+1}, respectively, and by writing $x_i = a_{i\,m} - a_{(i+1)\,m}$, and $c_i = -b_{i\,m}$, we obtain

$$A_i - A_{i+1} = x_i \begin{pmatrix} c_i^{m-1} \\ c_i^{m-2} \\ \vdots \\ c_i \\ 1 \end{pmatrix},$$

and $x_i \neq 0$. From the trivial fact that $\sum_{i=1}^{m}(A_i - A_{i+1})$ is a zero vector, where A_{m+1} is A_1, we have

$$\begin{pmatrix} c_1^{m-1} & c_2^{m-1} & \cdots & c_m^{m-1} \\ c_1^{m-2} & c_2^{m-2} & \cdots & c_m^{m-2} \\ \cdots & & & \cdots \\ c_1 & c_2 & \cdots & c_m \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \\ x_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Note that the left-hand side matrix is a Vandermonde matrix, and the ith column is a linear combination of the others since $x_i \neq 0$. Thus $c_i = c_j$ and hence $b_{i\,m} = b_{j\,m}$ for some $j \neq i$ by Lemma 7.18. □

Corollary 7.5 *The graph $H_m(q)$ contains no C_{2m} for $m = 2, 3, 5$.*

Proof. If $H_2(q)$ contains a cycle $C_4 = (A_1, B_1, A_2, B_2)$ with $A_i \in X$ and $B_i \in Y$. By Lemma 7.19, $b_{12} = b_{22}$. However, both B_1 and B_2 are adjacent to A_1, implying the last coordinates b_{12} and b_{22} are distinct by Lemma 7.17. This leads to a contradiction.

If $H_3(q)$ contains a cycle $C_6 = (A_1, B_1, A_2, B_2, A_3, B_3)$ with $A_i \in X$ and $B_i \in Y$. By Lemma 7.19, $b_{13} = b_{23}$ or $b_{13} = b_{33}$. However, B_1 and B_2 have a neighbor A_2 in common, and B_1 and B_3 have a neighbor A_1 in common, which leads to a contradiction from Lemma 7.17.

If $H_5(q)$ contains a cycle $C_{10} = (A_1, B_1, \ldots, A_5, B_5)$ with $A_i \in X$ and $B_i \in Y$. By Lemma 7.19, there exist three distinct vertices B_i, B_j and B_k such that $b_{i5} = b_{j5} = b_{k5}$. Two of these vertices must be consecutive in the cyclic sequence $B_1, B_2, \ldots, B_5, B_1$, so they have a neighbor in common, which again leads to a contradiction from Lemma 7.17. □

Proof of Theorem 7.21. Note that $H_m(q)$ has $n = 2q^m$ vertices and $q^{m+1} = (n/2)^{1+1/m}$ edges. The proof of Theorem 7.21 now follows immediately from the above corollary and a result that two consecutive prime numbers p, p' satisfying $p \sim p'$ (Siegel-Walfisz Theorem, see Walfisz (1936) and Prachar (1957, pp. 144)). □

Let us remark that we cannot expect that $H_m(q)$ gives more exact orders. It is shown by Shao, He and Shan (2008) that $H_m(q)$ contains all even cycles of order $2m$ for $m = 4$ or $m \geq 6$.

A further result due to Füredi, Naor and Verstraëte (2006) is as follows:

$$c_1 n^{4/3} + O(n) < ex(n, C_6) < c_2 n^{4/3} + O(n),$$

where $c_1 = 3(\sqrt{5} - 2)/(\sqrt{5} - 1)^{4/3} = 0.53\ldots$, and $c_2 = 0.62\ldots$ is the real root of $16x^3 - 4x^2 + x - 3 = 0$.

Moreover, as we have mentioned in the beginning of this section, Lazebnik, Ustimenko and Woldar (1995) proved that for fixed $m \geq 3$,

$$ex(n, C_{2m}) \geq \Omega\left(n^{1+2/(3m-3)}\right).$$

As we have known that for $m = 2, 3, 5$, $ex(n, C_{2m}) = \Theta(n^{1+1/m})$. The following conjecture proposed by Bondy and Simonovits (1974) is still open.

Conjecture 7.1 For $m = 4$ or fixed $m \geq 6$, $ex(n, C_{2m}) = \Theta(n^{1+1/m})$.

Let us see a generalization of the Turán number of even cycle as follows. Let $\theta_{k,\ell}$ be the graph consisting of ℓ internally disjoint paths of length k, each with the same endpoints. We see that $\theta_{k,2} = C_{2k}$. The problem of determining $ex(n, \theta_{k,\ell})$ was first studied by Faudree and Simonovits (1983), who showed that $ex(n, \theta_{k,\ell}) = O(n^{1+1/k})$ for all fixed $k, \ell \geq 2$. The lower bounds for $ex(n, C_{2k})$ imply that this bound is tight when $k = 2, 3$ or 5. Additionally, a result of Mellinger and Mubayi (2005) shows that it is tight for $k = 7$ and $\ell \geq 3$. Conlon (2019) obtained that for any fixed integer $k \geq 2$, there exists an integer ℓ such that

$$ex(n, \theta_{k,\ell}) = \Omega(n^{1+1/k}).$$

In the following, we will consider Ramsey numbers of bipartite graphs and large K_n. Recall a result in Chapter 3 that for any graph G of order N and average degree d, if the maximum degree of any subgraph induced by a neighborhood is less than an integer m, then

$$\alpha(G) \geq N f_m(d), \tag{7.11}$$

where $f_m(x) > (\log(x/m) - 1)/x$ for $x > 0$. The inequality (7.11) holds if any subgraph of G induced by a neighborhood contains no path of m edges.

The following result is due to Li and Zang (2003).

Theorem 7.22 *For any fixed integers $s \geq t \geq 2$,*

$$r(K_{t,s}, K_n) \leq (1 + o(1))(s - t + 1)\left(\frac{n}{\log n}\right)^t.$$

Proof. Let $N = r(K_{t,s}, K_n) - 1$, and let G be a graph on N vertices with no $K_{t,s}$ and $\alpha(G) \leq n - 1$. By the upper bound of Turán numbers in Section 7.2, we have

$$d(G) \le (1 + o(1))(s - t + 1)^{1/t} N^{1-1/t}, \tag{7.12}$$

where $d(G)$ is the average degree of G.

Let v be a vertex of G, and let G_v be the subgraph of G induced by the neighborhood of v with maximum degree $\Delta(G_v)$. Then

$$\Delta(G_v) \le s - 1 \text{ if } t = 2 \tag{7.13}$$

and

$$\Delta(G_v) \le r(K_{t-2,s}, K_n) - 1 \text{ if } t \ge 3. \tag{7.14}$$

Indeed, for $t = 2$, (7.13) follows immediately from the fact that G contains no $K_{2,s}$. For $t \ge 3$, suppose to the contrary that the degree of some vertex u in G_v is at least $r(K_{t-2,s}, K_n)$. Since G contains no K_n, $G_u \cap G_v$ must contain $K_{t-2,s}$, which together with u and v yield a $K_{t,s}$ in G, a contradiction.

Now let us apply induction on $t \ge 2$. For $t = 2$, in view of (7.12) and (7.13), from (7.11) we have,

$$\begin{aligned} n > N f_s(d(G)) &\ge (1 - o(1)) \frac{N \log(\sqrt{(s-1)N}/s)}{\sqrt{(s-1)N}} \\ &= (1 - o(1)) \sqrt{\frac{N}{s-1}} \frac{\log N}{2}. \end{aligned}$$

It follows that $N \le (s - 1 + o(1))(n/\log n)^2$ since otherwise, there exists a constant $\delta > 0$ such that $N \ge (s - 1 + \delta)(n/\log n)^2$ for infinitely many n, which will lead to a contradiction. Hence the assertion holds for $t = 2$.

For $t = 3$, our proof begins at Chvátal's discovery on $r(T, K_n)$ mentioned in Chapter 1,

$$r(K_{1,s}, K_n) = s(n - 1) + 1.$$

Let $m = s(n-1)+1 \sim \Omega(N^{1/3})$. Since any subgraph of G induced by a neighborhood has maximum degree less than m, a similar argument as that for $t = 2$ gives

$$\begin{aligned} n > N f_m(d(G)) &\ge (1 - o(1)) \frac{N \log((s-2)^{1/3} N^{2/3}/m)}{(s-2)^{1/3} N^{2/3}} \\ &= (1 - o(1)) \left(\frac{N}{s-2}\right)^{1/3} \frac{\log N}{3}. \end{aligned}$$

It follows that $N \le (s - 2 + o(1))(n/\log n)^3$, and the assertion holds for $t = 3$.

Let us proceed to the induction step for $t \ge 4$. For any fixed $0 < \epsilon < 1$, set

$$d = (1 + \epsilon)(s - t + 2) \left(\frac{n}{\log n}\right)^{t-1},$$

and

$$m = \left\lfloor (1+\epsilon)(s-t+3)\left(\frac{n}{\log n}\right)^{t-2} \right\rfloor.$$

The induction hypothesis assumes that for large n,

$$r(K_{t-1,s}, K_n) < d, \quad \text{and} \quad r(K_{t-2,s}, K_n) < m.$$

Using (7.11) as before, we have

$$\begin{aligned} n &\geq (1-o(1))\left(\frac{N}{s-t+1}\right)^{1/t} \log\frac{d}{m} \\ &= (1-o(1))\left(\frac{N}{s-t+1}\right)^{1/t} \log\left(\frac{n}{\log n}\right). \end{aligned}$$

It follows that $N \leq (s-t+1+o(1))(n/\log n)^t$ as desired. □

Combining the above theorem and the lower bounds obtained in Chapter 5 by using the Local Lemma, we have that for fixed $s \geq t \geq 2$,

$$c\left(\frac{n}{\log n}\right)^{(st-1)/(s+t-2)} \leq r(K_{t,s}, K_n) \leq (1+o(1))(s-t+1)\left(\frac{n}{\log n}\right)^{t},$$

where $c = c(s,t) > 0$ is a constant. Note that the exponent $(st-1)/(s+t-2)$ in the lower bound can be arbitrarily close to the exponent t when s is much larger than t. So it is natural to ask if the order of upper bound is sharp. Assuming yes, we shall be able to establish a somehow unexpected result: asymptotically, all the extremal graphs for $r(K_{t,s}, K_n)$ come from those for $ex(N, K_{t,s})$. Thus it is very interesting to estimate the independence numbers of known extremal graphs for $ex(n, K_{t,s})$. No doubt, this assumption is a bold adventure.

Proposition 7.2 *For any fixed integers $s \geq t \geq 2$, if there exists a constant $c > 0$ such that*

$$r(K_{t,s}, K_n) \geq (c-o(1))\left(\frac{n}{\log n}\right)^{t} \tag{7.15}$$

as $n \to \infty$, then

$$ex(N, K_{t,s}) \geq \frac{1}{2}(c-o(1))^{1/t} N^{2-1/t} \tag{7.16}$$

for all sufficiently large N of the form $N = r(K_{t,s}, K_n) - 1$. Furthermore, the extremal graphs yielding (7.15) *also yield* (7.16).

Proof. To prove it, assume the contrary: there exists $\delta > 0$ such that $ex(N, K_{t,s}) \leq \frac{1}{2}(c-\delta)^{1/t} N^{2-1/t}$ for infinitely many n, where $N = r(K_{t,s}, K_n) - 1$. Imitating the previous proof, we have

$$n \geq (1-o(1))\left(\frac{N}{c-\delta}\right)^{1/t} \log\frac{N^{1-1/t}}{r(K_{t-2,s}, K_n)},$$

which implies $N \leq (c - \delta + o(1))(\frac{n}{\log n})^t$, contradicting (7.15). □

The main idea for the proof of the above theorem is simple: if the average degree of a graph is small, then its independence number must be big. The main result in the last section $ex(N, C_{2m}) \leq c(m)N^{1+1/m}$ will be used to improve the result of Erdős, Faudree, Rousseau and Schelp (1978) that $r(C_{2m}, K_n) \leq cn^{m/(m-1)}$ with a factor $\log n$.

Theorem 7.23 *For any fixed $m \geq 2$, there exists a constant $c = c(m) > 0$ such that for all sufficiently large n,*

$$r(C_{2m}, K_n) \leq c\left(\frac{n}{\log n}\right)^{m/(m-1)}.$$

Proof. Let G be a graph of order $N = r(C_{2m}, K_n) - 1$ which contains no C_{2m} and $\alpha(G) < n$. For any vertex v of G, consider the subgraph G_v induced by the neighborhood of v. The subgraph G_v does not contain path P_{2m-2} since G does not contain C_{2m}, where P_{2m-2} is a path of $2m - 2$ edges. Also since $ex(N, C_{2m}) \leq c_1 N^{1+1/m}$ for some constant $c_1 = c_1(m) > 0$ thus the average degree of G is at most $2c_1 N^{1/m}$. Note that for fixed a, the function $f_a(x)$ is asymptotic equal to $\log x/x$ as $x \to \infty$. By Theorem 3.6,

$$n > \alpha(G) \geq (1 - o(1))N \frac{\log 2c_1 N^{1/m}}{2c_1 N^{1/m}} > c_2 N^{1-1/m} \log N$$

for some constant $c_2 > 0$. Now if for any large $c > 0$, there are infinitely many n such that $N \geq c(n/\log n)^{m/(m-1)}$, then $\log N \geq c_3 \log n$, where $c_3 > 0$ is a constant increasing as c increasing, and

$$n \geq c_2 c^{(m-1)/m} \frac{n}{\log n} c_3 \log n = c_2 c_3 c^{(m-1)/m} n$$

which would lead to a contradiction if c is large and $n \to \infty$. □

We have discussed the Ramsey number of cycle and K_n when n is large. For large m, Erdős, Faudree, Rousseau and Schelp (1978) conjectured that for every $m \geq n \geq 3$, except for $m = n = 3$,

$$r(C_m, K_n) = (m - 1)(n - 1) + 1. \tag{7.17}$$

Bondy and Erdős (1973) verified it for $n > 3$ and $m \geq n^2 - 2$, which was slightly improved by Schiermeyer (2003) and further by Nikiforov (2005) for $m \geq 4n + 2$. Recently, Keevash, Long and Skokan (2021) confirmed this conjecture in a stronger form by proving (7.17) holds for $m \geq c \log n/\log\log n$, where $c > 0$ is constant. This is best possible up to the constant factor c since we can prove that for any $\epsilon > 0$, there exists $n_0(\epsilon)$ such that $r(C_m, K_n) > n \log n \gg (m - 1)(n - 1) + 1$ for all $n \geq n_0(\epsilon)$ and $3 \leq m \leq (1 - \epsilon) \log n/\log\log n$ (see Exercise 14). It is challenging to determine the asymptotical order of $r(C_m, K_n)$ for each fixed $m > 3$. In particular, Erdős asked if there exists a constant $\epsilon > 0$ such that

$$r(C_4, K_n) = o(n^{2-\epsilon}).$$

We conclude this section with the following problem.

Problem 7.2 For fixed $s \geq t \geq 2$, determine the order of $r(K_{t,s}, K_n)$. Is it $(n/\log n)^t$. If yes, does it grow linear on s?

7.8 Exercises

1. Prove that $ex(n, H) = \binom{n}{2}$ if $n < |V(H)|$. What can we say about $ex(n, H)$ if $n = |V(H)|$?

2. Directly prove $\alpha(G) \geq n/(1+d)$ by the method proving the upper bound for $ex(n, K_k)$ in Section 7.1 as the former can be proved by the latter.

3. Let T_m be a graph of m edges. Show that

(i) $ex(n, T_m) \leq (m-1)n$.
(ii) If $n = sm + r$ with $0 \leq r < m$, then $ex(n, T_m) \geq s\binom{m}{2} + \binom{r}{2}$.

4. What are $ex(n, P_m)$ and $ex(n, K_{1,m})$?

5. Prove that $ex(n, C_{2m+1}) = \lfloor n^2/4 \rfloor$ for large n.

6.* An H-free graph G is called to be *critical* if any graph obtained from G by adding any edge in the complement of G contains H. A critical H-free graph is also called to be H-saturated. So the *saturation number* $sat(n; H)$ is defined as the minimum number of edges of an H-saturated graph of order n.

(i) Show $sat(n; K_3) = n - 1$.
(ii) Show generally $sat(n; K_t) = (t-2)(n-1) - \binom{t-2}{2}$ arising from the graph $K_{t-2} + \overline{K}_{n-t+2}$. (See Erdős, Hajnal and Moon, 1964)

7. Let G and H be graphs of order n. Prove that G contains a subgraph with at least $e(G)e(H)/\binom{n}{2}$ edges that is isomorphic to a subgraph of H. (See the proof of Theorem 7.5)

8. Let G be a graph with n vertices and m edges. By considering random bipartition of $V(G)$ of sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, show that G contains a bipartite graph with at least $2m\lfloor n^2/4 \rfloor / n(n-1)$ edges.

9. Let H be a bipartite graph, and let $ex_b(n; H)$ be the maximum number of edges in a *bipartite* graph of order n. Prove that $ex_b(n; H) \leq ex(n, H) \leq 2\, ex_b(n; H)$.

10. Show that if G is a subgraph of $K_{n,n}$ with average degree d that contains neither C_4 nor C_6, then $d < n^{1/3} + 1$.

11. Let q be a prime power. Prove that

$$r(C_4, K_{1,q^2}) \ge q^2 + q + 1, \quad \text{and} \quad r(C_4, K_{1,q^2+1}) \ge q^2 + q + 2.$$

12.* Let $n \ge 2$ be an integer. Prove that

$$r(C_4, K_{1,n}) \le n + \sqrt{n-1} + 2.$$

13.* Prove that for any $\epsilon > 0$, there exists $n_0(\epsilon)$ such that $r(C_m, K_n) > n \log n \gg (m-1)(n-1) + 1$ for all $n \ge n_0(\epsilon)$ and $3 \le m \le (1-\epsilon) \log n / \log \log n$. (Hint: Keevash, Long and Skokan, 2021)

14.* Extending Theorem 7.18 to that for each $t \ge 4$ and $s \ge t! + 1$, $ex(n, K_{t,s}) > cn^{2-1/t}$ for some $c > 0$. (Hint: Alon, Rónyai, and Szabó, 1999)

Chapter 8
Communication Channels

Ramsey theory has been applied to information theory in various ways. In this chapter, we shall see that the connection between Ramsey theory and communication channel is natural. The first section is on Shannon capacity, and the second section is on that of cycles, which contains a result of Lovász for Shannon capacity of C_5. The third section set an equalities for classical Ramsey numbers and functions from communication channels.

8.1 Introduction

A communication channel consists of a finite input set X, and an output set Y. For each input $x \in X$, there is a nonempty fan-shaped output $S_x \subseteq Y$, which is the set of outputs that may be received for the input x by the receiver. In each use of the channel, a sender transmits an input $x \in X$, and receiver receives an arbitrary output $y \in S_x$. For distinct inputs u and v, they can be received as the same output if and only if $S_u \cap S_v \neq \emptyset$. Suppose that the sender and receiver agree in advance on an input set $I \subseteq X$. In order to avoid error, the outputs of any two distinct inputs in I cannot intersect. In a noiseless channel, there is no intersect between two outputs. Shannon (1956) first studied the amount that an information channel can communicate without error. He formulated the problem to a problem of graph theory.

Let X be the input set of a channel and let G be a graph with vertex set X in which two distinct vertices are adjacent if and only if their outputs intersect. The graph G is called the *characteristic graph* of the channel. The characteristic graph of a completely noisy channel is K_k; and that of a noiseless channel is an empty graph (a graph with edge set empty).

In most situations, a channel is in repeated uses. When the channel is used n times, the sender transmits a sequence $x = (x_1, x_2, \ldots, x_n)$, where $x_i \in X$, and receiver receives a sequence $y = (y_1, y_2, \ldots, y_n)$, where $y_i \in S_{x_i} \subseteq Y$. The repeated use of the channel can be viewed as a single use of a larger channel. The large channel has an input set X^n, the Cartesian product of X. For $x = (x_1, x_2, \ldots, x_n) \in X^n$, its output

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set is

$$S_x = S_{x_1} \times S_{x_2} \times \cdots \times S_{x_n} = \{(y_1, y_2, \ldots, y_n) : y_i \in S_{x_i}, 1 \le i \le n\}.$$

Let $\mathcal{G}$ denote the characteristic graph of the large channel. Then its vertex set is X^n. Two distinct vertices $x = (x_1, x_2, \ldots, x_n)$ and $x' = (x'_1, x'_2, \ldots, x'_n)$ can be received as the same output if and only if for each $1 \le i \le n$, $S_{x_i} \cap S_{x'_i} \ne \emptyset$, when $x_i \ne x'_i$. Thus x and x' are adjacent in $\mathcal{G}$ if and only if for each $1 \le i \le n$, x_i and x'_i are adjacent in G, when $x_i \ne x'_i$. Hence the edge set of $\mathcal{G}$ is defined.

We now define the nth *AND product* of graph $G = (X, E)$, denoted by $\wedge^n G$. Its vertex set is X^n, two vertices $x, x' \in X^n$ are adjacent in $\wedge^n G$ if and only if for each $1 \le i \le n$, x_i and x'_i are adjacent in G, when $x_i \ne x'_i$; namely, either x_i and x'_i are adjacent in G, or $x_i = x'_i$. (It is slightly more convenient to give the definition if we admit any vertex in G is adjacent to itself.) Clearly, if the characteristic graph of a channel is G, and when its repeated use is viewed as a single use of a large channel, the characteristic graph of the large channel is $\wedge^n G$.

When we consider to transmit the sequence $x = (x_1, x_2, \ldots, x_n)$ of length n, where x_i may come from different input X_i, the characteristic graph of the channel is the AND product of the graphs G_i defined as follows. Let $G_1, G_2, \ldots, G_n$ be graphs, and let $V_1, V_2, \ldots, V_n$ be their vertex sets, respectively. Define their AND product $G_1 \wedge G_2 \wedge \cdots \wedge G_n$ as a graph with vertex set $V_1 \times V_2 \times \cdots \times V_n$, two distinct vertices $x = (x_1, x_2, \ldots, x_n)$ and $x' = (x'_1, x'_2, \ldots, x'_n)$ are adjacent if and only if for each $1 \le i \le n$, either x_i and x'_i are adjacent in G_i or $x_i = x'_i$. In the following proposition, the closed neighborhood $N[v]$ of a vertex v is $\{v\} \cup N(v)$.

Proposition 8.1 *Let $G_1, G_2, \ldots, G_n$ be graphs and $\mathcal{G} = G_1 \wedge G_2 \wedge \cdots \wedge G_n$. Then for any vertex $x = (x_1, x_2, \ldots, x_n)$ of $\mathcal{G}$, its closed neighborhood $N_{\mathcal{G}}[x]$ satisfies*

$$N_{\mathcal{G}}[x] = N_{G_1}[x_1] \times N_{G_2}[x_2] \times \cdots \times N_{G_n}[x_n].$$

Note the above equality does not hold for neighborhoods in general. Let $G = K_2 \cup K_1$ on vertex set $V = \{u, v, w\}$ with only one edge uv. Consider $\mathcal{G} = \wedge^2 G$. Take vertex $x = (u, w)$ of $\mathcal{G}$, its neighborhood $N_{\mathcal{G}}(x)$ is singleton $\{(v, w)\}$, but $N_G(u) \times N_G(w) = \emptyset$ since $N_G(w) = \emptyset$.

Proposition 8.2 *Let I_i be an independent set of G_i. Then $I_1 \times I_2 \times \cdots \times I_n$ is an independent set of $G_1 \wedge G_2 \wedge \cdots \wedge G_n$. Consequently,*

$$\alpha(G_1 \wedge G_2 \wedge \cdots \wedge G_n) \ge \alpha(G_1)\alpha(G_2) \cdots \alpha(G_n),$$

and

$$\alpha(\wedge^{m+n} G) \ge \alpha(\wedge^m G)\alpha(\wedge^n G).$$

Proof. For two distinct vertices $x = (x_1, x_2, \ldots, x_n)$ and $x' = (x'_1, x'_2 \ldots, x'_n)$ of $I_1 \times I_2 \times \cdots \times I_n$, there is some $1 \le i \le n$, $x_i \ne x'_i$. Since I_i is independent in G_i, so x_i and x'_i are non-adjacent in G_i. So x and x' are non-adjacent in $G_1 \wedge G_2 \wedge \cdots \wedge G_n$, hence $I_1 \times I_2 \times \cdots \times I_n$ is an independent set. □

Since $\alpha(\wedge^n G)$ is super-multiplicative, hence the limit

$$\lim_n (\alpha(\wedge^n G))^{1/n} = \sup_n (\alpha(\wedge^n G))^{1/n}$$

exists, which is denoted by $\Theta(G)$, called the *Shannon capacity* of G (or of the corresponding channel). Then we have an easy lower bound for $\Theta(G)$ as follows.

Theorem 8.1 *For any graph G, $\Theta(G) \geq \alpha(G)$.*

The inequality can be strict. The first such graph is C_5, which we will encounter in the next section.

We now define *OR product* of graphs. Let $V_1, V_2, \ldots, V_n$ be vertex sets of graphs $G_1, G_2, \ldots, G_n$, respectively. The OR product of $G_1, G_2, \ldots, G_n$, denoted by $G_1 \vee G_2 \vee \cdots \vee G_n$, is defined a graph on vertex set $V_1 \times V_2 \times \cdots \times V_n$, in which two distinct vertices $x = (x_1, x_2, \ldots, x_n)$ and $x' = (x'_1, x'_2, \ldots, x'_n)$ are adjacent if and only if for *some* $1 \leq i \leq n$, $x_i \neq x'_i$, and they are adjacent in G_i. For $G_1 = G_2 = \cdots = G_n = G$, this OR product is denoted by $\vee^n G$. By the definition, we have

$$\overline{G_1 \vee G_2 \vee \cdots \vee G_n} = \overline{G_1} \wedge \overline{G_2} \wedge \cdots \wedge \overline{G_n},$$

and hence

$$\overline{G_1 \wedge G_2 \wedge \cdots \wedge G_n} = \overline{G_1} \vee \overline{G_2} \vee \cdots \vee \overline{G_n}.$$

Combining these with the fact that $\alpha(G) = \omega(\overline{G})$, we obtain

$$\alpha(G_1 \wedge G_2 \wedge \cdots \wedge G_n) = \omega(\overline{G_1} \vee \overline{G_2} \vee \cdots \vee \overline{G_n}). \tag{8.1}$$

8.2 Shannon Capacities of Cycles

This section is devoted to compute the Shannon capacities of cycles.

Lemma 8.1 *The independence number of $\wedge^2 C_5$ is* 5.

Proof. Setting the vertex set of C_5 as $\{0, 1, 2, 3, 4\}$, we arrange all vertices of $\wedge^2 C_5$ as follows.

$$\begin{array}{ccccc}
(0,0)^* & (0,1) & (0,2) & (0,3) & (0,4) \\
(1,0) & (1,1) & (1,2)^* & (1,3) & (1,4) \\
(2,0) & (2,1) & (2,2) & (2,3) & (2,4)^* \\
(3,0) & (3,1)^* & (3,2) & (3,3) & (3,4) \\
(4,0) & (4,1) & (4,2) & (4,3)^* & (4,4)
\end{array}$$

It is easy to check that the set $\{(0,0), (1,2), (2,4), (3,1), (4,3)\}$, each element of which is marked by a star as above, forms an independent set in $\wedge^2 C_5$, it follows that $\alpha(\wedge^2 C_5) \geq 5$. For each $i = 0, 1, 2, 3, 4$, the consecutive two rows

$$\begin{array}{ccccc}
(i,0) & (i,1) & (i,2) & (i,3) & (i,4) \\
(i+1,0) & (i+1,1) & (i+1,2) & (i+1,3) & (i+1,4),
\end{array}$$

with $5 \equiv 1 \pmod 4$, contain only two non-adjacent vertices. Thus we have $\alpha(\wedge^2 C_5) \leq 5$, proving the lemma. □

Corollary 8.1 $\Theta(C_5) \geq \sqrt{5}$.

We know that there is graph G with the chromatic number $\chi(G)$ is considerably larger than its clique number $\omega(G)$, see Chapter 4. A graph G is called *perfect* if any induced subgraph H of G satisfies that $\chi(H) = \omega(H)$. Any even cycle is a perfect graph and any odd cycle with length at least 5 is not a perfect graph. Shannon (1956) proved that when G is a perfect graph, then $\Theta(G) = \alpha(G)$. The equality may not hold in general as $\Theta(C_5) \geq \sqrt{5}$. However, Lovász (1979) proved that $\Theta(C_5) = \sqrt{5}$. The knowledge on the Shannon capacity of imperfect graphs is very limited. By using stochastic search methods, Mathew and Östergård (2017) obtained that $\Theta(C_7) \geq 350^{1/5} > 3.2271$, and $\Theta(C_{15}) \geq 381^{1/3} > 7.2495$.

We adopt a simpler way from *Proofs from THE BOOK* by Aigner and Ziegler to obtain the Shannon capacities of even cycles and a general upper bound for odd cycles. We shall introduce the Lovász theta function briefly later.

Call a real vector $X = \{x_v : v \in V\}$ as a probability distribution or simply a *distribution* on the set V if $x_v \geq 0$ and $\sum_{v \in V} x_v = 1$. Denote by $\mathcal{T}$ for the set of all cliques of G. For a fixed distribution X, we write

$$\lambda(X) = \max_{T \in \mathcal{T}} \sum_{v \in T} x_v,$$

and $\lambda(G) = \inf_X \lambda(X)$. If the distribution X is viewed as weights of vertices in V, then $\sum_{v \in T} x_v$ is the weight of T, and $\lambda(X)$ is the maximum weight of a clique. Since the inf is achievable as $\lambda(X)$ is continuous on the compact set consisting of all distributions, so

$$\lambda(G) = \min_X \lambda(X) = \min_X \max_{T \in \mathcal{T}} \sum_{v \in T} x_v, \tag{8.2}$$

where the min runs through all distributions X on vertex set $V(G)$.

In order to get another expression for $\lambda(G)$, we need a basic result in the Game Theory or Linear Programming, called Minimax Theorem.

Theorem 8.2 *Let $A = (a_{ij})$ be a real $n \times m$ matrix, and let $X = (x_1, \ldots, x_m)$ and $Y = (y_1, \ldots, y_n)$ be probability distributions. Then*

$$\min_X \max_Y YAX^T = \max_Y \min_X YAX^T,$$

where the min *runs through all probability distributions X and the* max *does through all such Y. Furthermore, there exist probability distributions X^* and Y^* such that*

$$\min_X Y^* AX^T = \max_Y YAX^{*T}.$$

The proof is based on Duality Theorem, which can be found in most of textbooks on Game Theory or Linear Programming. We thus omit it.

Set $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$ and $V(G) = \{v_1, v_2, \ldots, v_m\}$. For any clique T_i and any vertex v_j of G, define

$$a_{ij} = \begin{cases} 1 \text{ if } v_j \in T_i, \\ 0 \text{ otherwise.} \end{cases}$$

Then $(a_{i1}, \cdots, a_{im})$ is the incident vector of T_i and we thus have an $n \times m$ real matrix $A = (a_{ij})$. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the vector of R^n of all zeros except for a one in the ith position. Then $e_i AX^T = \sum_{v \in T_i} x_v$, so the expression (8.2) is

$$\lambda(G) = \min_X \max_{1 \le i \le n} e_i AX^T.$$

We now consider the left-hand side in the first equality in Minimax Theorem, for a given distribution X, suppose in the ith position that the vertex AX^T has the maximum component, then

$$\max_Y YAX^T = e_i AX^T = \max_{1 \le i \le n} e_i AX^T,$$

and thus

$$\min_X \max_Y YAX^T = \min_X \max_{1 \le i \le n} e_i AX^T.$$

Denote by f_j for the vector of R^m of all zeros except for a one in the jth position. So

$$\min_X \max_{T \in \mathcal{T}} \sum_{v \in T} x_v = \min_X \max_{1 \le i \le n} e_i AX^T = \max_Y \min_{1 \le j \le m} YAf_j^T = \max_Y \min_{v \in V} \sum_{T \ni v} y_T,$$

where in the last expression the sum is taken on T over $\mathcal{T}$, and the max runs through all distributions $Y = \{y_T : T \in \mathcal{T}\}$ on $\mathcal{T}$.

We thus obtain the second expression for $\lambda(G)$ as

$$\lambda(G) = \max_Y \min_{v \in V} \sum_{T \ni v} y_T. \tag{8.3}$$

Let $U \subseteq V(G)$ be an independent set of G with $|U| = \alpha(G) = \alpha$, and define a distribution $X(U) = \{x_v : v \in V\}$ by

$$x_v = \begin{cases} 1/\alpha \text{ if } v \in U, \\ \;\;0 \quad \text{otherwise.} \end{cases}$$

Since each clique contains at most one vertex of U, we have that $\lambda(X(U)) = 1/\alpha$ and $\lambda(G) \le 1/\alpha$. We then have the following lemma.

Lemma 8.2 *For any graph G,*

$$\alpha(G) \le \frac{1}{\lambda(G)}.$$

Lemma 8.3 *For any graphs G_1 and G_2,*

$$\lambda(G_1 \wedge G_2) = \lambda(G_1)\lambda(G_2).$$

Proof. We now have two expressions for $\lambda(G)$ as

$$\lambda(G) = \min_X \max_{T\in\mathcal{T}} \sum_{v\in T} x_v = \max_Y \min_{v\in V} \sum_{T\ni v} y_T.$$

Let X and X' be the distributions which achieve the minima for $\lambda(G_1)$ and $\lambda(G_2)$, that is to say,

$$\lambda(X) = \lambda(G_1) \quad \text{and} \quad \lambda(X') = \lambda(G_2).$$

We define a distribution Z on vertex set of $G_1 \wedge G_2$ as $z_{(u,v)} = x_u x'_v$ for a vertex (u,v) of $G_1 \wedge G_2$. The fact that $Z = \{z_{(u,v)} : (u,v) \in V(G_1) \times V(G_2)\}$ is truly a probability distribution can be seen by $\sum z_{(u,v)} = \sum x_u \sum x'_v = 1$.

Claim A clique of $G_1 \wedge G_2$ is maximal if and only if it has the form $T_1 \times T_2$ with T_i is a maximal clique of G_i for $i = 1, 2$.

Proof. Indeed, note the "clique" in the statement cannot be replaced by "independent set". The dual form of the claim is " an independent set of $G_1 \vee G_2$ is maximal if and only if it has the form $S_1 \times S_2$ with S_i is a maximal independent set of G_i for $i = 1, 2$". □

Using the above claim, we have

$$\begin{aligned}\lambda(G_1 \wedge G_2) \le \lambda(Z) &= \max_{X\times X'} \sum_{(u,v)\in X\times X'} z_{(u,v)} \\ &= \max_{X\times X'} \sum_{u\in X} x_u \sum_{v\in X'} x'_v \\ &= \lambda(G_1)\lambda(G_2).\end{aligned}$$

On the other hand, denote still by T for clique of $G_1 \wedge G_2$, and Y for distribution on the set of cliques of $G_1 \wedge G_2$, then

$$\begin{aligned}\lambda(G_1 \wedge G_2) &= \max_Y \min_{(u,v)\in V(G_1)\times V(G_2)} \sum_{T\ni(u,v)} y_T \\ &\ge \lambda(Z) = \max_{X\times X'} \sum_{(u,v)\in X\times X'} z_{(u,v)} \\ &= \max_{X\times X'} \sum_{u\in X} x_u \sum_{v\in X'} x'_v \\ &= \lambda(G_1)\lambda(G_2),\end{aligned}$$

proving the lemma. □

Theorem 8.3 *For any graph G, we have*

$$\alpha(G) \le \Theta(G) \le \frac{1}{\lambda(G)}.$$

Proof. From Lemma 8.3, we see $\lambda(\wedge^n G) = \lambda^n(G)$. This and Lemma 8.2 give

$$\alpha(\wedge^n G) \leq \frac{1}{\lambda(\wedge^n G)} = \frac{1}{\lambda^n(G)},$$

hence $\Theta(G) \leq \frac{1}{\lambda(G)}$. □

Lemma 8.4 *Let $k \geq 4$ be an integer and let C_k be a cycle of length k. Then $\lambda(C_k) = \frac{2}{k}$, hence*

$$\Theta(C_k) \leq \frac{k}{2}.$$

Proof. Let $X_0 = (1/k, \dots, 1/k)$ be the uniform distribution on the vertex set. Since any clique T meets C_k at most two vertices, we obtain that

$$\lambda(C_k) = \min_X \lambda(X) \leq \lambda(X_0) = \max_{T \in \mathcal{T}} \sum_{v \in T} \frac{1}{k} = \frac{2}{k}.$$

On the other hand, $\mathcal{T}$ consists of all k vertices and all k edges. Defining a distribution Y_0 by choosing a component as $1/k$ for an edge and 0 for a vertex, and using the expression (8.3) for $\lambda(G)$, a similar argument yields that $\lambda(C_k) \geq 2/k$, and hence $\lambda(C_k) = 2/k$. □

Theorem 8.4 *Let $m \geq 2$ be an integer. Then $\Theta(C_{2m}) = m$.*

The above discuss is not sufficient to obtain the Shannon capacity of any odd cycle. The first one, $\Theta(C_5)$, was obtained by Lovász with an elegant solution. Recall the proof of a theorem in Chapter 5, in which the representation of vertices of hypergraps in R^{s+1} plays an important role.

In order to find the exact value of $\Theta(C_5)$, the idea of Lovász was to represent the vertices $VG) = \{v_1, v_2, \dots, v_m\}$ by real vectors (points in an Euclidean space) of *length one* such that any pair of vectors presenting two *non-adjacent vertices* are orthogonal. Let us call such a representation an *orthogonal representation* of G. Note that such a representation always exists: just take unit vectors

$$e_1 = (1, 0, \dots, 0), \; e_2 = (0, 1, 0 \dots, 0), \; \dots, \; e_m = (0, 0, \dots, 1)$$

in R^m.

Let $T = \{v^{(1)}, v^{(2)}, \dots, v^{(m)}\}$ be an orthogonal representation of G in R^s with $v^{(i)}$ corresponding to the vertex v_i. Denote by

$$\overline{v} = \frac{1}{m}(v^{(1)} + v^{(2)} + \dots + v^{(m)}).$$

When *any vector* $v^{(i)}$ has the same angle ($\neq \pi/2$) with $\overline{v}$, or equivalently that any inner product $v^{(i)} \cdot \overline{v}$ has the same non-zero value, denoted by $\sigma_T(G)$, we shall say that the representation T has *constant* $\sigma_T(G) = v^{(i)} \cdot \overline{v}$.

Denote by $|v| = \sqrt{v \cdot v}$ for the length of the vector v. For a probability distribution $X = (x_1, \dots, x_m)$ on vertex set V, set

$$\mu(X) = |x_1 v^{(1)} + x_2 v^{(2)} + \cdots + x_m v^{(m)}|^2$$

and

$$\mu(G) = \inf_X \mu(X) = \min_X \mu(X).$$

Lemma 8.5 *If $T = \{v^{(1)}, v^{(2)}, \ldots, v^{(m)}\}$ is an orthogonal representation of G with constant $\sigma_T(G)$, then*

$$\sigma_T(G) = \mu_T(G).$$

Proof. From the Cauchy-Schwarz inequality $|a \cdot b| \le |a||b|$, we have

$$\left[(x_1 v^{(1)} + x_2 v^{(2)} + \cdots + x_m v^{(m)}) \cdot \overline{v}\right]^2 \le \mu(X)|\overline{v}|^2.$$

However, since $v^{(i)} \cdot \overline{v} = \sigma_T(G)$ and $\sum x_i = 1$, we have

$$(x_1 v^{(1)} + x_2 v^{(2)} + \cdots + x_m v^{(m)}) \cdot \overline{v} = \sigma_T(G).$$

The above can be applied for the uniform distribution $X_0 = (1/m, \ldots, 1/m)$, giving $|\overline{v}|^2 = \sigma_T(G)$. We then have $\sigma_T^2(G) \le \mu(X)\sigma_T(G)$, or $\sigma_T(G) \le \mu(X)$ for any X, thus $\sigma_T(G) \le \min \mu(X) = \mu_T(G)$. On the other hand, we have

$$\mu_T(G) \le \mu(X_0) = \left|\frac{1}{m}(v^{(1)} + v^{(1)} + \cdots + v^{(m)})\right|^2 = |\overline{v}|^2 = \sigma_T(G),$$

so $\mu_T(G) = \sigma_T(G)$ follows. □

Lemma 8.6 *If $T = \{v^{(1)}, v^{(2)}, \ldots, v^{(m)}\}$ is an orthogonal representation of G, then*

$$\alpha(G) \le \frac{1}{\mu_T(G)} = \frac{1}{\sigma_T(G)}.$$

Proof. It suffices to show the first inequality. Let U be an independent set of G with $|U| = \alpha(G) = \alpha$. Define a distribution $X(U)$ on $V(G)$ by

$$x_i = \begin{cases} 1/\alpha \text{ if } v_i \in U, \\ 0 \quad \text{otherwise} \end{cases}$$

Since $v^{(i)} \cdot v^{(j)} = 0$ for any pair of two non-adjacent vertices and $v^{(i)} \cdot v^{(i)} = 1$, we have that

$$\mu_T(G) \le \mu(X(U)) = \left|\sum_{i=1}^m x_i v^{(i)}\right|^2 = \left|\sum_{v_i \in U} \frac{1}{\alpha} v^{(i)}\right|^2 = \frac{1}{\alpha},$$

yielding $\alpha \le 1/\mu_T(G)$ as desired. □

Let G_1 and G_2 be graphs with orthogonal representations T and S in R^t and R^s, respectively. We now do not distinguish the vertices and their representations. For

$u = (u_1, \ldots, u_t) \in T$ and $v = (v_1, \ldots, v_s) \in S$, the *tensor product* of u and v is defined as the vector

$$u \circ v = (u_1v_1, \ldots, u_1v_s, u_2v_1, \ldots, u_2v_s, \ldots\ldots, u_tv_1, \ldots, u_tv_s)$$

of R^{ts}. Denote by $T \circ S$ for the set $\{u \circ v : u \in T, v \in S\}$.

Lemma 8.7 *Let $u, x \in R^t$ and $v, y \in R^s$ be vectors. Then*

$$(u \circ v) \cdot (x \circ y) = (u \cdot x)(v \cdot y).$$

Proof. Directly from the definitions of tensor product and inner product. □

Lemma 8.8 *If T and S are orthogonal representations of G_1 and G_2, respectively, then $T \circ S$ is an orthogonal representation of $G_1 \wedge G_2$ with*

$$\mu_{T\circ S}(G_1 \wedge G_2) = \mu_T(G_1)\mu_S(G_2).$$

Proof. In fact, for any $u \in T$ and $v \in S$,

$$|u \circ v|^2 = \sum_{i,j}(u_iv_j)^2 = \sum_i u_i^2 \sum_j v_j^2 = 1.$$

If (u, v) and (x, y) are two non-adjacent vertices of $G_1 \wedge G_2$, then either u and x are non-adjacent in G_1 or v and y are non-adjacent in G_2, so $u \cdot x = 0$ or $v \cdot y = 0$. Thus

$$(u \circ v) \cdot (x \circ y) = (u \cdot x)(v \cdot y) = 0,$$

and the claimed follows. □

Theorem 8.5 *If $T = \{v^{(1)}, v^{(2)}, \ldots, v^{(m)}\}$ is an orthonormal representation of G with constant $\sigma_T(G)$, then*

$$\Theta(G) \leq \frac{1}{\sigma_T(G)}.$$

Proof. Repeatedly using Lemma 8.8, we know that T^n is an orthonormal representation of $\wedge^n G$ with constant $\mu_{T^n}(\wedge^n G) = \mu_T(G)^n$. Thus by Lemma 8.6,

$$\alpha(\wedge^n G) \leq \frac{1}{\mu_{T^n}(\wedge^n G)} = \frac{1}{\sigma_T(G)^n},$$

which yields $\alpha(\wedge^n G)^{1/n} \leq 1/\sigma_T(G)$ as desired. □

An "umbrella" used by Lovász in the proof of the following theorem is called "Lovász umbrella".

Theorem 8.6 $\Theta(C_5) = \sqrt{5}$.

Proof. For the graph C_5, Lovász obtained an orthogonal representation T in R^3 by considering an "umbrella" with five ribs $v_1, v_2, \ldots, v_5$ of unit length. Now open the umbrella (with tip at the origin o) to the point where the angles between alternate ribs are $\pi/2$, namely, it is an orthogonal representation of C_5. This can be done as the umbrella opens, in which the angle of alternate ribs varies from zero to $4\pi/5$ with $4\pi/5 > \pi/2$. After we have the orthogonal representation T of C_5 as $T = \{v^{(1)}, v^{(2)}, \cdots, v^{(5)}\}$, then a simple calculation shows that $h^2 = \frac{1}{\sqrt{5}}$, where h is distance from the origin to the plan determined by end-vertices of ribs. So $\overline{v} = (0, 0, h) = (0, 0, 5^{-1/4})$ hence

$$\sigma_T(C_5) = v^{(i)} \cdot \overline{v} = h^2 = \frac{1}{\sqrt{5}},$$

which and Theorem 8.5 prove that $\Theta(C_5) \leq \sqrt{5}$. The inverse inequality has been obtained. □

In order to improve the obtained upper bound $\Theta(C_k) \leq k/2$ for odd k, we are going to find the eigenvalues of the adjacency matrix of C_k first.

Lemma 8.9 *Let $k = 2m + 1 \geq 3$ be an integer. Then*

$$2\cos\frac{2\ell\pi}{k} \quad (\ell = 0, 1, \ldots, k-1)$$

are all eigenvalues of the adjacency matrix of C_k, in which the maximum and minimum are 2 and $-2\cos\frac{\pi}{k}$, respectively.

Proof. Let $A = (a_{ij})$ be the adjacency matrix of C_k. Then

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

in which each row contains two ones and $k-2$ zeros. Let $\zeta = e^{2\pi i/k}$. Note that $1, \zeta, \cdots, \zeta^{k-1}$ are all kth roots of unity. Denote by β for ζ^ℓ with $0 \leq \ell \leq k-1$. We shall show that $X^{(\ell)} = (1, \beta, \beta^2, \ldots, \beta^{k-1})^T$ is an eigenvector of A corresponding to eigenvalue $\beta + \beta^{-1} = \zeta^\ell + \zeta^{-\ell}$. In fact,

$$AX^{(\ell)} = \begin{pmatrix} \beta + \beta^{k-1} \\ \beta^2 + 1 \\ \beta^3 + \beta \\ \vdots \\ 1 + \beta^{k-2} \end{pmatrix} = (\beta + \beta^{-1})X^{(\ell)}.$$

Since $X^{(0)}, X^{(1)}, \ldots, X^{(k-1)}$ are independent (seen from the Vandermonde matrix formed by them), it follows that

$$\zeta^{\ell} + \zeta^{-\ell} = 2\cos\frac{2\ell\pi}{k} \quad (\ell = 0, 1, \ldots, k-1)$$

are all eigenvalues of A, which are decreasing from $\ell = 0$ to m and the increasing, proving the lemma. □

We need to recall some results in linear algebra:

Facts from Linear Algebra. If $M = (m_{ij})$ is an $m \times m$ real symmetric matrix, then it has m real eigenvalues. Furthermore, if all such eigenvalues are non-negative, then there are vectors $v^{(1)}, v^{(2)}, \ldots, v^{(m)}$ in R^s with $s = rank(M)$ such that $m_{ij} = v^{(i)} \cdot v^{(j)}$.

Let $A = (a_{ij})$ be the adjacency matrix of graph G of order k with eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k.$$

Since $\sum \lambda_i = \sum d_{ii} = 0$, we have $\lambda_k < 0$ (unless G has no edge). Let $p = |\lambda_k| = -\lambda_k$. Then the matrix

$$M = I + \frac{1}{p}A,$$

where I is the identity matrix, has k eigenvalues

$$1 + \frac{\lambda_1}{p} \geq 1 + \frac{\lambda_2}{p} \geq \cdots \geq 1 + \frac{\lambda_k}{p} = 0.$$

Therefore, we obtain a set of vectors $T = \{v^{(1)}, v^{(2)}, \ldots, v^{(k)}\}$ in R^s with $s = rank(M)$ such that

$$v^{(i)} \cdot v^{(i)} = m_{ii} = 1 \quad \text{and} \quad v^{(i)} \cdot v^{(j)} = m_{ij} = \frac{a_{ij}}{p} \quad (i \neq j).$$

If two vertices v_i and v_j are not adjacent, then $a_{ij} = 0$, thus *T forms an orthonormal representation of G*. Now for an odd cycle C_k, we have

$$\begin{aligned}\sigma_T(C_k) &= v^{(i)} \cdot \overline{v} = \frac{1}{k} v^{(i)} \cdot \left(v^{(1)} + v^{(2)} + \cdots + v^{(k)}\right) \\ &= \frac{1}{k}\left(1 + \frac{2}{p}\right) = \frac{1 + \cos(\pi/k)}{k\cos(\pi/k)}.\end{aligned}$$

Theorem 8.7 *Let $k \geq 3$ be an odd integer. Then*

$$\Theta(C_k) \leq \frac{k\cos(\pi/k)}{1 + \cos(\pi/k)}.$$

Note the equality holds for $k = 3$ in the above theorem with $\Theta(C_3) = 1$, and it does also for $k = 5$ with $\Theta(C_5) = \sqrt{5}$ and $\cos(\pi/5) = (\sqrt{5}+1)/4$. Whether or not it holds for $k \geq 7$ is unknown. We also refer the reader to Bohman and Holzman (2003) for a nontrivial lower bound on the Shannon capacities of the complements of odd cycles.

Let $T = \{v^{(1)}, v^{(2)}, \ldots, v^{(m)}\}$ with $v^{(i)} \in R^r$ be an orthonormal representation of graph G of order m. Define

$$val_T(G) = \min_c \max_{1\leq i\leq m} \frac{1}{(c \cdot v^{(i)})^2},$$

where the min runs over all unit vectors c in R^r. The vector c yielding the minimum is called the *handle* of the representation. The name "handle" comes from the Lovász umbrella in the proof for $\Theta(C_5) = \sqrt{5}$. The theta function introduced by Lovász is defined as

$$\vartheta(G) = \min_T val_T(G),$$

where T runs over all representations of G. Call a representation T to be *optimal* if it achieves the minimum value.

Lemma 8.10 $\vartheta(G_1 \wedge G_2) \leq \vartheta(G_1)\vartheta(G_2)$.

Proof. Let $T = \{u^{(1)}, u^{(2)}, \ldots, u^{(m)}\}$ in R^t and $S = \{v^{(1)}, v^{(2)}, \ldots, v^{(n)}\}$ in R^s be optimal orthonormal representations of G_1 and G_2 with handles c and d, respectively. Then $T \circ S$ is an orthonormal representation of $G_1 \wedge G_2$ and $c \circ d$ is a unit vector of R^{ts}. Hence

$$\begin{aligned}\vartheta(G_1 \wedge G_2) &\leq \max_{i,j} \frac{1}{((c \circ d)\cdot(u^{(i)} \circ v^{(j)}))^2}\\ &= \max_{i,j} \frac{1}{(c \cdot u^{(i)})^2}\frac{1}{(d \cdot v^{(j)})^2}\\ &= \vartheta(G_1)\vartheta(G_2),\end{aligned}$$

as claimed. □

In fact, the equality holds in the above lemma.

Lemma 8.11 $\alpha(G) \leq \vartheta(G)$.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_m\}$ be the vertex set of G, and let $T = \{v^{(1)}, v^{(2)}, \ldots, v^{(m)}\}$ be an optimal orthonormal representation of G in R^t with handle c. Suppose that $\{v_1, v_2, \ldots, v_k\}$ be an independent set of G with $k = \alpha(G)$. Then the vectors in $S = \{v^{(1)}, v^{(2)}, \ldots, v^{(k)}\}$ are pairwise orthogonal, thus they can be extended to a base of R^t by adding some unit vectors $w^{(k+1)}, \ldots, w^{(t)}$. Therefore,

$$1 = |c|^2 = \sum_{i=1}^{k}(c \cdot v^{(i)})^2 + \sum_{i=k+1}^{t}(c \cdot w^{(i)})^2 = \sum_{i=1}^{k}(c \cdot v^{(i)})^2 \geq \frac{k}{\vartheta(G)},$$

where the last inequality holds as by definition $\vartheta(G) \geq \frac{1}{(c \cdot v^{(i)})^2}$ for each i. □

Theorem 8.8 $\Theta(G) \leq \vartheta(G)$.

Proof. By the above two lemmas,

$$\alpha(\wedge^n G) \leq \vartheta(\wedge^n G) \leq \vartheta(G)^n,$$

which follows by the inequality as required immediately. □

Using linear algebra extensively, with a special case in a general upper bound, Lovász obtained that for odd $k \geq 3$,

$$\vartheta(C_k) = \frac{k \cos(\pi/k)}{1 + \cos(\pi/k)},$$

which is an upper bound for $\Theta(C_k)$ as we have seen.

8.3 Connection with Ramsey Numbers

The independence number of a graph product is important for the corresponding communication channel. As the capacity of a channel is described by the independence number of the graph, the following definition comes naturally. For integers $k_1 \geq 2, k_2 \geq 2, \ldots, k_n \geq 2$, define

$$\rho(k_1, \ldots, k_n) = \max\left\{\alpha(G_1 \wedge \cdots \wedge G_n) : \alpha(G_i) < k_i, i = 1, \ldots, n\right\}.$$

From the equality (8.1), we have

Proposition 8.3 *Let* $k_1 \geq 2, k_2 \geq 2, \ldots, k_n \geq 2$ *be integers. Then*

$$\rho(k_1, \ldots, k_n) = \max\{\omega(G_1 \vee \cdots \vee G_n) : \omega(G_i) < k_i, i = 1, \ldots, n\}.$$

When $k_1 = k_2 = \cdots = k_n = k \geq 2$,

$$\rho(k, \ldots, k) = \max\{\alpha(G_1 \wedge \cdots \wedge G_n) : \alpha(G_i) < k, i = 1, \ldots, n\},$$

where $G_1, G_2, \ldots, G_n$ are formally independent in the definition. However, Erdős, McEliece, and Taylor (1971), and later Alon and Orlitsky (1995) proved that $G_1, G_2, \ldots, G_n$ can be taken to one graph G. That is to say, $\rho(k, k, \ldots, k) = \rho_n(k)$, where

$$\rho_n(k) = \max\{\alpha(\wedge^n G) : \alpha(G) < k\}.$$

Theorem 8.9 *Let* $k \geq 2$ *be an integer. Then* $\rho_n(k) = \rho(k, \ldots, k)$, *i.e.,*

$$\max\{\alpha(\wedge^n G) : \alpha(G) < k\} = \max\{\alpha(G_1 \wedge \cdots \wedge G_n) : \alpha(G_i) < k, i = 1, \ldots, n\}.$$

Proof. Set

$$\rho = \rho(k, \cdots, k) = \max\{\alpha(G_1 \wedge \cdots \wedge G_n) : \alpha(G_i) < k,\, i = 1, \ldots, n\}.$$

Clearly $\rho_n(k) \le \rho$. It suffices to show that there exists a graph with $\alpha(G) < k$ such that $\alpha(\wedge^n G) \ge \rho$. By definition, there are graphs $G_1, \ldots, G_n$ with $\alpha(G_i) < k$ such that $\alpha(G_1 \wedge \cdots \wedge G_n) = \rho$, where the vertex sets of $G_1, \ldots, G_n$ are distinct. Let $G = G_1 + \cdots + G_n$ be the graph by adding edges connecting any pair of G_i and G_j completely. Then $\alpha(G) < k$. Clearly,

$$\alpha(\wedge^n G) \ge \alpha(G_1 \wedge \cdots \wedge G_n) = \rho$$

as desired. □

More importantly, Erdős, McEliece and Taylor (1971) obtained a relation of function $\rho(k_1, \ldots, k_n)$ and Ramsey number $r(k_1, \ldots, k_n)$ as follows.

Theorem 8.10 *Let* $k_1 \ge 2, k_2 \ge 2, \ldots, k_n \ge 2$ *be integers. Then*

$$\rho(k_1, \ldots, k_n) = r(k_1, \ldots, k_n) - 1.$$

Proof. Set $\rho = \rho(k_1, \ldots, k_n)$, and $r = r(k_1, \ldots, k_n) - 1$. We first prove that $\rho \ge r$. Recall the definition that r is the largest integer for which there exists a coloring of edges of K_r with colors $\{1, 2, \ldots, n\}$ such that any monochromatic clique in color i has size less than k_i for $1 \le i \le n$. Let

$$V = \{1, 2, \cdots, r\}$$

be the vertex set of this K_r, and let G_i be the subgraph with vertex set V whose edge set consists of all edges in color i. Then $\omega(G_i) < k_i$. By considering the OR product graph $G = G_1 \vee G_2 \vee \cdots \vee G_n$, we have $\rho \ge \omega(G)$. On the other hand, G contains a set

$$S = \{(1, 1, \ldots, 1), (2, 2, \ldots, 2), \cdots, (r, r, \ldots, r)\}.$$

For any pair of vertices i and j of K_r with $1 \le i \ne j \le r$, they are adjacent in some G_ℓ, so the vertices $(i, i, \ldots, i)$ and $(j, j, \ldots, j)$ of S are adjacent in G. Thus S is a clique and hence $\rho \ge \omega(G) \ge r$.

We then prove that $r \ge \rho$. From the definition, there are graphs $G_1, G_2, \ldots, G_n$ with $\omega(G_i) < k_i$ such that $G_1 \vee G_2 \vee \cdots \vee G_n$ contains a clique T of size ρ,

$$T = \{(x_1^1, x_2^1, \ldots, x_n^1), (x_1^2, x_2^2, \ldots, x_n^2), \ldots, (x_1^\rho, x_2^\rho, \ldots, x_n^\rho)\}.$$

Define a coloring for the edges of K_ρ on vertex set

$$U = \{1, 2, \ldots, \rho\}$$

with colors $\{1, 2, \ldots, n\}$ in following way. For an edge ab in this K_ρ, consider two vertices in T

$$(x_1^a, x_2^a, \ldots, x_n^a) \quad \text{and} \quad (x_1^b, x_2^b, \ldots, x_n^b),$$

which are adjacent in G. There is some i so that $x_i^a \neq x_i^b$ and $x_i^a x_i^b$ is an edge of G_i. We color the edge ab of K_ρ with the such smallest color i, then all edges of K_ρ are colored.

We claim that there is no monochromatic clique of size k_i in any fixed color i. In fact, if

$$A \subseteq U = \{1, 2, \dots, \rho\}$$

is a monochromatic clique in color i, consider

$$X = \{x_i^a : a \in A\} \subseteq V(G_i).$$

For any distinct x_i^a and x_i^b of X, the edge ab of K_ρ is colored i, thus $x_i^a x_i^b$ is an edge of G_i. Hence X forms a clique of G_i. Therefore

$$|A| = |X| \leq \omega(G_i) < k_i,$$

yielding the fact that $r \geq \rho$ and completing the proof. □

It is very interesting to study the behavior of $\rho_n(m)$. From the above theorem, we have

$$\lim_{n\to\infty} \rho_n(3)^{1/n} = \lim_{n\to\infty} (r_n(3) - 1)^{1/n}.$$

The later limit was proved to exist in Chapter 2 that is at least $321^{1/5}$, conjectured to be infinity.

8.4 Exercises

1. Determine $\alpha(\wedge^{2n} C_5)$. What can we say about $\alpha(\wedge^{2n+1} C_5)$?
2.* Prove $\alpha(\wedge^4 \overline{C}_7) \geq 17$. (Hint: Bohman and Holzman, 2003)
3. Show that the length of handle of the Lovász umbrella is $5^{-1/4}$.
4. Prove that $\Theta(C_{2m}) = m$.
5.* Prove that $\Theta(C_5) = \sqrt{5}$. (Hint: Lovász, 1979)
6.* For integers $k_1 \geq 2, k_2 \geq 2, \dots, k_n \geq 2$, define

$$\rho(k_1, \dots, k_n) = \max\left\{\alpha(G_1 \wedge \cdots \wedge G_n) : \alpha(G_i) < k_i,\ i = 1, \dots, n\right\}.$$

Prove that

$$\rho(k_1, \dots, k_n) = r(k_1, \dots, k_n) - 1.$$

(Hint: Erdős, McEliece and Taylor, 1971)

Chapter 9
Dependent Random Choice

The method of dependent random choice has many applications, particularly for extremal problems that deal with embedding a small or sparse graph into a dense graph, in which the most of embedded graphs are bipartite. To obtain such an embedding, it is sometimes convenient to find a large vertex subset U in a dense graph, in which all (or almost all) small subsets of U have many common neighbors. Using this U, one can greedily embed vertices of a desired subgraph one by one.

9.1 The Basic Lemma

For a graph G and a vertex set T, let $d(G)$ be the average degree of G, and $J(T)$ the set of common neighbors of vertices of T as

$$J(T) = \bigcap_{x \in T} N(x).$$

The following lemma is basic for dependent random choice, which appeared in different forms in Kostochka and Rödl (2001), Alon, Krivelevich and Sudakov (2003), Sudakov (2003), and the survey by Fox and Sudakov (2011).

Lemma 9.1 *Let m, r and t be positive integers. If G is a graph of order n and average degree $d = d(G)$, then there exists a subset U of G such that*

$$|U| \geq n \left(\frac{d}{n}\right)^t - \binom{n}{r} \left(\frac{m}{n}\right)^t$$

and every $R \subseteq U$ of size $|R| = r$ has $|J(R)| \geq m + 1$.

Proof. Pick a set T of t vertices uniformly at random with repetitions. For a vertex v, observe that a vertex v is a common neighbor of T, i.e., $v \in J(T)$ if and only if $T \subseteq N(v)$, so

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$$\Pr(v \in J(T)) = \Pr(T \subseteq N(v)) = \left(\frac{d(v)}{n}\right)^t.$$

Let $X = |J(T)|$. Then $E(X) = \sum_v \Pr(v \in J(T))$ and thus

$$E(X) = \sum_v \left(\frac{d(v)}{n}\right)^t = \frac{1}{n^{t-1}}\left(\frac{1}{n}\sum_v d^t(v)\right) \geq \frac{d^t}{n^{t-1}} = n\left(\frac{d}{n}\right)^t,$$

in which we use the convexity of the function $f(z) = z^t$.

Similarly, for a given set R, we have

$$\Pr(R \subseteq J(T)) = \Pr(T \subseteq J(R)) = \left(\frac{|J(R)|}{n}\right)^t.$$

Let Y be the number of subsets R of $J(T)$ with $|R| = r$ and $|J(R)| \leq m$. Then

$$E(Y) = \sum_{\substack{R \subseteq J(T),\\ |R|=r,\ |J(R)|\leq m}} \left(\frac{|J(R)|}{n}\right)^t \leq \binom{n}{r}\left(\frac{m}{n}\right)^t,$$

and thus

$$E(X - Y) \geq n\left(\frac{d}{n}\right)^t - \binom{n}{r}\left(\frac{m}{n}\right)^t. \tag{9.1}$$

Therefore, there is a choice T_0 of T for which the corresponding $X - Y$ has the lower bound as the right hand side of (9.1). Delete one vertex from each such R of $J(T_0)$. Let U be the remaining subset in $J(T_0)$. Then U satisfies the claimed property. □

The above result asserts the *size* of a set U such that there is a $K_{r,m+1}$ for *any* $R \subseteq U$ with $|R| = r$. The terms $n\left(\frac{d}{n}\right)^t$ and $\binom{n}{r}\left(\frac{m}{n}\right)^t$ are like some expectations, and m, r, t will be chosen according to requirements in applications. Note that Lemma 9.1 makes sense only if

$$|U| \geq r, \quad \text{and} \quad n\left(\frac{d}{n}\right)^t - \binom{n}{r}\left(\frac{m}{n}\right)^t > 0.$$

9.2 Applications

In this section, we will include several applications. The d-cube Q_d is a graph of order 2^d whose vertex set consists of all binary vectors of $\{0,1\}^d$ and two distinct vertices are adjacent if they differ in exactly one coordinate. Clearly, Q_d is d-regular and bipartite. Let us write $\Delta(S) = \max\{d_G(v) : v \in S\}$ for a subset S of vertices of G. The following result is a general upper bound, in which the constant is slightly larger than that for $H = K_{t,s}$ obtained by double counting method due to Füredi

(1991). This is best possible for every fixed t, as shown by the constructions due to Kollár, Rónyai, and Szabó (1996) and Alon, Rónyai and Szabó (1999).

Theorem 9.1 *Let $t \geq 2$ be an integer. If H is a bipartite graph on parts A and B with $\Delta(B) \leq t$, then*

$$ex(n, H) \leq cn^{2-1/t},$$

where $c = c(H) > 0$ is a constant. In particular, $ex(n, Q_d) \leq c\, n^{2-1/d}$.

Proof. Let $a = |A|, b = |B|, m = a+b, r = t$ and $c = \max(a^{1/t}, \frac{em}{t})$. Let G be a graph of order n and $e(G) \geq cn^{2-1/t}$. Hence the average degree $d = d(G) \geq 2cn^{1-1/t}$. Using the fact that $\binom{n}{r} < (en/r)^r$, we find that

$$n\left(\frac{d}{n}\right)^t - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq (2c)^t - \left(\frac{em}{t}\right)^t \geq (2c)^t - c^t \geq c^t.$$

Thus, by Lemma 9.1, there is a subset U in G with $|U| \geq c^t \geq a$ such that any t vertices of U have more than $m = a + b$ common neighbors.

Now we shall embed H into G as a subgraph, in which we first embed A to an arbitrary subset of size a in U. Without loss of generality, we may assume that $|U| = a$ and the embedding of A is U. For any $v \in B$, let M_v be the image of $N_H(v)$ in U. Thus $|M_v| \leq t = r$ and $|J_G(M_v)| \geq m = a + b$, and so we can embed v into $J_G(M_v) \setminus U$ as it contains at least b vertices. □

Recall a result that every graph G contains an induced subgraph H with minimum degree $\delta(H) \geq d(G)/2$ without knowing the order of H. Inversely, the aim of the following result of Sudakov (2005) is to find a sparse subgraph in a graph that contains no large book graph $B_a = K_2 + \overline{K_a}$, namely, the maximum degree of any subgraph induced by a neighborhood is less than a.

Lemma 9.2 *Let G be a graph of order n and $d = d(G)$. For any integers $t \geq 2$ and $a \geq 0$, if G contains no B_{a+1}, then it contains an induced subgraph H with*

$$|V(H)| \geq \frac{n}{2}\left(\frac{d}{n}\right)^t, \quad \text{and} \quad d(H) \leq 2d\left(\frac{a}{d}\right)^t.$$

Proof. Let T be a subset of t vertices, chosen uniformly at random with repetitions, and let $X = |J(T)|$. Similar to that in Lemma 9.1, we have

$$E(X) = \sum_v \left(\frac{d(v)}{n}\right)^t = \frac{1}{n^{t-1}}\left(\frac{1}{n}\sum_v d^t(v)\right) \geq \frac{d^t}{n^{t-1}} = n\left(\frac{d}{n}\right)^t,$$

For an edge $e = uv$, write $J(e) = J(\{u, v\})$ for the set of common neighbors of u and v. Clearly, $|J(e)| \leq a$ as G contains no B_a. Since e is an edge in $J(T)$ if and only if T is contained in $J(e)$, we have

$$\Pr(e \subseteq J(T)) = \Pr(T \subseteq J(e)) = \left(\frac{|J(e)|}{n}\right)^t \leq \left(\frac{a}{n}\right)^t.$$

Let Y be the number of edges in $J(T)$. Thus

$$E(Y) \le e(G)\left(\frac{a}{n}\right)^t = \frac{da^t}{2n^{t-1}}.$$

We shall find a choice T_0 such that the subgraph H induced by $J(T_0)$ satisfying the claimed properties.

If $a = 0$, then Y is identically 0. Therefore, there is a choice T_0 such that

$$|J(T_0)| \ge \frac{d^t}{n^{t-1}} \ge \frac{d^t}{2n^{t-1}}$$

and the number Y_0 of edges in $J(T_0)$ is 0. For the case $a \ge 1$, let

$$Z = X - \frac{d^{t-1}}{a^t}Y - \frac{d^t}{2n^{t-1}}.$$

Hence $E(Z) \ge 0$. It follows that there exists a choice T_0 such that $Z_0 = Z(T_0) \ge 0$. This implies that $X_0 = |J(T_0)| \ge \frac{d^t}{2n^{t-1}}$ and $X_0 \ge \frac{d^{t-1}}{a^t}Y_0$. Thus the subgraph H induced by $J(T_0)$ has X_0 vertices and average degree $2Y_0/X_0 \le \frac{2a^t}{d^{t-1}}$ as claimed. □

Li and Rousseau (1996) obtained that for sufficiently large n,

$$\frac{n^3}{44(\log n)^2} < r(B_n, K_n) < \frac{n^3}{\log(n/e)}.$$

Sudakov (2005) improved the above upper bound by a factor $\sqrt{\log n}$, in which Sudakov also conjectured that the order of $r(B_n, K_n)$ is $n^3/\log^2 n$.

Theorem 9.2 *For all large n,*

$$r(B_n, K_n) \le \frac{3n^3}{(\log n)^{3/2}}.$$

Proof. Let G be a graph of order $N = 3n^3/(\log n)^{3/2}$ that contains no B_n. We shall prove that $\alpha(G) \ge n$. We separate the proof into two cases depending on the magnitude of the average degree $d = d(G)$.

Case 1 $d \le 2.5n^2/\sqrt{\log n}$.

Note that the maximum degree of the subgraph induced by the neighborhood of a vertex in G is at most $n - 1$. By Theorem 3.4, we have $\alpha(G) \ge Nf_n(d)$, where $f_n(d) \ge \frac{\log(d/n)-1}{d}$, which implies that $\alpha(G) > n$.

Case 2 $d > 2.5n^2/\sqrt{\log n}$.

In this case, applying Lemma 9.2 with $t = 2$, we obtain that G contains an induced subgraph H of order

$$h \ge \frac{d^2}{2N} > n\sqrt{\log n} \quad \text{and} \quad d(H) \le \frac{2(n-1)^2}{d} < 0.8\sqrt{\log n}.$$

Thus $\alpha(G) \geq \alpha(H) \geq \frac{h}{1+d(H)} > n$. □

Another immediate application of Lemma 9.1 is a result of Sudakov (2003) on a Ramsey-Turán type problem. Let G_n be a graph of order n, and define

$$RT(n; H, f(n)) = \max\{e(G_n) : \ G_n \text{ is } H\text{-free and } \alpha(G_n) < f(n)\}.$$

Note that $RT(n; H, f_1(n)) \leq RT(n; H, f_2(n))$ if $f_1(n) \leq f_2(n)$. For a survey on Ramsey-Turán theory, see Simonovits and Sós (2001).

Trivially, $RT(n; K_3, o(n)) = o(n^2)$ since a triangle-free graph G_n has maximum degree less than $\alpha(G_n)$. A celebrated result in this area is

$$RT(n; K_4, o(n)) = (1 + o(1))\frac{n^2}{8},$$

in which the upper bound was proved by Szemerédi (1972) while the lower bound was given by Bollobás and Erdős (1976). To clarify, the above result states that every K_4-free graph G_n with independence number $\alpha(G_n) = o(n)$ has at most $(1 + o(1))n^2/8$ edges, and this bound is tight. It is natural to ask whether or not $RT(n; K_4, n^{1-\epsilon})$ is $\Omega(n^2)$ for some $\epsilon > 0$? A negative answer to this question was given by Sudakov (2003). For any fixed $\epsilon > 0$, the function $f(n)$ in the following result is larger than $n^{1-\epsilon}$ if ω tends to infinity sufficiently slowly.

Theorem 9.3 *Let* $f(n) = e^{-\omega\sqrt{\log n}}n$. *If* $\sqrt{\log n} \geq \omega \to \infty$, *then*

$$RT(n; K_4, f(n)) < e^{-\omega^2/2}n^2$$

for large n.

Proof. Suppose that there exists a K_4-free graph G of order n with edge number $e(G) \geq e^{-\omega^2/2}n^2$ and $\alpha(G) < f(n)$ for large n. It is clear that the average degree d of G is at least $2e^{-\omega^2/2}n$. For $r = 2$, $m = f(n)$ and $t = 2\sqrt{\log n}/\omega$, we have $t \geq 2$ and

$$n\left(\frac{d}{n}\right)^t \geq n\left(2e^{-\omega^2/2}\right)^t = 2^t n e^{-t\omega^2/2} = 2^t f(n),$$

and

$$\binom{n}{2}\left(\frac{m}{n}\right)^t = \frac{n(n-1)}{2}e^{-t\omega\sqrt{\log n}} = \frac{n(n-1)}{2}e^{-2\log n} < \frac{1}{2},$$

which implies that

$$n\left(\frac{d}{n}\right)^t - \binom{n}{2}\left(\frac{m}{n}\right)^t \geq 2^t f(n) - \frac{1}{2} \geq f(n).$$

From Lemma 9.1, we can find a subset U with $|U| \geq f(n)$ such that every pair of vertices in U has at least $m = f(n)$ common neighbors. The condition $\alpha(G) < f(n)$ implies that U contains an adjacent vertices u and v, which have common neighborhood W with $|W| \geq f(n)$. Since G is K_4-free, W must form an independent set and thus $\alpha(G) \geq |W| \geq f(n)$. This is a contradiction. □

The following lemma is due to Fox and Sudakov (2009), which can be used to give a better bound for Ramsey number of bipartite graphs with bounded maximum degree than that from the regularity lemma (see Chapter 11).

Lemma 9.3 *Let integers $n \geq d \geq 1$, and let $\epsilon > 0$. If G is a graph of order $N \geq 4dn\epsilon^{-d}$ and $e(G) \geq \epsilon N^2/2$, then there is a subset U in G with $|U| > 2n$ such that the number of d-subsets D in U with $|J(D)| < n$ is less than $\frac{1}{(2d)^d}\binom{|U|}{d}$. That is to say, the fraction of such d-subsets in U is less than $(2d)^{-d}$.*

Proof. Pick a set T of d vertices from $V(G)$ uniformly at random with repetitions. Let $X = |J(T)|$, and let Y be the number of d-sets D in $J(T)$ with $|J(D)| < n$. Similar to the proof of Lemma 9.1 by noting the average degree of G is at least ϵN, we obtain

$$E(X) \geq \frac{(\epsilon N)^d}{N^{d-1}} \geq \epsilon^d N, \quad \text{and} \quad E(Y) < \binom{N}{d}\left(\frac{n}{N}\right)^d.$$

If $E(Y) = 0$, then Y is identically zero. We are done by taking $U = J(T_0)$ for some choice T_0 of T such that $|J(T_0)| \geq E(X) \geq 4dn$ from the assumption. So we assume that $E(Y) > 0$ in the remaining proof. As $E(X^d) \geq E^d(X)$ from convexity, we obtain

$$E\left(X^d - \frac{E^d(X)}{2E(Y)}Y - \frac{1}{2}E^d(X)\right) \geq 0.$$

Therefore, there is a choice T_0 of T such that the expression in the bracket is nonnegative. Let $X_0 = |J(T_0)|$ and $Y_0 = Y(T_0)$. Thus

$$X_0^d \geq \frac{1}{2}E^d(X) \geq \frac{1}{2}(\epsilon^d N)^d,$$

and hence $|J(T_0)| = X_0 > 2n \geq 2d$. Note that

$$X_0^d = \frac{X_0^d\, d!}{X_0(X_0-1)\cdots(X_0-d+1)}\binom{X_0}{d} < 2^{d-1}d!\binom{X_0}{d},$$

so we have

$$Y_0 \leq \frac{2X_0^d E(Y)}{E^d(X)} < \frac{2^d d!}{(\epsilon^d N)^d}\binom{X_0}{d}\binom{N}{d}\left(\frac{n}{N}\right)^d < \frac{1}{(2d)^d}\binom{X_0}{d}.$$

Now we can take $U = J(T_0)$, which satisfies the asserted properties. □

Lemma 9.4 *Let H be a bipartite graph of order n with $\Delta(H) \leq d$. If a graph G contains a subset U with $|U| > 2n$ such that the fraction of subsets D in U with $|D| = d$ and $|J(D)| < n$ is less than $(2d)^{-d}$, then G contains H as a subgraph.*

Proof. We may assume that $d \geq 1$. We say that a d-subset D of U is *good* if $|J(D)| \geq n$. Generally, if S is a subset of U with $s = |S| \leq d$, then we say that S is *good* if S is contained in more than

$$\left(1 - \frac{1}{(2d)^{d-s}}\right)\binom{|U| - s}{d - s}$$

good d-subsets of U. For a good set S with $|S| < d$ and a vertex $w \in U \setminus S$, we say that w is *good* with respect to S if $S \cup \{w\}$ is good.

Clearly, a subset S of U with $s = |S| \leq d$ is good if the fraction of bad d-sets containing S is less than $\frac{1}{(2d)^{d-s}}$. For example, the empty set is good since the fraction of bad d-sets in U is at most $(2d)^{-d}$ from the assumption.

For a good set S, let B_S denote the set of vertices $w \in U \setminus S$ that are bad with respect to S. The following claim is crucial for the proof.

Claim If S is good with $s = |S| < d$, then $|B_S| \leq \frac{|U|-s}{2d}$.

Proof. Indeed, suppose to the contrary that $|B_S| > \frac{|U|-s}{2d}$. For any $w \in B_S$, the set $S \cup \{w\}$ is bad and thus the number of bad d-sets that contains $S \cup \{w\}$ is at least

$$\frac{1}{(2d)^{d-s-1}}\binom{|U| - s - 1}{d - s - 1}.$$

Let us count these bad d-sets over w of B_S. Note that each such d-set is counted at most $d - s$ times, thus the number of these bad d-sets is at least

$$\frac{|B_S|}{(d-s)(2d)^{d-s-1}}\binom{|U| - s - 1}{d - s - 1} > \frac{1}{(2d)^{d-s}}\binom{|U| - s}{d - s},$$

contradicting to the fact that S is good. □

Let V_1 and V_2 be the two parts of the bipartite graph H with

$$V_1 = \{v_1, v_2, \ldots, v_m\}.$$

Denote $L_i = \{v_1, v_2, \ldots, v_i\}$, and we shall find an embedding ϕ of H into G such that $\phi(V_1)$ is contained in U and

- $\phi(N(w) \cap V_1)$ is good for each $w \in V_2$,

where and henceforth $N(w) = N_H(w)$. This ϕ is constructed such that $\phi(N(w) \cap L_i)$ is good for any $w \in V_2$ and any $i \leq m$ by induction on i.

As mentioned, the empty set $\emptyset$ is good, and hence by the claim, the number of bad vertices respect to $\emptyset$ is at most $|U|/(2d)$. Any good vertex in U with respect to $\emptyset$ forms a good singleton set. Let us pick such a good vertex to be $\phi(v_1)$. Note that for any $w \in V_2$, $\phi(N(w) \cap L_1)$ is an empty set or a singleton $\{\phi(v_1)\}$, so $\phi(N(w) \cap L_1)$ is good as desired.

Suppose that we have embedded L_i into U such that $\phi(N(w) \cap L_i)$ is good for any $w \in V_2$. We then shall find a vertex in U to be $\phi(v_{i+1})$. Note that if w and v_{i+1}

are non-adjacent, then $N(w) \cap L_{i+1} = N(w) \cap L_i$ hence $\phi(N(w) \cap L_i)$ is good. Since $\Delta(H) \leq d$, there are at most d subsets S of the form $S = N(w) \cap L_{i+1}$ among neighbors w of v_{i+1}. By the induction hypothesis, for each such subset S, the set $\phi(S \setminus \{v_{i+1}\}) = \phi(N(w) \cap L_i)$ is good and therefore there are at most $|U|/(2d)$ bad vertices in U with respect to it. In total this gives at most $|U|/2$ bad vertices. The remaining at least $|U|/2 - i > 0$ vertices in $U \setminus \phi(L_i)$ are good with respect to all the above sets $\phi(S \setminus \{v_{i+1}\})$ and we can pick any of them to be $\phi(v_{i+1})$. Thus the set $\phi(N(w) \cap L_{i+1})$ is good for every $w \in V_2$.

Once we have found ϕ satisfying the mentioned property, we then embed vertices of V_2 one by one. Suppose that the current vertex to embed is $w \in V_2$. Then $\phi(N(w)) = \phi(N(w) \cap L_m)$ is good and hence $\phi(N(w))$ has at least n common neighbors. Since less than n of them were so far occupied, we still have an available vertex to embed w. We thus complete the embedding of H into G. □

Theorem 9.4 *Let H be a bipartite graph of order n with $\Delta = \Delta(H) \geq 1$. For any $\epsilon > 0$, if G is a graph of order $N \geq 8\Delta\epsilon^{-\Delta}n$ and $e(G) \geq \epsilon\binom{N}{2}$, then G contains H as a subgraph.*

Proof. Let $\epsilon' = (1 - 1/N)\epsilon$. Thus, we have $N \geq 8\Delta n/\epsilon^{\Delta} \geq 4\Delta n/\epsilon'^{\Delta}$ and $e(G) \geq \epsilon' N^2/2$. Therefore, Lemma 9.3 implies that G contains a subset U with $|U| > 2n$ such that the fraction of sets D in U with $|D| = \Delta$ and $|J(D)| < n$ is less than $1/(2\Delta)^{\Delta}$. Now, Lemma 9.4 guarantees that G contains every bipartite graph H of order n with $\Delta(H) \leq \Delta$ as desired. □

The following result follows from Theorem 9.4 immediately.

Theorem 9.5 *Let H be a bipartite graph of order n. If the maximum degree of H is at most $\Delta \geq 1$, then*

$$r(H) \leq 8\Delta 2^{\Delta} n.$$

In particular, $r(K_{n,n}) \leq 18n^2 2^n$ and $r(Q_d) \leq 8d4^d$.

Proof. Taking $\epsilon = 1/2$ together with the majority color in a 2-coloring of edges of K_N, where $N = 8\Delta 2^{\Delta} n$, we have the asserted upper bound from Theorem 9.4. □

Note that the upper bound for $r(K_{n,n})$ in the above theorem has been improved to $O(2^n \log n)$ by Conlon (2008). Conlon, Fox and Sudakov (2014) gave an upper bound as $r(H) \leq \Delta 2^{\Delta+5} n$ through a different and short proof. Furthermore, Conlon, Fox and Sudakov (2016) obtained an upper bound as $r(H) \leq 2^{\Delta+6} n$.

For the lower bound, Graham, Rödl and Ruciński (2001) proved that there is a constant $a > 1$ such that for each $\Delta \geq 2$ and $n \geq \Delta + 1$, there is a bipartite graph H of order n and $\Delta(H) = \Delta$ satisfying that $r(H) \geq a^{\Delta} n$. For the cube Q_d, Burr and Erdős conjectured that $\{Q_d : d \geq 1\}$ is a Ramsey linear family, i.e., $r(Q_d)$ is at most $c2^d$ for some constant c. This conjecture has been confirmed by Conlon, Fox, Lee and Sudakov (2013).

A graph G is called d-degenerate if every subgraph of G has a minimum degree at most d. Let us turn to the degenerate bipartite graphs. Erdős conjectured that $ex(n, H) = O(n^{2-1/r})$ if H is r-degenerate and bipartite, and for any graph H, if

H has no isolated vertices, then $r(H) \le 2^{c\sqrt{m}}$, where $m = e(H)$. A progress has been made for the first conjecture due to Alon, Krivelevich and Sudakov (2003), and the second conjecture was verified for bipartite graphs by Alon, Krivelevich and Sudakov (2003) and completely solved by Sudakov (2011).

Lemma 9.5 *Let $G = G(U_1, U_2)$ be a bipartite graph. If any r vertices in each of U_1 and U_2 have at least n common neighbors, then G contains every r-degenerate bipartite graph of order n.*

Proof. Let $V(H) = \{v_1, v_2, \dots, v_n\}$, where every vertex v_i has at most r neighbors v_j with $j < i$. Let A_1 and A_2 be two parts of H. We shall find an embedding ϕ of H into G such that $\phi(A_k) \subseteq U_k$ for $k = 1, 2$. Suppose that we have embedded $v_1, v_2, \dots, v_{i-1}$ and the current vertex to embed is v_i, where $v_i \in A_1$, say. Consider the set $\{\phi(v_j) : j < i,\ v_j v_i \in E(H)\}$. This set is contained in U_2, and it has cardinality at most r and hence at least n common neighbors in U_1. All these neighbors can be used to embed v_i and at least one of them is not occupied yet, which can be picked to be $\phi(v_i)$. □

Lemma 9.6 *Let $r, s \ge 2$ be integers. If G is a graph of order N with $e(G) \ge N^{2-1/(s^3 r)}$, then G contains disjoint subsets U_1 and U_2 such that $|U_1| \ge N^{1-1/s}$ and in each of which every r vertices have at least $N^{1-1.8/s}$ common neighbors in the other.*

Proof. Let $q = 1.75rs$, $d = 2e(G)/N \ge 2N^{1-1/(s^3 r)}$, $m = N^{1-1.8/s}$ and $t = s^2 r$. Thus

$$N\left(\frac{d}{N}\right)^t - \binom{N}{q}\left(\frac{m}{N}\right)^t \ge 2^t N^{1-t/(s^3 r)} - \frac{N^{q-1.8t/s}}{q!}$$
$$\ge 2^t N^{1-1/s} - \frac{1}{q!} \ge N^{1-1/s}.$$

Applying Lemma 9.1, we obtain a set U_1 with $|U_1| \ge N^{1-1/s}$ such that every q vertices in U_1 has at least $m = N^{1-1.8/s}$ common neighbors in G.

Let T be a subset of U_1 consisting of $q - r$ vertices chosen from U_1 randomly and uniformly with repetitions. If R is a fixed subset with $|R| = r$ and $|J(R) \cap U_1| \le m$, then

$$\Pr(T \subseteq J(R)) = \left(\frac{|J(R) \cap U_1|}{|U_1|}\right)^{q-r} \le \left(\frac{m}{|U_1|}\right)^{q-r}.$$

Note that the event $R \subseteq J(T)$ is exactly that $T \subseteq J(R)$. Thus the probability that $J(T)$ contains a subset R with $|R| = r$ and $|J(R) \cap U_1| \le m$ is at most

$$\binom{N}{r}\left(\frac{m}{|U_1|}\right)^{q-r} \le \frac{N^r}{r!} N^{-0.8(q-r)/s} < 1,$$

where we used that $q - r > 1.25rs$ and $|U_1| \ge N^{1-1/s}$.

Therefore, there is a choice T_0 of T such that every r vertices of $J(T_0)$ have at least m common neighbors in U_1. Let $U_2 = J(T_0)$. Consider now an arbitrary subset

S of U_1 with $|S| = r$. Since $S \cup T_0$ is a subset of U_1 of size at most q, this set has at least m common neighbors in G. Observe that $J(S \cup T_0) \subseteq J(T_0) = U_2$. Hence S has at least m common neighbors in U_2, and the statement follows as asserted. □

From the above two lemmas, we get the following corollary immediately.

Corollary 9.1 *Let $r, s \geq 2$ be integers. If G is a graph of order N and $e(G) \geq N^{2-1/(s^3 r)}$, then G contains every r-degenerate bipartite graph of order at most $N^{1-1.8/s}$.*

Theorem 9.6 *Let H be an r-degenerate bipartite graph of order h. For all $n \geq h^{10}$,*

$$ex(n, H) \leq n^{2-1/(8r)}.$$

Proof. Let G be a graph of order n and $e(G) \geq 2n^{2-1/(8r)}$. Substituting $s = 2$ in Corollary 9.1, we have that G contains H since it is r-degenerate with order $h \leq n^{1-1.8/s} = n^{0.1}$. □

The following result is due to Alon, Krivelevich and Sudakov (2003).

Theorem 9.7 *For any bipartite graph H with m edges and no isolated vertices,*

$$r(H) \leq 2^{16\sqrt{m}+1}.$$

Proof. We shall first prove that H is $\sqrt{m}$-degenerate. If not, H has a subgraph H' with $\delta(H') > \sqrt{m}$. Let (U, W) be the bipartition of H'. Thus, $|U| > \sqrt{m}$ and

$$e(H') = \sum_{v \in U} d_{H'}(v) \geq |U|\delta(H') > m,$$

which is a contradiction.

Let $N = 2^{16\sqrt{m}+1}$ and consider a red/blue coloring of the edges of K_N. We claim that at least $N^{2-1/(8\sqrt{m})}$ edges have been colored in red, say. To see this, it suffices to show $\frac{1}{2}\binom{N}{2} \geq N^{2-1/(8\sqrt{m})}$, which is $(N-1)N^{1/(8\sqrt{m})} \geq 4N$ and follows from the fact that

$$N^{1/8\sqrt{m}} = 2^{2+1/8\sqrt{m}} = 4\exp\left\{\frac{\log 2}{8\sqrt{m}}\right\} > 4\left(1 + \frac{\log 2}{8\sqrt{m}}\right)$$

immediately. These edges induce a red graph, denoted by G, which satisfies Corollary 9.1 with $r = \sqrt{m}$ and $s = 2$. Thus G contains every $\sqrt{m}$-degenerate bipartite graph of order at most $N^{1-1.8/s}$. Note that $N^{1-1.8/s} = N^{0.1} > 2^{1.6\sqrt{m}} > 2m$, and the order of H is at most $2m$, so H is a subgraph of G. □

9.3 Exercises

1. Prove that any graph with maximum degree Δ is Δ-degenerate. When will we say that it is $(\Delta - 1)$-degenerate?

2. Prove that P_n is 1-degenerate. How about $K_{1,n}$, C_n, T_n and $K_{m,n}$?

3. Prove that every graph with m edges is $\sqrt{m}$-degenerate.

4. Let G be a graph of order n and $d = d(G)$. For any integers $t \geq 2$ and $a \geq 0$, prove that if G contains no B_{a+1}, then it contains an induced subgraph H with

$$|V(H)| \geq \frac{n}{2}\left(\frac{d}{n}\right)^t, \quad \text{and} \quad d(H) \leq 2d\left(\frac{a}{d}\right)^t.$$

5. Improve the constant 3 in Theorem 9.2 to $2 + o(1)$. How do it further?

6. Sudakov (2005) conjectured that the order of $r(B_n, K_n)$ is $n^3/\log^2 n$. For each $k \geq 2$, estimate $r(B_n^{(k)}, K_n)$.

7. Let $G = G(U_1, U_2)$ be a bipartite graph. Prove that if any r vertices in each of U_1 and U_2 have at least n common neighbors, then G contains every r-degenerate bipartite graph of order n.

8.* Prove that $RT(n; K_4, o(n)) \leq (1 + o(1))\frac{n^2}{8}$. (Hint: Szemerédi, 1972)

9.* Prove that for any bipartite graph H with m edges and no isolated vertices, $r(H) \leq 2^{c\sqrt{m}}$ for some constant $c > 0$. (Hint: Alon, Krivelevich and Sudakov, 2003)

Chapter 10
Quasi-Random Graphs

Random graphs have been proven to be one of the most important tools in modern graph theory. Their tremendous triumph raises the following general question: what are the essential properties and how can we tell when a given graph behaves like a random graph G_p in $\mathcal{G}(n, p)$? Here a typical property of random graphs is what a.a.s. G_p satisfies. This leads us to a concept of *quasi-random graphs* (also called pseudo-random graphs). It was Thomason (1987) who introduced the notation of jumbled graphs in order to measure the similarity between the edge distribution of quasi-random graphs and random graphs. An important result of Chung, Graham and Wilson (1989) showed that many properties of different nature are equivalent. The *quasi-random graph* is in fact a family of graphs, which satisfy any of those equivalent properties. For a survey on this topic, see Krivelevich and Sudakov (2006).

10.1 Properties of Dense Graphs

Roughly speaking, a quasi-random graph G of order n is a graph that behaves like a random graph G_p with $p = e(G)/\binom{n}{2}$. For $0 < p < 1 \le \alpha$, a graph G is called (p, α)-*jumbled* if each induced subgraph H on h vertices of G satisfies that

$$\left| e(H) - p\binom{h}{2} \right| \le \alpha h.$$

For given graphs G and H, let $N_G^*(H)$ be the number of labeled occurrences of H as an induced subgraph of G, which is the number of adjacency-preserving injections from $V(H)$ to $V(G)$ whose image is the set of vertices of an induced copy of H of G. Namely, these injections are both adjacency-preserving and non-adjacency-preserving. Let $N_G(H)$ be the number of labeled copies of H as a subgraph (not necessarily induced) of G. Thus

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$$N_G(H) = \sum_{\substack{H':H'\supseteq H\\ V(H')=V(H)}} N_G^*(H'),$$

that is to say, H' ranges over all graphs on $V(H)$ obtained from H by adding a set of edges out of H.

For example, if $G = H = C_t$, then $N_G^*(H) = N_G(H) = 2t$, and if $G = K_n$ and $n \geq t \geq 4$, then $N_G^*(C_t) = 0$ and $N_G(C_t) = N_G^*(K_t) = (n)_t$. If $G = K_{n/2,n/2}$ and n is even, then $N_G^*(C_4) = N_G(C_4) = 2\left[\frac{n}{2}\left(\frac{n}{2} - 1\right)\right]^2$.

Let G be a (p, α)-jumbled graph of order n, where $\alpha = \alpha_n = o(n)$ as $n \to \infty$. As shown by Thomason, for fixed p and graph H of order h,

$$N_G^*(H) \sim p^{e(H)}(1-p)^{\binom{h}{2}-e(H)}n^h.$$

For distinct vertices x and y of G, denote by $s(x, y)$ the number of vertices of G that adjacent to x and y the same way: either to both or none. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of G with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. Denote $\lambda = \lambda(G) = |\lambda_2|$. For two subsets B and C, denote $e(B, C)$ by the number of edges between B and C, in which each edge in $B \cap C$ is counted twice. If $B \cap C = \emptyset$, then $e(B, C)$ is simply the number of edges between B and C.

The *quasi-random graph* defined by Chung, Graham and Wilson is in fact a family of simple graphs, which satisfy any (hence all) of those equivalent properties in the following theorem. It is remarkable that these properties ignore "small" local structures. The expressions of the properties are related to the edge density p, in particular the case $p = 1/2$ is as follows.

Theorem 10.1 *Let $\{G\}_{n=1}^{\infty}$ be a sequence of graphs, where $G = G_n$ is a graph of order n. The following properties are equivalent:*

$P_1(h)$**:** *For all graphs H of order $h \geq 4$, $N_G^*(H) \sim (\frac{1}{2})^{\binom{h}{2}}n^h$.*

$P_2(t)$**:** $e(G) \sim \frac{n^2}{4}$ *and* $N_G(C_t) \leq \left(\frac{n}{2}\right)^t + o(n^t)$ *for any* $t \geq 4$.

P_3**:** $e(G) \geq \frac{n^2}{4} + o(n^2)$, $\lambda_1 \sim \frac{n}{2}$ *and* $\lambda_2 = o(n)$.

P_4**:** *For each* $U \subseteq V(G)$, $e(U) = \frac{1}{2}\binom{|U|}{2} + o(n^2)$.

P_5**:** *For any two subsets* $U, V \subseteq V(G)$, $e(U, V) = \frac{1}{2}|U||V| + o(n^2)$.

P_6**:** $\sum_{x,y}\left||N(x) \cap N(y)| - \frac{n}{4}\right| = o(n^3)$.

P_7**:** $\sum_{x,y}\left|s(x, y) - \frac{n}{2}\right| = o(n^3)$.

Proof. In order to simplify the proof and catch the main idea, we assume that G is d-regular with $d = (1/2 + o(1))n$. The steps of the proof are as follows,

$$P_1(h) \Rightarrow P_2(h) \Rightarrow P_2(4) \Rightarrow P_3 \Rightarrow P_4 \Rightarrow P_5 \Rightarrow P_6 \Rightarrow P_7 \Rightarrow P_1(h).$$

Fact 1 $P_1(h) \Rightarrow P_2(h)$ $(h \geq 4)$.

Proof. We will show that

$$N_G(C_h) = \sum_{\substack{H' : H' \supseteq C_h \\ V(H')=V(C_h)}} N_G^*(C_h) \le (1 + o(1)) \left(\frac{n}{2}\right)^h.$$

As H' ranges over all graphs on $V(C_h)$ obtained from C_h by adding to it a set of edges out of C_h, we obtain that the number of such H' is $2^{\binom{h}{2}-h}$. $P_1(h)$ states that $N_G^*(H) \sim (\frac{1}{2})^{\binom{h}{2}} n^h$ for any graph H of order h. Therefore, $N_G(C_h) = (1+o(1)) \left(\frac{n}{2}\right)^h$ and so $P_2(h)$ follows.

Fact 2 $P_2(4) \Rightarrow P_3$.

Proof. Since we suppose G is regular, which together with the condition that $e(G) \sim \frac{n^2}{4}$ yield $\lambda_1 = \frac{n}{2} + o(n)$. Now, consider the trace of A^4. Clearly,

$$tr(A^4) = \sum_{i=1}^{n} \lambda_i^4 \ge \lambda_1^4 \ge (1 + o(1))\frac{n^4}{16}. \qquad (10.1)$$

On the other hand, as this trace is precisely the number of labeled and closed walks of length 4 in G, i.e., the number of sequences $v_0, v_1, v_2, v_3, v_4 = v_0$ such that $v_i v_{i+1}$ is an edge for $i = 0, 1, 2, 3$. This number is $N_G(C_4)$ plus the number of such sequences in which $v_2 = v_0$, and plus the number of such sequences in which $v_2 \ne v_0$ and $v_3 = v_1$. Thus by the condition of $P_2(4)$,

$$\sum_{i=1}^{n} \lambda_i^4 = N_G(C_4) + o(n^4) \le (1 + o(1))\frac{n^4}{16}. \qquad (10.2)$$

It follows from (10.1) and (10.2) that $\lambda_1 \sim \frac{n}{2}$ and $\sum_{i=2}^{n} \lambda_i^4 = o(n^4)$, and hence $\lambda_2 = o(n)$ follows as desired.

Fact 3 $P_3 \Rightarrow P_4$.

Proof. It follows from Corollary 10.2 in the next section by noting that G is regular.

Fact 4 $P_4 \Rightarrow P_5$.

Proof. If U and V are disjoint, then

$$\begin{aligned} e(U, V) &= e(U \cup V) - e(U) - e(V) \\ &= \frac{1}{4}(|U| + |V|)^2 - \frac{1}{4}|U|^2 - \frac{1}{4}|V|^2 + o(n^2) \\ &= \frac{1}{2}|U||V| + o(n^2). \end{aligned}$$

We now suppose that U and V are not disjoint, and we write $|U| = a$, $|V| = b$ and $|U \cap V| = c$. From P_4 and what we have just proved,

$$\begin{aligned} e(U,V) &= e(U\setminus V, V\setminus U) + e(U\cap V, U\setminus V) + e(U\cap V, V\setminus U) + 2e(U\cap V)\\ &= \frac{1}{2}(a-c)(b-c) + \frac{1}{2}c(a-c) + \frac{1}{2}c(b-c) + 2\cdot\frac{1}{4}c^2 + o(n^2)\\ &= \frac{1}{2}ab + o(n^2), \end{aligned}$$

which is P_5 as desired.

Fact 5 $P_5 \Rightarrow P_6$.

Proof. Let x be a fixed vertex of G, and let V_1 be the set of all neighbors of x in G. We have $|V_1| = (1/2 + o(1))n$ under the assumption that G is d-regular with $d = (1/2 + o(1))n$. Define

$$U_1 = \left\{y \in V(G), y \neq x : \; |N(x)\cap N(y)| \geq \frac{n}{4}\right\},$$

and

$$U_2 = \left\{y \in V(G), y \neq x : \; |N(x)\cap N(y)| < \frac{n}{4}\right\}.$$

Observe that

$$\begin{aligned} \sum_{y\in U_1}\left||N(x)\cap N(y)| - \frac{n}{4}\right| &= \sum_{y\in U_1}|N(x)\cap N(y)| - |U_1|\frac{n}{4}\\ &= e(U_1, V_1) - |U_1|\frac{n}{4}\\ &= \frac{1}{2}|U_1||V_1| + o(n^2) - |U_1|n/4\\ &= o(n^2), \end{aligned}$$

in which the third equality follows from P_5. A similar argument implies that

$$\sum_{y\in U_2}\left||N(x)\cap N(y)| - \frac{n}{4}\right| = o(n^2).$$

Therefore, for every vertex x of G,

$$\sum_{y\in V(G), y\neq x}\left||N(x)\cap N(y)| - \frac{n}{4}\right| = o(n^2).$$

Summing over all vertices x we conclude that G satisfies property P_6 as desired.

Fact 6 $P_6 \Rightarrow P_7$.

Proof. Since G is d-regular with $d = (1/2 + o(1))n$, it follows that

$$\begin{aligned} s(x,y) &= |N(x)\cap N(y)| + (n - |N(x)\cup N(y)|)\\ &= |N(x)\cap N(y)| + n - (2d - |N(x)\cap N(y)|)\\ &= 2|N(x)\cap N(y)| + o(n), \end{aligned}$$

which together with property P_6 yield that

$$\sum_{x,y} \left| s(x,y) - \frac{n}{2} \right| = o(n^3).$$

Fact 7* $P_7 \Rightarrow P_1(h)$.

Proof. Suppose that P_7 holds:

$$\sum_{x,y} \left| s(x,y) - \frac{n}{2} \right| = o(n^3). \tag{10.3}$$

For any fixed graph H on h vertices, denote by $N_h = N_G^*(H)$ and we shall show that

$$N_h = (1 + o(1))n^h 2^{-\binom{h}{2}}.$$

Let $\{v_1, v_2, \ldots, v_h\}$ denote the vertex set of H. For each $1 \le r \le h$, put $V_r = \{v_1, v_2, \ldots, v_r\}$, and let $H(r)$ be the induced subgraph of H on V_r. Denote $N_r = N_G^*(H(r))$. It suffices to prove that for $1 \le r \le h$,

$$N_r = (1 + o(1))n_{(r)} 2^{-\binom{r}{2}},$$

where $n_{(r)} = n(n-1)\cdots(n-r+1)$. The proof is by induction on r.

This is trivial for $r = 1$. Assuming it holds for r, where $1 \le r < h$, we prove it for $r+1$. For any two distinct vertices u and v of G, let $a(u,v)$ be 1 if $uv \in E(G)$ and 0 otherwise. For a vector $\alpha = (\alpha_1, \ldots, \alpha_r)$ of labeled distinct vertices of G, and for a vector $\epsilon = (\epsilon_1, \ldots, \epsilon_r)$ of $(0,1)$-entries, define

$$f_r(\alpha, \epsilon) = |\{v \in V : v \ne \alpha_1, \ldots, \alpha_r \text{ and } a(v, \alpha_j) = \epsilon_j, \text{ for all } 1 \le j \le r\}|.$$

Note that if $\alpha_1, \ldots, \alpha_r$ induce a copy of $H(r)$, then $f_r(\alpha, \epsilon)$ is just the number of vertices such that each of such a vertex together with $\alpha_1, \ldots, \alpha_r$ induce a copy of $H(r+1)$. Thus N_{r+1} is the sum of the N_r quantities $f_r(\alpha, \epsilon)$ in which $\epsilon_j = a(v_{r+1}, v_j)$ and α ranges over all N_r induced copies of $H(r)$ in G.

Observe that altogether there are precisely $n_{(r)} 2^r$ quantities $f_r(\alpha, \epsilon)$ since there are $(n)_r$ ways to choose α and 2^r possibilities of ϵ. It is convenient to view $f_r(\alpha, \epsilon)$ as a random variable defined on a sample space of $n_{(r)} 2^r$ points, each having an equal probability. To complete the proof we compute the expectation and the variance of this random variable. We show that the variance is so small that most of the quantities $f_r(\alpha, \epsilon)$ are very close to the expectation, and thus obtain a sufficiently accurate estimate for N_{r+1} which is the sum of N_r such quantities.

We start with the simple computation of the expectation $E[f_r]$ of $f_r(\alpha, \epsilon)$. Note that every vertex $v \ne \alpha_1, \ldots, \alpha_r$ defines ϵ uniquely, that is to say, for each such fixed vertex v, there is exactly one ϵ such that $f_r(\alpha, \epsilon)$ contributes 1. Therefore,

$$E[f_r] = \frac{1}{n_{(r)}2^r} \sum_{\alpha,\epsilon} f_r(\alpha,\epsilon) = \frac{1}{n_{(r)}2^r} \sum_{\alpha} \sum_{\epsilon} f_r(\alpha,\epsilon)$$
$$= \frac{1}{n_{(r)}2^r} \sum_{\alpha} (n-r) = \frac{n-r}{2^r}.$$

Next, we estimate the quantity S_r defined by

$$S_r = \sum_{\alpha,\epsilon} f_r(\alpha,\epsilon)(f_r(\alpha,\epsilon)-1).$$

Claim $S_r = \sum_{x \neq y} s(x,y)_{(r)}$.

Proof. We count s_r on two ways. Observe that S_r can be interpreted as the number of ordered triples $(\alpha, \epsilon, (x, y))$, where $\alpha = (\alpha_1, \ldots, \alpha_r)$ is an ordered set of r distinct vertices of G, $\epsilon = (\epsilon_1, \ldots, \epsilon_r)$ is a binary vector of length r, and (x, y) is an ordered pair of additional vertices of G so that

$$a(x,\alpha_k) = a(y,\alpha_k) = \epsilon_k \text{ for all } 1 \le k \le r.$$

Now we count s_r on another way. Given x and y, the required additional r vertices $\alpha_1, \ldots, \alpha_r$ must come exactly from $\{u \in V(G) : a(x,u) = a(y,u)\}$. Therefore, there are $s(x, y)_{(r)}$ ways to choose them, which completes the proof of the claim. □

We next assert that (10.3) implies

$$S_r = \sum_{x \neq y} s(x,y)_{(r)} = (1+o(1))n^{r+2}2^{-r}. \tag{10.4}$$

To this end, we first define

$$\epsilon_{xy} = s(x,y) - n/2.$$

By (10.3), $\sum_{x \neq y} |\epsilon_{xy}| = o(n^3)$. Clearly, $|\epsilon_{xy}| \le n$. Thus for any fixed a,

$$\sum_{x \neq y} |\epsilon_{xy}|^a \le n^{a-1} \sum_{x \neq y} |\epsilon_{xy}| = o(n^{a+2}).$$

This implies that for some appropriate constants c and c_k depending on r,

$$\sum_{x \neq y} s(x,y)_{(r)} = \sum_{x \neq y} \left(\frac{n}{2} + \epsilon_{xy}\right)_{(r)} = \left(\frac{n}{2}\right)^r n_{(2)} + \sum_{k=0}^{r-1} \sum_{x \neq y} c_k \left(\frac{n}{2}\right)^k \epsilon_{xy}^{r-k}.$$

Note that

$$\left|\sum_{x \neq y} c_k \left(\frac{n}{2}\right)^k \epsilon_{xy}^{r-k}\right| \le \sum_{x \neq y} |c_k| n^k |\epsilon_{xy}|^{r-k} \le cn^k \sum_{x \neq y} |\epsilon_{xy}|^{r-k}$$
$$\le cn^k \cdot o(n^{r-k+2}) = o(n^{r+2}),$$

which implies (10.4).

By the above claim and (10.4), $S_r = (1 + o(1))n^{r+2}2^{-r}$. Therefore,

$$\begin{aligned}\sum_{\alpha,\epsilon}(f_r(\alpha,\epsilon) - E[f_r])^2 &= \sum_{\alpha,\epsilon} f_r(\alpha,\epsilon)^2 - \sum_{\alpha,\epsilon} E[f_r]^2\\ &= \sum_{\alpha,\epsilon}(f_r(\alpha,\epsilon)^2 - f_r(\alpha,\epsilon)) + \sum_{\alpha,\epsilon} f_r(\alpha,\epsilon) - n_{(r)}2^r(n-r)^2 2^{-2r}\\ &= S_r + n_{(r)}2^r E[f_r] - n_{(r)}2^{-r}(n-r)^2\\ &= S_r + n_{(r+1)} - (1+o(1))n^{r+2}2^{-r}\\ &= o(n^{r+2}).\end{aligned}$$

Recall that N_{r+1} is the summation of N_r quantities of $f_r(\alpha,\epsilon)$. Thus

$$\left|N_{r+1} - N_r E[f_r]\right|^2 = \left|\sum_{N_r \text{ terms}} (f_r(\alpha,\epsilon) - E[f_r])\right|^2.$$

By Cauchy-Schwarz inequality, the last expression is at most

$$\begin{aligned}N_r \sum_{N_r \text{ terms}} (f_r(\alpha,\epsilon) - E[f_r])^2 &\le N_r \sum_{\alpha,\epsilon}(f_r(\alpha,\epsilon) - E[f_r])^2\\ &= N_r \cdot o(n^{r+2})\\ &= o(n^{2r+2}).\end{aligned}$$

It follows that

$$|N_{r+1} - N_r E[f_r]| = o(n^{r+1}),$$

and hence, by the induction hypothesis,

$$\begin{aligned}N_{r+1} &= N_r E[f_r] + o(n^{r+1})\\ &= (1+o(1))n_{(r)}2^{-\binom{r}{2}}(n-r)2^{-r} + o(n^{r+1})\\ &= (1+o(1))n_{(r+1)}2^{-\binom{r+1}{2}}.\end{aligned}$$

This completes the proof of the induction step of Fact 7 and hence establishes Theorem 10.1. □

A property is called *a quasi-random property* for $p = 1/2$ if it is equivalent to any property in Theorem 10.1. It is surprised that $P_2(4)$, which seems to be weaker, is a quasi-random property.

Theorem 10.2 *The property*

$$P_5'\text{: For each } U \subseteq V(G) \text{ with } |U| = \left\lfloor \frac{n}{2} \right\rfloor,\ e(U) \sim \frac{n^2}{16}.$$

is a quasi-random property for $p = 1/2$.

Proof. The implication $P_4 \Rightarrow P_5'$ is immediate, so we show $P_5' \Rightarrow P_4$. By ignoring one vertex possibly, we assume that n is even so that $n/2$ is an integer. Suppose that for any subset S with $|S| = n/2$, $\left|e(S) - \frac{n^2}{16}\right| < \epsilon n^2$, where $\epsilon > 0$ is fixed. We shall show that for any subset T,

$$\left|e(T) - \frac{1}{2}\binom{t}{2}\right| < 20\epsilon n^2, \tag{10.5}$$

where $t = |T|$.

Case 1 $t = |T| \geq n/2$.

By averaging over all $S \subseteq T$ with $|S| = n/2$, we have

$$e(T) = \frac{1}{\binom{t-2}{n/2-2}} \sum_{S \subseteq T,\ |S|=n/2} e(S)$$

as each edge of T appears in exactly $\binom{t-2}{n/2-2}$ such $n/2$-sets. Thus

$$e(T) \leq \frac{\binom{t}{n/2}}{\binom{t-2}{n/2-2}} \left(\frac{n^2}{16} + \epsilon n^2\right) \leq \binom{t}{2}\left(\frac{1}{2} + 9\epsilon\right).$$

Similarly,

$$e(T) \geq \binom{t}{2}\left(\frac{1}{2} - 9\epsilon\right).$$

Therefore, (10.5) follows.

Case 2 $t = |T| < n/2$.

We shall show that the assumption

$$e(T) \geq \frac{1}{2}\binom{t}{2} + 20\epsilon n^2$$

leads to a contradiction. Set $\overline{T} = V \setminus T$. Note that $|\overline{T}| = n - t > n/2$. By Case 1, we have

$$\binom{n-t}{2}\left(\frac{1}{2} - 9\epsilon\right) < e(\overline{T}) < \binom{n-t}{2}\left(\frac{1}{2} + 9\epsilon\right).$$

Consider the average value of $e(T \cup T')$, denoted by A, where T' ranges over all subsets of $\overline{T}$ with $|T'| = n/2 - t$ so that $|T \cup T'| = n/2$, we have that

$$A = \frac{1}{\binom{n-t}{n/2-t}} \sum_{\substack{T' :\ T' \subseteq \overline{T} \\ |T'| = n/2-t}} e(T \cup T')$$

as there are $\binom{n-t}{n/2-t}$ such $T \cup T'$-sets. Counting how much different edges contribute to the sum, we know that the sum equals to

$$e(T)\binom{n-t}{n/2-t} + e(\overline{T})\binom{n-t-2}{n/2-t-2} + e(T,\overline{T})\binom{n-t-1}{n/2-t-1}$$

since each edge in T appears in exactly $\binom{n-t}{n/2-t}$ such $T \cup T'$, each edge in $\overline{T}$ appears in exactly $\binom{n-t-2}{n/2-t-2}$ such $T \cup T'$ and each edge in $(T,\overline{T})$ appears in exactly $\binom{n-t-1}{n/2-t-1}$ such $T \cup T'$, respectively. Note that $e(T,\overline{T}) = e(G) - e(T) - e(\overline{T})$, we obtain that

$$\begin{aligned} A &= \frac{n/2}{n-t}e(T) - \frac{(n/2-t)n/2}{(n-t)(n-t-1)}e(\overline{T}) + \frac{n/2-t}{n-t}e(G) \\ &\geq \frac{n/2}{n-t}\left\{\frac{1}{2}\binom{t}{2} + 20\epsilon n^2\right\} - \frac{(n/2-t)n/2}{(n-t)(n-t-1)}\binom{n-t}{2}\left(\frac{1}{2}+9\epsilon\right) \\ &+ \frac{n/2-t}{n-t}\binom{n}{2}\left(\frac{1}{2}-9\epsilon\right) \\ &> \frac{n^2}{16} + \epsilon n^2, \end{aligned}$$

which contradicts the property P_5.

Similarly, the assumption

$$e(T) < \frac{1}{2}\binom{t}{2} - 20\epsilon n^2$$

will lead to a contradiction too. This completes the proof. □

The analogous results of Theorem 10.1 can be established by a similar argument for general edge density p, where $0 < p < 1$ is fixed.

Theorem 10.3 *Let* $\{G\}_{n=1}^{\infty}$ *be a sequence of graphs, where* $G = G_n$ *is a graph of order* n. *If* $0 < p < 1$ *is fixed, then the following properties are equivalent:*

$P_1(h)$**:** *For any fixed* $h \geq 4$ *and graph* H *of order* h,

$$N_G^*(H) \sim p^{e(H)}(1-p)^{\binom{h}{2}-e(H)}n^h.$$

$P_2(t)$**:** $e(G) \sim \frac{pn^2}{2}$ *and* $N_G(C_t) \leq (pn)^t + o(n^t)$ *for any even* $t \geq 4$.

P_3**:** $e(G) \geq \frac{pn^2}{2} + o(n^2)$, $\lambda_1 \sim pn$ *and* $\lambda_2 = o(\lambda_1)$.

P_4**:** *For each* $U \subseteq V(G)$, $e(U) = p\binom{|U|}{2} + o(n^2)$.

P_5**:** *For any two subsets* $U, V \subseteq V(G)$, $e(U,V) = p|U||V| + o(n^2)$.

P_6**:** $\sum_{x,y}\big||N(x)\cap N(y)| - p^2 n\big| = o(n^3)$.

P_7**:** $\sum_{x,y}\big|s(x,y) - (p^2 + (1-p)^2)n\big| = o(n^3)$.

For the quasi-randomness of the sparse graphs, the situation is much different. Under some certain conditions, there are several equivalent properties, see Chung and Graham (2002).

Return to the Paley graph P_q. It satisfies that

$$e(P_q) = \frac{q(q-1)}{4} \sim \frac{q^2}{4},$$
$$\lambda_1 = \frac{q-1}{2} \sim \frac{q}{2},$$
$$\lambda = \left| -\frac{\sqrt{q}+1}{2} \right| = o(q).$$

Thus P_q satisfies quasi-random property P_3 hence all other quasi-random properties with $p = 1/2$.

10.2 Graphs with Small Second Eigenvalues

The last section was devoted to the quasi-random graphs with fixed edge density. Let us now switch to the case of density $p = p(n) = o(1)$, which is more important for some applications.

In applications, we shall allow the graphs to be semi-simple, that is, each vertex is attached with at most one loop. When p tends to zero, the situation is significantly more complicated as revealed by Chung and Graham (2002). The first remarked fact is that the properties defined for quasi-random graphs with fixed edge density may be not equivalent anymore.

Recall the Erdős-Rényi graph ER_q, which has order $n = q^2+q+1$ and each vertex of ER_q has degree q or $(q+1)$. So the edge density $p \sim \frac{1}{\sqrt{n}}$. We know from Lemma 7.6 that $\lambda_1 = q+1$, and $\lambda \sim \sqrt{q} = o(\lambda_1)$. So the property P_3 holds. However,

$$p^4(1-p)^2n^4 \sim n^2,$$

and thus the property $P_1(4)$ of Theorem 10.3 does not hold as ER_q does not contain C_4 hence $N^*_G(C_4) = 0$.

Recall that the quasi-random property P_3, the magnitude of $\lambda = \lambda(G)$ is a measure of quasi-randomness. For sparse graphs with $p = o(1)$, Chung and Graham (2002) found some equivalent properties under certain conditions. One of the properties is that $\lambda_1 \sim pn$ and $\lambda = o(\lambda_1)$.

In this section, we shall focus on (n, d, λ)-graphs defined by Alon. We say a graph G is an (n, d, λ)-graph if G is d-regular with n vertices and

$$\lambda = \lambda(G) = \max\{|\lambda_i| : \ 2 \le i \le n\},$$

where $\lambda_1 = d$, and $\lambda_2, \ldots, \lambda_n$ are all eigenvalues of G. For an (n, d, λ)-graph, the spectral gap between d and λ is a measure for its quasi-random property. The smaller the value of λ compared to d, the closer is edge distribution to the ideal uniform distribution. How small λ can be?

Theorem 10.4 *Let G be an (n, d, λ)-graph and let $\epsilon > 0$. If $d \le (1-\epsilon)n$, then*

$$\lambda \ge \sqrt{\epsilon d}.$$

Proof. Let A be the adjacency matrix of G. Note that G is d-regular, so we obtain that

$$nd = tr(A^2) = \sum_{i=1}^{n} \lambda_i^2 \le d^2 + (n-1)\lambda^2 \le (1-\epsilon)nd + n\lambda^2,$$

which follows by what claimed. □

On this estimate, we can say, not precisely, that an (n, d, λ)-graph with $\lambda = \Omega(\sqrt{d})$ has good quasi-randomness. Recall that if G is a strongly regular graph $srg(n, d, \mu_1, \mu_2)$ with $n \ge 3$, then all eigenvalues except $\lambda_1 = d$ are solutions of the equation

$$\lambda^2 + (\mu_2 - \mu_1)\lambda + (\mu_2 - d) = 0.$$

Thus when $\mu_2 - \mu_1$ is small compared to d, which implies that λ is close to $\sqrt{d}$ and G has good quasi-randomness. For example, the Paley graph P_q has good quasi-randomness since $\mu_2 - \mu_1 = 1$.

For a simple graph G with vertex set V and for two subsets $B, C \subseteq V$, $e(B, C)$ counts each edge from $B \setminus C$ to $C \setminus B$ once, and each edge in $B \cap C$ twice. When G is semi-simple, it also counts each loop in $B \cap C$ once. For disjoint subsets B and C in a random graph, $e(B, C)$ is expected to be $\frac{d}{n}|B||C|$, which is close to the expectation as shown in the following when λ is much smaller than d.

Some graphs G constructed by algebraic method are nearly regular with $\Delta(G) - \delta(G) \le 1$, which will be regular if we attach some vertices with a loop. Hence, to get a regular graph, we always add a loop to some vertices when necessary.

Theorem 10.5 *Let G be a semi-simple (n, d, λ)-graph with vertex set V and edge set E. For each partition of V into disjoint subsets B and C,*

$$e(B, C) \ge \frac{(d-\lambda)|B||C|}{n}.$$

Proof. Let A be the adjacency matrix of G and I the identity matrix of order n. Observe that for any real vector x of dimension n (as a real valued function on V), we have the coordinate $(x^T A)_u = \sum_{v:\, uv \in E} x_v$. Thus the inner product

$$((dI - A)x, x) = dx^T x - x^T Ax = \sum_{u \in V} \left(dx_u^2 - \sum_{v:\, uv \in E} x_v x_u \right) = \sum_{uv \in E} (x_u - x_v)^2.$$

Set $b = |B|$ and $c = |C| = n - b$. Define a vector $x = (x_v)$ by

$$x_v = \begin{cases} -c & \text{if } v \in B, \\ b & \text{if } v \in C. \end{cases}$$

Note that $dI - A$ and A have the same eigenvectors, and that the eigenvalues of $dI - A$ are precisely $d - \mu$ as μ ranges over all eigenvalues of A. Also, d is the largest eigenvalue of A corresponding to the eigenvector $\mathbf{1} = (1, 1, \ldots, 1)^T$ and $(x, \mathbf{1}) = 0$. Hence x is orthogonal to the eigenvector of the smallest eigenvalue (zero) of $dI - A$.

Since $dI - A$ is a real symmetric matrix, its eigenvectors are orthogonal each other and form a basis of the n-dimensional space and x is a linear combination of these eigenvectors other than that of $\mathbf{1}/\sqrt{n}$. This together with the fact that $d - \lambda$ is the second smallest eigenvalue of $dI - A$, we have

$$((dI - A)x, x) \geq (d - \lambda)(x, x) = (d - \lambda)(bc^2 + cb^2) = (d - \lambda)bcn. \tag{10.6}$$

However, as B and C form a partition of V,

$$\sum_{uv \in E} (x_u - x_v)^2 = e(B, C)(b + c)^2 = e(B, C)n^2,$$

implying the desired inequality. □

In a random d-regular graph, we expect that a vertex v has $\frac{d}{n}|B|$ neighbors in B. The theorem below shows that if λ is small, then $|N_B(v)|$ is not too far from the expectation for most vertices v, where $N_B(v) = N(v) \cap B$.

Theorem 10.6 *If G is a semi-simple (n, d, λ)-graph with vertex set V, then for each $B \subseteq V$,*

$$\sum_{v \in V} \left(|N_B(v)| - \frac{d}{n}|B| \right)^2 \leq \lambda^2 \frac{|B|(n - |B|)}{n}.$$

Proof. Let A be the adjacency matrix of G. Define a vector $f : V \to R$ by

$$f_u = \begin{cases} 1 - \frac{b}{n} & \text{if } u \in B, \\ -\frac{b}{n} & \text{if } u \notin B, \end{cases}$$

where $b = |B|$. Therefore, $\sum_u f_u = 0$ and f is orthogonal to the eigenvector $\mathbf{1} = (1, 1, \ldots, 1)^T$ of the largest eigenvalue d of A. Thus f is a linear combination of eigenvectors other than $\mathbf{1}$, and

$$(Af, Af) = f^T A^2 f \leq \lambda^2 (f, f) = \lambda^2 \frac{b(n - b)}{n}.$$

Now, let A_v be the row of A corresponding to vertex v. Note that the coordinate $(Af)_v$ of Af at v is

$$A_v f = \left(1 - \frac{b}{n}\right) |N_B(v)| - \frac{b}{n}\big(d - |N_B(v)|\big) = |N_B(v)| - \frac{db}{n},$$

it follows that

$$(Af, Af) = \sum_v \left(|N_B(v)| - \frac{db}{n} \right)^2,$$

the desired inequality follows. □

Corollary 10.1 *Let G be a semi-simple (n, d, λ)-graph with vertex set V. For every two subsets B and C of V,*

$$\left| e(B, C) - \frac{d}{n}|B||C| \right| \le \lambda\sqrt{|B||C|}.$$

Proof. Set $b = |B|$ and $c = |C|$. Note that

$$\begin{aligned}\left| e(B, C) - \frac{dbc}{n} \right| &= \left| \sum_{v \in C} \left(|N_B(v)| - \frac{db}{n} \right) \right| \\ &\le \sum_{v \in C} \left| |N_B(v)| - \frac{db}{n} \right| \le \sqrt{c} \left[\sum_{v \in C} \left(|N_B(v)| - \frac{db}{n} \right)^2 \right]^{1/2},\end{aligned}$$

where the Cauchy-Schwarz inequality is used. By Theorem 10.6,

$$\begin{aligned}\left| e(B, C) - \frac{dbc}{n} \right| &\le \sqrt{c} \left[\sum_{v \in V} \left(|N_B(v)| - \frac{db}{n} \right)^2 \right]^{1/2} \\ &\le \lambda\sqrt{c}\sqrt{b\left(1 - \frac{b}{n}\right)} \le \lambda\sqrt{bc}\end{aligned}$$

as desired. □

Let $e(B)$ and $\ell(B)$ be the number of edges and loops in B, respectively. Note that

$$e(B, B) = 2\,e(B) + \ell(B),$$

and $\ell(B) \le |B|$ if G is semi-simple.

Corollary 10.2 *Let G be a semi-simple (n, d, λ)-graph with vertex set V. For any subset $B \subset V$,*

$$\left| e(B) - \frac{d}{2n}|B|^2 \right| \le \frac{\lambda + 1}{2}|B|.$$

By setting $e(B) = 0$, together with the Turán bound (Theorem 3.2), we have

$$\frac{n}{d+1} \le \alpha(G) \le \frac{\lambda + 1}{d} n.$$

For an (n, d, λ)-graph $G = (V, E)$ and $B \subseteq V$, define C as the set of vertices u so that the proportion of its neighborhood $N(u)$ in B, which is $|N_B(u)|/|B|$, is at most half of that in V. The following result implies that $|B||C|$ is at most $\Theta(n^2/d)$ if $\lambda = \Theta(\sqrt{d})$.

Corollary 10.3 *Let G be a semi-simple (n, d, λ)-graph with vertex set V. For any subset $B \subset V$, define*

$$C = \left\{ u \in V : \ |N_B(u)| \le \frac{d}{2n}|B| \right\}.$$

We have

$$|B||C| \le \left(\frac{2\lambda n}{d} \right)^2.$$

Consequently, $|B \cap C| \le \frac{2\lambda n}{d}$.

Proof. By Theorem 10.6,

$$\sum_{v \in V} \left(|N_B(v)| - \frac{d}{n}|B| \right)^2 \le \lambda^2 \frac{|B|(n-|B|)}{n} \le \lambda^2 |B|.$$

Note that each $v \in C$ contributes to the left-hand side more than $(\frac{d|B|}{2n})^2$, thus we obtain that

$$|C| \left(\frac{d|B|}{2n} \right)^2 \le \lambda^2 |B|,$$

implying what as claimed. □

10.3 Some Multicolor Ramsey Numbers

For $H_1 = \cdots = H_k = H$, let us write the multicolor Ramsey numbers as

$$r_{k+1}(H; H_{k+1}) = r(H_1, \ldots, H_k, H_{k+1}).$$

For $k \ge 2$, Alon and Rödl (2005) gave sharp bounds for $r_{k+1}(H; K_n)$ when H is a (some kind) bipartite graph or K_3. Their main idea is to estimate the number of independent sets of given size in a quasi-random graph G, which contains no H, and then consider the random shifting of G. The number of shifts is k. The bigger of k we choose, the tighter of the bound follows. When $k = 1$, there are no shifting actually.

Theorem 10.7 *Let G be a semi-simple (N, d, λ)-graph with vertex set V. For any $n \ge n_0 = \frac{2N \log N}{d}$, the number M of independent sets of size n in G satisfies that*

$$M \le \left(\frac{edn}{2\lambda n_0} \right)^{n_0} \left(\frac{2e\lambda N}{dn} \right)^n.$$

Proof. Consider the number of ways to choose an ordered set $v_1, v_2, \ldots, v_n$ of n vertices which form an independent set. Starting with $B_0 = V$, we choose v_1 arbitrarily. Define

$$B_i = V \setminus \cup_{j=1}^{i} N[v_j],$$

i.e., B_i is the set of vertices by deleting $\{v_1, v_2, \ldots, v_i\}$ and their neighbors, where $v_1, \ldots, v_i$ are the vertices that have been chosen. Clearly the size of B_i is decreasing since $B_i \supseteq B_{i+1}$, and v_{i+1} has to lie in B_i. Define

$$C_i = \left\{ u \in V : \ |N_{B_i}(u)| \le \frac{d}{2N}|B_i| \right\}.$$

If the next chosen vertex v_{i+1} from B_i does not lie in C_i, then B_{i+1} is obtained by deleting v_{i+1} and at least $\frac{d}{2N}|B_i|$ neighbors of v_{i+1} from B_i and so

$$|B_{i+1}| < \left(1 - \frac{d}{2N}\right)|B_i|.$$

Hence throughout the process there cannot be more than $n_0 = \frac{2N \log N}{d}$ choices like that, since otherwise the corresponding set of non-neighbors will be empty before the process terminates from

$$\left(1 - \frac{d}{2N}\right)^{n_0} = \left(1 - \frac{d}{2N}\right)^{(2N \log N)/d} < \frac{1}{N}.$$

It follows that with at most n_0 possible exceptions, each vertex v_{i+1} has to lie in $B_i \cap C_i$. By Corollary 10.3, we have

$$|B_i \cap C_i| \le \frac{2\lambda N}{d}.$$

Therefore, the total number of choices for the ordered set $v_1, v_2, \ldots, v_n$ is at most

$$\binom{n}{n_0} N^{n_0} \left(\frac{2\lambda N}{d}\right)^{n-n_0} \le \left(\frac{edn}{2\lambda n_0}\right)^{n_0} \left(\frac{2\lambda N}{d}\right)^{n}.$$

Indeed, there are $\binom{n}{n_0}$ possibilities to choose a set of indices covering all indices i for which the vertex v_{i+1} has not been chosen in $B_i \cap C_i$. Moreover, there are at most N ways to choose each such vertex v_i, and at most $\frac{2\lambda N}{d}$ ways to choose each vertex v_{j+1} for each other index j.

Now, dividing the above bound by $n!$, we obtain an upper bound for the number of unordered independent sets of size n as claimed. □

Lemma 10.1 *Let G be a graph of order N that contains no H, and let M be the number of independent sets of size n in G. If*

$$M^k < \binom{N}{n}^{k-1},$$

then $r_{k+1}(H; K_n) > N$.

Proof. For each i, $1 \le i \le k$, let G_i be a random copy of G on the same vertex set V, that is, a graph obtained from G by mapping its vertices to those of V according to a random one to one mapping. The probability that a fixed set of n vertices of V will be an independent set in each G_i is

$$\left(\frac{M}{\binom{N}{n}}\right)^k < \frac{1}{\binom{N}{n}},$$

implying that with a positive probability there is no such independent set. This gives the existence of the graphs G_i as required.

Now we color each edge of K_N on $V(G)$ by the minimum i if the edge belongs to G_i. Otherwise, color the edge by $k+1$. Therefore, there is no monochromatic H in the first k colors and no K_n in the last color $k+1$, so the lower bound follows. □

One can also find the second part of the following result in Lin and Li (2011).

Theorem 10.8 *The Ramsey number $r_{k+1}(C_4; K_n)$ satisfies the following:*
(1) For any fixed $k \geq 3$, $r_{k+1}(C_4; K_n) = \Theta(\frac{n^2}{\log^2 n})$.
(2) There are positive constants c_1 and c_2 such that

$$c_1 \left(\frac{n \log\log n}{(\log n)^2}\right)^2 \leq r(C_4, C_4, K_n) \leq c_2 \left(\frac{n}{\log n}\right)^2.$$

We know that for every fixed bipartite graph H, there exists some real number $t > 1$ such that the Turán number $ex(n, H) \leq O(N^{2-1/t})$ by Theorem 7.5. The upper bound of the above theorem follows from the following general result.

Lemma 10.2 *Let H be a fixed bipartite graph such that $ex(n, H) \leq O(N^{2-1/t})$ for some real $t > 1$. For every fixed $k \geq 1$,*

$$r_{k+1}(H; K_n) \leq O\left[\left(\frac{n}{\log n}\right)^t\right].$$

Proof. Let $N = r_{k+1}(H; K_n) - 1$. Given an edge-coloring of K_N by $k+1$ colors with no monochromatic copy of H in each of the first k colors, and no monochromatic K_n in the last color. Let T be the graph whose edges are all edges of K_N colored by one of the first k colors. Thus, the total number of edges of T is clearly at most $k \cdot ex(n, H) \leq c_1 N^{2-1/t}$, where c_1 is a constant depending only on k and H. Moreover, the neighborhood of any vertex of degree d in T contains at most $k \cdot ex(d; H) \leq c_2 d^{2-1/t}$ edges of T, where c_2 is a constant.

Ajtai, Komlós and Szemerédi (1981) proved that if a graph on N vertices with average degree at most D contains at most $ND^{2-\eta}$ triangles, then it contains an independent set of size at least $c(\eta)N\frac{\log D}{D}$. Therefore, if D is the average degree of T then, as T contains an induced subgraph, denoted by T', with $N/2$ vertices and maximum degree at most $2D$ and hence at most $O(D^{2-1/t})$ edges in any neighborhood of a vertex in T'. Thus

$$\alpha(T') \geq \Omega\left(N\frac{\log D}{D}\right) \geq \Omega(N^{1/t} \log N).$$

Since $\alpha(T') \le \alpha(T) < n$ it follows that $n > \Omega(N^{1/t}\log N)$, implying the desired upper bound. □

Proof of Theorem 10.8. It remains to prove the lower bounds. Consider the Erdős-Rényi graph ER_q^o (we attach a loop to each vertex $\langle a_1, a_2, a_3\rangle$ if $a_1^2 + a_2^2 + a_3^2 = 0$) of order $N = q^2 + q + 1$ and let M be the number of independent sets of size n. It suffices to show that $M^k < \binom{N}{n}^{k-1}$ by Lemma 10.1. Note that the graph ER_q^o is d-regular, where $d = q + 1$. Set $n_0 = \frac{2N\log N}{d}$. Thus for large q,

$$4q\log q < n_0 < 4(q+1)\log q.$$

From Theorem 10.7, if $n \ge n_0$ which can be seen as follows then

$$M \le \left(\frac{edn}{2\lambda n_0}\right)^{n_0}\left(\frac{2e\lambda N}{dn}\right)^n,$$

where $\lambda = \sqrt{q}$.

(1) For $k \ge 3$, it suffices to show that $r_4(C_4; K_n) \ge \Omega(\frac{n^2}{(\log n)^2})$. Set $n = cq\log q$, where $c > 4$ is a large constant to be chosen later. We shall show that

$$M^{3/n} < \binom{N}{n}^{2/n}. \tag{10.7}$$

Substituting d, λ, n_0, n, N by values in terms of q, we have

$$M^{3/n} \le \left(\frac{ce(q+1)}{8\sqrt{q}}\right)^{\frac{12}{c}(1+1/q)}\left(\frac{2e\sqrt{q}}{c\log q}\right)^3 \sim c_1\frac{q^{12/c+3/2}}{(\log q)^3},$$

where c_1 is a positive constant, and

$$\binom{N}{n}^{2/n} \sim \left(\frac{eN}{n}\right)^2 \sim \left(\frac{eq}{c\log q}\right)^2.$$

Thus the inequality (10.7) holds if we choose c such that $12/c + 3/2 \le 2$, which is $c \ge 24$. Then we have $n > n_0$ and $N \sim q^2 \sim n^2/(c\log n)^2$ as $q \to \infty$, completing the proof for $k \ge 3$.

(2) For $r(C_4, C_4, K_n)$, set $n = cq\log^2 q/\log\log q$, where c is a positive constant will be chosen later. It suffices to show that $M^{2/n} < \binom{N}{n}^{1/n}$ by Lemma 10.1. Note that for some constant $c_i > 0$,

$$\begin{aligned} M^{2/n} &\le c_1\left(\frac{\sqrt{q}\log q}{\log\log q}\right)^{\frac{8\log\log q}{c\log q}}\left(\frac{\sqrt{q}\log\log q}{\log^2 q}\right)^2 \\ &\le c_2 q^{1+\frac{4\log\log q}{c\log q}}\left(\frac{\log\log q}{\log^2 q}\right)^2 = c_2\frac{q(\log\log q)^2}{(\log q)^{4-4/c}}, \end{aligned}$$

and

$$\binom{N}{n}^{1/n} \geq c_3 \frac{eN}{n} \geq c_4 \frac{q \log\log q}{\log q}.$$

We are done by taking $c > 4/3$ so that $4 - 4/c > 1$. □

Note that we have found the spectrum of the projective norm graph $G_{q,t}$ in the last section and it contains no $K_{t,s}$ for $s \geq (t-1)! + 1$. Similar argument as above gives the following result.

Theorem 10.9 *For any fixed $t \geq 2$ and $s \geq (t-1)! + 1$,*
(1) For any $k \geq 3$,

$$r_{k+1}(K_{t,s}; K_n) = \Theta\left(\frac{n}{\log n}\right)^t.$$

(2) There are positive constants c_1 and c_2 such that

$$c_1 \left(\frac{n \log\log n}{(\log n)^2}\right)^t \leq r(K_{t,s}, K_{t,s}, K_n) \leq c_2 \left(\frac{n}{\log n}\right)^t.$$

Alon and Rödl (2005) also solved a conjecture of Erdős and Sós that

$$\lim_{n\to\infty} \frac{r(K_3, K_3, K_n)}{r(K_3, K_n)} = \infty.$$

The r-blow-up G' of a graph G is the graph obtained by replacing each vertex v of G by an independent set S_v of size r, and each edge uv of G by the set of all edges xy with $x \in S_u$ and $y \in S_v$.

Lemma 10.3 *There is a constant $c = c_k > 0$ such that*

$$r_{k+1}(K_3; K_n) \geq \frac{c\, n^{k+1}}{(\log n)^{2k}}$$

for all large n.

Proof. Let $N = c_1 s^2/\log s < r(K_3, K_{s+1})$, where $c_1 > 0$ is a fixed constant. Thus there is a graph F of order N with no K_3 and its independence number $\alpha(F) \leq s$. Let G be the r-blow up of F, where $r = r(s)$ will be chosen later. Denote M by the number of independent sets of size n in G. Note that there are at most $\binom{N}{s}$ ways to choose these blocks, and each independent set of size n can be chosen from at most s blocks, so we obtain that

$$M \leq \frac{\binom{N}{s}(rs)^n}{n!} \leq \left(\frac{eN}{s}\right)^s \left(\frac{ers}{n}\right)^n,$$

where we use the fact that $t! \geq (\frac{t}{e})^t$.

Since G has rN vertices and it contains no K_3, by Lemma 10.1, we can deduce that $r_{k+1}(K_3; K_n) > rN$ if

$$M^k < \binom{rN}{n}^{k-1}.$$

We now take $r = s^{k-1}(\log s)^{2-k}$ and $n = cs \log s$, where $c > 0$ is a constant to be chosen. Thus

$$M^{k/n} < \left(\frac{c_1 es}{\log s}\right)^{k/(c \log s)} \left(\frac{es^{k-1}}{c(\log s)^{k-1}}\right)^k \leq \frac{c_2}{c^k} \left(\frac{s}{\log s}\right)^{k(k-1)},$$

where c_2 and henceforth c_3 and c_4 are positive constants that is independent of c, and

$$\binom{rN}{n}^{(k-1)/n} > c_3 \left(\frac{erN}{n}\right)^{k-1} \geq \frac{c_4}{c^{k-1}} \left(\frac{s}{\log s}\right)^{k(k-1)}.$$

Thus the condition is satisfied if we take large c such that $c_2/c^k < c_4/c^{k-1}$, and hence

$$r_{k+1}(K_3; K_n) > rN = \frac{c_1 s^{k+1}}{(\log s)^{k-1}} = \Theta\left(\frac{n^{k+1}}{(\log n)^{2k}}\right),$$

completing the proof. □

Theorem 10.10 *For each fixed* $k \geq 1$*, there are constants* $c_i = c_i(k) > 0$ *such that*

$$\frac{c_1\, n^{k+1}}{(\log n)^{2k}} \leq r_{k+1}(K_3; K_n) \leq \frac{c_2\, n^{k+1}}{(\log n)^k}$$

for all large n*.*

Proof. It remains to show the upper bound. The proof is by induction on $k \geq 1$. For $k = 1$, it is already implied by Theorem 3.5. Now we suppose $k \geq 2$ and assume that the upper bound holds for $k - 1$, we will prove it also holds for k. Let $N = r_{k+1}(K_3; K_n) - 1$. There is an edge-coloring of K_N by colors $1, 2, \ldots, k+1$ with no monochromatic K_3 in any of the first k colors, and no monochromatic K_n in the last color. Consider the graph T induced by all edges of the first k colors. The neighborhood $N(v)$ of a vertex v in T is $\cup_{i=1}^k N_i(v)$, where $N_i(v)$ is the set of neighbors of v that are connected to v by an edge in the color i, $1 \leq i \leq k$. Note that for $1 \leq i \leq k$, $|N_i(v)| < r_k(K_3; K_n)$ as there is no monochromatic K_3 in any of the first k colors, and no monochromatic K_n in the last color. Let D be the maximum degree of T. Thus

$$D \leq k(r_k(K_3; K_n) - 1) < kr_k(K_3; K_n).$$

For a vertex u in $N(v)$, we consider the neighborhood of u in the subgraph induced by $N(v)$ in T. Suppose $u \in N_1(v)$, say. Such neighbors are these in

$$N(u) \cap N(v) = \cup_{j=1}^k \left(\cup_{i=1}^k \left(N_i(u) \cap N_j(v)\right)\right).$$

First of all, $N_1(u) \cap N_1(v) = \emptyset$ since there is no monochromatic triangle in the color 1. For $2 \le i \le k$, $N_i(u) \cap N_1(v)$ contains no edge in the colors i and 1, thus $|N_i(u) \cap N_1(v)| \le r_{k-1}(K_3; K_n) - 1$ which implies that

$$\left|\cup_{i=1}^{k} (N_i(u) \cap N_1(v))\right| < (k-1) r_{k-1}(K_3; K_n).$$

Similarly, for $2 \le j \le k$,

$$\left|\cup_{i=1}^{k} \left(N_i(u) \cap N_j(v)\right)\right| < (k-1) r_{k-1}(K_3; K_n).$$

Thus the maximum degree of the subgraph induced by $N(v)$ in T is less than $m = k^2 r_{k-1}(K_3; K_n)$. By Theorem 3.4, we obtain that

$$n \ge \alpha(T) \ge N \frac{\log(D/m) - 1}{D}.$$

Therefore, using the induction hypothesis for D and m, the desired upper bound follows. □

10.4 A Related Lower Bound of $r(s,t)$

Constructions of (n, d, λ)-graphs arise from a number of sources, including Cayley graphs, projective geometry and strongly regular graphs – we refer the reader to Krivelevich and Sudakov (2006) for a survey of (n, d, λ)-graphs. Sudakov, Szabo and Vu (2005) proved that a K_s-free (n, d, λ)-graph satisfies

$$\lambda = \Omega(d^{s-1}/n^{s-2}) \tag{10.8}$$

as $n \to \infty$. For $s = 3$, if G is any triangle-free (n, d, λ)-graph with adjacency matrix A, then

$$0 = tr(A) \ge d^3 - \lambda^3(n-1). \tag{10.9}$$

If $\lambda = \sqrt{d}$, then this gives $d = O(n^{2/3})$ matching (10.8). Alon (1994b) constructed a triangle-free pseudorandom graph attaining this bound, and Conlon (2017) more recently analyzed a randomized construction with the same average degree. A similar argument to (10.9) shows that a K_s-free (n, d, λ)- graph with $\lambda = \sqrt{d}$ has $d = O(n^{1-\frac{1}{2s-3}})$. The Alon-Boppana Bound (see Nilli 1991, 2004) shows that $\lambda = \sqrt{d}$ for every (n, d, λ)-graph provided d/n is bounded away from 1. Sudakov, Szabo and Vu (2005) raised the question of the existence of optimal pseudorandom K_s-free graphs for $s \ge 4$, namely (n, d, λ)-graphs achieving the bound in (10.8) with $\lambda = \sqrt{d}$ and $d = \Omega(n^{1-\frac{1}{2s-3}})$. The following result due to Mubayi and Verstraëte (2019+) shows that a positive answer to this question will give the exponent of the Ramsey numbers $r(s,t)$.

Theorem 10.11 *Let F be a graph, n, d, λ be positive integers with $d \geq 1$ and $\lambda > 1/2$ and let $t = \lceil 2n \log^2 n/d \rceil$. If there exists an F-free (n, d, λ)-graph, then*

$$r(F, K_t) > \frac{n}{20\lambda} \log^2 n.$$

Proof. Let G be an F-free (n, d, λ)-graph and let U be a random set of vertices of G where each vertex is chosen independently with probability $p = \log^2 n/2e^2\lambda$. Let Z be the number of independent sets of size $t = \lceil 2n \log^2 n/d \rceil$ in the induced subgraph $G[U]$. By Theorem 10.7, the number of independent sets of size t in G is at most

$$\left(\frac{ed \log n}{2\lambda}\right)^{2n \log n/d} \left(\frac{2e\lambda n}{dt}\right)^t \leq (2e)^t \left(\frac{e\lambda}{\log^2 n}\right)^t = \left(\frac{2e^2\lambda}{\log^2 n}\right)^t$$

by noting $(d/2\lambda)^{1/\log n} < e$ and $(e \log n)^{1/\log n} < 1.1$. Therefore,

$$E(|U| - |Z|) \geq pn - p^t \left(\frac{2e^2\lambda}{\log^2 n}\right)^t = pn - 1,$$

which implies that there is a subset $U \subset V(G)$ such that if we remove one vertex from every independent set in U, the remaining set T has $|T| \geq pn - 1$ and $G[T]$ has no independent set of size t. It follows that

$$r(F, K_t) \geq pn > \frac{n}{20\lambda} \log^2 n,$$

completing the proof. □

Theorem 10.11 provides good bounds if there exists an F-free (n, d, λ)-graph with many edges and good pseudorandom properties (meaning that d is large and λ is small). For example, we immediately obtain the following consequence.

Corollary 10.4 *If there exists a K_s-free (n, d, λ)-graph with $d = \Omega(n^{1-\frac{1}{2s-3}})$ and $\lambda = O(\sqrt{d})$, then*

$$r(s, t) \geq \Omega\left(\frac{t^{s-1}}{\log^{2s-4} t}\right)$$

as $t \longrightarrow \infty$.

Proof. From $t = \lceil 2n \log^2 n/d \rceil$, we have $n = \Omega(dt/\log^2 t)$. Apply Theorem 10.11 with $F = K_s$, d and $\lambda = O(\sqrt{d})$ we obtain that the lower bound holds as desired. □

Alon and Krivelevich (1997) gave a construction of K_s-free (n, d, λ)-graphs with $d = \Omega(n^{1-\frac{1}{s-2}})$ and $\lambda = O(\sqrt{d})$ for all $s \geq 3$, and this was slightly improved by Bishnoi, Ihringer and Pepe (2020+) to obtain $d = \Omega(n^{1-\frac{1}{s-1}})$. This is the current record for the degree of a K_s-free (n, d, λ)-graph with $\lambda = O(\sqrt{d})$. The problem of obtaining optimal K_s-free pseudorandom constructions in the sense (10.8) with $\lambda = O(\sqrt{d})$ for $s \geq 4$ seems difficult and is considered to be a central open problem in pseudorandom graph theory. The problem of determining the growth rate of $r(s, t)$ is classical and much older, and it wasn't completely clear whether the upper bound in Theorem 3.5 or the lower bound in (5.10) was closer to the truth. Based on Theorem 1, it seems reasonable to conjecture that if $s \geq 4$ is fixed, then $r(s,t) = t^{s-1+o(1)}$ as $t \to \infty$.

10.5 A Lower Bound for Book Graph

Let $B_n^{(m)}$ denote the book graph that consists of n copies of K_{m+1} sharing a common K_m. The study of Ramsey numbers of books goes back to Erdős, Faudree, Rousseau and Schelp (1978), and Rousseau and Sheehan (1978). For convenience, we also denote $r(G)$ instead of $r(G,G)$. It is shown by Erdős et al. (1978) that for fixed $m \geq 2$,

$$r(B_n^{(m)}) \geq (2^m - o(1))n$$

by using the elementary probabilistic method, see Theorem 3.16. For the upper bound, Thomason (1982) conjectured that

$$r(B_n^{(m)}) \leq 2^m(n + m - 2) + 2.$$

For $m = 2$, Rousseau and Sheehan (1978) verified this conjecture and proved that it is tight for infinitely many values of n. Using the refined regularity lemma, Conlon (2019) proved that

$$r(B_n^{(m)}) \leq (2^m + o(1))n,$$

and thus $r(B_n^{(m)}) \sim 2^m n$ as $n \to \infty$. This anwsers a question of Erdős et al. (1978) and confirms a conjecture of Thomason (1982) asymptotically. Recently, the upper bound was improved further by Conlon, Fox and Wigderson (2021) as $r(B_n^{(m)}) \leq 2^m n + O(\frac{n}{(\log\log\log n)^{1/25}})$. For more Ramsey numbers on books, see e.g. Nikiforov, Rousseau and Schelp (2005, three papers) and other related references.

Let us point out that the lower bound $r(B_n^{(m)}) \geq (2^m - o(1))n$ by Erdős et al. (1978) follows from considering the random graph space of edge density $1/2$. We shall improve this by a constructive bound by using the Paley graph as follows, one can see Thomason (1982).

Theorem 10.12 *If $q \equiv 1 \pmod 4$ is a prime power and $m \geq 2$, then*

$$r(B_n^{(m)}) > 2^m n - m2^{3m/2}\sqrt{n}$$

for $\frac{q}{2^m} + (m-1)\sqrt{q} \leq n \leq \frac{q}{2^m} + m\sqrt{q}$.

We will apply Weil bound in the proof. The characters of a finite field F_q are group homomorphisms from F_q or $F_q^* = F_q \setminus \{0\}$ to

$$S^1 = \{e^{i\theta} : 0 \leq \theta < 2\pi\},$$

respectively, where S^1 is viewed as a multiplicative group of complex numbers.

A *multiplicative character* of F_q is a function $\chi : F_q^* \to S^1$ such that for any $x, y \in F_q^*$,

$$\chi(xy) = \chi(x)\chi(y).$$

We often extend the domain of a multiplicative character χ from F_q^* to F_q by defining $\chi(0) = 0$. The trivial function χ_0 with $\chi_0(x) \equiv 1$ is called the *principal multiplicative character* of F_q. The order of a multiplicative character χ is the smallest positive integer d such that $\chi^d = \chi_0$.

Let $F_q[x]$ be the set of all polynomials over F_q. Let χ be the multiplicative character of F_q of order $d > 1$ and $f(x) \in F_q[x]$. If $f(x)$ has precisely s distinct zeros and it is not the form $c(g(x))^d$ for some $c \in F_q$ and $g(x) \in F_q[x]$, then

$$\left|\sum_{x \in F_q} \chi(f(x))\right| \le (s-1)\sqrt{q}, \tag{10.10}$$

which is known as the Weil bound (1948).

Proof of Theorem 10.12. Let $U \subseteq F_q$ be a subset of vertices of the Paley graph P_q with $|U| = m$ which forms a clique. Denote by $J(U)$ for the common neighbors of the vertices of U. If $|J(U)| < n$ for any such clique U, then $r(B_n^{(m)}) > q$ since the Paley graph P_q is self-complementary. For a fixed clique U with $|U| = m$, define a function $f(x)$ depending on U as that

$$f(x) = \prod_{u \in U} (1 + \chi(x-u)), \quad x \in F_q,$$

where χ is the quadratic residue character defined by (2.2). Note that

$$\sum_{x \in U} f(x) = \sum_{x \in U} \prod_{u \in U} (1 + \chi(x-u)) = m2^{m-1}.$$

For $x \in F_q \setminus U$, if $x \in J(U)$ then $f(x) = 2^m$, and if $x \notin J(U)$ then $f(x) = 0$ as $\chi(x-u) = -1$ for some $u \in U$. Therefore,

$$\sum_{x \in F_q \setminus U} f(x) = 2^m |J(U)|.$$

Suppose that $U = \{u_1, u_2, \ldots, u_m\}$. We have

$$\begin{aligned}2^m|J(U)| &= \sum_{x\in F_q\setminus U} f(x) = \sum_{x\in F_q} f(x) - \sum_{x\in U} f(x)\\ &= \sum_{x\in F_q}\sum_{i_1,\ldots,i_m\in\{0,1\}} \chi\Big((x-u_1)^{i_1}\cdots(x-u_m)^{i_m}\Big) - m2^{m-1}\\ &= \sum_{x\in F_q}\Bigg(1+\sum_{\substack{i_1,\ldots,i_m\in\{0,1\}\\ i_1+\cdots+i_m\ge 1}} \chi\Big((x-u_1)^{i_1}\cdots(x-u_m)^{i_m}\Big)\Bigg) - m2^{m-1}\\ &= q+\sum_{\substack{i_1,\ldots,i_m\in\{0,1\}\\ i_1+\cdots+i_m\ge 1}}\sum_{x\in F_q} \chi\Big((x-u_1)^{i_1}\cdots(x-u_m)^{i_m}\Big) - m2^{m-1}.\end{aligned}$$

Note that the polynomial $(x-u_1)^{i_1}\cdots(x-u_m)^{i_m}$ is not the form $a(g(x))^2$ with $a\in F_q$ and $g(x)\in F_q[x]$ as $i_1,\ldots,i_m\in\{0,1\}$, it follows from Weil bound (10.10) that

$$\begin{aligned}\sum_{x\in F_q\setminus U} f(x) &\le q+\sum_{\substack{i_1,\ldots,i_m\in\{0,1\}\\ i_1+\cdots+i_m\ge 1}} (m-1)\sqrt{q}-m2^{m-1}\\ &= q+(m-1)(2^m-1)\sqrt{q}-m2^{m-1}.\end{aligned}$$

Consequently,

$$|J(U)| = \frac{1}{2^m}\sum_{x\in F_q\setminus U} f(x) < \frac{q}{2^m}+(m-1)\sqrt{q}.$$

If we suppose that $\frac{q}{2^m}+(m-1)\sqrt{q}\le n\le \frac{q}{2^m}+m\sqrt{q}$, then $|J(U)|<n$ and

$$r(B_n^{(m)})\ge q+1>2^m n-m2^m\sqrt{q}>2^m n-m2^{3m/2}\sqrt{n}$$

as claimed. □

10.6 Exercises

1. Let G be a graph of order n satisfying that for any vertex v and subset B, $|N(v)\cap B| = p|B|+o(n)$. Is G quasi-random for fixed p?

2. Define a property: For any $U\subseteq V$ with $|U|=\lfloor n/2\rfloor$, $e(U,V\setminus U)\sim p\frac{n^2}{4}$. Is this a quasi-random property for fixed edge density p?

3. Prove that the inequality holds in (10.6).

4. Estimate $br_k(C_4;K_{n,n})$ for $k\ge 3$ by modifying the argument for $r_k(C_4;K_n)$.

5. Let G be a finite group and S be an inverse-closed subset of G that contains no identity. Prove that each eigenvalue of the Cayley graph $\Gamma(G,S)$ is of the form

$\sum_{s\in S}\chi(s)$ with respective to the eigenvector is $(\chi(g) : g \in G)$, where χ is a character of G.

6. Prove that if ψ is a complex-valued function such that $\psi(x) \neq 0$ and $\psi(x+y) = \psi(x)\psi(y)$, then ψ is an additive character of F_q.

7. Prove that if ψ is an additive character of F_q, where $q = p^m$ with $p \geq 3$, then $\psi(x) \neq -1$ for any $x \in F_q$.

8. Let G be a semi-simple (N, d, λ)-graph with vertex set V. Prove that for any $n \geq \frac{4N\log N}{d}$, the number M of $K_{n,n}$ in $\overline{G}$ satisfies that

$$M \leq \left(\frac{ed^2n}{4\lambda N\log N}\right)^{\frac{4N\log N}{d}} \left(\frac{2e\lambda N}{dn}\right)^{2n}.$$

(Hint: Consider the number of ways to choose ordered subsets $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ of G which form a $K_{n,n}$ in $\overline{G}$, where $u_i \neq v_j$ for $1 \leq i, j \leq n$. Starting with choosing a pair of distinct and non-adjacent vertices u_1 and v_1. See Lin and Li (2011) or Liu and Li (2021))

Chapter 11
Regularity Lemma and van der Waerden Number

Bartel L. van der Waerden (February 2, 1903–January 12, 1996) was a Dutch mathematician, who published his *Algebra*, an influential two-volume treatise on abstract algebra at age 27. Before this, in 1927, he proved the following result, conjectured by Baudet in 1926, which is now called van der Waerden theorem: For any positive integers k and t, there exists a positive integer N such that if the set $\{1, 2, \ldots, N\}$ is partitioned into k classes, then at least one class contains an arithmetic progression of t terms. This is one of the most important Ramsey-type results on the integers, and another such result is that of Schur, discussed in Chapter 2. Let $w_k(t)$ be the smallest N for the van der Waerden theorem, which has a very huge upper bound in the original proof. Much later, in 1988, a substantially improved upper bound was given by Israeli mathematician Saharon Shelah (born on July 3, 1945, recipient of the 2001 Wolf prize). He gave the first primitive recursive upper bound for $w_k(t)$, which is proved in Section 11.2 as it is almost transparent even the bound is still enormous. The bound was further improved greatly by Timothy Gowers (born on November 20, 1963, recipient of the 1998 Fields Medal) to a tower of height 6. He is a British mathematician, and his research is connecting the fields of functional analysis and combinatorics in surprise.

In the 1930s, Erdős and Turán conjectured that if a set A of positive integers satisfies $\overline{\lim\limits_{n\to\infty}} |A \cap [n]|/n > 0$, then A contains arbitrarily long arithmetic progressions. The conjecture in case of length 3 was proved by Roth in 1953 and 1954. Klaus F. Roth (born on 29 October 1925, recipient of the 1958 Fields Medal) is a German-born British mathematician. The full conjecture was proved by Szemerédi in 1975 with a deep and complicated combinatorial argument, which thus becomes Szemerédi theorem. Endre Szemerédi (born on August 21, 1940, recipient of the 2012 Abel prize) is a Hungarian-American mathematician, working in the field of combinatorics and theoretical computer science. In the proof, he used a result which is called the Regularity Lemma (in bipartite version). In 1978, he proved the full lemma, which is a powerful tool in extremal graph theory. Sometimes the Regularity Lemma is called uniformity lemma, see e.g., Bollobás (1998). The proof of Szemerédi theorem is beyond our book, but we shall have an clear and detailed proof for the Regularity Lemma in this chapter. Let us remark that fully understanding of the

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lemma is important for its applications. For many applications, we refer the reader to the survey of Komlós and Simonovits (1996) and other related references.

A further problem of Erdős and Turán (1936) is still open, who conjectured that if $A = \{a_i\}$ is a set of positive integers such that $\sum 1/a_i = \infty$, then A contains arbitrarily long arithmetic progression. Note that $\sum_{p \le n} 1/p \sim \log \log n$, where the sum is taken over all primes no more than n. The most important special case by Green and Tao (2008) states that there are arbitrarily long arithmetic progression of primes. Terence Tao (Chinese name Chi-Shen Tao, born on July 17, 1975, recipient of the 2006 Fields Medal) is an Australian-born American mathematician working in many mathematical fields with excellent results.

11.1 van der Waerden Number

Let t-AP denote an arithmetic progression of t terms. If no specifying, a t-AP always means non-trivial one (with distinct t terms). The existence of an arithmetic progression in partition of integers by van der Waerden (1927) is as follows.

Theorem 11.1 *Let k and t be positive integers. If an integer w is sufficiently large and the set $[w] = \{1, 2, \ldots, w\}$ is partitioned into k classes, then one of the classes must contain a t-AP.*

As usual, a partition of $[w]$ is called a coloring of $[w]$. Define $w_k(t)$ as the smallest integer w so that the mentioned property holds, and write $w(t) = w_2(t)$. We call $w_k(t)$ the van der Waerden number. It is trivial to see that $w_k(1) = 1$, $w_k(2) = k + 1$ and $w_1(t) = t$. However, for general k and t, the functions $w(t) = w_2(t)$ and $w_k(3)$ are not trivial at all. The following data for $w_k(t)$ was announced by Heule in his own web page, which improves that in the paper of Herwig, Heule, Lambalgen and Maaren (2007).

$k \backslash t$	3	4	5	6	7	8
2	9	35	178	1132	> 3703	> 11495
3	27	> 292	> 2173	> 11191	> 48811	> 238400
4	76	> 1048	> 17705	> 91331	> 420217	
5	> 170	> 2254	> 98740	> 540025		
6	> 223	> 9778	> 98748	> 816981		

Table 12.1 Small van der Waerden numbers $w_k(t)$.

The original proof of van der Waerden (1927) gave an extremal large upper bound for $w(t)$. Shelah (1998) improved this with a cerebrated upper bound, it is still a tower, in which the height of the tower on t is somehow like the value of the tower on $t - 1$. The current best upper bound for $w(t)$ is a striking result of Gowers (see Gowers (2001), Corollary 18.7) as a tower of height 6 as

$$w(t) < 2^{2^{2^{2^{2^{(t+9)}}}}}.$$

These upper bounds may be far away from the truth. However, it is hard to show the existence of finite $w_k(t)$. In fact, nobody has found an easy proof for the statement: If all natural numbers are k-colored, then there exists an arbitrary long AP, where an arbitrary length signifies that the length of AP can be any given $t > 0$, as this statement implies $w_k(t) < \infty$ immediately, see Exercise 12.2.

Proof of Theorem 11.1. We shall prove $w_k(t) < \infty$ by induction on t. For $t = 2$, we have $w_k(2) = k + 1$. To conduct the induction step for positive integers r, s and $N = 4rs$, we partition $[N]$ into $2r$ blocks $B_0, B_1, \ldots, B_{2r-1}$ as

$$\begin{aligned} B_0 &= \{1, 2, \ldots, 2s\}, \\ B_1 &= \{2s + 1, 2s + 2, \ldots, 4s\}, \\ B_{2r-1} &= \{(2r - 1)2s + 1, (2r - 1)2s + 2, \ldots, 4rs\}, \end{aligned}$$

where each B_i is a block containing $2s$ consecutive integers, and

$$B_i = i \cdot 2s + B_0.$$

We shall call $a + td$ the *continuation* of a t-AP $\{a, a + d, \ldots, a + (t - 1)d\}$. Note that a partition $\{C_1, \ldots, C_k\}$ of $[N]$ induces an equivalence relation on the sets of all colorings of blocks $B_0, B_1, \ldots, B_{2r-1}$, for which B_i and B_j with $i \le j$ are in the same equivalence class if and only if

$$C_\ell \cap B_j = (j - i)2s + (C_\ell \cap B_i),$$

for all $1 \le \ell \le k$. In this case, we say that B_i and B_j have the same pattern. i.e., $b_{ih} \in C_\ell \cap B_i$ if and only if $b_{ih} + (j - i)2s \in C_\ell \cap B_j$. It follows that there are at most k^{2s} equivalence classes (different patterns) since there are $2s$ elements of B_i and each element of B_i has k choices of C_ℓ.

By the induction hypothesis, the number $w_{k^{2s}}(t)$ exists, which means that in case $r \ge w_{k^{2s}}(t)$, there exists some t-AP

$$\{a, a + d, \ldots, a + (t - 1)d\} \subseteq \{0, 1, \ldots, r - 1\},$$

such that all blocks $B_a, B_{a+d}, \ldots, B_{a+(t-1)d}$ have the same pattern. Each k^{2s}-coloring on $\{0, 1, \ldots, r-1\}$ can be referred to as a k^{2s}-coloring on $B_0, B_1, \ldots, B_{r-1}$. Note that the block B_{a+td} is still contained in $[N]$ as $r - 1 + d < 2r$.

The induction step will be completed by verifying the following **claim**: For each ℓ with $1 \le \ell \le k$, there exists some $N(\ell)$ such that the following assertion holds: If $[N(\ell)]$ is partitioned into k classes, then either one class contains a $(t + 1)$-AP or there are ℓ APs $A_1, \ldots, A_\ell$ with $|A_i| = t$ such that

(i) All A_i are monochromatic but of different colors, namely, $A_i \subseteq C_{j_i}$, $1 \le i \le \ell$, where $j_1, \ldots, j_\ell$ are distinct.

(ii) All A_i have the same continuation that are still in $[N(\ell)]$.

Suppose first the claim is proved, we then choose that $N \geq N(k)$. If no $(t+1)$-AP exists, then we have k APs $A_1, \ldots, A_k$ of all colors with the same continuation in $[N]$. Since this element is also colored by one of the k-colors, one AP can be extended.

The proof of our claim is by induction on ℓ. For $\ell = 1$, we have $N(1) \leq w_k(t)$. For the induction step, the set $[N]$ is partitioned into blocks $B_0, B_1, \ldots, B_{2r-1}$ as above with $s = N(\ell - 1)$ and $r = w_{k^{2s}}(t)$. If there is no monochromatic $(t+1)$-AP, then we conclude from the above two induction hypotheses with the following two facts:

(1) There is an AP of blocks $B_a, B_{a+d}, \ldots, B_{a+(t-1)d}$, all of the same pattern as described above;

(2) In the block B_a, without loss of generality, we may assume that there are $\ell - 1$ APs $A'_1, \ldots, A'_{\ell-1}$ such that $A'_i \subseteq C_i$, $1 \leq i \leq \ell - 1$, which all have the same continuation $c \in B_a$. We may assume that $c \in C_\ell$.

Let $A_\ell = \{c, c+2sd, \ldots, c+(t-1)2sd\}$. From (1), we have $A_\ell \subseteq C_\ell$. Indeed, we have that $c+2sd \in C_\ell$ since $C_\ell \cap B_{a+d} = (a+d-a)2s+(C_\ell \cap B_a)$ and $c \in C_\ell \cap B_a$. Inductively, we can get that $A_\ell \subseteq C_\ell$. Moreover, $c+t \cdot 2sd$ is the continuation of A_ℓ.

If $A'_i = \{\alpha, \alpha+\delta, \ldots, \alpha+(t-1)\delta\} \subseteq C_i$, then $c = \alpha + t\delta \in B_a$ and a similar argument as above yields that $A_i = \{\alpha, \alpha+\delta+2sd, \ldots, \alpha+(t-1)(\delta+2sd)\} \subseteq C_i$ is a t-AP with continuation $\alpha + t(\delta + 2sd) = c + t \cdot 2sd$. □

What about the lower bound of $w_k(t)$? The first lower bound $w_k(t) \geq \Omega(\sqrt{tk^t})$ was due to Erdős and Rado (1952) by a counting method, see the exercises. Szabó (1990) proved that if $k \geq 2$ is fixed, then $w_k(t) \geq k^{t-1}/(et)$, which can be slightly improved as follows.

Theorem 11.2 *Let $k \geq 2$ be fixed and $t \geq 2$. Then*

$$w_k(t) \geq \frac{k^t}{e(k-1)t}.$$

Proof. Randomly k-color $[N]$, each $x \in [N]$ being colored by a color with probability $1/k$. For each S of t-AP, let A_S^i be the event that S is monochromatic in color i. Then $\Pr(A_S^i) = 1/k^t$. We try to use the local lemma. Define a graph whose vertex set consists of all events A_S^i, which contains edges from joining events A_S^i and A_T^j if and only if $S \cap T \neq \emptyset$ and $i \neq j$. For fixed S of t-AP, the number of T with $S \cap T \neq \emptyset$ is at most Nt, and thus the maximum degree d of the dependency graph satisfies $d < (k-1)Nt$.

If $N = \lfloor k^t/(e(k-1)t) \rfloor$, then $ep(d+1) \leq 1$. When the symmetric form of the local lemma is applied, we know $\Pr(\cap_S \overline{A_S}) > 0$, which implies that there is a k-coloring of $[N]$ without monochromatic t-AP. Therefore, $w_k(t) \geq N + 1$, and the assertion holds. □

Let us call a prime of the form $2^p - 1$ to be a *Mersenne prime*, where p is necessarily a prime. Most known top primes are such ones and an old conjecture is that there are infinitely many of them. The following bound is due to Berlekamp (1968).

Theorem 11.3 *If* $2^p - 1$ *is a prime, then* $w(p+1) \ge p(2^p - 1)$.

Proof. Let $F(2^p)$ be the finite field of 2^p elements. As $2^p - 1$ is a prime, any $\alpha \in F(2^p) \setminus \{0, 1\}$ is a primitive element, which generates the cyclic multiplicative group $F^*(2^p)$. Fix a primitive element $\alpha \in F(2^p)$. Since $F(2^p)$ is a linear space of dimension p over Z_2, we can have a basis $v_1, v_2, \ldots, v_p$. For any integer j with $1 \le j \le p(2^p - 1)$, set

$$\alpha^j = a_{1j}v_1 + a_{2j}v_2 + \cdots + a_{pj}v_p, \qquad a_{ij} \in Z_2.$$

We shall partition the set $[p(2^p - 1)]$ of integers into C_0 and C_1 according to the first coordinate of α^j as

$$C_0 = \{j : \ a_{1j} = 0, \ 1 \le j \le p(2^p - 1)\},$$

and

$$C_1 = \{j : \ a_{1j} = 1, \ 1 \le j \le p(2^p - 1)\}.$$

We then *claim* that (C_0, C_1) is a 2-coloring of $[p(2^p - 1)]$ with no monochromatic $(p+1)$-AP. For $k = 0$ or $k = 1$, suppose that $a, a+b, a+2b, \ldots, a+pb$ are integers from the same C_k, where $a, b \ge 1$. We shall show that this leads to a contradiction. Let $\beta = \alpha^a$, and $\gamma = \alpha^b$. Then the vectors in

$$\{\alpha^a, \alpha^{a+b}, \alpha^{a+2b}, \ldots, \alpha^{a+pb}\} = \{\beta, \beta\gamma, \beta\gamma^2, \ldots, \beta\gamma^p\}$$

have the same first coordinates when they are expressed as linear combinations of $v_1, v_2, \ldots, v_p$. Since $1 \le a < a + pb \le p(2^p - 1)$, we have $1 \le b \le 2^p - 2$, and $\gamma = \alpha^b$ is a primitive element as $2^p - 1$ is a prime.

Case 1 $k = 0$.

Then p vectors $\beta, \beta\gamma, \ldots, \beta\gamma^{p-1}$ are linearly dependent as they are linear combination of $\{v_2, v_3, \ldots, v_p\}$ as all first coordinates are 0 over Z_2. Thus there exist $c_0, c_1, \ldots, c_{p-1}$ in Z_2, not all 0, such that

$$\sum_{i=0}^{p-1} c_i(\beta\gamma^i) = 0, \quad \text{and hence} \quad \sum_{i=0}^{p-1} c_i\gamma^i = 0.$$

But $\gamma = \alpha^b \in F(2^p)$, $\gamma \ne 0, 1$, so γ satisfies a non-trivial polynomial of degree at most $p - 1$ over $F(2^p)$, a contradiction.

Case 2 $k = 1$.

For this case, each of $\{\beta, \beta\gamma, \ldots, \beta\gamma^p\}$ has the first coordinate 1 and thus $\beta(\gamma - 1), \beta(\gamma^2 - 1), \ldots, \beta(\gamma^p - 1)$ are linearly dependent. i.e.,

$$\sum_{i=1}^{p} c_i[\beta(\gamma^i - 1)] = 0,$$

where $c_i \in Z_2$, and some $c_i \neq 0$. Dividing by $\beta(\gamma - 1)$, we have

$$c_1 + c_2(\gamma + 1) + \cdots + c_p(\gamma^{p-1} + \gamma^{p-2} + \cdots + 1) = 0.$$

Since γ cannot be a root of any polynomial of degree at most $p - 1$, we have $c_p = 0$, and consequently $c_{p-1} = c_{p-2} = \cdots = c_1 = 0$, a contradiction. □

For two colors, the lower bounds in the above two theorems are very close, but the proofs are completely different. This may suggest that the lower bounds are much closer to the truth than the known upper bounds.

Let $\nu(n)$ be the maximum cardinality of a subset in $[n]$ that does not contain any 3-AP. The first non-trivial upper bound concerning the size of 3-AP-free sets was given by Roth (1953) who showed $\nu(n) \le n/\log\log n$. Subsequently, it was refined by several researchers, see Heath-Brown (1987), Szemerédi (1990), Bourgain (1999, 2008), Sanders (2011,2012), Bloom (2016), Bloom and Sisask (2019), and Schoen (2021(b)) etc. To our best knowledge, the current bounds for ν are

$$ne^{-c_1\sqrt{\log n}} < \nu(n) \ll n/(\log n)^{1+c_2},$$

in which the lower and upper bounds were given by Behrend (1946) and Bloom and Sisask (2021+) respectively, where c_1 and c_2 are positive constants.

What about bounds of $w_k(t)$ for fixed t as $k \to \infty$? The first nontrival case is $w_k(3)$. Let $n = w_k(3) - 1$. Then there exists a k-coloring on $[n]$ such that there is no monochromatic 3-AP, which means each color class has size at most $\nu(n)$. It follows from that $n \le k \cdot \nu(n)$, which combines the above upper bound of $\nu(n)$ yield that for some constant $c > 0$,

$$w_k(3) < e^{k^{1/(1+c)}}.$$

For the lower bound, Brown (2008) obtained that $w_k(3) > e^{\Omega(k^{\log_3 2})}$.

The off-diagonal van der Waerden number $w(m, n)$ in two colors is the smallest positive integer w such that if $[w]$ is red-blue colored, then there is either a red m-AP or a blue n-AP. It is easy to see that $w(1, n) = n$, and for $n \ge 2$, $w(2, n) = 2n$ if n is odd and $2n - 1$ otherwise. However, it is also hard to prove the existence of $w(m, n)$ for fixed $m \ge 3$ as that is similar to that of $w(n, n)$.

For the van der Waerden function $w(3, n)$, it is known that

$$n^{2-1/\log\log n} \le w(3, n) \le n^{cn^2}.$$

The upper bound is due to Bourgain (1999), and the lower bound is a special case of that for $w(m, n)$ of Brown, Landman and Robertson (2008), whose proof is from the symmetric form of the local lemma. This result was improved by Li and Shu (2010) by using the local lemma as $w(m, n) \ge cn^{m-1}/\log^{m-1} n$. Recently, Guo and Warnke (2021+) show that $w(m, n) \ge \Omega(n^{m-1}/\log^{m-2} n)$ by using a different method. A breakthrough due to Green (2021+) states that there is a constant $c > 0$ such that

$$w(3,n) \geq n^{c(\frac{\log n}{\log\log n})^{1/3}} = \exp\left(\log^{4/3-o(1)} n\right).$$

For the upper bound, Schoen (2021(a)) obtain that there is a small positive ϵ such that

$$w(3,n) \leq e^{cn^{1-\epsilon}}.$$

11.2 Recursive Bounds for $w_k(t)^\star$

Rather than proving van der Waerden theorem directly, we shall introduce a remarkable extension of the theorem by Hales and Jewett in 1963, which leads to the proof of Shelah (1988). We introduce Shelah's proof in this section because it is transparent.

A *cube* C_t^n is a set of sequences of length n formed from t symbols. The symbols are ordered, usually as $[t] = \{1, 2, \ldots, t\}$ if no specified, or in example as $\{A, B, C, \ldots, Z\}$ with $t = 26$. The symbol in the ith position of an element is called its ith coordinate. The elements are called *points* often.

Some points of C_{26}^5: P_1 : $AAAAB$, P_2 : $AXYCN$, P_3 : $PPPPP$. The third coordinate of P_2 is Y.

The points of C_t^n can be viewed as the points of the n-dimensional discrete cube. For example, the points of C_4^2 with symbols $1, 2, 3, 4$ can be arranged in a matrix.

11	12	13	14
21	22	23	24
31	32	33	34
41	42	43	44

A set of t points $L = \{P_1, P_2, \ldots, P_t\}$ in C_t^n is called a *line* if there is a non-empty set I of indices such that for any $i \in I$, the ith coordinate of any point $P_j \in L$ is just the jth symbol, and for $i \notin I$, the ith coordinates of all points of L are the same. The ith coordinates for $i \in I$ are called *moving* coordinates, and those for $i \notin I$ are called *constant* coordinates.

Let us take a look at two lines L_1 and L_2 in C_{26}^5 as follows.

$$\begin{array}{rcc}
 & L_1 & L_2 \\
P_1: & XAQAB & AAAAA \\
P_2: & XBQBB & BABAA \\
P_3: & XCQCB & CACAA \\
\vdots & \vdots & \vdots \\
P_{25}: & XYQYB & YAYAA \\
P_{26}: & XZQZB & ZAZAA
\end{array}$$

In line L_1, the first, third, and fifth coordinates are constant coordinates; the second and fourth are moving coordinates.

In C_4^2, there are totally 9 lines, namely, each row and each column, and one diagonal line $\{11, 22, 33, 44\}$. The last line does not have constant coordinates.

Clearly, every line has t points, and there are

$$\sum_{J\subseteq[n],J\neq[n]} t^{|J|} = \sum_{j=0}^{n-1} \binom{n}{j} t^j = (t+1)^n - t^n$$

lines in C_t^n. For example, the cube C_t^2 contains $2t + 1$ lines: t horizontal lines, t vertical lines, and one diagonal line. Note that a line is not determined completely by the set of indices of moving coordinates.

The following is an important result of Hales and Jewett in 1963.

Theorem 11.4 *For any positive integers k and t, there exists an integer N such that if the points of C_t^N are k-colored in any fashion, then there is a monochromatic line.*

The *Hales-Jewett function $HJ(k, t)$* is defined as the minimal value of N that will satisfy the above theorem. To see that the function makes sense, one needs to make sure that $N + 1$ will do the theorem. This is clear if we consider the derived coloring χ' on C_t^N from a given coloring χ on C_t^{N+1} by $\chi'(P) = \chi(Pt)$, where P is a point of C_t^N, and Pt is a point of C_t^{N+1} with the last coordinate t. More importantly, we need to show that such finite N exists.

The theorem of van der Waerden follows from Theorem 11.4 immediately.

Proof of Theorem 11.1. For given k and t, let N be an integer such that any k-coloring of C_t^N contains a monochromatic line, where the set of symbols of C_t^N is $\{0, 1, \ldots, t-1\}$. Then each point can be viewed as a representation of a nonnegative integer in base t. The largest such integer is $t^N - 1$. This representation establishes a bijection from C_t^N to $\{0, 1, \ldots, t^N - 1\}$. A key observation is that a line in C_t^N is exactly a t-AP. Now any coloring of $\{1, 2, \ldots, t^N\}$ with k colors yields a natural coloring for $\{0, 1, \ldots, t^N-1\}$ and hence a coloring of C_t^N with k colors. By Theorem 11.4, we have a monochromatic line in C_t^N hence a monochromatic t-AP. □

The above proof in fact gives an upper bound for $w_k(t)$.

Theorem 11.5 *Let k and t be positive integers and let $N = HJ(k, t)$. Then*

$$w_k(t) \le t^N.$$

The remaining part of this section is Shelah's proof of Theorem 11.4. However, in order to make the proof clearer, we shall give some examples and details to explain the concepts in the context.

A line L in C_t^n is called a *Shelah line* if for some i, j with $1 \le i \le j \le n$, the following holds:

(i) Coordinates $1, \ldots, i-1$ are all equal to $t-1$ (constant);

(ii) Coordinates $i, \ldots, j$ are moving;
(iii) Coordinates $j+1, \ldots, n$ are all equal to t (constant).

Namely, a line is a Shelah line if the positions of moving coordinates are consecutive, and all constant coordinates are $t-1$ and t before and after moving coordinates, respectively. In the above definition, the first condition disappears if $i = 1$, and the third condition disappears if $j = n$. A point is called *Shelah point* if it is on some Shelah line.

Let us take a look at some Shelah lines in C_{26}^5 as follows.

$$\begin{array}{ccc} L_1 & L_2 & L_3 \\ YAAZZ & AAZZZ & AAAAA \\ YBBZZ & BBZZZ & BBBBB \\ YCCZZ & CCZZZ & CCCCC \\ \vdots & \vdots & \vdots \\ YYYZZ & YYZZZ & YYYYY \\ YZZZZ & ZZZZZ & ZZZZZ \\ i=2, j=3 & i=1, j=2 & i=1, j=5 \end{array}$$

The points $YYQZZ$, $YCCCZ$, $HHHHZ$, $SSSSS$ are Shelah points, but $YFGZZ$, $AAYYY$ are not.

The following result is clear from the fact that a Shelah line is uniquely determined by i and j in the definition.

Lemma 11.1 *The cube C_t^n contains $\binom{n+1}{2}$ Shelah lines and at most $\binom{n+1}{2}t$ Shelah points.*

Assume $n = n_1 + n_2 + \cdots + n_s$. Then

$$C_t^n = C_t^{n_1} \times C_t^{n_2} \times \cdots \times C_t^{n_s}.$$

Let L_j be a Shelah line of $C_t^{n_j}$ for $j = 1, 2, \ldots, s$. We call

$$L_1 \times L_2 \times \cdots \times L_s$$

a *Shelah s-space* of C_t^n, which contains t^s points of C_t^n.

For example, if $n_1 = 5$ and $n_2 = 6$, then

$$\{Y\alpha\alpha ZZ \,|\, YY\beta\beta ZZ\},$$

where α and β are letters in alphabet and the vertical line is added for clarity of conjunction, forms a Shelah 2-space in C_{26}^{11}. It contains 26^2 points. If $n_1 = 3$, $n_2 = 4$ and $n_3 = 4$, then

$$\{Y\alpha Z \,|\, Y\beta\beta Z \,|\, \gamma\gamma ZZ\}$$

forms a Shelah 3-space in C_{26}^{11}. It contains 26^3 points.

We now define a canonical isomorphism ϕ between a Shelah s-space of C_t^n and C_t^s as follows.

$$\phi: \ L_1 \times L_2 \times \cdots \times L_s \to C_t^s, \qquad \phi(\eta) = \alpha_1\alpha_2 \ldots \alpha_s,$$

where α_j is the common value of the moving coordinate in L_j, which can be viewed as the index of the corresponding point in L_j. Using the previous examples,

$$\phi(Y\alpha\alpha ZZ \,|\, YY\beta\beta ZZ) = \alpha\beta \in C_{11}^2,$$
$$\phi(Y\alpha Z \,|\, Y\beta\beta Z \,|\, \gamma\gamma ZZ) = \alpha\beta\gamma \in C_{11}^3.$$

It is easy to see that the map ϕ is really a bijection.

During the rest of the proof, χ is used as a coloring function, thus $\chi(P)$ is the color of the point $P \in C_t^n$. A coloring χ on C_t^n is called a *flip-top* if $P = p_1p_2 \ldots p_n, \ Q = q_1q_2 \ldots q_n \in C_t^n$, for which there exists $I \subseteq [n]$ such that $p_i = t-1, \ q_i = t$ for $i \in I$, and $p_j = q_j$ for $j \notin I$, then $\chi(P) = \chi(Q)$, where $t-1$ and t are last two elements of the set of symbols. Equivalently, $\chi(P) = \chi(Q)$ if P and Q are the last two points of some line. For example, a flip-top coloring χ of C_{26}^5 satisfies

$$\chi(BA\mathbf{YY}O) = \chi(BA\mathbf{ZZ}O),$$
$$\chi(\mathbf{Z}E\mathbf{Z}AK) = \chi(\mathbf{Y}E\mathbf{Y}AK),$$
$$\chi(\mathbf{YYYYY}) = \chi(\mathbf{ZZZZZ}).$$

It is easy to see that $\chi(\mathbf{Z}E\mathbf{Y}AK) = \chi(\mathbf{Y}E\mathbf{Z}AK)$ as both of them are equal to $\chi(\mathbf{Z}E\mathbf{Z}AK)$.

Let χ be a coloring of the points of $L_1 \times L_2 \times \cdots \times L_s$, and let ϕ be the canonical isomorphism from $L_1 \times L_2 \times \cdots \times L_s$ to C_t^s. Define a coloring χ' on C_t^s as for $P \in C_t^s$,

$$\chi'(P) = \chi(\phi^{-1}(P)),$$

which is called the *derived coloring* of χ. The coloring χ is called *flip-top* on the Shelah s-space if χ' is flip-top on C_t^s.

For example, if χ is a flip-top coloring of the Shelah 2-space in the form of $\{Y\alpha\alpha ZZ \,|\, YY\beta\beta\beta Z\}$, then

$$\chi(Y\mathbf{QQ}ZZ \,|\, YY\mathbf{YYY}Z) = \chi(Y\mathbf{QQ}ZZ \,|\, YY\mathbf{ZZZ}Z)$$

since $\chi'(QY) = \chi'(QZ)$ by noting that χ' is flip-top on C_{26}^2.

If $s = 1$, then the above definition gives flip-top coloring of a Shelah line. In this case ϕ maps a Shelah line L in C_t^n onto C_t^1, and the derived coloring on C_t^1 must be flip-top. However, a flip-top coloring of C_t^1 simply says that the color of $t-1$ equals to the color of t. This gives the following remark.

Let us remark that the points of a Shelah line $L = \{P_1, P_2, \ldots, P_t\}$ in C_t^n are colored flip-top if and only if the "last two points" P_{t-1} and P_t have the same color.

Lemma 11.2 (Shelah Cube Lemma) *Let positive integers k, s, t be fixed. Then there exist $n_1, n_2, \ldots, n_s$ with the following property: Any k-coloring of the points of $C_t^n = C_t^{n_1+n_2+\cdots+n_s}$ is flip-top on a suitable Shelah s-space $L_1 \times L_2 \times \cdots \times L_s$,* *where L_j is a Shelah line in $C_t^{n_j}$. In fact, the numbers n_j can be explicitly given by $n_1 = k^{t^{s-1}}$ and $n_{i+1} = k^{A_i}$ for $1 \le i < s$, where*

$$A_i = \left[\Pi_{1\le j\le i}\binom{n_j+1}{2}\right] t^{s-1}.$$

Proof. We shall prove the lemma by induction on s. For $s = 1$, take $n_1 = n = k$. We have to show that in an arbitrary k-coloring of C_t^k there is a flip-top colored Shelah line. Consider the following points in C_t^k:

$$\begin{array}{rccccc}
P_1: & t-1 & t-1 & \cdots & t-1 & t-1 & t-1 \\
P_2: & t-1 & t-1 & \cdots & t-1 & t-1 & t \\
\vdots & & & & & & \\
P_k: & t-1 & t & \cdots & t & t & t \\
P_{k+1}: & t & t & \cdots & t & t & t
\end{array}$$

Each of these points consists of a block of $t-1$ followed by a block of t. Two of them, say P_i and P_j with $i < j$, are colored with the same color in this k-coloring. Note that P_i and P_j are the last two points of the following Shelah line determined by themselves:

$$\begin{array}{r|c|c|c}
Q_1: & t-1 \cdots t-1 & 1 \ \ \ldots \ \ 1 & t \cdots t \\
Q_2: & t-1 \cdots t-1 & 2 \ \ \ldots \ \ 2 & t \cdots t \\
\vdots & & & \\
Q_{t-2}: & t-1 \cdots t-1 & t-2 \ldots t-2 & t \cdots t \\
P_i: & t-1 \cdots t-1 & \mathbf{t-1} \ldots \mathbf{t-1} & t \cdots t \\
P_j: & t-1 \cdots t-1 & \mathbf{t} \ \ \ldots \ \ \mathbf{t} & t \cdots t
\end{array}$$

By the previous remark, the above line is exactly what we want: a flip-top colored Shelah line.

For $s = 2$, let $n_1 = k^t$ and $n_2 = k^{A_1}$ with $A_1 = \binom{n_1+1}{2}t$. Let χ be a k-coloring on

$$C_t^n = C_t^{n_1+n_2} = C_t^{n_1} \times C_t^{n_2}.$$

In order to find a Shelah 2-space $L_1 \times L_2$ such that χ is flip-top on it, we shall define new colorings χ_1 and χ_2 on $C_t^{n_1}$ and $C_t^{n_2}$, respectively. Let us write points of C_t^n as XY with $X \in C_t^{n_1}$ and $Y \in C_t^{n_2}$.

Let us define χ_2 on $C_t^{n_2}$ first. Denote by m the number of Shelah points in $C_t^{n_1}$ and label these Shelah points as $X_1, X_2, \ldots, X_m$. For any $Y \in C_t^{n_2}$, we assign a vector for its color as

$$\chi_2(Y) = (\chi(X_1Y), \chi(X_2Y), \ldots, \chi(X_mY)).$$

Clearly two points Y and Y' of $C_t^{n_2}$ have same color in χ_2 if and only if $\chi(XY) = \chi(XY')$ for any Shelah point X of $C_t^{n_1}$.

Since $m \leq A_1 = \binom{n_1+1}{2} t$ as mentioned, and each coordinate of $\chi_2(Y)$ is one of the k colors, so the coloring χ_2 uses at most $k^m \leq k^{A_1} = n_2$ colors. Referring to the case $s = 1$, the coloring χ_2 is flip-top on a suitable Shelah line L_2 in $C_t^{n_2}$.

The Shelah line L_2 contains t points, say $L_2 = \{Y_1, Y_2, \ldots, Y_t\}$. We then define a coloring χ_1 on any point X of $C_t^{n_1}$ as a vector

$$\chi_1(X) = (\chi(XY_1), \chi(XY_2), \ldots, \chi(XY_t)).$$

Thus two points X and X' of $C_t^{n_1}$ have same color in χ_1 if and only if $\chi(XY_\beta) = \chi(X'Y_\beta)$ for any point Y_β of L_2.

Clearly χ_1 uses at most $k^t = n_1$ colors. Again, referring the case $s = 1$, the coloring χ_1 is flip-top on a suitable Shelah line L_1 in $C_t^{n_1}$.

Next we verify that χ is flip-top on $L_1 \times L_2$: the derived coloring χ' of χ is a flip-top coloring on C_t^2. That is to say, we need to verify

$$\begin{aligned}
\chi'(\alpha\,(t-1)) &= \chi'(\alpha\,t),\\
\chi'((t-1)\,\beta) &= \chi'(t\,\beta),\\
\chi'((t-1)\,(t-1)) &= \chi'(t\,t)
\end{aligned}$$

for any symbols α and β. We verify the first equality, and omit the similar proofs for the others. By relabelling the Shelah points of $C_t^{n_1}$, we may assume that $L_1 = \{X_1, X_2, \ldots, X_t\}$. Note that points of $L_1 \times L_2$ have form $X_\alpha Y_\beta$, where

$$X_\alpha = \; t-1 \ldots t-1\, \alpha \ldots \alpha\, t \ldots t, \quad Y_\beta = \; t-1 \ldots t-1\, \beta \ldots \beta\, t \ldots t.$$

Since χ_2 is flip-top on L_2, we have $\chi_2(Y_{t-1}) = \chi_2(Y_t)$, implying $\chi(PY_{t-1}) = \chi(PY_t)$ for any Shelah point P of $C_t^{n_1}$ from the definition of χ_2. In particular, we have $\chi(X_\alpha Y_{t-1}) = \chi(X_\alpha Y_t)$, which yields $\chi'(\alpha\,(t-1)) = \chi'(\alpha\,t)$.

The proof for $s \geq 3$ is similar to the case $s = 2$, we thus omit it. □

Lemma 11.3 (Induction Lemma) *Assume that $s = HJ(k, t-1)$ is defined, namely, in any k-coloring of C_{t-1}^s there is a monochromatic line. Then under any flip-top k-coloring of C_t^s, there is a monochromatic line.*

Proof. Consider a flip-top k-coloring χ on C_t^s. Since C_{t-1}^s is a subset of C_t^s, χ is a k-coloring on C_{t-1}^s. From the definition of s, there is a monochromatic line L in C_{t-1}^s, say $L = \{P_1, P_2, \ldots, P_{t-1}\}$ in color 1, then the following line

$$\begin{array}{rc|c|l}
P_1: & \cdots\ \alpha\ \cdots & 1 & \cdots\ \text{color 1}\\
P_2: & \cdots\ \alpha\ \cdots & 2 & \cdots\ \text{color 1}\\
\vdots & & & \\
P_{t-1}: & \cdots\ \alpha\ \cdots & t-1 & \cdots\ \text{color 1}\\
P_t: & \cdots\ \alpha\ \cdots & \mathbf{t} & \cdots\ \text{new point}
\end{array}$$

is a monochromatic line, where P_t is a new point, and α is a constant coordinate. The point P_t is colored in 1 since χ is flip-top on C_t^s. □

Proof of Theorem 11.4. Now it is easy to see the existence of $HJ(k,t)$. We use the induction on t for fixed k.

Basis It is a trivial fact that $HJ(k,1) = 1$ for any k.

Inductive Step Assume $s := HJ(k,t-1)$ exists, i.e., $HJ(k,t-1) < \infty$. Define n as above which satisfies the cube lemma for the given k, t and s.

Claim For any k-coloring of C_t^n, there is a monochromatic line.

Proof. Let χ be a k-coloring of C_t^n, the cube lemma shows that χ is flip-top on a suitable Shelah s-space $L_1 \times L_2 \times \cdots \times L_s$. By definition, the derived coloring χ' is flip-top on C_t^s. By the definition of $s = HJ(k,t-1)$, we can apply the induction lemma to χ' to obtain that there is a monochromatic line L in C_t^s. The $\phi^{-1}(L)$ is a monochromatic line in C_t^n under the coloring χ, each point of which is formed from the corresponding point of L by adding constant coordinates. □

Let us remark a bit how Shelah's bound increases. Define a sequence of functions $\mathcal{N} = \{1, 2, \ldots\} \to \mathcal{N}$ as $f_1(n) = 2n$ and

$$f_{m+1}(n) = \underbrace{f_m \circ f_m \circ \cdots \circ f_m}_{n}(1).$$

Some small values of $f_m(n)$ are listed in the following table.

$f_m(n)$	$n=1$	$n=2$	$n=3$	$n=4$	$\cdots$	n
$f_1(n)$	2	4	6	8	$\cdots$	$2n$
$f_2(n)$	2	2^2	2^3	2^4	$\cdots$	2^n
$f_3(n)$	2	2^2	2^{2^2}	$2^{2^{2^2}}$	$\cdots$	$\left.2^{2^{\cdot^{\cdot^{\cdot^2}}}}\right\}n$
$f_4(n)$	2	2^2	$2^{2^{2^2}}$	$\left.2^{2^{\cdot^{\cdot^{\cdot^2}}}}\right\}f_4(3)$	$\cdots$	$\left.2^{2^{\cdot^{\cdot^{\cdot^2}}}}\right\}f_4(n-1)$
$f_5(n)$	2	2^2	$f_4(4)$	$f_4(f_4(4))$		

Table 12.2 Some values of $f_m(n)$

The function $f_3(n)$ is called a "tower", whose height is n. The function $f_4(n)$ grows much faster, which is called a *wowzer*, wow! Note that $f_4(n)$ is a tower of height $f_4(n-1)$. For example $f_4(3) = 65536$, which is the height of the tower of $f_4(4)$. The huge value of $f_4(f_4(4)-1)$ is just the height of the tower of $f_5(4)$. The *Ackerman function* $A(n)$ is defined as the diagonal value $f_n(n)$.

No reasonable upper bound for $w_k(t)$ or $w_2(t)$ has been found. Shelah's proof gives "wowzer" bound for $HJ(k,t)$ hence $w_k(t)$ because it iterates the cube lemma, which gives a "tower" bound by iterating the exponential functions. In fact, $HJ(k,t) \le f_4(c(k+t))$ for some constant c by noticing that n_s in the cube lemma is a tower of height around s. Even so, this proof for the van der Waerden theorem still reduces the original bound of van der Waerden greatly, who used double induction on k and t, even for $w(t) = w_2(t)$. The original upper bound of van der Waerden in Section 11.1 is a Ackerman function.

Shelah's proof for upper bound of $w(t)$ is celebrated, but it is far away from the truth from Gower's upper bound of tower of height 6. Graham offered 1000 USD for a proof or disproof of $w(t) < 2^{k^2}$, see Chung, Erdős and Graham (2000).

11.3 Szemerédi's Regularity Lemma

Let G be a graph with vertex set V and let X and Y be nonempty disjoint subsets of V. Denote by $e(X, Y)$ the number of edges between X and Y in G. The ratio

$$d_G(X, Y) = \frac{e(X, Y)}{|X||Y|}$$

is called the edge density of (X, Y). It can be seen as the probability that any pair (x, y) selected randomly from $X \times Y$ is an edge. It is easy to see $0 \le d_G(X, Y) \le 1$, and

$$d_G(X, Y) + d_{\overline{G}}(X, Y) = 1,$$

where $\overline{G}$ is the complement of G. We always simply denote $d_G(X, Y)$ by $d(X, Y)$ if the context is clear.

The density $d(X, Y)$ behaves in a fair continuous fashion.

Lemma 11.4 *Suppose X and Y are disjoint subsets of $V(G)$, and $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| > (1 - \eta)|X|$ and $|Y'| > (1 - \eta)|Y|$. Then*

$$|d(X', Y') - d(X, Y)| < 2\eta \quad \textit{and} \quad |d^2(X', Y') - d^2(X, Y)| < 4\eta.$$

Proof. Note that

$$\begin{aligned} 0 &\le e(X, Y) - e(X', Y') \\ &= e(X \setminus X', Y) + e(X, Y \setminus Y') - e(X \setminus X', Y \setminus Y') \\ &\le e(X \setminus X', Y) + e(X, Y \setminus Y') \\ &< 2\eta|X||Y|, \end{aligned}$$

so $d(X, Y) - d(X', Y') < 2\eta$. Similarly,

$$d(X', Y') - d(X, Y) = d_{\overline{G}}(X, Y) - d_{\overline{G}}(X', Y') < 2\eta.$$

Hence the assertion follows immediately. □

Let $\epsilon > 0$ be a real number. We say a disjoint pair (X, Y) is *ϵ-regular* if any $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| > \epsilon|X|$ and $|Y'| > \epsilon|Y|$ satisfy that

$$|d(X, Y) - d(X', Y')| < \epsilon.$$

A partition $V_0, V_1, \ldots, V_k$ of V is said to be *equitable* with exceptional set V_0 if $|V_1| = |V_2| = \cdots = |V_k|$. Furthermore, we say a partition $V_0, V_1, \ldots, V_k$ of V is ϵ-regular if the following two conditions hold:

(1) $|V_1| = |V_2| = \cdots = |V_k|$ and $|V_0| \le \epsilon |V|$;
(2) All but at most ϵk^2 pairs (V_i, V_j) with $1 \le i < j \le k$ are ϵ-regular.

Theorem 11.6 (Regularity Lemma) *For any real $\epsilon > 0$ and any integer $m \ge 1$, there exist $n_0 = n_0(\epsilon, m)$ and $M = M(\epsilon, m) > m$ such that every graph G on $n \ge n_0$ vertices has an ϵ-regular partition $V_0, V_1, \ldots, V_k$ with $m \le k \le M$.*

The above theorem is trivial for $|V(G)| \le M$ since a partition that each class contains at most one vertex is 0-regular. The crucial point for the lemma is that the number of classes of the partition can be bounded.

The defect form of Cauchy-Schwarz inequality is as follows, which can be applied to sequences that the average is greater than some local average.

Lemma 11.5 *Let d_i be reals and $s > t \ge 1$ be integers. If*

$$\frac{1}{s}\sum_{i=1}^{s} d_i = \frac{1}{t}\sum_{i=1}^{t} d_i + \delta,$$

then

$$\frac{1}{s}\sum_{i=1}^{s} d_i^2 \ge \left(\frac{1}{s}\sum_{i=1}^{s} d_i\right)^2 + \frac{t\delta^2}{s-t} \ge \left(\frac{1}{s}\sum_{i=1}^{s} d_i\right)^2 + \frac{t\delta^2}{s}.$$

Proof. Let $D_s = \frac{1}{s}\sum_{i=1}^{s} d_i$. The Cauchy-Schwarz inequality implies that

$$\sum_{i=1}^{s} d_i^2 = \sum_{i=1}^{t} d_i^2 + \sum_{i=t+1}^{s} d_i^2 \ge tD_t^2 + (s-t)\left(\frac{1}{s-t}\sum_{i=t+1}^{s} d_i\right)^2$$
$$= tD_t^2 + (s-t)\left(\frac{sD_s - tD_t}{s-t}\right)^2 = s\left(D_s^2 + \frac{t(D_s - D_t)^2}{s-t}\right),$$

so the assertion follows. □

Given an equitable partition $\mathcal{P} = \{V_0, V_1, \ldots, V_k\}$ with exceptional set V_0, define

$$q(\mathcal{P}) = \frac{1}{k^2}\sum_{1 \le i < j \le k} d^2(V_i, V_j).$$

It is easy to see that $0 \le q(\mathcal{P}) < 1/2$ since $d(V_i, V_j) \le 1$.

The function $q(\mathcal{P})$ is a cornerstone in the proof of the Regularity Lemma. We will show that if $\mathcal{P}$ is not ϵ-regular, then there is a partition $\mathcal{P}'$ with the new exceptional class a bit larger than the old one, but $q(\mathcal{P}') \ge q(\mathcal{P}) + \epsilon^5/2$. Continue this procedure until we obtain the partition as desired. The number of iterations is thus at most $1/\epsilon^5$ to guarantee the occurrence of an ϵ-regular partition.

Note that for $\epsilon < \epsilon'$, an ϵ-regular pair is also ϵ'-regular. Thus, without loss of generality, we may assume that $0 < \epsilon \le 1/2$ since if $\epsilon > 1/2$, then one can take $M(\epsilon, m) = M(1/2, m)$.

Lemma 11.6 *Let G be a graph with vertex set V, where $|V| = n$. Suppose $\mathcal{P} = \{V_0, V_1, \ldots, V_k\}$ is a partition of V with exceptional class V_0,*

$$|V_1| = |V_2| = \cdots = |V_k| = n_1 \ge 2^{3k+1}, \quad \text{and} \quad 2^k \ge 8/\epsilon^5.$$

If $\mathcal{P}$ is not ϵ-regular, then there is an equitable partition $\mathcal{P}' = \{V_0', V_1', \ldots, V_\ell'\}$ with exceptional class $V_0' \supseteq V_0$ and $\ell = k(4^k - 2^{k-1})$ such that

(1) $|V_0'| \le |V_0| + n/2^k$,
(2) $q(\mathcal{P}') \ge q(\mathcal{P}) + \epsilon^5/2$.

Proof.* For each pair (V_i, V_j) with $1 \le i < j \le k$, put the corresponding indices pair (i, j) into S if (V_i, V_j) is ϵ-regular, and put (i, j) into T otherwise. If $(i, j) \in S$, then set $V_{ij} = V_{ji} = \emptyset$. If $(i, j) \in T$, i.e., (V_i, V_j) is not ϵ-regular, then we can choose $V_{ij} \subseteq V_i$ and $V_{ji} \subseteq V_j$ with $|V_{ij}| > \epsilon n_1$, $|V_{ji}| > \epsilon n_1$ such that

$$|d(V_{ij}, V_{ji}) - d(V_i, V_j)| \ge \epsilon.$$

Fix i for $1 \le i \le k$, consider an equivalence relation $\equiv$ on V_i as $x \equiv y$ if and only if both x and y belong to the same subset V_{ij} for every $j \ne i$. (Clearly, there are vertices that may not lie in any such V_{ij}.) Thus V_i has at most 2^{k-1} equivalent classes. Set $d = \lfloor n_1/4^k \rfloor$. Clearly,

$$d \ge 2^{k+1}, \quad \text{and} \quad 4^k d \le n_1 < 4^k(d+1).$$

Cut V_i into pairwise disjoint d-subsets such that each d-subset belongs to some equivalent class of V_i. Denote z by the maximal number of these d-subsets that one can take. It follows that

$$zd + 2^{k-1}(d-1) \ge n_1 \ge 4^k d,$$

yielding $z \ge 4^k - 2^{k-1}$. Set $H = 4^k - 2^{k-1}$. Take *exactly* H such d-subsets and put the remainder into the "rubbish bin" to get a new exceptional set V_0'. Label all these d-subsets in V_i as $D_{i1}, D_{i2}, \ldots, D_{iH}$. Set

$$V_0' = V_0 \cup \left[\cup_{i=1}^k \left(V_i \setminus \cup_{h=1}^H D_{ih} \right) \right].$$

Note that $|V_0'| = |V_0| + k(n_1 - Hd)$, and

$$Hd \ge \left(4^k - 2^{k-1}\right)\left(\frac{n_1}{4^k} - 1\right) > n_1 - \frac{n_1}{2^{k+1}} - 4^k \ge n_1 - \frac{n_1}{2^k}$$

as $n_1 \ge 2^{3k+1}$. Hence $n_1 - Hd < n_1/2^k$ and

$$|V_0'| \le |V_0| + k(n_1 - Hd) \le |V_0| + n/2^k.$$

Rename D_{ih} as V'_j for $1 \le j \le \ell$, where $\ell = kH$. Set $\mathcal{P}' = \{V'_0, V'_1, \dots, V'_\ell\}$. All that remains is to show $q(\mathcal{P}') \ge q(\mathcal{P}) + \epsilon^5/2$.

For $i, j = 1, 2, \dots, k$, set

$$\overline{V_i} = \bigcup_{h=1}^{H} D_{ih} \quad \text{and} \quad \overline{V_{ij}} = \bigcup_{D_{ih} \subseteq V_{ij}} D_{ih}.$$

Set $\overline{\mathcal{P}} = \{V'_0, \overline{V_1}, \dots, \overline{V_k}\}$ with exceptional class V'_0.

Claim 1 $q(\overline{\mathcal{P}}) \ge q(\mathcal{P}) - \epsilon^5/4$.

Proof. For any pair (V_i, V_j),

$$\frac{|V_i \setminus \overline{V_i}|}{|V_i|} = \frac{n_1 - Hd}{n_1} < \frac{1}{2^k} \le \frac{\epsilon^5}{8}. \tag{11.1}$$

By Lemma 11.4,

$$|d(\overline{V_i}, \overline{V_j}) - d(V_i, V_j)| \le \frac{\epsilon^5}{4}. \tag{11.2}$$

Consequently, $|d^2(\overline{V_i}, \overline{V_j}) - d^2(V_i, V_j)| \le \frac{\epsilon^5}{2}$, and so $d^2(\overline{V_i}, \overline{V_j}) \ge d^2(V_i, V_j) - \epsilon^5/2$, which implies that $q(\overline{\mathcal{P}}) \ge q(\mathcal{P}) - \epsilon^5/4$ as claimed. □

To illustrate what can be obtained in the proof, we shall use $A \subset\approx B$ to represent for "$A \subseteq B$ and $|A| = (1 + o(1))|B|$", and $A \subset\ll B$ for "$A \subseteq B$ and $|A| = o(|B|)$". For a pair $(i, j) \in T$, we have

$$\begin{array}{llll} (\overline{V_{ij}}, \overline{V_{ji}}) \subset\approx (V_{ij}, V_{ji}) & \subset\ll & & (V_i, V_j) \\ (\overline{V_{ij}}, \overline{V_{ji}}) & \subset\ll & (\overline{V_i}, \overline{V_j}) \subset\approx & (V_i, V_j). \end{array}$$

So for $(i, j) \in T$, it is expected that $|d(\overline{V_{ij}}, \overline{V_{ji}}) - d(\overline{V_i}, \overline{V_j})|$ is almost as large as $|d(V_{ij}, V_{ji}) - d(V_i, V_j)|$.

Claim 2 If $(i, j) \in T$, then $|d(\overline{V_{ij}}, \overline{V_{ji}}) - d(\overline{V_i}, \overline{V_j})| > \frac{15}{16}\epsilon$.

Proof. Note that $V_{ij} \setminus \overline{V_{ij}} \subseteq V_i \setminus \overline{V_i}$, so from (11.1),

$$\frac{|V_{ij} \setminus \overline{V_{ij}}|}{|V_{ij}|} \le \frac{|V_i \setminus \overline{V_i}|}{|V_i|} \frac{|V_i|}{|V_{ij}|} \le \frac{\epsilon^5}{8} \cdot \frac{1}{\epsilon} = \frac{\epsilon^4}{8}, \tag{11.3}$$

which and Lemma 11.4 give

$$|d(\overline{V_{ij}}, \overline{V_{ji}}) - d(V_{ij}, V_{ji})| \le \frac{\epsilon^4}{4}. \tag{11.4}$$

The definition of V_{ij}, the bounds (11.2) and (11.4) with the assumption that $0 < \epsilon \le 1/2$ yield

$$\begin{aligned}
&|d(\overline{V_{ij}}, \overline{V_{ji}}) - d(\overline{V_i}, \overline{V_j})| \\
\geq &|d(V_{ij}, V_{ji}) - d(V_i, V_j)| - |d(\overline{V_{ij}}, \overline{V_{ji}}) - d(V_{ij}, V_{ji})| \\
&- |d(\overline{V_i}, \overline{V_j}) - d(V_i, V_j)| \\
\geq &\epsilon - \frac{\epsilon^4}{4} - \frac{\epsilon^5}{4} \geq \frac{15}{16}\epsilon,
\end{aligned}$$

as claimed. □

Let us return to the partition $\mathcal{P}'$ in which each class is a d-subset D_{ih} except the class V_0'. Note that $\overline{V_i} = \cup_{h=1}^{H} D_{ih}$, we obtain that

$$d(\overline{V_i}, \overline{V_j}) = \frac{e(\overline{V_i}, \overline{V_j})}{|\overline{V_i}||\overline{V_j}|} = \frac{\sum_{1 \leq h,h' \leq H} e(D_{ih}, D_{jh'})}{H^2 |D_{ih}||D_{jh'}|} = \frac{1}{H^2} \sum_{1 \leq h,h' \leq H} d(D_{ih}, D_{jh'})$$

since $|\overline{V_i}| = |\overline{V_j}| = Hd$. Set

$$A(i, j) = \frac{1}{H^2} \sum_{1 \leq h,h' \leq H} d^2(D_{ih}, D_{jh'}).$$

For any pair (V_i, V_j), from Cauchy-Schwarz inequality, we have

$$A(i, j) \geq \left(\frac{1}{H^2} \sum_{1 \leq h,h' \leq H} d(D_{ih}, D_{jh'}) \right)^2 = d^2(\overline{V_i}, \overline{V_j}). \tag{11.5}$$

If $(i, j) \in T$, we have some gain. Let $R = R(i, j)$ be the set of indices (h, h') such that $D_{ih} \subseteq \overline{V_{ij}}$ and $D_{jh'} \subseteq \overline{V_{ji}}$. Then

$$d(\overline{V_{ij}}, \overline{V_{ji}}) = \frac{1}{|R|} \sum_{(h,h') \in R} d(D_{ih}, D_{jh'}).$$

So for $(i, j) \in T$, from Lemma 11.5 and Claim 2,

$$\begin{aligned}
A(i, j) &\geq d^2(\overline{V_i}, \overline{V_j}) + \frac{|R|}{H^2} \left(d(\overline{V_i}, \overline{V_j}) - d(\overline{V_{ij}}, \overline{V_{ji}}) \right)^2 \\
&\geq d^2(\overline{V_i}, \overline{V_j}) + \frac{|R|}{H^2} \left(\frac{15\epsilon}{16} \right)^2. \qquad (11.6)
\end{aligned}$$

Let H_{ij} be the number of $D_{ih} \in \overline{V_{ij}}$. Then by (11.1),

$$\begin{aligned}
H_{ij} d = |V_{ij}| - |V_{ij} \setminus \overline{V_{ij}}| &\geq |V_{ij}| - |V_i \setminus \overline{V_i}| \\
&\geq (\epsilon - \epsilon^5/8)|V_i| \geq (1 - 2^{-7})\epsilon |V_i|.
\end{aligned}$$

Note that $|R| = H_{ij} H_{ji}$ and $|V_i||V_j| \geq (Hd)^2$, so for $(i, j) \in T$,

$$\frac{|R|}{H^2} = \frac{H_{ij}H_{ji}d^2}{H^2d^2} \geq \frac{H_{ij}H_{ji}d^2}{|V_i||V_j|} \geq \left((1-2^{-7})\epsilon\right)^2,$$

and thus recall (11.6) we obtain that

$$A(i,j) \geq d^2(\overline{V_i},\overline{V_j}) + \left((1-2^{-7})\epsilon\right)^2\left(\frac{15\epsilon}{16}\right)^2 \geq d^2(\overline{V_i},\overline{V_j}) + \frac{3}{4}\epsilon^4. \tag{11.7}$$

Note that $\ell = kH$ and so we have

$$\begin{aligned} q(\mathcal{P}') &= \frac{1}{\ell^2}\sum_{1\leq t<s\leq \ell} d^2(V'_t, V'_s) \\ &\geq \frac{1}{k^2H^2}\sum_{1\leq i<j\leq k}\sum_{1\leq h,h'\leq H} d^2(D_{ih}, D_{jh'}) = \frac{1}{k^2}\sum_{1\leq i<j\leq k} A(i,j), \end{aligned} \tag{11.8}$$

where the summands of form $d^2(D_{ih}, D_{jh'})$ with $i = j$ are ignored in the inequality. Now combining (11.5), (11.7) and (11.8) we obtain that

$$\begin{aligned} q(\mathcal{P}') &\geq \frac{1}{k^2}\left(\sum_{(i,j)\in S} A(i,j) + \sum_{(i,j)\in T} A(i,j)\right) \\ &\geq \frac{1}{k^2}\left(\sum_{(i,j)\in S} d^2(\overline{V_i},\overline{V_j}) + \sum_{(i,j)\in T}\left(d^2(\overline{V_i},\overline{V_j}) + \frac{3}{4}\epsilon^4\right)\right) \\ &= q(\overline{\mathcal{P}}) + \frac{3|T|}{4k^2}\epsilon^4 \geq q(\overline{\mathcal{P}}) + \frac{3}{4}\epsilon^5, \end{aligned}$$

where we used the fact that $|T| \geq \epsilon k^2$ as $\mathcal{P}$ is not ϵ-regular. From this fact and Claim 1, we have

$$q(\mathcal{P}') \geq q(\mathcal{P}) - \frac{\epsilon^5}{4} + \frac{3}{4}\epsilon^5 = q(\mathcal{P}) + \frac{\epsilon^5}{2}.$$

This completes the proof of Lemma 11.6. □

Now, we give the proof for Theorem 11.6.

Proof of Theorem 11.6. We shall use Lemma 11.6 repeatedly by showing that at most $t = \lfloor \epsilon^{-5} \rfloor$ iterations will yield a required partition. Let k_0 be an integer such that $k_0 \geq m$ and $2^{-k_0} \leq \epsilon^5/8$, and define $k_{i+1} = k_i(4^{k_i} - 2^{k_i-1})$. Set $M_i = k_i 4^{k_i}$ and $M = k_t$.

Let G be a graph of order n. We may assume that $n > M$ since otherwise the partition with each class being a singleton with empty exceptional set will do. Let $\mathcal{P}_0 = \{V_0^{(0)}, V_1^{(0)}, \ldots, V_{k_0}^{(0)}\}$ be an equitable partition with $|V_1^{(0)}| = \cdots = |V_{k_0}^{(0)}| = \lfloor n/k_0 \rfloor$. So

$$|V_0^{(0)}| < k_0 \leq M_0/4^{k_0} < \frac{\epsilon n}{2}.$$

If $\mathcal{P}_0$ is ϵ-regular, then we are done. Otherwise, by Lemma 11.6, there is an equitable partition $\mathcal{P}_1 = \{V_0^{(1)}, V_1^{(1)}, \ldots, V_{k_1}^{(1)}\}$ such that

$$|V_0^{(1)}| \le |V_0^{(0)}| + n/2^{k_0} < \epsilon n, \quad \text{and} \quad q(\mathcal{P}_1) \ge q(\mathcal{P}) + \epsilon^5/2.$$

If $\mathcal{P}_1$ is not ϵ-regular yet, then we continue the procedure to obtain $\mathcal{P}_2$. Suppose we have obtained an equitable partition $\mathcal{P}_j$ with exceptional class $V_0^{(j)}$ in the jth step. Thus,

$$|V_0^{(j)}| \le |V_0^{(0)}| + n(2^{-k_0} + 2^{-k_1} + \cdots) < \epsilon n,$$

and $q(\mathcal{P}_j) \ge q(\mathcal{P}_0) + j\frac{\epsilon^5}{2}$. It follows that $j \le t$ since $q(\mathcal{P}_j) < 1/2$. This completes the proof of Theorem 11.6. □

The bound on $M(\epsilon, m)$ given in the Regularity Lemma is enormous, it is a tower of the height up to ϵ^{-5} since the number of iterations in the proof. This seems to be very bad at the first glance. However, a celebrated result of Gowers (1997) proved that $M(\epsilon, 2)$ grows at least as a such tower of height about $\epsilon^{-1/16}$. His argument is powerful. Subsequently, Conlon and Fox (2012) estimated the number of irregular pairs in the Regularity Lemma, and Moshkovitz and Shapira (2016) gave a simpler proof of a tower-type lower bound. By using the *mean square density*, i.e., $q(\mathcal{P}) = \sum_{i,j=1}^{k} \frac{|V_i||V_j|}{|V|^2} d^2(V_i, V_j)$, Fox and Lovász (2017) showed that the bound on the number of parts is at most a tower of height at most $2 + \epsilon^{-2}/16$. They also gave a tight lower bound on the tower height in the Regularity Lemma, which addresses a question of Gowers.

The size of the exceptional class may be larger than that of normal ones. To see this, if m is much larger than $1/\epsilon$, then the size of normal classes is around n/k, which is much less than ϵn.

For graph $G = (V, E)$, we say that a partition $V_1, \ldots, V_k$ of V is an equipartition if $|V_i|$ and $|V_j|$ differ by no more than 1 for all $1 \le i < j \le k$. One of reformulations of the Regularity Lemma is as follows.

Theorem 11.7 *For any $\epsilon > 0$ and integer $m \ge 1$, there exist $n_0 = n_0(\epsilon, m)$ and $M = M(\epsilon, m) > m$ such that every graph G on $n \ge n_0$ vertices has an equipartition $V_1, V_2, \ldots, V_k$ with $m \le k \le M$, in which all but at most ϵk^2 pairs (V_i, V_j), $1 \le i < j \le k$, are ϵ-regular.*

Proof. By Theorem 11.6, for any real $\epsilon > 0$ and $m \ge 1$, there exist $n_0 = n_0(\epsilon, m)$ and $M = M(\epsilon, m) > m$ such that every graph G on $n \ge n_0$ vertices has an $\frac{\epsilon^2}{4}$-regular partition $\mathcal{P} = \{V_0, V_1, \ldots, V_k\}$ with $m \le k \le M$. Note that $|V_0| < \frac{\epsilon^2}{4}n$, we have $\lfloor (1 - \epsilon^2/4)n/k \rfloor \le |V_i| \le n/k$. Partition V_0 into k classes $B_1, B_2, \ldots, B_k$ such that $|B_i| = \lfloor |V_0|/k \rfloor$ or $|B_i| = \lceil |V_0|/k \rceil$. Set

$$V_i' = V_i \cup B_i.$$

Thus the sizes of any V_i' and V_j' differ at most by one. We aim to prove that the partition $\mathcal{P}' = \{V_1', V_2', \ldots, V_k'\}$ is as desired. It suffices to verify that if a pair (V_i, V_j) is $\frac{\epsilon^2}{4}$-regular, then (V_i', V_j') is ϵ-regular.

Indeed, suppose $X_i' \subseteq V_i'$ and $X_j' \subseteq V_j'$ with $|X_i'| > \epsilon|V_i'|$ and $|X_i'| > \epsilon|V_j'|$, respectively. Set $X_i = X_i' \setminus B_i \subseteq V_i$. Note that $|V_i'| = \lfloor n/k \rfloor$ or $|V_i'| = \lceil n/k \rceil$, so for sufficiently large n,

$$\frac{|X_i|}{|X_i'|} \geq 1 - \frac{|B_i|}{|X_i'|} \geq 1 - \frac{\lceil \epsilon^2 n/(4k) \rceil}{\epsilon \lfloor n/k \rfloor} > 1 - \frac{\epsilon}{3}.$$

Hence by Lemma 11.4,

$$|d(X_i', X_j') - d(X_i, X_j)| \leq \frac{2\epsilon}{3}. \tag{11.9}$$

The sizes of V_i and V_i' are close. Indeed,

$$\frac{|V_i|}{|V_i'|} \geq \frac{\lfloor (1 - \epsilon^2/4)n/k \rfloor}{\lceil n/k \rceil} > 1 - \frac{\epsilon^2}{2},$$

and thus

$$|d(V_i, V_j) - d(V_i', V_j')| \leq \epsilon^2. \tag{11.10}$$

Also, since (V_i, V_j) is $\frac{\epsilon^2}{4}$-regular and

$$\frac{|X_i|}{|V_i|} \geq \frac{|X_i|}{|X_i'|} \frac{|X_i'|}{|V_i'|} \geq \left(1 - \frac{\epsilon}{3}\right) \epsilon > \frac{\epsilon^2}{4},$$

it follows that

$$|d(X_i, X_j) - d(V_i, V_j)| \leq \frac{\epsilon^2}{4}. \tag{11.11}$$

Consequently, by inequalities (11.9), (11.10) and (11.11),

$$\begin{aligned}
&|d(X_i', X_j') - d(V_i', V_j')| \\
&\leq |d(X_i', X_j') - d(X_i, X_j)| + |d(X_i, X_j) - d(V_i, V_j)| \\
&\quad + |d(V_i, V_j) - d(V_i', V_j')| \\
&\leq \frac{2\epsilon}{3} + \epsilon^2 + \frac{\epsilon^2}{4} < \epsilon.
\end{aligned}$$

This completes the proof of Theorem 11.7. □

A similar proof yields another formulation of the Regularity Lemma as following.

Theorem 11.8 *For any $\epsilon > 0$ and integer $m \geq 1$, there exist $n_0 = n_0(\epsilon, m)$ and $M = M(\epsilon, m) > m$ such that every graph G on $n \geq n_0$ vertices has a partition $V_0, V_1, \ldots, V_k$ with $|V_0| \leq k - 1$, $|V_1| = |V_2| = \cdots = |V_k|$, and $m \leq k \leq M$, in which all but at most ϵk^2 pairs (V_i, V_j), $1 \leq i < j \leq k$, are ϵ-regular.*

The following is the multicolor Regularity Lemma.

Theorem 11.9 (Multicolor Regularity Lemma) *For any $\epsilon > 0$ and any positive integers m and s, there exist $n_0 = n_0(\epsilon, m, s)$ and $M = M(\epsilon, m, s)$ such that if the edges of a graph G on $n \geq n_0$ vertices are colored in s colors, all monochromatic graphs have a same partition $V_0, V_1, \ldots, V_k$ that is ϵ-regular with exceptional set V_0 and $m \leq k \leq M$.*

Proof. Using the original proof, but modify the definition of $q(\mathcal{P})$ by summing the indices over all colors: for a partition $\mathcal{P} = \{V_0, V_1, \ldots, V_k\}$ of $V(G)$ with exceptional set V_0, let

$$q(\mathcal{P}) = \frac{1}{k^2} \sum_{\ell=1}^{s} \sum_{1 \leq i < j \leq k} d_\ell^2(V_i, V_j),$$

where $d_\ell(V_i, V_j)$ is the edge density in the ℓ-th color. □

Similar to Theorem 11.7, we have the following multicolor Regularity Lemma.

Theorem 11.10 *For any $\epsilon > 0$ and integer $m \geq 1$, there exist $n_0 = n_0(\epsilon, m, s)$ and $M = M(\epsilon, m, s)$ such that if the edges of a graph G on $n \geq n_0$ vertices are colored in s colors, all monochromatic graphs have a partition $V_1, \ldots, V_k$ with $|V_1| \leq |V_2| \leq \cdots \leq |V_k| \leq |V_1| + 1$, and $m \leq k \leq M$, in which all but at most ϵk^2 pairs (V_i, V_j), $1 \leq i < j \leq k$, are ϵ-regular.*

In some applications, the following degree form of the Regularity Lemma is more applicable.

Theorem 11.11 *For any $\epsilon > 0$, there exist an $M = M(\epsilon)$ such that for any graph $G = (V, E)$ and any $d \in [0, 1]$, there exists a partition $V_0, V_1, \ldots, V_k$ of V with $k \leq M$, $V_0 \leq \epsilon|V|$, each V_i has the same size $m \leq \epsilon|V|$, and there exists a subgraph $G' \subseteq G$ with the following properties:*

(1) $\deg_{G'}(v) > \deg_G(v) - (d + \epsilon)|V|$ *for all* $v \in V$,

(2) $e(G'(V_i)) = 0$ *for all* $i \geq 1$,

(3) all pairs $G'(V_i, V_j)$, $1 \leq i < j \leq k$, are ϵ-regular, each with a density either 0 *or greater than d.*

11.4 Two Applications

Recall that the Ramsey-Turán number $RT(n; K_4, o(n))$ is the maximum number of edges among all K_4-free graph whose independence number is $o(n)$. We know that $RT(n; K_4, o(n)) > \frac{1}{8}n^2 + o(n^2)$ by Bollobás and Erdős (1976). Now, let us have an application of the regularity lemma on the upper bound of $RT(n; K_4, o(n))$ given by Szemerédi (1972).

Theorem 11.12 *We have $RT(n; K_4, o(n)) < \frac{1}{8}n^2 + o(n^2)$.*

Proof. Let G be a graph on n vertices satisfying $e(G) > (1/8 + 4\epsilon)n^2$ where $\epsilon > 0$ is sufficiently small and n is large. Let $d = 2\epsilon$. We apply Theorem 11.11 to obtain an $M = M(\epsilon)$ and a partition $V_0, V_1, \ldots, V_k$ of V with $k \le M$, $V_0 \le \epsilon|V|$, each V_i has the same size $m \le \epsilon|V|$, and there exists a subgraph $G' \subseteq G$ with the following properties:

(1) $\deg_{G'}(v) > \deg_G(v) - (d + \epsilon)|V|$ for all $v \in V$,

(2) $e(G'(V_i)) = 0$ for all $i \ge 1$,

(3) all pairs $G'(V_i, V_j)$, $1 \le i < j \le k$, are ϵ-regular, each with a density either 0 or greater than d.

We assume that

$$\alpha(G) \le \frac{\epsilon^2}{M} n - 1, \quad \text{and} \quad n \ge \frac{M}{\epsilon}.$$

We aim to show that $K_4 \subset G$.

Let $G'' = G' - V_0$. We have $e(G'') > (1/8 + \epsilon)n^2$. Also note that

$$\alpha(G) < \epsilon^2 \left(\frac{n}{M} - 1\right) \le \epsilon^2 \left(\frac{n}{k} - 1\right) < \epsilon^2 m.$$

Case 1. The reduced graph contains more than $k^2/4$ edges.

For this case, by Turan's theorem, the reduced graph contains a triangle which corresponding to sets V_1, V_2, V_3 without loss of generality. By Lemma 11.7, we obtain a subset $V_1' \subseteq V_1$ with $|V_1'| \ge (1 - 2\epsilon)|V_1|$ such that each vertex in V_1' is adjacent to at least $(d - \epsilon)m$ vertices in V_2 and V_3 respectively. Fix a vertex $v_1 \in V_1'$, we consider the pair $(N(v_1) \cap V_2, N(v_1) \cap V_3)$. We again apply Lemma 11.7 to obtain a vertex $v_2 \subset N(v_1) \cap V_2$ such that $|N(v_2) \cap N(v_1) \cap V_3| \ge \epsilon^2 m$. Thus $N(v_2) \cap N(v_1) \cap V_3$ contains an edge by noting that $\alpha(G) < \epsilon^2 m$, and so we can get a $K_4 \subset G$ as desired.

Case 2. The reduced graph contains at most $k^2/4$ edges.

Note that

$$\sum_{1 \le i < j \le k} d(V_i, V_j) = \frac{e(G'')}{m^2} \ge \frac{e(G'')k^2}{n^2} > \left(\frac{1}{8} + \epsilon\right) k^2.$$

For this case, their average density of pairs is greater than $1/2 + 4\epsilon$. Thus, at least one pair, say (V_1, V_2), has a density greater than $1/2 + 4\epsilon$. A similar argument by using Lemma 11.7 we can show that the subgraph induced by (V_1, V_2) together with the edges inside the two clusters contains a copy of K_4. □

In the following, let us have another application of the regularity lemma.

A family $\mathcal{G} = \{G_n\}$, where G_n is a graph of order n, is said to be *Ramsey linear* if there exists a constant $c = c(\mathcal{G}) > 0$ such that $r(G_n) \le cn$ for any G_n in $\mathcal{G}$, where $r(G_n) = r(G_n, G_n)$.

For dense graph G_n, $r(G_n)$ may tend to grow exponentially in n. For example, the extreme case $r(K_n)$ is lying roughly between $2^{n/2}$ and 4^n as discussed in previous

Chapters. However, for relatively sparse graphs, $r(G_n)$ grows much more modestly. A special class which has been investigated from this aspect is the class of graphs G_n with maximum degree $\Delta(G_n) \le \Delta$, where Δ is fixed. Denote

$$\mathcal{G}(\Delta) = \{G_n : \ n \ge 1 \text{ and } \ \Delta(G_n) \le \Delta\},$$

and

$$\mathcal{D}(d) = \{G_n : \ n \ge 1 \text{ and } G_n \text{ is } d\text{-degenerate}\}.$$

Conjecture 11.1 (Burr-Erdős, 1975) For any fixed positive integers Δ and d,

(1) the family $\mathcal{G}(\Delta)$ is Ramsey linear.

(2) the family $\mathcal{D}(d)$ is Ramsey linear.

A well known application of the Regularity Lemma by Chvátal, Rödl, Szemerédi and Trotter (1983) tells us that $\mathcal{G}(\Delta)$ is Ramsey linear. Given a graph G, let $N_G(x)$ be the neighborhood of x in G and for $X, Y \subseteq V(G)$, $N_Y(X) = (\cup_{x\in X} N_G(x)) \cap Y$. A graph G of order n is called p-arrangeable if there exists an ordering $v_1, v_2, \dots, v_n$ of the vertices of G such that for each $1 \le i \le n-1$, $|N_{L_i}(N_{R_i}(v_i))| \le p$, where $L_i = \{v_1, v_2, \dots, v_i\}$ and $R_i = \{v_{i+1}, v_{i+2}, \dots, v_n\}$. Set a family of graphs as

$$\mathcal{G}_p = \{G_n \mid G_n \text{ is } p\text{-arrangeable}\}.$$

Chen and Schelp (1993) showed that the family of p-arrangeable graphs is also Ramsey linear, and so does the family of planar graphs. Recently, a celebrate result of Lee (2017) confirms the second conjecture of Burr and Erdős (1975), in which one of the main ingredients of the proof is the dependent random choice introduced in Chapter 9.

In this section, we will mainly give the proof of the first conjecture by Chvátal, Rödl, Szemerédi and Trotter (1983).

Theorem 11.13 *For any fixed integer $\Delta \ge 1$, $\mathcal{G}(\Delta)$ is Ramsey linear.*

Before giving a proof for the above result, let us have some properties of an ϵ-regular pair as follows.

Lemma 11.7 *Let (A, B) be an ϵ-regular pair of density $d \in (0, 1]$. If $Y \subseteq B$ with $|Y| \ge \epsilon|B|$, then there exists a subset $A' \subseteq A$ with $|A'| \ge (1-\epsilon)|A|$ such that each vertex in A' is adjacent to at least $(d-\epsilon)|Y|$ vertices in Y.*

Proof. Let X be the set of vertices with fewer than $(d-\epsilon)|Y|$ neighbors in Y. Thus $e(X, Y) < (d-\epsilon)|X||Y|$, which implies that $d(X, Y) < d - \epsilon$. Since (A, B) is ϵ-regular, we have that $|X| \le \epsilon|A|$. □

Lemma 11.8 *Let (A, B) be an ϵ-regular pair of density $d \in (0, 1]$. If $X \subseteq A$ and $Y \subseteq B$ with $|X| \ge \alpha|A|$ and $|Y| \ge \alpha|B|$ for some $\alpha > \epsilon$, then (X, Y) is ϵ'-regular satisfying $|d(A, B) - d(X, Y)| < \epsilon$, where $\epsilon' = \max\{\frac{\epsilon}{\alpha}, 2\epsilon\}$.*

Proof. Let $X' \subseteq X$ and $Y' \subseteq Y$. If $|X'| > \epsilon'|X|$ and $|Y'| > \epsilon'|Y|$, then $|X'| > \epsilon|A|$ and $|Y'| > \epsilon|B|$. Hence the assertion follows from the assumption that (A, B) is ϵ-regular. □

Proof of Theorem 11.13. Let $\Delta \geq 1$ be fixed. Let $m \geq r(\Delta+1, \Delta+1)$ and $0 < \epsilon \leq \frac{1}{6m}$ such that

$$\left((1/2-\epsilon)^{\Delta} - \Delta\epsilon\right) m \geq 1 \quad \text{hence} \quad (1/2-\epsilon)^{\Delta} - \Delta\epsilon > \epsilon.$$

Let $M = M(\epsilon, 2m) > 2m$ be the integer determined by ϵ and $2m$ in Theorem 11.7. Finally, let $c = mM$, which is a constant depends only on Δ. We shall show that $r(G) \leq cn$ for any graph G of order n with $\Delta(G) \leq \Delta$.

Consider a red/blue edge coloring of the complete graph K_{cn} with vertex set V. Let R be the graph spanned by all red edges on V.

From Theorem 11.7, there is a partition $\{V_1, V_2, \ldots, V_k\}$ of V with $||V_i| - |V_j|| \leq 1$ and $2m \leq k \leq M$ such that all but at most ϵk^2 pairs (V_i, V_j), $1 \leq i < j \leq k$, are ϵ-regular. Clearly, $|V_i| \geq mn$ for $1 \leq i \leq k$.

Let F be the reduced graph on vertex set $\{1, 2, \ldots, k\}$, in which (i, j) is an edge if and only if (V_i, V_j) is ϵ-regular. Note that the number of edges of F is at least

$$\binom{k}{2} - \epsilon k^2 = \binom{k}{2} - \frac{k^2}{6m} \geq \frac{m-1}{m}\frac{k^2}{2} > t_{m-1}(k).$$

By Turán's theorem, i.e., Theorem 7.2, the subgraph of F spanned by all red edges contains a complete graph K_m. Color an edge (i, j) of K_m red if $d_R(V_i, V_j) \geq 1/2$, and blue otherwise. Since $m \geq r(\Delta+1, \Delta+1)$, this will yield a monochromatic $K_{\Delta+1}$. Without loss of generality, suppose that $K_{\Delta+1}$ is red for otherwise we consider the complement graph $\overline{R}$.

Relabeling the sets of the partition if necessary, we may assume that the vertex set of $K_{\Delta+1}$ is $\{1, 2, \ldots, \Delta+1\}$. Thus we have

(i) all pairs (V_i, V_j) for $1 \leq i < j \leq \Delta+1$ are ϵ-regular, and
(ii) $d_R(V_i, V_j) \geq 1/2$.

Assume that $V(G) = \{u_1, u_2, \ldots, u_n\}$. Since the chromatic number of G is at most $\Delta+1$, we can define a map $\chi\colon \{1, \ldots, n\} \to \{1, \ldots, \Delta+1\}$, where $\chi(i)$ is the color of the vertex u_i, such that $\chi(i) \neq \chi(j)$ if $u_i u_j$ is an edge of G. In order to prove that the red graph R contains G as a subgraph, we will define an embedding $\varphi : u_i \to v_i$ for $1 \leq i \leq n$ such that $v_i \in V_{\chi(i)}$ and $v_i v_j$ is an edge of R whenever $u_i u_j$ is an edge of G.

Our plan is to choose the vertices $v_1, \ldots, v_n$ inductively. Throughout the induction, we shall have a *target set* $Y_i \subseteq V_{\chi(i)}$ assigned to each i. Initially, Y_i is the entire $V_{\chi(i)}$. As the embedding proceeds, Y_i will get smaller and smaller since some vertices will be deleted. However, for each $i = 1, \ldots, \Delta+1$, the number that $V_{\chi(i)}$ will have some vertices deleted is at most Δ times, and each time there are majority of vertices remaining for us. This guarantees that each Y_i will not get too small to make this approach work.

Let us begin the initial step. Set

$$Y_1^0 = V_{\chi(1)},\ Y_2^0 = V_{\chi(2)}, \ldots, Y_n^0 = V_{\chi(n)}.$$

Note that Y_i^0 and Y_j^0 are not necessarily distinct sets.

We first consider $u_1 \in V(G)$ with degree $d_G(u_1) = d$. Denote its neighbors by $u_{\alpha_1}, \ldots, u_{\beta_1}$. Since (Y_1^0, Y_j^0) is ϵ-regular for $j = \alpha_1, \ldots, \beta_1$, by using Lemma 11.7 repeatedly, there exists a subset $Y_1^1 \subseteq Y_1^0$ with

$$|Y_1^1| \geq (1 - d\epsilon)|Y_1^0| \geq n$$

such that each vertex in Y_1^1 has at least $(1/2 - \epsilon)|Y_j^0|$ neighbors in Y_j^0. Choose a vertex v_1 from Y_1^1 arbitrarily. For $j = \alpha_1, \ldots, \beta_1$, define Y_j^1 as the neighborhood of v_1 in Y_j^0. For $j \geq 2, j \neq \alpha_1, \ldots, \beta_1$, define $Y_j^1 = Y_j^0$, that is, no vertices are deleted from such Y_j^0. In this step, v_1 has been chosen which is completely adjacent to Y_j^1 in R whenever u_1 and u_j are adjacent in G. See Fig. 13.1 for $i = 1$.

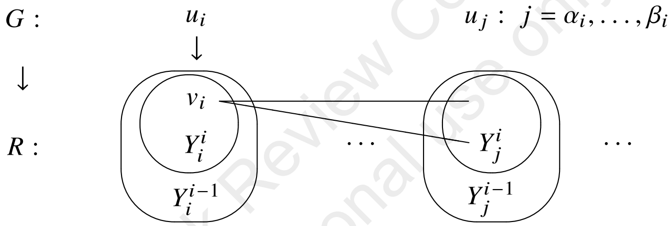

Fig. 13.1 Embed G into R

Generally, we consider u_i and its neighbors. We will choose v_i from Y_i^{i-1}. Suppose that u_i has d_1 neighbors in $\{u_1, \ldots, u_{i-1}\}$, and d_2 neighbors, say $u_{\alpha_i}, \ldots, u_{\beta_i}$, in $\{u_{i+1}, \ldots, u_n\}$. It is clear that $d_1 + d_2 \leq \Delta$, and

$$|Y_i^{i-1}| \geq (1/2 - \epsilon)^{d_1}|Y_i^0|,$$

i.e., the current set Y_i^{i-1} are obtained from Y_i^0 by deleting some vertices d_1 times before this step. By using Lemma 11.7 repeatedly again, we have a subset $Y_i^i \subseteq Y_i^{i-1}$ with

$$|Y_i^i| \geq |Y_i^{i-1}| - d_2\epsilon|Y_i^0|$$

so that each vertex in Y_i^i has at least $(1/2 - \epsilon)|Y_j^{i-1}|$ neighbors in Y_j^{i-1}, where $j = \alpha_i, \ldots, \beta_i$. Note that

$$\begin{aligned}|Y_i^i| &\geq |Y_i^{i-1}| - d_2\epsilon|Y_i^0| \geq \left((1/2 - \epsilon)^{d_1} - d_2\epsilon\right)|Y_i^0| \\ &\geq \left((1/2 - \epsilon)^{\Delta} - \Delta\epsilon\right)|Y_i^0| \geq n,\end{aligned}$$

thus we can choose a vertex v_i from Y_i^i, which is distinct from $v_1, \ldots, v_{i-1}$ that have been chosen before this step. For $j = \alpha_i, \ldots, \beta_i$, define Y_j^i as the neighborhood of v_i in Y_j^{i-1}. For $j \geq i+1, j \neq \alpha_i, \ldots, \beta_i$, define $Y_j^i = Y_j^{i-1}$, i.e., no vertices are deleted from such Y_j^{i-1}. Note that for $j < i$, v_i is adjacent to v_j in R whenever u_j is adjacent to u_i in G. Moreover, v_i is completely connected with each set Y_j^i, in which a neighbor of v_i will be selected after this step. This finishes the general step and hence the proof of Theorem 11.13. □

More generally, we have the following result, in which the proof will use the multicolor Regularity Lemma, i.e., Theorem 11.10.

Theorem 11.14 *For any positive integers k and* Δ, *there is a constant* $c = c(k, \Delta) > 0$ *such that* $r_k(G) \leq cn$ *for any graph G of order n with maximum degree* $\Delta(G) \leq \Delta$.

A problem needed to consider is that the huge constant obtained in Theorem 11.13 by using the Regularity Lemma. Some improvements on the constant have been done by Eaton (1998), and further by Graham, Rödl, and Ruciński (2000, 2001), in which they showed that the constant can be bounded from above by $2^{c\Delta(\log\Delta)^2}$, and particularly $2^{(\Delta+c)\log\Delta}$ for bipartite graphs, one can see Chapter 9 for more better bounds of that for bipartite graphs. The authors also showed that there is a positive constant c' such that for each Δ and n sufficiently large there is a bipartite graph H on n vertices with maximum degree Δ for which $r(H) > 2^{c'\Delta}n$. Therefore, one can only improve the constant of the exponent by noting Theorem 9.5.

The constant for the d-degenerate graphs due to Lee (2017) is as follows, which confirms a conjecture of Burr and Erdős. Let us point out that the following result does not apply the regularity method.

Theorem 11.15 *There exists a constant c such that the following holds for every natural number d and r. For every edge two-coloring of the complete graph on at least* $2^{d2^{cr}}n$ *vertices, one of the colors is universal for the family of d-degenerate r-chromatic graphs on at most n vertices.*

11.5 Extensions on the Regularity Lemma

There are many generalizations of Szemerédi's Regularity Lemma. In this section, we will introduce more forms on the Regularity Lemma.

At first, we would like to introduce the sparse Regularity Lemma. Let $G = (V, E)$ be a graph. Let $0 < p \leq 1$, $\eta > 0$ and $K > 1$. For two disjoint subsets X, Y of V, let

$$d_{G,p}(X, Y) = \frac{e_G(X, Y)}{p|X||Y|},$$

which is referred to as the *p-density* of the pair (X, Y). We say that G is an (η, λ)-bounded graph with respect to the p-density if any disjoint subsets X, Y of V with

$|X| \geq \eta|V|, |Y| \geq \eta|V|$ satisfy

$$e_{G,p}(X,Y) \leq \lambda p|X||Y|.$$

For fixed $\epsilon > 0$, we say such a pair (X,Y) is (ϵ, p)-*regular* if for all $X' \subseteq X$ and $Y' \subseteq Y$ with

$$|X'| \geq \epsilon|X| \text{ and } |Y'| \geq \epsilon|Y|,$$

we have

$$|d_{G,p}(X,Y) - d_{G,p}(X',Y')| \leq \epsilon.$$

The following is a variant of the Szemerédi's Regularity Lemma for sparse graphs, developed by Kohayakawa and Rödl (1997, 2003).

Theorem 11.16 *For any fixed $\epsilon > 0$, $\lambda > 1$ and $t_0 \geq 1$, there exist T_0, η and N_0, such that each graph $G = (V,E)$ with at least N_0 vertices that is (η, λ)-bounded with respect to density p with $0 < p \leq 1$, has a partition $\{V_0, V_1, \ldots, V_t\}$ with $t_0 \leq t \leq T_0$ such that*

(i) $|V_0| \leq \epsilon N$ and $|V_1| = |V_2| = \cdots = |V_t|$;

(ii) all but at most ϵt^2 pairs (V_i, V_j) are (ϵ, p)-regular for $1 \leq i \neq j \leq t$.

Denote

$$\mathcal{H}(\Delta, n) = \{H \subseteq K_n : \Delta(H) \leq \Delta\},$$

where $H \subseteq K_n$ means that H is a spanning subgraph of K_n. We say that a graph F is *partition universal* for $\mathcal{H}(\Delta, n)$, if $F \to (H, H)$ for each $H \in \mathcal{H}(\Delta, n)$. The well-known result of Chvátal, Rödl, Szemerédi and Trotter (1983) implies that K_N is partition universal for $\mathcal{H}(\Delta, n)$. Applying the above sparse Regularity Lemma Kohayakawa, Rödl, Schacht and Szemerédi (2011) strengthened this result by replacing K_N with sparse graphs as follows.

Theorem 11.17 *For fixed $\Delta \geq 2$, there exist constants $B = B(\Delta)$ and $C = C(\Delta)$ such that if $N \geq Bn$ and $p = C(\log N/N)^{1/\Delta}$, then*

$$\lim_{n\to\infty} \Pr(G(N,p) \textit{ is partition universal for } \mathcal{H}(\Delta, n)) = 1.$$

The above result implies the following result.

Corollary 11.1 *For fixed $\Delta \geq 2$, there exist constants $B = B(\Delta)$ and $C = C(\Delta)$ such that for every n and $N \geq Bn$ there exists a graph F on N vertices and at most $CN^{2-1/\Delta}(\log N)^{1/\Delta}$ edges that is partition universal for $\mathcal{H}(\Delta, n)$).*

The size Ramsey number $\hat{r}(H)$ is defined as $\min\{e(G) : G \to (H,H)\}$ in Erdős, Faudree, Rousseau and Schelp (1978). Rödl and Szemerédi (2000) conjectured that, for every $\Delta \geq 3$, there exists $\epsilon = \epsilon(\Delta) > 0$ such that

$$n^{1+\epsilon} \leq \widehat{r}_{\Delta,n} := \max\{\hat{r}(H) : H \in \mathcal{H}(\Delta, n)\} \leq n^{2-\epsilon}.$$

For the lower bound, Rödl and Szemerédi proved that there exists a constant $c > 0$ such that $\hat{r}(H) \geq n \log^c n$ for some graphs H of order n with maximum degree three.

From Corollary 11.1, we have $\hat{r}(H) \le \Theta(n^{2-1/\Delta}(\log n)^{1/\Delta})$ for $H \in \mathcal{H}(\Delta, n)$. Hence this confirms the upper bound of the conjecture in a stronger form.

Note that, for $p = C(\log N/N)^{1/\Delta}$, we have a.a.s. the chromatic number $\chi(G(N,p)) = \Theta((N/\log N)^{1-1/\Delta})$, see e.g. Bollobás (2000, Theorem 11.29) and Łuczak (1991). A natural problem is that do there exist some sparse graphs with small chromatic number such that they are also partition universal for $\mathcal{H}(\Delta, n)$. The answer is yes. Indeed, as an application of the sparse Regularity Lemma we obtain that the r-partite random graph $G_r(N,p)$ is also partition universal for $\mathcal{H}(\Delta, n)$, where $r = r(\Delta)$ is a constant, see Lin and Li (2018).

Another extension of the Regularity Lemma is the following multi-partite Regularity Lemma, one can find a detailed proof in Lin and Li (2015).

Lemma 11.9 *For any $\epsilon > 0$ and integers $m \ge 1$, $p \ge 2$ and $r \ge 1$, there exists an $M = M(\epsilon, m, p, r)$ such that if the edges of a p-partite graph $G(V^{(1)}, \dots, V^{(p)})$ with $|V^{(s)}| \ge M$, $1 \le s \le p$ are r-colored, then all monochromatic graphs have the same partition $\{V_1^{(s)}, \cdots, V_k^{(s)}\}$ for each $V^{(s)}$, where k is same for each part $V^{(s)}$ and $m \le k \le M$, such that*

(1) $\big||V_i^{(s)}| - |V_j^{(s)}|\big| \le 1$ for each s;

(2) All but at most $\epsilon k^2 r \binom{p}{2}$ pairs $(V_i^{(s)}, V_j^{(t)})$, $1 \le s < t \le p$, $1 \le i, j \le k$, are ϵ-regular in each monochromatic graph.

Proof of Lemma 11.9. A similar proof as Theorem 11.6, but modify the definition of index by summing the indices for each color,

$$q(\mathcal{P}) = \frac{1}{k^2} \sum_{1\le h\le r} \sum_{1\le s<t\le p} \sum_{1\le i,j\le k} d^2\big(V_i^{(s)}, V_j^{(t)}\big).$$

Then we have analogy of Lemma 11.6, and the proof follows. □

Define a family $\mathcal{F}(G; p)$ of graphs as

$$\mathcal{F}(H; p) = \{F : F \to (H, H) \text{ and } \omega(F) \le p\},$$

where $\omega(F)$ is the order of the maximum clique of graph F, and define $f(H; p) = \min\{|V(F)| : F \in \mathcal{F}(H; p)\}$, which is called the *Folkman number*. We admit that $f(H; p) = \infty$ if $\mathcal{F}(H; p) = \emptyset$, and thus $f(H; p) = \infty$ if $p < \omega(H)$.

Let us call a family $\mathcal{H}$ of graphs H_n of order n *Ramsey linear* if there exists a constant $c = c(\mathcal{H}) > 0$ such that $R(H_n) \le cn$ for any $H_n \in \mathcal{H}$. Similarly, we call $\mathcal{H}$ to be Folkman p-linear if $f(H_n; p) \le cn$ for any $H_n \in \mathcal{G}$, where p is a constant.

The multicolor multi-partite Regularity Lemma has many applications. A classic result of Chvátal, Rödl, Szemerédi and Trotter (1983) tells that the family $\mathcal{H}(\Delta, n)$ is Ramsey linear. Lin and Li (2015) extended this result to that the family $\mathcal{H}(\Delta, n)$ is Folkman p-linear, where $p = p(\Delta)$.

Theorem 11.18 *Let $\Delta \ge 3$ be an integer and $p = R(K_\Delta)$. Then there exists a constant $c = c(\Delta) > 0$ such that $K_p(cn) \to (H, H)$ for any graph with n vertices and $\Delta(H) \le \Delta$. Particularly, the family $\mathcal{H}(\Delta, n)$ is Folkman p-linear.*

The above result can easily generalized to multicolor case. For more applications, see e.g. Böttcher, Heinig and Taraz (2010), Shen, Lin and Liu (2018), etc.

For graph $G = (V, E)$, recall that a partition $\mathcal{A} = \{V_i : 1 \le i \le k\}$ of V is an equipartition if $|V_i|$ and $|V_j|$ differ by no more than 1 for all $1 \le i < j \le k$. A refinement of such an equipartition is an equipartition of the form $\mathcal{B} = \{V_{i,j} : 1 \le i \le k, 1 \le j \le \ell\}$ such that $V_{i,j}$ is a subset of V_i for every $1 \le i \le k$ and $1 \le j \le \ell$. Now, we will introduce another variant of Szemerédi's Regularity Lemma due to Alon, Fischer, Krivelevich and Szegedy (2000), which can be used to find induced subgraphs in graph G.

Lemma 11.10 *For every natural number m and function $0 < \epsilon(r) < 1$, there exists a natural number $S = S(m, \epsilon)$ with the following property.*

For any graph G on $n \ge S$ vertices, there is an equipartition $\mathcal{A} = \{V_i : 1 \le i \le k\}$ and a refinement $\mathcal{B} = \{V_{i,j} : 1 \le i \le k, 1 \le j \le \ell\}$ that satisfy:

(1) $|\mathcal{A}| = k \ge m$ but $|\mathcal{B}| = k\ell \le S$.

(2) For all $1 \le i < i' \le k$ but at most $\epsilon(0)\binom{k}{2}$ of them the pair $(V_i, V_{i'})$ is $\epsilon(0)$-regular.

(3) For all $1 \le i < i' \le k$, for all $1 \le j, j' \le \ell$ but at most $\epsilon(k)\ell^2$ of them the pair $(V_{i,j}, V_{i',j'})$ is $\epsilon(k)$-regular.

(4) All $1 \le i < i' \le k$ but at most $\epsilon(0)\binom{k}{2}$ of them are such that for all $1 \le j, j' \le \ell$ but at most $\epsilon(0)\ell^2$ of them $|d_G(V_i, V_{i'}) - d_G(V_{i,j}, V_{i',j'})| < \epsilon(0)$ holds.

The following lemma implies that for any graph G, there exists an induced subgraph having an equipartition in which all pairs are regular.

Lemma 11.11 *For every m and $0 < \epsilon(r) < 1$, there exist $S = S(m, \epsilon)$ and $\delta = \delta(m, \epsilon)$ with the following property.*

For any graph G on $n \ge S$ vertices, there is an equipartition $\mathcal{A} = \{V_i : 1 \le i \le k\}$ and an induced subgraph G' of G, with an equipartition $\mathcal{A}' = \{V_i' : 1 \le i \le k\}$ of vertices of G', that satisfy:

(1) $m \le k \le S$.

(2) $V_i' \subset V_i$ for all $i \ge 1$, and $V_i' \ge \delta n$.

(3) In the equipartition $\mathcal{A}'$, **all pairs** *are $\epsilon(k)$-regular.*

A graph H is *ϵ-unavoidable* in G if no adding and removing more than $\epsilon|G|^2$ edges results in G not having an induced subgraph isomorphic to H. H is called *δ-abundant* if G contains at least $\delta|G|^{|H|}$ (distinct) induced subgraphs isomorphic to H.

As an application of Lemma 11.11, Alon, Fischer, Krivelevich and Szegedy (2000) showed that a certain degree of unavoidability also implies a certain degree of abundance.

Theorem 11.19 *For every ℓ and ϵ, there is $\delta = \delta(\ell, \epsilon)$, such that for any graph H with ℓ vertices, if H is ϵ-unavoidable in a graph G, then it is also δ-abundant in G.*

We conclude this section with a refined Regularity Lemma. For graph G defined on vertex set $V = V(G)$ and $X, Y \subseteq V$, recall that the edge density $d_G(X, Y) = \frac{e(X,Y)}{|X||Y|}$, we admit that if $X \cap Y \neq \emptyset$, then edges in $X \cap Y$ are counted twice. We say a subset U is ϵ-regular if the pair (U, U) is ϵ-regular.

The following lemma due to Conlon and Fox (2012) tells that every graph contains a large ϵ-regular subset.

Lemma 11.12 *For every $0 < \epsilon < 1$, there exists a constant δ such that every graph G contains an ϵ-regular vertex subset U with $|U| \geq \delta|V(G)|$.*

The following is a refined version of the Regularity Lemma by Conlon (2019).

Lemma 11.13 *For every $0 < \eta < 1$ and natural number m_0, there exists a natural number M such that every graph G with at least m_0 vertices has an equipartition $V(G) = \cup_{i=1}^m V_i$ with $m_0 \leq m \leq M$ parts and subsets $W_i \subset V_i$ such that W_i is η-regular for all i and, for all but ηm^2 pairs (i, j) with $1 \leq i \neq j \leq m$, (V_i, V_j), (W_i, V_j) and (W_i, W_j) are η-regular with $|d_G(W_i, V_j) - d_G(V_i, V_j)| \leq \eta$ and $|d_G(W_i, W_j) - d_G(V_i, V_j)| \leq \eta$.*

Proof. Apply the Regularity Lemma (Theorem 11.7) to G with $\epsilon = \eta \cdot \delta(\eta)$, with δ as in Lemma 11.12. This yields an equitable partition $V(G) = \cup_{i=1}^m V_i$ where all but $\epsilon m^2 \leq \eta m^2$ pairs (V_i, V_j) with $1 \leq i \neq j \leq m$ are ϵ-regular. Within each piece V_i, now apply Lemma 11.12 to find a set W_i of order at least $\delta(\eta)|V_i|$ which is η-regular. Note that if (V_i, V_j) is ϵ-regular, then, since $|W_i| \geq \delta(\eta)|V_i|$ and $\epsilon = \eta \cdot \delta(\eta)$, the pairs (W_i, V_j) and (W_i, W_j) are η are η-regular with $|d_G(W_i, V_j) - d_G(V_i, V_j)| \leq \epsilon \leq \eta$ and $|d_G(W_i, W_j) - d_G(V_i, V_j)| \leq \eta$. □

Using the above refined Regularity Lemma, Conlon (2019) proved that the following result.

Theorem 11.20 *For each fixed integer $m \geq 2$, $r(B_n^{(m)}) \leq (2^m + o(1))n$.*

This upper bound together with the lower bound obtained by Erdős et al. (1978) yield that $r(B_n^{(m)}) \sim 2^m n$ as $n \to \infty$. This answers a question of Erdős et al. (1978) and confirms a conjecture of Thomason (1982) asymptotically.

The following refined Regularity Lemma due to Conlon, Fox and Wigderson (2021) is a further strengthening of that due to Conlon (2019), which guarantees that each part itself is regular.

Lemma 11.14 *For every $\epsilon > 0$ and $M_0 \in \mathbb{N}$, there is some $M = M(\epsilon, M_0) > M_0$ such that for every graph G, there is an equitable partition $V(G) = \cup_{i=1}^k V_i$ into $M_0 \leq k \leq M$ parts so that the following hold:*

(1) Each part V_i is ϵ-regular.

(2) For every $1 \leq i \leq k$, there are at most ϵk values $1 \leq j \leq k$ such that the pair (V_i, V_j) is not ϵ-regular.

By using the above refined Regularity Lemma, Conlon, Fox and Wigderson (2021) further improved the upper bound of $r(B_n^{(m)}, B_n^{(m)})$ as that for each $m \geq 2$,

$$r(B_n^{(m)}, B_n^{(m)}) \le 2^m n + O\left(\frac{n}{(\log\log\log n)^{1/25}}\right).$$

For hypergraph Regularity Lemma, one can see Frankl and Rödl (1992) and later Chung (1991) in which the author discussed the problems of quasi-random hypergraphs.

11.6 Exercises

1. Show that $w(3) = 9$.

2. Show that $w(4) > 34$ by red/blue coloring $\{0, 1, \ldots, 33\}$, in which x is red if $x = 0, 11$ or a quadratic non-residue $(\mathrm{mod}\ 11)$.

3. Prove that for $n \ge 2$, the off-diagonal van der Waerden numbers $w(1, n) = n$ and $w(2, n)$ is $2n - 1$ if n is even and $2n$ otherwise.

4.* The following conjecture was due to Baudet, which is a weaker version of van der Waerden Theorem "$w_k(t) < \infty$". If all natural numbers are k-colored, then there is a monochromatic and arbitrarily long AP. Assuming the conjecture is true, give a short proof for "$w_k(t) < \infty$". (Hint: See Schreier (1926), reported in van der Waerden (1971). For k and t, let us call a k-partition to be *bad* if no class contains a t-AP. Suppose the statement "$w_k(t) < \infty$" is not true for some k and t, we shall find a bad k-partition of $\mathcal{N} = \{1, 2, \ldots\}$ for this t. Assume $k = 2$. Suppose for each w, there is a bad partition $[w]$ as $[w] = C_1^{(w)} \cup C_2^{(w)}$. For any w_1 and $w > w_1$, $C_1^{(w)} \cap [w_1]$ and $C_2^{(w)} \cap [w_1]$ form a bad partition of $[w_1]$. Since the number of bad partitions of $[w_1]$ is finite, there is $w_2 > w_1$ such that

$$C_1^{(w_2)} \cap [w_1] = C_1^{(w_1)},\ C_2^{(w_2)} \cap [w_1] = C_2^{(w_1)}.$$

Generally, we can find a sequence $w_1 < w_2 < w_3 < \cdots$ such that

$$C_1^{(w_j)} \cap [w_i] = C_1^{(w_i)},\ C_2^{(w_j)} \cap [w_i] = C_2^{(w_i)}$$

for all $i < j$. Then, we define $C_1, \ldots, C_k$ by $x \in C_i$ if and only if $x \in C_i^{(w_n)}$ for all $w_n \ge x$, and $\{C_1, \ldots, C_k\}$ is a bad k-partition of $\mathcal{N}$.)

5. Give a 2-coloring for natural numbers such that there exists no monochromatic AP of infinite length.

6. Prove that $w_k(t+1) > \sqrt{2tk^t}$ as follows. (Hint: Show the number of $(t+1)$-APs in $[N]$ is less than $\frac{N^2}{2t}$ from the fact that the number of APs with common difference d in $[N]$ is at most $N - dt$ for $d \le N/t$. Then let X be the number of monochromatic $(t+1)$-APs in a uniform random k-coloring of $[N]$. Then $E(X) < \frac{N^2}{2tk^t}$. (Erdős-Rado, 1952))

7. Let $N = w_k(t^2 + 1)$, and let χ be a coloring of $[N]$ in two colors. Show that there exists a t-AP $\{a + id : 0 \leq i \leq t - 1\}$ which together with d is monochromatic. (Hint: Consider a monochromatic t^2-AP with difference d, and $\{d, 2d, \ldots, td\}$.)

8. Let (X, Y) be a pair of disjoint subsets of a graph G and let $\epsilon' \geq \epsilon > 0$. If (X, Y) is ϵ-regular, then they are ϵ'-regular.

9. Let $G = (A, B, E)$ be a bipartite graph with $|A| = |B| = n$ and $d = d(A, B)$, where (A, B) is an ϵ-regular pair.

(1) There are $A_1 \subseteq A$ and $B_1 \subseteq B$ such that $|A_1| \geq (1 - \epsilon)n$ and $|B_1| \geq (1 - \epsilon)n$, and the subgraph of G induced by $A_1 \cup B_1$ has minimum degree at least $(d - 2\epsilon)n$.

(2) There are $A_2 \subseteq A$ and $B_2 \subseteq B$ such that $|A_2| \geq (1 - \epsilon)n$ and $|B_2| \geq (1 - \epsilon)n$, and the subgraph of G induced by $A_2 \cup B_2$ has maximum degree at most $(d + 2\epsilon)n$.

(3) What can we say about the subgraph of G induced by $A_1 \cap A_2$ and $B_1 \cap B_2$?

10.* Sketch the proof of Theorem 11.6. In particular, define the function $q(\mathcal{P})$ for a partition $\mathcal{P}$.

Chapter 12
More Ramsey Linear Functions

From an application of the regularity lemma by Chvátal, Rödl, Szemerédi and Trotter (1983) introduced in the last chapter, we know that the family of graphs with bounded maximum degree is Ramsey linear. The Ramsey linearity of families of graphs with bounded degeneracy is confirmed by Lee (2017), which confirms a conjecture of Burr and Erdős (1975). In this chapter, we shall discuss more Ramsey linear functions. The first section discusses the linearity of subdivided graphs, and the second is on a special linearity: so called Ramsey goodness. All results on Ramsey goodness may be viewed as the generalizations of Chvátal's result (1977).

12.1 Subdivided Graphs

The following result is due to Alon (1994), whose original constant was 12. The improved constant 6 is due to Li, Rousseau, and Šoltés (1997). Note that if G is an essential subdivided graph, then all vertices on G of degree at least three are independent.

Theorem 12.1 *If G is a graph of order $n \geq 3$ in which all vertices of degree at least three are independent, then $r(G) \leq 6n - 12$.*

Corollary 12.1 *The family of essential subdivided graphs is Ramsey linear.*

We need two lemmas for the proof.

Lemma 12.1 *Let G be a graph without isolated vertices. Then*

$$r(K_3, G) \leq 3q,$$

where q is the number of edges of G.

Proof. For $q = 1$, the assertion is trivial. It is easy to see that

$$r(K_3, G_1 \cup G_2) \leq r(K_3, G_1) + r(K_3, G_2),$$

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so we may assume that G is connected and $q \geq 2$.

Suppose to the contrary, the assertion fails, then there is a connected graph G with q edges, where q is minimal such that $r(K_3, G) > 3q$. Let us write n for the order of G and N for $3q$. From the definition, there is a red-blue edge coloring of K_N on vertex set V such that there is neither red K_3 nor blue G. Denote by R and B for the subgraphs on V with edge sets consisting of all red edges or all blue edges, respectively. Note that for any vertex $u \in V$, $d_R(u) \leq n-1$ since R is triangle-free thus the subgraph induced by $N_R(u)$ is completely blue.

Let $\delta = \delta(G) \geq 1$ be the minimum degree of G, and let G' be the subgraph of G by deleting a vertex v of degree δ. By the minimality of q, we have $r(K_3, G') \leq 3(q-\delta) < 3q$ and thus B contains G' as a subgraph. Let $X \subseteq V$ be the subset of vertices not belonging to $V(G')$ in B. Since B contains no G and $d_G(v) = \delta \geq 1$, we have that each vertex $u \in X$ is adjacent to at least one vertex in $N_G(v)$ in R. However, each vertex in $N_G(v)$ can be adjacent to at most $n-1$ vertices of X in R, so $|X| = N-(n-1) \leq \delta(n-1)$, implying $N \leq (\delta+1)(n-1)$, which with the facts that $\delta n \leq 2q$ and $n-1 \leq q$ yield

$$N \leq 3q - \delta \leq 3q - 1,$$

leading to a contradiction. □

The above lemma has been improved as $r(K_3, G) \leq 2q+1$ by Sidorenko (1993), and Goddard and Kleitman (1994), which was conjectured by Harary.

Another lemma for the proof of main result is as follows.

Lemma 12.2 *Let $m \geq 1$ and $n \geq 2$ be integers. Then*

$$r(K_n, mK_2) = n + 2m - 2.$$

Proof. The graph $K_{n-2} + \overline{K}_{2m-1}$ does not contain K_n and its complement, $\overline{K}_{n-2} \cup K_{2m-1}$ does not contain m independent edges. Hence $r(K_n, mK_2) \geq n + 2m - 2$.

On the other hand, let G be a graph of order $n + 2m - 2$ that does not contain K_n. We shall prove that $\overline{G}$ contains m independent edges by induction on m. This is clear if $m = 1$. For general $m \geq 2$, by deleting a pair of non-adjacent vertices u and v, which form an edge in $\overline{G}$, we have a graph H of order $n + 2(m-1) - 2$, which does not contain K_n. By induction hypothesis, the complement of H contains $m-1$ independent edges, which with the edge uv give m independent edges of $\overline{G}$. □

We will need that $r(K_{2,n}) \leq 4n-2$ and the equality holds for infinitely many n, proved in Chapter 8. This indicates that the constant in Theorem 12.1 cannot be replaced by one less than 4.

Proof of Theorem 12.1. With the fact $r(K_3) = 6$, we may assume that $n \geq 4$. We also assume that G is not a subgraph of $K_{2,n-2}$ by the upper bound just mentioned. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G, without loss of generality, let

$$I = \{v_1, v_2, \ldots, v_m\} \subseteq V$$

be a maximal independent set containing all vertices of degree at least three, where $m < n$. Since I is maximal, each vertex in $V \setminus I$ has at least one neighbor in I. Thus each component C of $G - I$ is a single vertex or an edge. If C is a single vertex, it has either one or two neighbors in I; and if C is an edge, then each of its end-vertices has exactly one neighbor in I.

We now define a new graph H with vertex set $V(H) = \{1, 2, \ldots, m\}$, and $\{i, j\}$ is an edge of H if v_i and v_j are commonly in the neighborhood of some component of $G - I$. Since there are at most $n - m$ components of $G - I$, we have $|E(H)| \le n - m$ as any component associates at most one such pair (v_i, v_j). Consider a red-blue edge coloring of the complete graph of order $6n - 12$ on vertex set $U = \{u_1, u_2, \ldots, u_{6n-12}\}$. Denote by R and B for the subgraph with vertex set U and edge set consisting of all red edges or all blue edges, respectively. By symmetry, we may assume that at least half of the vertices in U have red degrees at least $3n-6$. Therefore there is a subset $U_1 \subseteq U$, say $U_1 = \{u_1, u_2, \ldots, u_{3n-6}\}$, such that $|N_R(u_i)| \ge 3n - 6$ for $1 \le i \le 3n - 6$. We then define a yellow graph F on vertex set U_1 with u_i and u_j are connected by a yellow edge if and only if $|N_R(u_i) \cap N_R(u_j)| \ge n - 3$.

We claim that the independent number of the graph F is at most two. In fact, if u_1, u_2 and u_3 are independent in F, then

$$\begin{aligned} \left|\cup_{i=1}^{3} N_R(u_i)\right| &= \sum_{i=1}^{3} |N_R(u_i)| - \sum_{1 \le i < j \le 3} |N_R(u_i) \cap N_R(u_j)| + \left|\cap_{i=1}^{3} N_R(u_j)\right| \\ &\ge 3(3n - 6) - 3(n - 4) > 6n - 12, \end{aligned}$$

a contradiction. Then by the fact from Lemma 12.1, we know that F contains H as a subgraph including all isolated vertices since F has enough vertices.

Therefore, by the definitions of H and F, we obtain a subset

$$I' = \{v'_1, v'_2, \ldots, v'_m\} \subseteq U_1$$

with $|N_R(v'_i)| \ge 3n - 6$ for $1 \le i \le m$, and $|N_R(v'_i) \cap N_R(v'_j)| \ge n - 3$ if both v_i and v_j are commonly in the neighborhood of some component of $G - I$.

We shall seek to embed G into the red graph R such that $v_i \longrightarrow v'_i$ for $1 \le i \le m$. Suppose that we have constructed the appropriate embedding of some components of $G - I$. We now wish to extend this embedding to one more component C.

Case 1 $C = \{v\}$ is a single vertex.

Subcase 1.1 v has two neighbors v_i and v_j in I.

Since we have chosen at most $n-3$ vertices previously that are not all in $N_R(v'_i) \cap N_R(v'_j)$ as G is not isomorphic to $K_{2,n-2}$, there is at least one vertex in $N_R(v'_i) \cap N_R(v'_j)$ left that can be chosen for v.

Subcase 1.2 v has only one neighbor v_i in I.

Simply take any vertex that has not been chosen in $N_R(v'_i)$.

Case 2 $C = e$ is an edge.

Subcase 2.1 Two end-vertices of e have a common neighbor v_i in I.

We can choose any red edge in $N_R(v_i')$ for e. If no such red edge available, then $N_R(v_i')$ induces a complete blue graph of order at least $3n - 6 \geq n$. We thus have a blue graph G.

Subcase 2.2 Each end-vertex of e has a (different) neighbor in I, say v_i and v_j with $1 \leq i < j \leq m$, respectively.

We shall seek a red path of length three with end-vertices v_i' and v_j', and one internal vertex in $N_R(v_i')$ and one in $N_R(v_j')$, which have not been chosen previously. Suppose that no such path exists, then any edge between $N_R(v_i')$ and $N_R(v_j')$ is blue unless one of its end vertices has been chosen previously. We shall construct a blue G in another way. Let W be the set of vertices chosen so far and all vertices of I in the embedding of G into red graph R, then $|W| \leq n - 2$. And let

$$X = (N_R(v_i') \cap N_R(v_j')) \setminus W,$$
$$Y = N_R(v_i') \setminus (W \cup N_R(v_j')),$$
$$Z = N_R(v_j') \setminus W.$$

We may assume that $|X| \leq n - 2$ since otherwise we have a complete blue graph induced by X and a vertex in Y of order at least n hence a blue G. Since $Y = N_R(v_i') \setminus (W \cup X)$ and $|W \cup X| \leq 2n - 4$, we have $|Y| \geq 3n - 6 - (2n - 4) = n - 2$. Also $|Z| \geq 3n - 6 - (n - 2) = 2n - 4$. By Lemma 12.2, $r(K_n, (\lfloor n/2 \rfloor - 1)K_2) \leq 2n - 4$, we assume that there is a blue matching M on $n - 2$ or $n - 3$ vertices in Z if n is even or odd, respectively. Take a subset $J \subseteq Y$ with $|J| = |I| = m < n$. Let M' be the set of union of M and one more vertex from Z, then all edges between J and M are blue. We can use J and M to construct a blue G with J corresponding to I. This completes the proof. □

Recently, a result of Chen, Yu and Zhao (2021) states that for any n,

$$4.5n - 5 \leq r(F_n) \leq 5.5n + 6.$$

where $F_n = K_1 + nK_2$ is an n-fan, or a friendship graph, see Chapter 7. The above lower bound follows from the following construction. Let t be the largest even number less than $3n/2$. Thus $t \geq 3n/2 - 2$. We construct a graph $G = (V, E)$ on $3t$ vertices as follows. Let $V_1 \cup V_2 \cup V_3$ be a partition of V such that $|V_1| = |V_2| = |V_3| = t$ and all $G[V_i]$ are complete graphs. For each $i \in [3]$, further partition V_i into two subsets X_i and Y_i with $|X_i| = |Y_i| = t/2$, and add edges between X_i and Y_{i+1} such that $G[X_i, Y_{i+1}]$ is an $\lceil n/2 \rceil$-regular bipartite graph, where we assume $Y_4 = Y_1$. It is not difficult to check that both G and $\overline{G}$ do not contain a copy of F_n.

Note that the graph F_n has at most one vertex of degree more than two, so F_n satisfies the condition of Theorem 12.1. Therefore, we cannot expect the upper bound of Theorem 12.1 to go down to less than $4.5n - 5$.

We conclude this section with the following problem.

Problem 12.1 Determine $r(F_n)$.

12.2 Ramsey Goodness

Recall a result of Chvávtal in Chapter 1 that for $k, n \geq 1$,

$$r(K_k, T_n) = (k-1)(n-1) + 1.$$

Burr (1981) generalized Chvátal's result in the following way. Denote by $s(G)$ for the minimum number of vertices in a color class among all proper vertex colorings of G by $\chi(G)$ colors and call $s(G)$ the *chromatic surplus*. For example, $s(K_k) = 1$, $s(C_{2m}) = m$ and $s(C_{2m+1}) = 1$.

Theorem 12.2 *Let G be a graph with $\chi(G) = k$ and let H be a connected graph of order $n \geq s(G)$, then*

$$r(G, H) \geq (k-1)(n-1) + s(G).$$

Proof. Let $s = s(G)$ and $N = (k-1)(n-1) + s - 1$. Color the edges of K_N red and blue such that the blue graph is isomorphic to $(k-1)K_{n-1} \cup K_{s-1}$, so it contains no H. The chromatic number of the red graph is k, and the smallest vertex color class has size $s-1$, so it contains no G. Thus $r(G, H) \geq N + 1$ as desired. □

Burr and Erdős (1983) initiated the study of Ramsey goodness problems. We say that the connected graph H is G-*good* if

$$r(G, H) = (\chi(G) - 1)(n-1) + s(G).$$

A K_k-good graph is also called a k-good graph. So any tree is k-good from Chvátal's result, but the edge density of a tree is less than 1. We shall prove F_n is 3-good. Let us recall a lemma in the last section, $r(K_k, mK_2) = k + 2m - 2$, thus $r(K_3, nK_2) = 2n+1$. The following result is due to Li and Rousseau (1996). Generally, we have that for any fixed graph F and G, $K_1 + nF$ is $(K_2 + G)$-good for large n by using stability lemma.

Theorem 12.3 *Let $n \geq 2$ be an integer, then F_n is 3-good, that is*

$$r(K_3, F_n) = 4n + 1.$$

Proof. Theorem 12.2 yields $r(K_3, F_n) \geq 4n+1$. We then verify the inverse inequality. For an arbitrary two-coloring of edges of K_{4n+1}, let R and B be the subgraph induced by all red edges and all blue edges, respectively. Suppose that R contains no K_3 and B contains no F_n. For any vertex u, the absence of K_3 implies that $N_R(u)$ induces a complete graph in B, and thus $|N_R(u)| \leq 2n$. Also, by $r(K_3, nK_2) = 2n + 1$ as mentioned, the absence of a blue F_n implies $|N_B(u)| \leq 2n$. It follows that both graphs R and B are regular of degree $2n$.

Suppose that u and v are adjacent in R. Then $N_R(u)$ and $N_R(v)$ are disjoint sets, each inducing a complete graph in B of order $2n$, and $N_B(u) \cap N_B(v)$ has a single vertex, which we denote by w. If w were adjacent to one or more vertices of $N_R(u)$

or to one or more vertices of $N_R(v)$, there would yield a blue F_n. It follows that the degree of w in B, already determined to be $2n$, is 2, so $n = 1$. This contradiction proves that $r(K_3, F_n) \le 4n + 1$. □

We have seen that any tree T is k-good and $F_n = K_1 + nK_2$ is 3-good if $n \ge 2$. For more Ramsey numbers on fans, we refer the reader to Lin and Li (2009, 2010), Zhang, Broersma and Chen (2015), and Chen, Yu and Zhao (2021), etc. We shall discuss Ramsey goodness more in this section. Before goodness was defined, Bondy and Erdős (1973) proved that a long cycle (hence a long path) is C_m-good and $K_r(t)$-good. Rosta (1973), and Faudree and Schelp (1974) independently proved that when m is odd, C_n is C_m-good for $n \ge m$ and $(m, n) \ne (3, 3)$. When m is even, C_n is C_m-good for $n \ge m$ and $(m, n) \ne (4, 4)$ unless n is odd and $3m/2 > n \ge m$, in which case $r(C_m, C_n) = 2m - 1$. This and $r(C_3, C_3) = r(C_4, C_4) = 6$ gave all Ramsey numbers $r(C_m, C_n)$. In particular, we have that

$$r_2(C_n) = \begin{cases} 2n - 1 & \text{for odd } n \ge 5, \\ 3n/2 - 1 & \text{for even } n \ge 6. \end{cases}$$

Lemma 12.3 *If $n \ge 3$ is an odd integer, then*

$$r_k(C_n) \ge 2^{k-1}(n - 1) + 1.$$

If $n \ge 2$ is an even integer, then

$$r_k(C_n) \ge \frac{k+1}{2}n - k + 1.$$

Proof. It is easy to see, as proved in Chapter 8, we have

$$r_k(G) \ge (\chi - 1)(r_{k-1}(G) - 1) + 1,$$

where $\chi = \chi(G)$, the first lower bound follows immediately. The second is also easy. For an even integer $n \ge 2$, let $N_k = r_k(C_n) - 1$. There is an edge-coloring of complete graph of order N_k by k colors such that there is no monochromatic C_n. Consider such a colored complete graphs and a new complete graph of order $n/2 - 1$. Color all edges of the new graph and all edges between the two complete graphs by a new color. Clearly, there is no monochromatic C_n, thus $N_{k+1} \ge N_k + n/2 - 1$, which and the fact that $N_1 = r_1(C_n) - 1 = n - 1$ imply the assertion. □

An application of the regularity lemma showed that for three colors, the above lower bounds are asymptotical equalities. Namely, it was shown that

$$r_3(C_n) = \begin{cases} (4 + o(1))n & \text{for odd } n, \\ (2 + o(1))n & \text{for even } n. \end{cases}$$

The result for the odd length n case was obtained by Luczak (1999), and Gyárfás, Ruszinkó, Sárközy and Szemerédi (2007), and the other by Figaj and Luczak (2007). Kohayakawa, Simonovits and Skokan (2005) used Luczak's method together with stability methods proved that $r_3(C_n) = 4n - 3$ for sufficiently large odd n. By using the regularity method together with nonlinear optimisation, Jenssen and Skokan (2021) established that

$$r_k(C_n) = 2^{k-1}(n-1) + 1$$

for all fixed $k \geq 2$ and sufficiently large odd n, which confirms a conjecture by Bondy and Erdős (1973). For each fixed $k \geq 3$ and large even n, we have

$$(k - 1 + o(1))n \leq r_k(C_n) \leq (k - 1/2 + o(1))n$$

where the lower bound is due to Yongqi, Yuansheng, Feng and Bingxi (2006) while the upper bound by Knierim and Su (2019) improves that of Davies, Jenssen and Roberts (2017) and an earlier upper bound by Sárközy (2016).

Burr (1981) generalized the above results for long cycle C_n to a graph H that contains a long suspended path. A path of H is called *suspended* if the degree of each internal vertex is two.

Lemma 12.4 *Let G be a graph of order m, and H a connected graph of order n that contains a suspended path of length ℓ. Let G_1 be a graph from G by deleting an independent set of t vertices, and let H_1 be a graph from H by shortening the suspended path by* 1. *If $\ell \geq (m-2)(m-t) + t + 1$, then*

$$r(G, H) \leq \max\{r(G, H_1),\ r(G_1, H) + n - 1\}.$$

Proof. For $m = 1$ or $m = 2$, the assertion holds trivially. So we assume that $m \geq 3$. Write the right-hand side of the above by N. Consider a red-blue edge coloring of K_N. We shall prove that there is either a red G or a blue H. Since $N \geq r(G, H_1)$, we are done unless there is a blue H_1. Delete $n - 1$ vertices of this blue H_1, there are at least $r(G_1, H)$ vertices left, so we may assume that there is a red G_1. Thus we obtain a blue H_1 and a red G_1. Let X and Y be their disjoint vertex sets with $|X| = n - 1$ and $|Y| = m - t$, respectively.

The blue H_1 has a suspended path with ℓ vertices, say

$$X' = \{x_1, x_2, \ldots, x_\ell\} \subseteq X$$

in order. Write $X'' = \{x_1, x_2, \ldots, x_{\ell-1}\} \subseteq X'$. For any $y \in Y$, consider all $\ell - 1$ edges between y and X''. We assume that no two consecutive edges yx_i and yx_{i+1} are both blue, since otherwise we have a blue H. Furthermore, suppose that $m - 1$ of these edges are blue, say $yx_{i_1}, yx_{i_2}, \ldots, yx_{i_{m-1}}$ are blue. Consider any edge $x_{i_j+1}x_{i_k+1}$, if this edge is blue, we have a blue H with the new suspended path

$$x_1 \ldots x_{i_j} y x_{i_k} x_{i_k+1} \ldots x_{i_j+1} \ldots x_\ell.$$

If all edges $x_{i_j+1}x_{i_k+1}$ are red, then $x_{i_1+1}, x_{i_2+1}, \ldots, x_{i_{m-1}+1}$ and y will form a red K_m hence a red G. Consequently, we may assume that any $y \in Y$ is connected with X'' in at most $m - 2$ blue edges. Therefore there are at most $(m-2)(m-t)$ blue edges between Y and X''. Hence at least

$$\ell - 1 - (m-2)(m-t) \geq t$$

vertices in X'' are connected with each vertex of Y in red completely. These vertices and Y yield a red G, completing the proof. □

Theorem 12.4 *Let G be any graph and let H be a connected graph of order p. Choose an edge of H and form a sequence of graphs H_n by putting $n-p$ extra vertices to that edge. If n is sufficiently large, then H_n is G-good.*

Proof. Set $k = \chi(G)$, $s = s(G)$ and $m = |V(G)|$. We shall use the induction on k to show that $r(G, H_n) = (k-1)(n-1) + s$ for sufficiently large n. The assertion is trivial for $k = 1$ since $r(\overline{K}_s, H_n) = s$ if $n \geq 2$, so we assume that $k \geq 2$ and the assertion holds for $k-1$. Note that H_n has a suspended path of length $n-p+1$, which would yield graph H_{n-1} by shortening the suspended path by 1. Let H_n and H_{n-1} play the roles of H and H_1 in Lemma 12.4. Consider a vertex coloring of G with k colors such that there is a color class with s vertices. Let $t \geq s$ be the number of vertices in some other color class and let G_1 be the graph from G by deleting those t vertices. Applying Lemma 12.4, we have

$$r(G, H_n) \leq \max\{r(G, H_{n-1}),\ r(G_1, H_n) + n - 1\}.$$

By the induction hypothesis, $r(G_1, H_n) = (k-2)(n-1) + s$ when $n \geq n_0$ for some $n_0 \geq p$. Consequently, for $n \geq n_0$,

$$r(G, H_n) \leq \max\{r(G, H_{n-1}),\ (k-1)(n-1) + s\}.$$

Using Lemma 12.4 repeatedly, we have

$$r(G, H_{n-1}) \leq \max\{r(G, H_{n-2}),\ (k-1)(n-1) + s\},$$

and

$$\begin{aligned} r(G, H_n) &\leq \max\{r(G, H_{n-2}),\ (k-1)(n-1) + s\} \\ &\leq \max\{r(G, H_{n_0}),\ (k-1)(n-1) + s\} \end{aligned}$$

for all $n \geq n_0$. Hence if $(k-1)(n-1) + s \geq r(G, H_{n_0})$, then

$$r(G, H_n) = (k-1)(n-1) + s,$$

completing the proof. □

Recall that a graph H is called a subdivision of F if it is obtained by replacing each edge of F by a path. We shall say that H_1 is *homeomorphic* to H_2 if they have isomorphic subdivisions.

Theorem 12.5 *Let G be a graph and let H be a connected graph. Let H_n be a graph of order n which is homeomorphic to H. Then if n is large enough, then H_n is G-good.*

Proof. Let $k = \chi(G)$, $s = s(G)$, $m = |V(G)|$ and $p = |V(H)|$. We shall proceed the proof by induction on k as that in the proof of Theorem 12.4. The assertion is trivial for $k = 1$, so we assume that $k \geq 2$ and the assertion holds for $k-1$. If H has q edges, then H_n has a suspended path of length at least $(n-p)/q + 1$. Let H_{n-1} be formed from H_n by shortening the suspended path by 1 and let G_1 be a graph as defined in the proof of Theorem 12.4. Therefore, if $(n-p)/q + 1 \geq (m-2)(m-t) + t + 1$, then Lemma 12.4 gives that

$$r(G, H_n) \leq \max\{r(G, H_{n-1}),\ r(G_1, H_n) + n - 1\}.$$

Applying the induction hypothesis on $r(G_1, H_n)$, we know that there is some $n_0 \geq p$, such that for all $n \geq n_0$,

$$r(G, H_n) \leq \max\{r(G, H_{n-1}),\ (k-1)(n-1) + s\}.$$

The proof concludes as before. □

Theorem 12.4 has been generalized to multi-color cases $r(B_1, \ldots, B_k, H_n)$ where B_i are bipartite graphs, and

$$r(K_i, \ldots, K_j, C_{2k+1}, \ldots, C_{2\ell+1}, H_n),$$

see Burr (1982).

One can find the following result in Lin, Li and Dong (2009).

Theorem 12.6 *Let G be a graph with $s(G) = 1$ and let T be a tree. If T is G-good, then it is $(K_1 + G)$-good.*

Proof. Let n be the order of T. As $\chi(K_1 + G) = \chi(G) + 1$ and $s(K_1 + G) = 1$ we have

$$r(K_1 + G, T) \geq \chi(G)(n-1) + 1,$$

so the assertion follows from

$$r(K_1 + G, T) \leq r(G, T) + n - 1.$$

Let $N = r(G, T) + n - 1$ and consider any red-blue edge coloring of K_N. Let T' be the maximum subtree of T in color blue. If $T' = T$, we are done. So we assume that the order of T' is at most $n-1$, and delete these vertices. There are at least $r(G, T)$ vertices left. Since there is no blue T, we have a red G on a vertex set X. Among deleted vertices, there is a vertex, say v, from which one blue edge to a vertex in X will yield a large blue subtree of T. By this reason, v is connected to X completely red, so we have a red $K_1 + G$. □

Let G be a fixed graph. We may wonder whether or not an enough sparse large connected graph is G-good. The answer for general graph G is negative such as $G = C_4$.

Recall graph ER_q constructed by Erdős, Rényi, Sós and Brown in Chapter 9. The order of ER_q is $q^2 + q + 1$ and ER_q contains no C_4 as a subgraph. From the fact that each vertex has a degree q or $q+1$, we know that the maximum degree of $\overline{E}_q$ is q^2. Hence for $n = q^2 + 1$,

$$r(C_4, K_{1,n}) \geq n + \sqrt{n-1} + 1.$$

The above lower bound means that $r(C_4, K_{1,n}) > n + 2$ for $n = q^2 + 1$.

Proposition 12.1 *The star $K_{1,n}$ is not C_4-good for $n = q^2 + 1$.*

For more non-goodness examples, we refer the reader to Lin, Li and Dong (2010), and Lin and Liu (2021).

However, we can find some star-like graphs are G-good for any G. Before giving a result, we define the *upper chromatic surplus* $\bar{s}(G)$ as maximum number of vertices in a color class among all proper vertex coloring of G using $\chi(G)$ colors. Clearly, $\bar{s}(G) \geq s(G)$. Also, a pendant edge of a graph H is an edge that has an end vertex of degree one.

Theorem 12.7 *Let G be any graph without isolates and let H be a connected graph of order $p \geq \bar{s}(G)$. For any $U \subseteq V(H)$ with $|U| = \bar{s}(G)$, let $\mathcal{H}_n$ denote the class of graphs obtained from H by adding $n-p$ pendent edges joining with vertices in U. If n is sufficiently large, then H_n is G-good for some $H_n \in \mathcal{H}_n$.*

Proof. Let $k = \chi(G)$ and $s = s(G)$. We shall prove that for some $H_n \in \mathcal{H}_n$,

$$r(G, H_n) \leq (k-1)(n-1) + s$$

if n is sufficiently large. For $k = 1$, $G = K_1$ and the assertion is trivial. Here we admit that the only vertex in K_1 is not an isolate. Suppose that $k \geq 2$ and the assertion holds for $k-1$. Consider a vertex coloring of G by k colors with color classes $C_1, C_2, \ldots, C_k$ satisfying

$$s = |C_1| \leq |C_2| \leq \cdots \leq |C_k| \leq \bar{s}(G).$$

Denote by G' for the graph from G by deleting C_k. Then $\chi(G') = k-1$ and $s(G') = s$. By the induction hypothesis, there exists $N > 0$ such that if $n \geq N$ we can find some specific $H' \in \mathcal{H}_n$ that is G'−good, namely

$$r(G', H') = (k-2)(n-1) + s.$$

Take $N_1 \geq N$ such that $(k-1)(N_1 - 1) + s \geq r(G, H)$. For $n \geq N_1$ set $q = (k-1)(n-1) + s$ and let (R, B) be an edge-coloring of K_q in red and blue. We want to show that either $G \subset \langle R \rangle$ or $H'' \subset \langle B \rangle$ for some $H'' \in \mathcal{H}_n$.

Suppose to the contrary, that $\langle R \rangle$ contains no G and $\langle B \rangle$ contains no any member of $\mathcal{H}_n$. Since $q \geq r(G, H)$, we have $H \subset \langle B \rangle$. We thus assume that there is some m with $p \leq m < n$ such that $\langle B \rangle$ contains some member of $\mathcal{H}_m$ but no member of $\mathcal{H}_{m+1}$. Then there is a partition $X \cup Y$ of vertex set K_q with $|X| = m$ and

$$|Y| = q - m \geq (k-2)(n-1) + s = r(G', H')$$

such that $\langle X\rangle_B$ (the blue graph induced by X) contains some member of $\mathcal{H}_m$ and all edges xy with $x \in U \subseteq V(H) \subseteq X$ and $y \in Y$ are red. Since $\langle Y\rangle_B$ contains no H', we have that $\langle Y\rangle_R$ contains G'. Finally from the fact $|U| = \bar{s}(G) \geq |C_k|$, we obtain a red G. This contradiction completes the proof. □

We have seen that any tree is k-good for any k, and a long path is G-good for any fixed G. Negatively, all large stars are not C_4-good. So we do not expect that a large sparse graph is G-good for a non-complete graph G. But we may believe that $r(G, K_{1,n})$ achieves the maximum value among all $r(G, T_n)$, where T_n is a tree on n vertices.

However, for fixed k and d, if H is d-degenerate, then $r(K_k, H)$ grows linearly on the order of H.

Theorem 12.8 *Let k and d be fixed positive integers. Then there exists a constant $c = c(k, d) > 0$ such that for any d-degenerate graph H of order n,*

$$r(K_k, H) \leq cn.$$

Proof. For $k = 1$, the assertion is trivial. Noting the fact that $r(K_2, H) = n$, we recursively define a sequence of constants $\{c_i\}$ by setting $c_2 = 1$ and $c_k = dc_{k-1} + 1$ for $k \geq 3$. We shall prove that $r(K_k, H) \leq c_k n$. Suppose to the contrary that there is a red-blue coloring of edges of K_N, where $N = c_k n$, that contains neither a red K_k nor a blue H. Moreover, suppose $k \geq 3$ is the smallest integer with this property. If the colored K_N has a vertex v incident with at least $c_{k-1}n$ red edges, since $r(K_{k-1}, H) \leq c_{k-1}n$ with the choice of k, then we obtain a blue H or a red K_{k-1} extendable to a red K_k by the addition of v. Both cases violate our assumption. Thus each vertex is incident with at most $c_{k-1}n - 1$ red edges.

Now we shall show that any d vertices have at least n common blue neighbors. To see this, fixed d vertices and remove them and their red neighbors. Since each vertex is incident with at most $c_{k-1}n - 1$ red edges, we delete at most $dc_{k-1}n$ vertices. The remaining set, which is the intersection of blue neighborhoods of fixed vertices, contains at least $c_k n - dc_{k-1}n = n$ vertices.

Since H is d-degenerate, we may set $V(H) = \{v_1, v_2, \ldots, v_n\}$ with the property that any vertex v_i has at most d neighbors in $\{v_1, \ldots, v_{i-1}\}$. Let H_m be the subgraph of H induced by $\{v_1, \ldots, v_m\}$. Since the colored K_N contains neither red K_k nor blue H, we may assume that m is the largest integer such that a blue H_m exists with $1 \leq m \leq n-1$. However, v_{m+1} has at most d neighbors in $\{v_1, \ldots, v_m\}$, we can easily obtain a blue H_{m+1} since any d vertices have at least n common blue neighbors, and fewer than n of them have been used. This is a contradiction. □

12.3 Large Books Are p-Good

Burr and Erdős (1983) asked a problem as follows. Let p and d be fixed. Let G be large connected graphs with $\Delta(G) \leq d$. Is G p-good? This was disproved by

Brandt (unpublished), see ftp://ftp.math.fu-berlin.de/pub/math/publ/pre/1996/pr-a-96-24.ps. Nikiforov and Rousseau (2009) mentioned that they showed that almost all 100-regular graphs are not 3-good. Their paper contains positive answers for almost all problems of Burr and Erdős on Ramsey goodness. Define the m-th power H^m of a graph H as a graph on vertex set $V(H)$ and uv is an edge of H^m if and only if the distance of u and v in H is at most m. Nikiforov and Rousseau (2009) showed that for fixed graph H and integers k, ℓ and m, if H_1 is large and homeomorphic to H, then $K_\ell + H_1^m$ is k-good; and the essential subdivision of large K_n is p-good. Allen, Brightwell and Skokan (2013) proved that for any fixed integer $m \geq 1$ and graph G, then P_n^m (hence the connected subgraph of order n) is G-good for large n. Their method is a mix of the regularity lemma and Turán type stability. Before these results, Nikiforov and Rousseau (2004) already proved that the large book graph $B_n^{(k)}$ are p-good. However, all of the bounds on n of these results are of tower type when the proofs rely on the regularity lemma. Recently, avoiding to use regularity lemma, Fox, He and Wigderson (2021+) obtain that if $n \geq 2^{k^{10p}}$, then the book graph $B_n^{(k)}$ is p-good. For more results on Ramsey goodness, we refer the reader to the survey by Conlon, Fox and Sudakov (2015, Section 2.5) and other related references.

Theorem 12.9 *If $n \geq 2^{k^{10p}}$, then $B_n^{(k)}$ is K_p-good.*

We first have the following lemmas. For convenience, we use $B_{k,n}$ to denote the book graph $B_{n-k}^{(k)}$.

Lemma 12.5 *Let k, r, s, t be positive integers with $s \leq t$ and $2k \leq t$, and let G be any graph. Let Γ be a G-free graph with $N \geq \binom{t}{s}^r \frac{t}{2ks} r(G, K_s)$ vertices which contains $K_r(t)$ as an induced subgraph, with parts $V_1, \ldots, V_r$. If $\overline{\Gamma}$ does not contain a book $B_{k,n}$ with $n \geq (1 - 4ks/t)N/r$ vertices, then Γ contains an induced copy of $K_{r+1}(s)$ with parts $W_0, \ldots, W_r$, where $W_i \subseteq V_i$ for every $1 \leq i \leq r$.*

Proof. Let $\epsilon = s/t$. Partition the vertex set of Γ into $r + 1$ parts $U_0, U_1, \ldots, U_r$, where, for each $i \in [r]$, every vertex in U_i has degree at most ϵt to V_i, and every vertex in U_0 has degree at least ϵt to each V_j. Note that by construction, $V_i \subseteq U_i$ for $i \in [r]$.

Suppose there is $i \in [r]$ such that $|U_i| \geq (1 - 2k\epsilon)N/r$. Let X denote the set of all vertices $v \in V_i$ with at most $2\epsilon|U_i \setminus V_i|$ neighbors in $U_i \setminus V_i$. Since each vertex in U_i has density at most ϵ to V_i, we have $|X| \geq |V_i| \setminus 2 = t/2 \geq k$. Let Q be any k vertices in X. Then all but at most a $2k\epsilon$ fraction of the vertices in $U_i \setminus V_i$ are empty to Q. So Q together with the vertices of U_i that have no neighbors in Q form a k-book in Γ with at least $(1 - 2k\epsilon)|U_i \setminus V_i| + |V_i| \geq (1 - 4k\epsilon)N/r$ vertices.

So we may assume that there is no $i \in [r]$ with $|U_i| \geq (1 - 2k\epsilon)N/r$. In this case, we have $|U_0| \geq N - r(1 - 2k\epsilon)N/r = 2k\epsilon N$. By the pigeonhole principle, there is a subset $T \subset U_0$ of size at least $\binom{t}{s}^{-r}|U_0| \geq r(G, K_s)$ such that there are subsets $W_i \subseteq V_i$ with $|W_i| = s$ for $i \geq 1$ such that every vertex in T is complete to each W_i. As Γ and hence the induced subgraph $\Gamma[T]$ is G-free and $|T| \geq r(G, K_s)$, we know that T contains an independent set W_0 of order s. Then $W_0, W_1, \ldots, W_r$ form a complete induced $(r + 1)$-partite subgraph of Γ with parts of size s. □

The next lemma shows that, once we find a large induced complete multipartite subgraph of Γ, we can find a large book in $\overline{\Gamma}$.

Lemma 12.6 *If a K_p-free graph Γ on n vertices contains $K_{p-1}(k)$ as an induced subgraph, then its vertex set can be partitioned into $p-1$ subsets that each span a k-book in $\overline{\Gamma}$.*

Proof. Let $V_1, \ldots, V_{p-1}$ be the $p-1$ parts of the induced $K_{p-1}(k)$. As Γ is K_p-free, each vertex in Γ has no neighbors in some V_i. Partition the vertex set of Γ into $p-1$ parts $U_1, \ldots, U_{p-1}$, where, for each $i \in [p-1]$, each vertex in U_i has no neighbors in V_i. Then each U_i spans a k-book in $\overline{\Gamma}$ with spine V_i. □

The next result is the main form in which we use Lemma 12.5, and follows from it by a simple inductive argument.

Lemma 12.7 *Let k, p, x be positive integers, and let $z = x \cdot (20k)^p$. Let Γ be a K_p-free graph on at least $N = (p-1)(n-1)+1$ vertices, and suppose $S \subseteq V(\Gamma)$ satisfies $|S| \geq z^z \cdot r(K_p, K_z)$. Then either Γ contains a copy of $B_{k,n}$, or else Γ contains $K_{p-1}(x)$ as an induced subgraph, one part of which is a subset of S.*

Proof. For $r = 1, \ldots, p-2$, let $\epsilon_r = (1 - r/(p-1))/(4k)$ so that $(1 - 4k\epsilon_r)/r = 1/(p-1)$. Let $t_{p-1} = x$ and $t_r = t_{r+1}/\epsilon_r$ for $r = p-2, \ldots, 1$. Observe that

$$t_1 = t_{p-1}/\Pi_{r=1}^{p-2}\epsilon_r = x(4k)^{p-2}(p-1)^{p-2}/(p-2)! < (20k)^p x = z.$$

Since $t_1 \geq t_2 \geq \cdots \geq t_{p-1}$, this implies that $t_r < z$ for all r. We now prove by induction on r for $r \in [p-1]$ that Γ contains $K_r(t_r)$ as an induced subgraph, with the first part of $K_r(t_r)$ being a subset of S. For the base case $r = 1$, we have $|S| \geq r(K_p, K_z) > r(K_p, K_{t_1})$, so Γ contains an independent set of order t_1, that is, $\Gamma[S]$ contains $K_r(t_r)$ with $r = 1$ as an induced subgraph. Now suppose Γ contains $K_r(t_r)$ as an induced subgraph, with the first part a subset of S. We apply Lemma 12.5 with $s = t_r + 1$, $t = t_r$, and $G = K_p$. Observe that

$$\binom{t_{r+1}}{t_r}\left(\frac{2kt_{r+1}}{t_r}\right)^{-1} r(K_p, K_{t_{r+1}}) \leq \left(\frac{e}{\epsilon_r}\right)^{rt_r}\left(\frac{2kt_{r+1}}{t_r}\right)^{-1} r(K_p, K_{t_{r+1}})$$
$$\leq z^z \cdot r(K_p, K_z) \leq |S|.$$

So either Γ contains a k-book with at least $(1 - 4k\epsilon r)N/r = N/(p-1) \geq n$ vertices, in which case we are done, or Γ contains an induced $K_{r+1}(t_{r+1})$ whose first r parts are subsets of the r parts of the $K_r(t_r)$. In particular, the first part of this induced $K_{r+1}(t_r)$ is a subset of S. This proves the claimed inductive statement. The desired statement is just then the case $r = p-1$. □

Proof of Theorem 12.9. Let $N = (p-1)(n-1)+1$. Our choice of n guarantees that if $z = k(20k)^p$, then $N \geq z^z \cdot r(K_p, K_z)$. Suppose for the sake of contradiction that there is a K_p-free graph on N vertices such that Γ does not contain a k-book with n vertices. By Lemma 12.7, applied with $S = V(\Gamma)$ and $x = k$, we see that Γ

must contain $K_{p-1}(k)$ as an induced subgraph. But then Lemma 12.5 implies that Γ contains a k-book with n vertices as a subgraph, completing the proof. □

Note that all bounds on the parameters are of tower types whenever the proofs depend on the regularity lemma. Fox, He and Wigderson (2021+) also obtain general goodness results involving books. As pointed out by Fox, He and Wigderson (2021+), it would be very interesting to see how far one can push these ideas. In particular, is it possible to completely eliminate the use of the regularity lemma from the proof of Nikiforov and Rousseau (Theorem 2.1, 2009)?

12.4 Exercises

1. Let a, b, k be positive integers and let $G_1, \cdots, G_k$ be graphs of order n. If in each G_i the number of vertices with degrees greater than a is at most b, then there exists a constant $c = c(a, b, k)$ such that $r(G_1, \cdots, G_k) \le cn$.

2. Prove that B_n is 3-good for $n \ge 2$.

3. Prove that $r_k(K_{1,\ell}, \ldots, K_{1,m}) = \ell + \cdots + m - k + \tau$, where $\tau = 1$ if the number of even integers in $\{\ell, \ldots, m\}$ is even and positive, and $\tau = 2$ otherwise.

4. Prove that $r(K_{1,\ell}, \ldots, K_{1,m}, K_p) = (r-1)(p-1)+1$, where $r = r(K_{1,\ell}, \ldots, K_{1,m})$.

5. Let $G_1, \ldots, G_k$ be connected graphs. Denote $r_1 = r(G_1, \ldots, G_k)$ and $r_2 = r(K_m, \ldots, K_n)$.

(1) Prove that if $r(G_1, \ldots, G_k, K_\ell) = (r_1 - 1)(\ell - 1) + 1$ for any $\ell \ge 2$, then $r(G_1, \ldots, G_k, K_m, \ldots, K_n) = (r_1 - 1)(r_2 - 1) + 1$. (Hint: Omidi and Raeisi, 2011)

(2) Prove taht $r(K_{1,\ell}, \ldots, K_{1,m}, K_p, \ldots, K_q) = (r_1 - 1)(r_2 - 1) + 1$, where $r_1 = r(K_{1,\ell}, \ldots, K_{1,m})$ and $r_2 = r(K_p, \ldots, K_q)$.

(3) Given $p, \ldots, q$, prove $r(C_n, K_p, \ldots, K_q) = (n-1)(r-1) + 1$ for large n, where $r = r(K_p, \ldots, K_q)$.

6.* Prove that for any $k \ge 1$ and large n, $r(K_k, F_n) = 2(k-1)n + 1$. (Hint: Li and Rousseau, 1996)

7.* Let G be a graph of order m, and H a connected graph of order n that contains a suspended path of length ℓ. Let G_1 be a graph from G by deleting an independent set of t vertices, and H_1 a graph from H by shortening the suspended path by 1. Prove that if $\ell \ge (m-2)(m-t) + t + 1$, then

$$r(G, H) \le \max\{r(G, H_1),\ r(G_1, H) + n - 1\}.$$

(Hint: Burr, 1981)

8.* Let k and d be fixed positive integers. Prove that there exists a constant $c = c(k, d) > 0$ such that for any d-degenerate graph H of order n, $r(K_k, H) \le cn$.

9.* Chen, Yu and Zhao (2021) obtained that $4.5n - 5 \le r(F_n) \le 5.5n + 6$. Prove the upper bound.

Chapter 13
Various Ramsey Problems

Ramsey theorem has inspired many striking and difficult problems with a lot of variations. We discuss some of them in this chapter, particularly size Ramsey numbers, bipartite Ramsey numbers, and Folkman numbers, etc.

13.1 Size Ramsey Numbers

For graphs G, G_1 and G_2, let

$$G \to (G_1, G_2)$$

signify that any red-blue edge coloring of G contains a red G_1 or a blue G_2. So Ramsey number $r(G_1, G_2)$ is the smallest N such that $K_N \to (G_1, G_2)$, namely

$$\begin{aligned} r(G_1, G_2) &= \min\{N : K_N \to (G_1, G_2)\} \\ &= \min\{|V(G)| : G \to (G_1, G_2)\}, \end{aligned}$$

where the second equality holds as $G \to (G_1, G_2)$ implies $K_N \to (G_1, G_2)$ with $N = |V(G)|$. As the number of edges $e(G)$ of a graph G is often called the size of G, Erdős, Faudree, Rousseau and Schelp (1978) introduced an idea of measuring minimality with respect to size rather than order of the graphs with $G \to (G_1, G_2)$. Recall the *size Ramsey number*

$$\hat{r}(G_1, G_2) = \min\{e(G) : G \to (G_1, G_2)\}.$$

Directly from the definition, we see that $\hat{r}(G_1, G_2) \le q$ is equivalent to the existence of a graph G with $q = e(G)$ such that $G \to (G_1, G_2)$. However, a statement $\hat{r}(G_1, G_2) > q$ is equivalent to that for any graph G with $e(G) = q$, there is a coloring of $E(G)$ in red and blue, such that there is neither red G_1 nor blue G_2

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that is denoted by $G \nrightarrow (G_1, G_2)$. Needless to say, an edge coloring in two colors is equivalent to a partition of edge set into two subsets.

Lemma 13.1 *Let G_1 and G_2 be graphs. Then*

$$\hat{r}(G_1, G_2) \le \binom{r(G_1, G_2)}{2}.$$

Proof. Set $r = r(G_1, G_2)$. Then $K_r \rightarrow (G_1, G_2)$, so

$$\hat{r}(G_1, G_2) \le e(K_r) = \binom{r}{2}$$

as claimed. □

The following result is due to Chvátal, reported in Erdős, Faudree, Rousseau and Schelp (1978), which indicates that the problem is not new if both G_1 and G_2 are complete graphs.

Theorem 13.1 *Let $r = r(m, n)$. Then*

$$\hat{r}(K_m, K_n) = \binom{r}{2}.$$

Furthermore, if G is a connected graph with $e(G) \le \binom{r}{2}$ such that $G \rightarrow (K_m, K_n)$, then $G = K_r$.

Proof. To avoid the trivial case, we assume that $m, n \ge 2$. Let us begin with an observation. Let the edge set of a graph G be colored by red and blue and let u and v be two non-adjacent vertices of G. Consider the induced edge colorings of graphs $G - u$ and $G - v$. If there is neither a monochromatic K_m in $G - u$ nor a monochromatic K_n in $G - v$, then the same is true in G. The reason is simple. Any assumed monochromatic complete graph in G cannot contain both u and v, since these two vertices are not adjacent. Hence, any monochromatic complete graph would appear in the induced coloring of either $G - u$ or $G - v$.

Set $R = \binom{r}{2}$. Let $G = (V, E)$ be a connected graph of size q with $q \le R$. We shall prove the following claim first.

Claim If $G \ne K_r$, then $G \nrightarrow (K_m, K_n)$.

Proof. Let p be the order of G. We shall prove the claim by induction on p. The claim is certainly true for $p < r = r(m, n)$, since $K_p \nrightarrow (K_m, K_n)$ induces an edge coloring of G for $G \nrightarrow (K_m, K_n)$. So we assume that $p \ge r$. Since $q \le R$ and $G \ne K_r$, we know that G is not a complete graph. Let u and v be two non-adjacent vertices of G and set

$$W = V \setminus \{u, v\},\ H = G - \{u, v\}.$$

If there exist red-blue edge colorings of $G - u$ and $G - v$, respectively, which agree on H, such that there is neither red K_m nor blue K_n, then, by the observation made before, $G \not\rightarrow (K_m, K_n)$. The proof is then to establish such an edge coloring.

Let $X = N_G(u) \cup N_G(v)$. Denote by H' for the graph obtained from H by adding a new vertex x and joining x and each vertex in X. That is to say, the vertices u and v are contracted to a new vertex x in G. Note that the order of H' is $p - 1$ and its size is at most q.

For the reasons which follow, we may assume $H' \neq K_r$, where $r = r(m, n)$. In fact, if $H' = K_r$, then $H = K_{r-1}$ and $X = W$. It is clear $N(u) \cap N(v) = \emptyset$ since $q \leq R$. By taking $w \in W$ with $dist(u, w) = 2$ so $w \in N(v)$, we may simply consider $G - u$ and $G - w$ at the beginning since u and w are not adjacent and $N(u) \cap N(w) \neq \emptyset$. Hence, we can apply the induction hypothesis on H'. The existence of the desired edge coloring which agree on H is manifest. Clearly, the deletion of edges so that H' returns to $G - u$ and $G - v$ spoils nothing so the desired edge coloring of G has been constructed. This proves the claim. □

Now we return to the main proof. Suppose that the size q of a graph G is less than R, then if G is complete, its order is less than r, so $G \not\rightarrow (K_m, K_n)$. If G is not complete, the claim shows that $G \not\rightarrow (K_m, K_n)$. Thus $\hat{r}(K_m, K_n) > q$, in particular $\hat{r}(K_m, K_n) > R - 1$, which and Lemma 13.1 yield the desired assertion. □

We shall see that the values of $\hat{r}(K_{1,m}, K_{1,n})$ and $r(K_{1,m}, K_{1,n})$ are very close.

Theorem 13.2 *For any positive integers m and n,*

$$\hat{r}(K_{1,m}, K_{1,n}) = m + n - 1.$$

Proof. Since in any edge coloring of $K_{1,m+n-1}$ by red and blue, there is either a red $K_{1,m}$ or a blue $K_{1,n}$, we have $\hat{r}(K_{1,m}, K_{1,n}) \leq m + n - 1$. In what follows, we suppose $m \leq n$. Let G be a graph of size $q \leq m + n - 2$. It is clear that G has at most one vertex v with $d(v) \geq n$. If this is the case, then every other vertex u satisfies $d(u) \leq m - 1$. We may color all edges of $G - v$ red. Then at most $m - 1$ edges incident with v have been colored. We shall color other edges incident with v in such a way that there is neither red $K_{1,m}$ nor blue $K_{1,n}$. If every vertex has degree at most $n - 1$, we may color all edges of G in blue. We thus reach a conclusion that $G \not\rightarrow (K_{1,m}, K_{1,n})$. Since G is arbitrary, we have $\hat{r}(K_{1,m}, K_{1,n}) \geq m + n - 1$. □

We shall write $\hat{r}(G)$ for $\hat{r}(G, G)$. The above theorem gives that $\hat{r}(K_{1,n}) = 2n - 1$. However, it is difficult to find the exact values of $\hat{r}(P_n)$. With an impressive proof and disproving a conjecture of Erdős (1981), Beck (1983) showed that $\hat{r}(P_n) \leq 900n$. In the proof he used transforms defined by Pósa (1976) for finding an upper threshold function $p = c \log n/n$ in random graph $G \in \mathcal{G}(n, p)$ of being Hamiltonian. Bollobás (2001) noted a better bound, and the current best upper bound due to Dudek and PraLat (2017) gives that $\hat{r}(P_n) \leq 74n$. In this section, we will include a slightly weak result by Letzter (2016).

The following lemma was obtained independently by Dudek and PraLat (2015) and Pokrovskiy (2014).

Lemma 13.2 *For every graph G there exist two disjoint subsets $U, W \subseteq V(G)$ of equal size such that there are no edges between them and $G \setminus (U \cup W)$ has a Hamilton path.*

Proof. In order to find sets with the desired properties, we apply the following algorithm, maintaining a partition of $V(G)$ into subsets U, W and a path P. Start with $U = V(G)$, $W = \emptyset$ and P an empty path. At each stage of the algorithm, do the following. If $|U| \leq |W|$, stop. Otherwise, if P is empty, move a vertex from U to P (note that $U \neq \emptyset$). If P is non-empty, let v be its endpoint. If v has a neighbor $u \in U$, then put u in P, otherwise move v to W.

Note that at any given point in the algorithm there are no edges between U and W. Moreover, $|U| - |W|$ is positive at the beginning of the algorithm and decreases by one at every stage, thus at some point the algorithm will stop and will produce sets U, W with the required properties. □

It is easier to use the following immediate consequence of Lemma 13.2.

Corollary 13.1 *If G is a balanced bipartite graph on n vertices with bipartition $\{V_1, V_2\}$ which has no path of length k, then there exist disjoint subsets $X_i \subseteq V_i$ such that $|X_1| = |X_2| \geq (n - k)/4$ and $e_G(X, Y) = 0$.*

Proof. By Lemma 13.2, there exist disjoint subsets $U, W \subseteq V(G)$ of equal size such that $e_G(U, W) = 0$ and $V(G) \setminus (U \cup W)$ has a Hamilton path P. Note that P must alternate between V_1 and V_2 and has an even number of vertices, implying that $|V_1 \cap V(P)| = |V_2 \cap V(P)|$. It follows that $|U_1| + |W_1| = |U_2| + |W_2|$, where $U_i = U \cap V_i$ and $W_i = W \cap V_i$. Since $|U| = |W|$, we conclude that $|U_1| = |W_2|$ and $|U_2| = |W_1|$. Without loss of generality, suppose that $|U_1| \geq |U_2|$. Then $|U_1| = |W_2| \geq (n - |V(P)|)/4 \geq (n - k)/4$. Take $X_1 = U_1$ and $X_2 = W_2$. □

The following is an easy consequence of Corollary 13.1.

Corollary 13.2 *If G is a graph on n vertices such that $G \nrightarrow P_{k+1}$, then there exist disjoint subsets $X, Y \subseteq V(G)$ of size at least $(n - 2k)/4$ such that $e_G(X, Y) = 0$.*

Proof. Consider a red-blue coloring of edges of G with no monochromatic P_{k+1}. Let G_R and G_B be the graphs induced by all red and blue edges, respectively. Since G_R contains no P_{k+1}, we can apply Lemma 13.2 to G_R to obtain disjoint sets U and W, both of size at least $(n - k)/2$, with no red edges between them. Now we consider the subgraph $G_B[U, W]$ induced by blue edges between U and W. Since $G_B[U, W]$ contains no P_{k+1}, it follows from Corollary 13.1 that there exist sets $X \subseteq U$, $Y \subseteq W$ of size at least $(n - 2k)/4$, with no blue edges between them. We conclude that there are no edges of G between X and Y. □

We now have the following lemma.

Lemma 13.3 *Let $c = 4.86$, $d = 7.7$ and $G = G(cn, d/n)$. Then w.h.p. (with high probability) the following two conditions hold.*

(i) $|E(G)| \leq (1 + o(1))c^2 dn/2$.

(ii) For every two disjoint sets $U, W \subseteq V(G)$ of size at least $(c - 2)n/4$, we have $e_G(U, W) > 0$.

Proof. The number of edges in G is a binomial random variable with mean

$$\binom{cn}{2}\cdot\frac{d}{n}=(1+o(1))\frac{c^2d}{2}n.$$

Thus (i) follows immediately from the concentration of binomial random variables around their mean.

For (ii), we will apply the first moment method. Let Z denote the number of pairs (U,W) of disjoint subsets of $V(G)$ of size $(c-2)n/4$ with $e_G(U,W)=0$. The expectation of Z satisfies the following, where $\alpha=(c-2)/4$:

$$\begin{aligned}E(Z)&=\binom{cn}{\alpha n}\binom{(c-\alpha)n}{\alpha n}\left(1-\frac{d}{n}\right)^{(\alpha n)^2}\\&\le\frac{(cn)!}{((\alpha n)!)^2((c-2\alpha)n)!}\exp\{-d\alpha^2n\}\le\exp\{\beta n\}.\end{aligned}$$

By Stirling's formula, we can take

$$\beta=c\log c-2\alpha-(c-2\alpha)\log(c-2\alpha)-d\alpha^2\le-0.0005.$$

It follows that $E(Z)\to 0$ as $n\to\infty$, implying that w.h.p. $Z=0$, and hence (ii) holds. □

The constants c,d in the above lemma were chosen to minimize the number of edges in G under condition (ii).

Theorem 13.3 *For all large n,*

$$\hat{r}(P_n)\le 91n.$$

Proof. Let $c=4.86$ and $d=7.7$. Take a graph $G\in\mathcal{G}(cn,d/n)$ such that it satisfies conditions (i) and (ii) in Lemma 13.3. If $G\not\to P_n$ then Corollary 13.2 implies that there exist disjoint subsets $X,Y\subseteq V(G)$ of size at least $(c-2)n/4$ such that $e_G(X,Y)=0$, contradiction condition (ii) from Lemma 13.3. We conclude that $G\to P_n$. Note that $|E(G)|\le 91n$ by condition (i) of Lemma 13.3, it follows that $\hat{r}(P_n)\le 91n$ for large n as desired. □

Friedman and Pippenger (1987) generalized Beck's linear bound of $\hat{r}(P_n)$ for proving that there is some constant $c=c(\Delta)>0$ such that $\hat{r}(T_n)\le cn$ for any tree T_n on n vertices and maximum degree Δ. Subsequently, this was improved by Ke (1993), and Haxell and Kohayakawa (1995). Let $V(T)=V_0(T)\cup V_1(T)$ be the partition determined by the unique proper two-coloring of the vertex set of $V(T)$. Set $\Delta_i=\max\{d_T(v):v\in V_i(T)\}$ and $n_i=|V_i(T)|$ for $i=0,1$ and let $\beta(T)=n_0\Delta_0+n_1\Delta_1$. Solving a conjecture of Beck (1990), Dellamonica (2012) proved that $\hat{r}(T)=\Theta(\beta(T))$.

Beck (1990) even asked if there is some constant $c=c(\Delta)>0$ such that $\hat{r}(G)\le cn$ for any graph with n vertices and maximum degree at most Δ. Rödl and Szemerédi (2000) answered the question of Beck negatively for even $\Delta = 3$. By applying the sparse regularity lemma, Kohayakawa, Rödl, Schacht and Szemeredi (2011) proved that $\hat{r}(G) \le cn^{2-1/\Delta}(\log n)^{1/\Delta}$ for any graph G with n vertices and maximum degree at most Δ. For the size Ramsey number of cycle, Haxell, Kohayahawa and Łuczak (1995) proved that $\hat{r}(C_n) \le cn$. This upper bound has been improved by Javadi, Khoeini, Omidi and Pokrovskiy (2019) to that $\hat{r}(C_n) \le 10^5 \times cn$, where $c = 6.5$ is n is even and $c = 1989$ otherwise.

The size Ramsey numbers of graphs with bounded degrees can not be bounded linearly as mentioned above even for maximum degree $\Delta = 3$. However, $\hat{r}(K_{m,n})$ has a linear upper bound if m is fixed. The following result was proved by Erdős, Faudree, Rousseau and Schelp (1978).

Theorem 13.4 *For any fixed positive integer m, if n is sufficiently large, then*

$$\frac{1}{2e}m2^m n \le \hat{r}(K_{m,n}) \le 4m^2 2^m n.$$

Let us have a lemma at first.

Lemma 13.4 *Suppose that G is a subgraph of $K_{M,N}$ with $e(G) \ge Np$ and*

$$N\binom{p}{m} > (n-1)\binom{M}{m},$$

then G contains $K_{m,n}$.

Proof. The proof is similar to that of Theorem 8.4. The key for the proof is so called "double counting argument". Without loss of generality, we assume that G is a spanning subgraph of $K_{M,N}$ that contains no $K_{m,n}$. Let the bipartition of $K_{M,N}$ be X and Y with $|X| = M$ and $|Y| = N$, and let $d_1, d_2, \ldots, d_N$ be the degree sequence of vertices in Y of G. For any vertex $v \in Y$, an m-set in neighborhood of v is covered by at most $n-1$ vertices in Y. So

$$\sum_{k=1}^{N}\binom{d_k}{m} \le (n-1)\binom{M}{m}.$$

The left hand side is at least $N\binom{p}{m}$ by the convexity of the function $\binom{x}{m}$ since $\left(\sum d_k\right)/N = e(G)/N \ge p$, which leads to a contradiction. □

Proof of Theorem 13.4. The assertion is obvious for $m = 1$ by Theorem 13.2, so we assume $m \ge 2$. For the upper bound, let us consider a complete bipartite graph $K_{M,N}$ on bipartition (A, B) and an edge partition (E_1, E_2). We may assume that $|E_1| \ge MN/2$. Hence, by setting $p = M/2$ in Lemma 13.4, the subgraph induced by E_1 contains $K_{m,n}$ if

$$N\binom{M/2}{m} > (n-1)\binom{M}{m}.$$

This will certainly be the case if we set $N = \left\lfloor \binom{M}{m} n / \binom{M/2}{m} \right\rfloor$. It follows that for all $M \geq 2m$,

$$\hat{r}(K_{m,n}) \leq MN \leq Cn,$$

where

$$C = \frac{M\binom{M}{m}}{\binom{M/2}{m}} = 2^{m-1} M \frac{(M-1)(M-2)\cdots[M-(m-1)]}{(M-2)(M-4)\cdots[M-2(m-1)]}.$$

By taking $M = \lfloor m^2/2 \rfloor$, we have $C \leq 4m^2 2^m$ for $m \leq 8$. For $m \geq 9$,

$$C \leq m^2 2^{m-2} \left(1 + \frac{2(m-1)}{(m-2)^2}\right)^{m-1} \leq m^2 2^{m-2} \exp\left(\frac{2(m-1)^2}{(m-2)^2}\right) < 4m^2 2^m,$$

where we use the facts that $1 + x \leq e^x$ and the minimum value of $\frac{2(m-1)^2}{(m-2)^2}$ attains at $m = 9$, so the desired upper bound follows.

The proof of the lower bound employs the probabilistic method. Suppose that $G = (V, E)$ is a graph in which every edge coloring in two colors produces a monochromatic $K_{m,n}$. Let

$$V_k = \{v \in V : d(v) \geq k\}.$$

Then $|V_k| \leq 2|E|/k$. If G contains $K_{m,n}$ on bipartition (A, B) as a subgraph, then, clearly, $A \subseteq V_n$ and $B \subseteq V_m$. Hence, setting $M = |V_n|$ and $N = |V_m|$, it must be true that every two-coloring of edges of $K_{M,N}$ produces a monochromatic $K_{m,n}$. However, in a random edge coloring of $K_{M,N}$ in red and blue in which

$$\Pr[e \text{ is red}] = \Pr[e \text{ is blue}] = \frac{1}{2},$$

the probability that there is a monochromatic $K_{m,n}$ is at most

$$\frac{2\binom{M}{m}\binom{N}{n}}{2^{mn}} \leq 2\left(\frac{eM}{m}\right)^m \left(\frac{eN}{n}\right)^n \frac{1}{2^{mn}} = 2\left(\frac{eM}{m}\right)^m \left(\frac{eN}{2^m n}\right)^n.$$

Now suppose that $e(G) < m2^{m-1}n/e$. Then $M = |V_n| < m2^m/e$ and $N = |V_m| < n2^m/e$ so that $eN/(2^m n) \leq 1 - \epsilon$ for some $\epsilon > 0$. It follows that if n is sufficiently large, then the probability there is a monochromatic $K_{m,n}$ is less than one. Hence, for any graph G if $e(G) < m2^{m-1}n/e$, then $G \not\rightarrow (K_{m,n})$, which follows by the desired lower bound follows. □

It is natural to believe that $\hat{r}(K_{n,n}) = (2 + o(1))^n$, which is exactly the case. The following result is due to Erdős and Rousseau (1993).

Theorem 13.5 *For all large integer n,*

$$\frac{1}{30} n^2 2^n < \hat{r}(K_{n,n}) < \frac{3}{2} n^3 2^n.$$

Proof. The proof for the upper bound is similar with that in Theorem 13.4 as if M and N satisfy

$$N\binom{M/2}{n} > (n-1)\binom{M}{n},$$

then $K_{M,N} \rightarrow K_{n,n}$. In particular, if we set $N = \lfloor n^2/2 \rfloor$ and $M = 3n2^n$, then the above holds for all $n \geq 6$ and we have the upper bound as desired.

The proof of the lower bound depends on the following counting result.

Lemma 13.5 *A graph with q edges contains at most*

$$\left(\frac{2eq}{n}\right)\left(\frac{2e^2q}{n^2}\right)^n$$

copies of $K_{n,n}$.

Proof. Let $G = (V, E)$ be a graph with q edges and let

$$m = \left\lceil \frac{n}{2} \log\left(\frac{2q}{n^2}\right) \right\rceil.$$

Without loss of generality, we assume that G contains no isolated vertices. To distinguish the magnitude of the degrees of vertices, we set

$$d_{-1} = 1,\ d_k = ne^{k/n}\ (k = 0, 1, \ldots, m),\ d_{m+1} = \infty,$$

and set

$$X_k = \{x \in V : d_k \leq d(x) < d_{k+1}\}$$

for $k = -1, 0, \ldots, m$. Then $X_{-1}, X_0, \ldots, X_m$ form a partition of V. Let

$$W_k = \cup_{j=k}^m X_j = \{x \in V : d(x) \geq d_k\}.$$

Let us say that a subgraph $K_{n,n}$ in G on vertex set $U \subseteq V$ is of *type* k if k is minimum such that $X_k \cap U \neq \emptyset$. Equivalently a copy of $K_{n,n}$ on U is of type k if and only if $d_k \leq \min\{d_G(v) : v \in U\} < d_{k+1}$, where $d_G(v)$ is the degree of v in G not that in $K_{n,n}$. Denote by M_k for the number of type k copies of $K_{n,n}$ in G. Note there is no $K_{n,n}$ of type -1 and $M = \sum_{k=0}^m M_k$ is the total number of copies of $K_{n,n}$ in G. In a type k copy of $K_{n,n}$ every vertex belongs to W_k and at least one vertex belongs to X_k. Thus one side of the $K_{n,n}$ is an n-element subset of the neighborhood of a vertex in X_k and the other side is an n-element subset of W_k. It follows that

$$M_k \leq |X_k|\binom{d_{k+1}}{n}\binom{|W_k|}{n}.$$

Note that $\binom{N}{n} \leq (eN/n)^n$ and $|W_k| \leq 2q/d_k$, so we have that for $k = 0, 1, \ldots, m-1$,

$$M_k \leq |X_k|\left(\frac{ed_{k+1}}{n}\right)^n\left(\frac{2eq}{d_k n}\right)^n = e|X_k|\left(\frac{2e^2q}{n^2}\right)^n.$$

Since $d_m \geq \sqrt{2q}$, we have that $|X_m| \leq \sqrt{2q}$, and each vertex of a type m subgraph must belong to X_m. Thus

$$M_m \leq \binom{|X_m|}{n}^2 \leq \left(\frac{2e^2q}{n^2}\right)^n.$$

If $|X_m| = 0$, then $M_m = 0$. Hence we can write

$$M_m \leq e|X_m|\left(\frac{2e^2q}{n^2}\right)^n,$$

which coincides with the upper bound of M_k with $0 \leq k \leq m-1$. Therefore we obtain

$$M = \sum_{k=0}^{m} M_k \leq \sum_{k=0}^{m} e|X_k|\left(\frac{2e^2q}{n^2}\right)^n = e|W_0|\left(\frac{2e^2q}{n^2}\right)^n \leq \left(\frac{2eq}{n}\right)\left(\frac{2e^2q}{n^2}\right)^n,$$

completing the proof. □

Proof for the lower bound in Theorem 13.5. Now let G be an arbitrary graph with q edges with $q \leq n^2 2^n/30$. Consider a random red-blue edge coloring of G in which each edge is red with probability $1/2$ and colorings of distinct edges are independent. In view of the lemma, we find that the probability P that such a random coloring yields a monochromatic $K_{n,n}$ satisfies

$$P < 2\left(\frac{2eq}{n}\right)\left(\frac{2e^2q}{n^2}\right)^n 2^{-n^2} \leq \frac{2en}{15}\left(\frac{4e^2q}{n^2}\right)^n 2^{-n^2} \leq n\left(\frac{2e^2}{15}\right)^n \to 0$$

as $n \to \infty$. Thus $G \not\to (K_{n,n})$, and the desired lower bound follows. □

13.2 Induced Ramsey Numbers⋆

In the definition of Ramsey number $r(K_n)$, we ask what is the smallest N such that in any red-blue edge coloring of K_N, there is either a red K_n or a blue K_n. Here the clique K_n is an induced subgraph. Let us change the problem for a general graph H and ask what for a graph G such that in any red-blue edge coloring of G, there is a monochromatic *induced* graph H. This slightly modification changes the problem dramatically. A substantial question is whether or not such graph G exists for an arbitrary given graph H. We call such a graph G a *Ramsey graph* for H. This should be different from a definition in Chapter 1, where a graph G was called a Ramsey graph for a Ramsey number $r(H)$ if the order of G is $r(H) - 1$ such that neither G nor $\bar{G}$ contains H as a subgraph, in which the subgraph is not necessarily induced.

The following existence theorem was proved independently by Deuber (1975), by Erdős, Hajnal and Pósa (1975), and by Rödl (1973).

Theorem 13.6 *Every graph has a Ramsey graph.*

The remainder of the section is the proof for this theorem, which begins with its version on bipartite graphs.

We shall write a bipartite graph B as triples (V_1, V_2, E), where V_1 and V_2 are bipartition of B and E is the edge set. Given another bipartite graph $\bar{B} = (\bar{V}_1, \bar{V}_2, \bar{E})$, if B is isomorphic to an *induced* subgraph of $\bar{B}$, in which V_i corresponds $\bar{V}_i$ for $i = 1, 2$, then there is an injective map $\phi : V_1 \cup V_2 \to \bar{V}_1 \cup \bar{V}_2$ so that $\phi(V_i) \subseteq \bar{V}_i$ for $i = 1, 2$, and $\phi(v_1)\phi(v_2) \in \bar{E}$ if and only $v_1 v_2 \in E$. We will call such a map ϕ as an *embedding* of B in $\bar{B}$, denoted by $\phi : B \to \bar{B}$.

As before, let $X^{(k)}$ denote the family of all k-subsets of X. Define a bipartite graph $(X, X^{(k)}, E_k)$, in which the edge set E_k contains all edges of the form xY with $Y \in X^{(k)}$ and $x \in Y$.

Lemma 13.6 *Every bipartite graph $B = (V_1, V_2, E)$ can be embedded in a bipartite graph of the form $\bar{B} = (X, X^{(k)}, E_k)$.*

Proof. Let B be a bipartite graph with vertex classes $V_1 = \{a_1, a_2, \ldots, a_n\}$ and $V_2 = \{b_1, b_2, \ldots, b_m\}$. Denote by X for a set of $2n + m$ vertices as

$$X = \{x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_m\}.$$

Let $\bar{B} = (X, X^{(n+1)}, E_{n+1})$. We shall show that there is an embedding ϕ of B in $\bar{B}$.

Set $\phi(a_i) = x_i$ for $i = 1, \ldots, n$ first. Then for any $j = 1, \ldots, m$, a member $Y \in X^{(n+1)}$ will be chosen as $\phi(b_j)$, which is denoted by Y_j. We in fact construct Y_i as follows. Since $\phi(b_j) = Y_j$, we have that the neighbors of b_j in B should be mapped into Y_j, that is to say,

$$\phi(N(b_j)) \subseteq Y_j.$$

In order to distinguish different Y_j, we put z_j into Y_j as a label. So far we have chosen

$$|\phi(N(b_j))| + 1 = d(b_j) + 1 \le n + 1$$

elements for Y_j. If the inequality is strict, we then simply fill Y_j with elements from $\{y_1, \ldots, y_n\}$ until Y_j has $n + 1$ elements.

Note that $\phi(V_1) = \{x_1, \ldots, x_n\}$ and $\phi(V_2) = \{Y_1, \ldots, Y_m\}$, and x_i is a neighbor of Y_j in $\bar{B}$ if and only if a_i is a neighbor of b_j in B. The other neighbors of Y_j are out of $\{x_1, \ldots, x_n\}$. Thus B is isomorphic to the subgraph induced by $\{x_1, \ldots, x_n\}$ and $\{Y_1, \ldots, Y_m\}$ in $\bar{B}$. It follows that the map ϕ is an embedding of B in $\bar{B}$ as desired. □

The next lemma is the bipartite case of Theorem 13.6, which says that every bipartite graph has a bipartite Ramsey graph.

Lemma 13.7 *Let B be a bipartite graph. Then there exists a bipartite graph $\bar{B}$ such that for any two-coloring of edges of $\bar{B}$ there is an embedding $\phi : B \to \bar{B}$ in which all edges of $\phi(B)$ are monochromatic.*

Proof. It is easy to see that if ϕ_1 is an embedding of B_1 in B_2 and ϕ_2 is an embedding of B_2 in B_3, then $\phi_2\phi_1$ is an embedding of B_1 in B_3. Thus by Lemma 13.6 we may assume that $B = (X, X^{(k)}, E_k)$. Let $n = |X|$, $\ell = 2k - 1$ and $s = 2\binom{\ell}{k}$. Let $\bar{X}$ be a set whose cardinality is an s-color Ramsey number for ℓ-uniform hypergraph as

$$|\bar{X}| = r_s^{(\ell)}(kn + k - 1).$$

We shall show the assertion in the lemma with $\bar{B} = (\bar{X}, \bar{X}^{(\ell)}, E_\ell)$.

Let us fix a coloring on E_ℓ with colors α and β. Among $\ell = 2k-1$ edges incident to a vertex $\bar{Y} \in \bar{X}^{(\ell)}$ in $\bar{B}$, at least k of them are monochromatic. Define $\bar{Z} \subseteq \bar{Y}$ with $|\bar{Z}| = k$ so that all edges $\bar{x}\bar{Y}$ for $\bar{x} \in \bar{Z}$ are in the same color. The color and the set $\bar{Z}$ are called as the color and the set associated with $\bar{Y}$, respectively.

Assign a linear order to $\bar{X}$ as

$$\bar{X} = \{\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots\}.$$

For every $\bar{Y} \in \bar{X}^{(\ell)}$ with $\bar{Y} = \{\bar{x}_{i_1}, \ldots, \bar{x}_{i_\ell}\}$, denote by $\sigma_{\bar{Y}}$ for the order-preserving map with $\sigma_{\bar{Y}}(\bar{x}_{i_j}) = j$, then $\sigma_{\bar{Y}}(\bar{Z}) \in [\ell]^{(k)}$.

We now color $\bar{X}^{(\ell)}$ with $s = 2\binom{\ell}{k}$ elements of the set $[\ell]^{(k)} \times \{\alpha, \beta\}$ as colors. For a given $\bar{Y} \in \bar{X}^{(\ell)}$, color $\bar{Y}$ with the pair $(\sigma_{\bar{Y}}(\bar{Z}), \gamma)$, where $\bar{Z}$ and $\gamma \in \{\alpha, \beta\}$ are the set and the color associated with $\bar{Y}$, respectively. By the definition of Ramsey number for uniform hypergraph, we know that there is $W \subseteq \bar{X}$ with $|W| = kn + k - 1$ such that all elements of $W^{(\ell)}$ are monochromatic. Thus there exists $S \in [\ell]^{(k)}$ and a color γ, say α, such that all $\bar{Y} \in W^{(\ell)}$ are colored (S, α). That is to say, all $\bar{Y} \in W^{(\ell)}$ satisfy that $\sigma_{\bar{Y}}(\bar{Z}) = S$ and they are all associated with the same color α.

We now construct the desired embedding ϕ of B in $\bar{B}$. The elements of W have an order preserved from that in $\bar{X}$. Without loss of generality, we assume that W contains the first $kn + k - 1$ elements of $\bar{X}$,

$$W = \{\bar{x}_i : \ i = 1, 2, \ldots, kn + k - 1\}.$$

Set

$$X = \{x_1, x_2, \ldots, x_n\},$$

and define $\phi(x_i) = \bar{x}_{ik}$. Denote by w_i for $\bar{x}_{ik}$, so $\phi(X) = \{w_1, \ldots, w_n\}$ and there are exactly $k - 1$ elements of W between w_i and w_{i+1} in the order.

We then define ϕ on $X^{(k)}$. Given $Y \in X^{(k)}$, we shall choose $\phi(Y) = \bar{Y} \in \bar{X}^{\ell}$ so that the neighbors of $\bar{Y}$ among the vertices in $\phi(X)$ are precisely the images of the neighbors of Y in B, i.e., the vertices $\phi(x)$ with $x \in Y$, and so that all these edges incident to $\bar{Y}$ in $\bar{B}$ are colored α. To find such $\bar{Y}$, we first construct its subset $\bar{Z}$ as $\{\phi(x) : x \in Y\}$, which are k vertices of type w_i. Then extend $\bar{Z}$ by $\ell - k$ further vertices $u \in W \setminus \phi(X)$ to a set $\bar{Y} \in W^{(\ell)}$, in such a way that $\sigma_{\bar{Y}}(\bar{Z}) = S$. This is possible since there are $k - 1 = \ell - k$ other vertices of W between w_i and w_{i+1}. Then

$$\bar{Y} \cap \phi(X) = \bar{Z} = \{\phi(x) : x \in Y\},$$

so $\bar{Y}$ has the right neighbors in $\phi(X)$, and all the edges between $\bar{Y}$ and these neighbors are colored α. The images $\bar{Y}$ of different vertices Y are distinct since their intersections with $\phi(X)$ differ, so ϕ is an injective map on $X^{(k)}$. Hence, the map ϕ is indeed an embedding of B in $\bar{B}$ satisfied the desired condition. □

Proof of Theorem 13.6. The idea of the proof is to reduce the general case of the theorem to the bipartite case, where Lemma 13.7 can be employed. Let H be a given graph of order s and let $n = r(s, s)$. Then in any edge coloring of K_n by two colors, there is a monochromatic K_s hence a monochromatic copy of H. Denote by K for this K_n. Note that the monochromatic subgraph H in K may be not an induced subgraph in that color.

We will construct a graph G_0. Let $\ell = \binom{n}{s}$. Arrange the vertices of K in a column, and replace every vertex by a row of ℓ vertices. In each of the ℓ columns, choose an $s-$set so that any pair of such sets contain vertices coming from different rows. Let us furnish each column in the chosen $s-$set with the edges of a copy of H. The graph G_0 consists of ℓ disjoint copies of H and $(n - s)\ell$ isolated vertices.

We define G_0 formally as follows. Assume that $V(K) = \{1, \ldots, n\}$ and choose copies $H_1, \ldots, H_\ell$ of H in K with pairwise distinct (not necessarily disjoint) vertex sets. We then define

$$V(G_0) = \{(i, j) : i = 1, \ldots, n; j = 1, \ldots, \ell\}$$
$$E(G_0) = \cup_{j=1}^{\ell}\{(i, j)(i', j) : ii' \in E(H_j)\},$$

where the end vertices of each edge $(i, j)(i', j)$ are in the same column.

Applying Lemma 13.7 iteratively to all the pairs of rows of G_0, we construct a large graph G such that for every edge coloring of G there is an induced copy of G_0 in G that is monochromatic on all the bipartite subgraph induced by its pairs of rows. By contracting its rows the projection of this $G_0 \subseteq G$ to $\{1, \ldots, n\}$, we define an edge coloring of K. Thus one of the $H_j \subseteq K$ will be monochromatic. But this H_j occurs with the same coloring in the jth column of G_0, and it is an induced subgraph of G_0 hence an induced subgraph of G. We omit the formal definition of desired embedding map described the above procedure. □

Define the *induced Ramsey number* $r_{ind}(H)$ to be the minimum n such that there exists a graph G on n vertices satisfying that every 2-edge-coloring of G contains a monochromatic induced copy of H in G. Theorem 13.6 implies that $r_{ind}(H)$ exists. Erdős (1975) conjectured that there is a constant c such that every graph H on k vertices satisfies $r_{ind}(H) \le 2^{ck}$. The result of Rödl (1973) implies that this conjecture holds if H is bipartite. Erdős and Hajnal (1984) proved that $r_{ind}(H) \le 2^{2^{k^{1+o(1)}}}$ holds for every graph H on k vertices. Kohayakawa, Prömel, and Rödl (1998) improved this bound substantially and showed that if a graph H has k vertices and chromatic number χ, then $r_{ind}(H) \le k^{ck\log\chi}$, where c is a constant. In particular, their result implies that $r_{ind}(H) \le k^{ck(\log k)^2}$ for any graph on k vertices. In their proof, the graph G which gives this bound is randomly constructed using projective planes. For more special classes of (sparse) graphs, see e.g. Beck (1990) in which the author considered the case when H is a tree; Haxell, Kohayakawa, and

Łuczak (1995) proved that the induced Ramsey number of cycle C_k is linear in k; Łuczak and Rödl (1996) showed that the induced Ramsey number of a graph with bounded degree is at most polynomial in the number of its vertices, which confirms a conjecture of Trotter; Fox and Sudakov (2008) obtained that there is a positive constant c such that $r_{ind}(H) \le k^{cd \log \chi}$ for every d-degenerate graph H with k vertices and chromatic number $\chi \ge 2$.

13.3 Bipartite Ramsey Numbers

It is likely that there are some similarities between Ramsey numbers of complete graphs and that of complete bipartite graphs, such as $r(K_n)$ and $r(K_{n,n})$. However, the Ramsey number of bipartite graphs has the bipartite version. Let B_1 and B_2 be bipartite graphs. We define the *bipartite Ramsey number* $br(B_1, B_2)$ to be the smallest integer N such that in any red-blue edge-coloring of $K_{N,N}$, there is a red B_1 or a blue B_2. As usual, we shall write $br(B, B)$ as $br_2(B)$ or $br(B)$. An obvious relation is as follows.

Lemma 13.8 *Let B_1 and B_2 be bipartite graphs. Then*

$$r(B_1, B_2) \le 2\, br(B_1, B_2).$$

Proof. Set $N = br(B_1, B_2)$. Consider a red-blue edge-coloring of K_{2N} on the vertex set $X \cup Y$, where X and Y are disjoint and $|X| = |Y| = N$. The coloring induces an edge-coloring of $K_{N,N}$ on the bipartition X and Y, thus we have a red B_1 or a blue B_2 from the definition for N. □

The following result is due to Thomason (1982).

Theorem 13.7 *For any integers $n \ge m \ge 1$,*

$$br(K_{m,n}) \le 2^m(n-1) + 1.$$

The result will follow the following lemma immediately. Let G be a bipartite graph, whose first vertex class is X and the second is Y. As mentioned in Section 2 of Chapter 7, we signify the fact that $K_{m,n}$ is a subgraph of G with m vertices in X and n vertices in Y by saying that $K_{(m,n)}$ is a subgraph of G.

Lemma 13.9 *Let $N = 2^m(n-1)$ with $n \ge m \ge 1$. Suppose that the edges of $K_{N,N}$ are red-blue colored such that there is neither a red $K_{(m,n)}$ nor a blue $K_{(n,m)}$. Then each vertex of $K_{N,N}$ is incident with exactly $N/2$ red edges and $N/2$ blue edges unless $m = n = 2$. Furthermore, any red-blue edge colored $K_{N,N+1}$ yields a red $K_{(m,n)}$ or a blue $K_{(n,m)}$.*

Proof. Observe that if the first part of the lemma is proven, then the second part follows. Suppose that the edges of $K_{N,N+1}$ are red-blue colored without a red $K_{(m,n)}$ nor a blue $K_{(n,m)}$. Choose $x \in X$ and $y_1, y_2 \in Y$ such that xy_1 is red and xy_2 is blue. Then for $i = 1, 2$, $K_{N,N+1} - \{y_i\}$ has a red-blue edge coloring with neither red $K_{(m,n)}$ nor blue $K_{(n,m)}$ so x is incident with exactly $N/2$ red edges in each. This is impossible since x is incident with more red edges in $K_{N,N+1} - \{y_2\}$ than that in $K_{N,N+1} - \{y_1\}$.

The proof for the first part is by induction on m. Let the vertex sets of $K_{N,N}$ be X and Y with $|X| = |Y| = N = 2^m(n-1)$. Suppose its edges are red-blue colored that contains neither a red $K_{(m,n)}$ nor a blue $K_{(n,m)}$. If $m = 1$, then $N = 2(n-1)$ and $K_{N,N}$ has at most $N(n-1)$ red edges and at most $N(n-1)$ blue edges. So

$$N^2 = \sum_{x\in X} d_R(x) + \sum_{y\in Y} d_B(y) \le 2N(n-1) = N^2,$$

hence the equalities hold throughout, implying that $d_R(x) = d_B(y) = n-1$ for any $x \in X$ and $y \in Y$.

Let $m \ge 2$ be fixed. Suppose that the theorem holds for smaller values of m. Let $t = N/2 = 2^{m-1}(n-1)$. Suppose that there is vertex $x_0 \in X$ with $d_R(x_0) > t$.

Case 1 There is a vertex $y_0 \in Y$ with $d_B(y_0) > t$.

We assume that the edge x_0y_0 is blue, say. Consider the subgraph induced by $N_R(x_0)$ and $N_B(y_0) \setminus \{x_0\}$. This is a complete bipartite graph. Its vertex class in X has at least t vertices and that in Y has at least $t + 1 = 2^{m-1}(n-1) + 1$ vertices. We thus have a complete bipartite graph $K_{t+1,t}$. From the induction hypothesis, we have either a red $K_{(m-1,n)}$ or a blue $K_{(n,m-1)}$. Together with the vertex x_0 or y_0, we get a red $K_{(m,n)}$ or a blue $K_{(n,m)}$. This is impossible. The situation is similar if we assume that the edge x_0y_0 is red.

Case 2 Each vertex $y \in Y$ satisfies that $d_B(y) \le t$.

We shall show that for $1 \le s \le m-1$, a red $K_{(s,2^{m-s}(n-1)+1)}$ yields a red $K_{(s+1,2^{m-s-1}(n-1)+1)}$. The existence of such a red $K_{(s,2^{m-s}(n-1)+1)}$ for $s = 1$ is just the condition $d_R(x_0) > t$ as given. Suppose that there is a red $K_{(s,2^{m-s}(n-1)+1)}$ on vertex classes P and Q with $P \subseteq X$, $|P| = s$, $Q \subseteq Y$, $|Q| = 2^{m-s}(n-1) + 1 = 2\lambda + 1$, where $\lambda = 2^{m-s-1}(n-1)$. Then the number of red edges between $X \setminus P$ and Q is at least $|Q|(N/2 - s)$ since each $y \in Y$ satisfies $d_R(y) \ge N/2$.

Suppose that each $x \in X \setminus P$ satisfies $|N_R(x) \cap Q| \le \lambda = 2^{m-s-1}(n-1)$. Then

$$|X \setminus P|\lambda \ge |Q|(N/2 - s),$$

which gives

$$(N - s)\lambda \ge (2\lambda + 1)(N/2 - s).$$

We thus have

$$s(\lambda + 1) \ge N/2 = 2^{s+1}\lambda,$$

which is impossible unless $s = 1$ and $m = n = 2$.

So we assume that some $x \in X \setminus P$ satisfies $|N_R(x) \cap Q| \ge 2^{m-s-1}(n-1) + 1$. Thus $N_R(x) \cap Q$ and $P \cup \{x\}$ induce a red $K_{(s+1,2^{m-s-1}(n-1)+1)}$ as desired. We hence obtain a red $K_{(m,n)}$ with $m = s + 1$.

Therefore, in any edge-coloring of $K_{N,N}$ that contains neither a red $K_{(m,n)}$ nor a blue $K_{(n,m)}$, there is no vertex x with $d_R(x) > N/2$. So each vertex $x \in X$ satisfies $d_R(x) \le N/2$. Similarly each $y \in Y$ satisfies $d_R(y) \le N/2$. The same argument will yield that each vertex v satisfies $d_B(v) \le N/2$, so the desired statement follows immediately. □

For fixed m and large n, the asymptotic formula of $br(K_{m,n})$ is $2^m n$, which is also that of $r(K_{m,n})$.

Theorem 13.8 *For fixed m, as* $n \to \infty$,

$$br(K_{m,n}) \sim 2^m n.$$

Proof. The upper bound follows from Theorem 13.7. The proof for the desired lower bound $br(K_{m,n}) \ge (1-o(1))2^m n$ is similar to that for $r(K_{m,n})$ in Theorem 3.16, so we omit it. □

Theorem 13.7 gives that $br(K_{n,n}) \le 2^n(n-1)+1$, which was improved by Conlon (2008).

Theorem 13.9 *For all large n,*

$$(1-o(1))\frac{\sqrt{2}}{e}\, n\, 2^{n/2} \le br(K_{n,n}) \le (1+o(1))2^{n+1}\log_2 n.$$

The lower bound can be proved by the symmetric form of Local Lemma in the same manner as that for the lower bound of $r(n,n)$ in Chapter 5, we thus omit it. For the upper bound, we shall establish a lemma first. From the upper bounds of the Zarankiewicz number in Chapter 7, we see that an $M \times N$ bipartite graph G contains $K_{m,n}$ if the density $p = \frac{e(G)}{MN}$ is positively bounded from below and M and N are large.

Lemma 13.10 *Let G be a bipartite graph on vertex sets X and Y, whose edge density* $p \ge a$ *for some constant* $a > 0$*. If for any* $\epsilon > 0$*, as* $t \to \infty$,

$$\frac{|X|}{t^2} \to \infty, \quad \text{and} \quad |Y| \ge (1+\epsilon)\frac{s-1}{p^t},$$

then G must contain $K_{(t,s)}$ *for large t.*

Proof. Let $M = |X|$ and $N = |Y|$. If G contains no $K_{(t,s)}$, from the double-counting argument used in Chapter 7 and Jensen's inequality, we have

$$N\binom{pM}{t} = N\binom{\frac{1}{N}\sum_{v\in Y} d(v)}{t} \le \sum_{v\in Y}\binom{d(v)}{t} \le (s-1)\binom{M}{t}.$$

Note that $\binom{M}{t}/\binom{pM}{t}$ can be bounded as

$$\frac{M(M-1)\cdots(M-t+1)}{pM(pM-1)\cdots(pM-t+1)} \le \left(\frac{M-t}{pM-t}\right)^t = \frac{1}{p^t}\left(\frac{M-t}{M-t/p}\right)^t.$$

Using the fact that $\log(1+x) = x + o(x)$ as $x \to 0$ and the condition that $t^2/M \to 0$ as $t \to \infty$, we have

$$\left(\frac{M-t}{M-t/p}\right)^t = \exp\left(t \log \frac{M-t}{M-t/p}\right) = \exp\left(t \log\left(1 + \frac{t(1-p)}{pM-t}\right)\right) \to 1.$$

Hence $N \le (1+o(1))(s-1)/p^t$, which is a contradiction. □

Proof of the upper bound in Theorem 13.9. For any $\epsilon > 0$, set

$$N = (1+3\epsilon)2^{n+1}\log_2 n.$$

Suppose that edges of $K_{N,N}$ on vertex sets X and Y are red/blue colored. Let

$$Y_R = \{v \in Y : d_R(v) \ge N/2\}, \quad Y_B = \{v \in Y : d_B(v) \ge N/2\}.$$

Then one of them, say Y_R, satisfies $|Y_R| \ge N/2$.

Now consider the red bipartite graph induced by X and Y_R, which has density $p \ge 1/2$. An application of Lemma 13.10 for $t = n - 2\log_2 n$ and $s = n^2 \log_2 n$ tells us that we can find a red $K_{(t,s)}$ as $|X|/t^2 \to \infty$ and

$$|Y_R| \ge (1+3\epsilon)2^n \log_2 n \ge (1+\epsilon)2^t n^2 \log_2 n \ge (1+\epsilon)\frac{s}{p^t}.$$

Let X_0 be the set of t vertices in X which are counted in our red $K_{(t,s)}$ and let $X' = X \setminus X_0$. Let Y' be the set of s vertices in Y_R which are counted in this red $K_{(t,s)}$. Consider the induced red sub-bipartite graph on X' and Y'. Each vertex in Y' is red-adjacent to at least $N/2 - t$ vertices in X', so that the density p' of the induced subgraph satisfies that

$$p' \ge \frac{N/2 - t}{|X'|} \ge \frac{N - 2t}{2(N-t)} \ge \frac{1}{2} - \frac{n}{2^n} > \frac{1}{3} = a.$$

Applying Lemma 13.10 to this bipartite graph, since

$$|X'| = N - t \ge (1+2\epsilon)2^{n+1}\log_2 n, \quad |Y'| = n^2 \log_2 n,$$

we can find a red $K_{(s',t')}$, where $s' = 2\log_2 n$ and $t' = n$, since $|Y'|/t'^2 \to \infty$ and

$$|X'| \ge (1+2\epsilon)\frac{s'}{(1/2)^n} \ge (1+\epsilon)\frac{s'-1}{p'^n}$$

Adding all t vertices of X_0 to this red $K_{(s',n)}$ (in the first vertex class) will produce a red $K_{n,n}$, as desired. □

It is also difficult to obtain a good asymptotic formula of $br(K_{m,m}, K_{n,n})$. The situation is similar to that for $r(m,n)$ that we have encountered. The following result is due to Caro and Rousseau (2001).

Theorem 13.10 *Let integer $m \ge 2$ be fixed. Then there are positive constant A and B such that*

$$A\left(\frac{n}{\log n}\right)^{(m+1)/2} \le br(K_{m,m}, K_{n,n}) \le B\left(\frac{n}{\log n}\right)^{m}.$$

Proof. The lower bound can be seen from Lemma 13.8 and a theorem in Section 3 of Chapter 5 as

$$br(K_{m,m}, K_{n,n}) \ge \frac{1}{2} r(K_{m,m}, K_{n,n}) \ge A\left(\frac{n}{\log n}\right)^{(m^2-1)/(2m-2)} = A\left(\frac{n}{\log n}\right)^{(m+1)/2}.$$

The upper bound is based on well-known results for the Zarankiewicz numbers $z(N, M; s, t)$, that is defined in Chapter 7. Let $z(N; s)$ denote $z(N, N; s, s)$. Then we have

$$z(N; s) \le \left(\frac{s-1}{N}\right)^{1/s} N(N - s + 1) + (s-1)N.$$

To prove $br(K_{m,m}, K_{n,n}) \le N$ it suffices to show that $z(N; m) + z(N; n) < N^2$. Take $\epsilon > 0$ and $N = c(n/\log n)^m$, where $c = (1+\epsilon)/(m-1)^{m-1}$. Then

$$\begin{aligned}\frac{z(N; m)}{N^2} &< \left(\frac{m-1}{N}\right)^{1/m}\left(1 - \frac{m-1}{N}\right) + \frac{m-1}{N}\\ &= \left(\frac{m-1}{c}\right)^{1/m} \frac{\log n}{n} + O\left(\left(\frac{\log n}{n}\right)^{m}\right).\end{aligned}$$

To bound $z(N; n)/N^2$, we first have

$$\left(\frac{n-1}{N}\right)^{1/n} = \left(\frac{(n-1)\log^m n}{cn^m}\right)^{1/n} = 1 - \frac{(m-1)\log n}{n} + O\left(\frac{\log\log n}{n}\right).$$

Hence

$$\begin{aligned}\frac{z(N; n)}{N^2} &\le \left(\frac{n-1}{N}\right)^{1/n}\left(1 - \frac{n-1}{N}\right) + \frac{n-1}{N}\\ &= 1 - \frac{(m-1)\log n}{n} + O\left(\frac{\log\log n}{n}\right).\end{aligned}$$

Adding the above bounds, we obtain

$$\begin{aligned}\frac{z(N; m) + z(N; n)}{N^2} =& 1 - \left(1 - \frac{1}{(1+\epsilon)^{1/m}}\right)\frac{(m-1)\log n}{n}\\ &+ O\left(\frac{\log\log n}{n}\right),\end{aligned}$$

so $(z(N; m) + z(N; n))/N^2 < 1$ for all large n as required. □

We believe the following problems are easier than the corresponding problems for $r(n, n)$ and $r(m, n)$.

Problem 13.1 Determine $\lim_{n\to\infty} br(K_{n,n})^{1/n}$ if it exists, and determine the order of $br(K_{m,m}, K_{n,n})$ for fixed $m \geq 2$.

For non-complete bipartite case, Faudree and Schelp (1975), and independently Gyárfás and Lehel (1973) proved that $br(P_n, P_n) = n-1$ for even n and $br(P_n, P_n) = n$ for odd n. By using the regularity lemma, Shen, Lin and Liu (2018) obtained that $br(C_{2n}, C_{2n}) = (2 + o(1))n$. For the three color case, Bucić, Letzter, and Sudakov (2019) obtained the asymptotic order of $br(C_{2n}, C_{2n}, C_{2n})$. Luo and Peng (2020) obtained the asymptotic order of $br(C_{2\lfloor \alpha_1 n\rfloor}, C_{2\lfloor \alpha_2 n\rfloor}, C_{2\lfloor \alpha_3 n\rfloor})$, where α_i, $1 \leq i \leq 3$, are constants.

13.4 Folkman Numbers

The Ramsey number $r(m, n)$ is the smallest N such that $K_N \to (K_m, K_n)$. It is difficult to determine the behavior of $r(m, n)$, and even more difficult if the graphs are restricted with smaller cliques instead of complete graphs.

Let G_1 and G_2 be graph. Define a family $\mathcal{F}(G_1, G_2; p)$ of graphs as

$$\mathcal{F}(G_1, G_2; p) = \{G : G \to (G_1, G_2) \quad \text{and} \quad \omega(G) \leq p\}.$$

Define the Folkman number $f(G_1, G_2; p)$ as

$$f(G_1, G_2; p) = \min\{|V(G)| : \; G \in \mathcal{F}(G_1, G_2; p)\}.$$

We admit that $f(G_1, G_2; p) = \infty$ if $\mathcal{F}(G_1, G_2; p) = \emptyset$, and thus $f(G_1, G_2; p) = \infty$ if $p < \max\{\omega(G_1), \omega(G_2)\}$. As we write $G \to (m, n)$ for $G \to (K_m, K_n)$, we write $\mathcal{F}(m, n; p)$ and $f(m, n; p)$ for $\mathcal{F}(K_m, K_n; p)$ and $f(K_m, K_n; p)$, respectively.

We list some elementary properties of $f(m, n; p)$ as follows, for which the similar properties of $f(G_1, G_2; p)$ can be given easily.

Lemma 13.11 *If $p < r(m, n)$ and $\mathcal{F}(m, n; p) \neq \emptyset$, then*

$$\{G : \; G \to (m, n) \quad \textit{and} \quad \omega(G) = p\} \neq \emptyset,$$

and $f(m, n; p) = \min\{|V(G)| : \; G \to (m, n) \quad \textit{and} \quad \omega(G) = p\}$.

Proof. Since $\mathcal{F}(m, n; p) \neq \emptyset$, we have $p \geq \max\{m, n\}$. Let G be a graph in $\mathcal{F}(m, n; p)$ of order $f(m, n; p)$. If $\omega(G) = p$, then we are done. Otherwise, G is not complete as there is no K_ω with $\omega < r(m, n)$ such that $K_\omega \to (m, n)$, and thus we can obtain a graph G' from G by adding an edge. Then the order of G' is still $f(m, n; p)$, and $\omega(G')$ is $\omega(G)$ or $\omega(G) + 1$ and $G' \to (m, n)$. Continuing the process, we will obtain a graph $\hat{G}$ of order $f(m, n; p)$ such that $\hat{G} \to (m, n)$ and $\omega(\hat{G}) = p$. □

Lemma 13.12 *For any positive integer* p*,* $\mathcal{F}(m,n;p) = \mathcal{F}(n,m;p)$*, and thus* $f(m,n;p) = f(n,m;p)$.

Proof. The first equality can be seen from the fact that $G \to (m,n)$ if and only if $G \to (n,m)$, and thus the second equality follows. □

Lemma 13.13 *If* $p < q$*, then* $f(m,n;p) \geq f(m,n;q)$*, and if* $p \geq r(m,n)$*, then* $f(m,n;p) = r(m,n)$.

Proof. The inequality is from the fact $\mathcal{F}(m,n;p) \subseteq \mathcal{F}(m,n;q)$. If $p \geq r(m,n)$, then $K_N \in \mathcal{F}(m,n;p)$, where $N = r(m,n)$, and there is no graph G of order smaller than N such that $G \in \mathcal{F}(m,n;p)$ from the definition of Ramsey number, and thus $f(m,n;p) = N = r(m,n)$. □

Folkman (1970) proved that $\mathcal{F}(m,n;p) \neq \emptyset$ if $p \geq \max\{m,n\}$, and thus $f(m,n;p) < \infty$, from which the name after. This investigation was motivated by a question of Erdős and Hajnal (1967) who asked what was the minimum p such that $\mathcal{F}(3,3;p) \neq \emptyset$. Folkman's result was generalized by Nešetřil and Rödl (1976) as follows.

Theorem 13.11 *If* $p \geq \max\{\omega(G_1), \omega(G_2)\}$*, then*

$$\mathcal{F}(G_1, G_2; p) \neq \emptyset.$$

Nešetřil and Rödl proved their result even in multi-color cases. Here, we shall only prove Folkman's result.

As usual, we signify the isomorphism of graphs G_1 and G_2 as $G_1 \cong G_2$. For a subset S of $V(G)$, denote by $G[S]$ the subgraph of G induced by S.

Recall that a graph H is Ramsey for G if any edge-coloring of H by two colors contains an induced monochromatic G. Correspondingly, we call H to be *n-vertex-Ramsey* for G if the vertex set of H is partitioned into $V_1, V_2, \cdots, V_n$, then G is an induced subgraph of $H[V_i]$ for some $i = 1, 2, \ldots, n$.

Lemma 13.14 *Let* $n \geq 2$ *be an integer and* G *a graph of order* r*. Then there is a graph* $H = H(n,G)$ *that is n-vertex-Ramsey for* G *and* $\omega(H) = \omega(G)$.

Proof. We first construct a graph $H = H(2,G)$ by induction on r such that H is 2-vertex Ramsey and $\omega(H) = \omega(G)$.

If $r = 1$, we simply take $H = G = K_1$. We then assume that $r \geq 2$.

Let V be the vertex set of G with $V = \{v_1, v_2, \ldots, v_r\}$. Let $u \in V$ be a fixed vertex, and $V' = V \setminus \{u\}$ and $V'' = N(u)$. Let $G' = G[V']$ and $G'' = G[V'']$. Then from the inductive assumption, a graph $H' = H(2,G')$ can be defined as asserted.

Note the facts that H' is 2-vertex-Ramsey for G' and G' contains a subset S such that $H'[S] \cong G''$. We shall find an additional vertex to play the role of u. Let W be the vertex set of H'. Define a family $\mathcal{X}$ of subsets of W as

$$\mathcal{X} = \left\{ S \subseteq W : \ H'[S] \cong G'' \right\}.$$

Since $V'' = N(u)$ is a subset of V' and thus $\mathcal{X} \neq \emptyset$. Let $I = \{1, 2, \ldots, 2^{|W|}r\}$, a set of integers, and $I^{(r)}$ the family of all r-element subsets of I. For any $T \in I^{(r)}$, when we write $T = \{t_1, t_2, \ldots, t_r\}$, we always admit the natural order $t_1 < t_2 < \cdots < t_r$.

We define a graph H as follows. For any $(S, T) \in (\mathcal{X}, I^{(r)})$, we have a copy $G_{S,T}$ of G, and for any $i \in I$, we have a copy H'_i of H'. Let H be the union of those copies of G and H' by adding edges between the certain copies of G and that of H'. The copy of v_j in $G_{S,T}$, where $T = \{t_1, t_2, \cdots, t_r\}$, is adjacent to each vertex in the copy of S in H'_{t_j}, $j = 1, 2, \ldots, r$. Fig. 12. 1 illustrates the edges between a copy $G_{S,T}$ of G and that of H'.

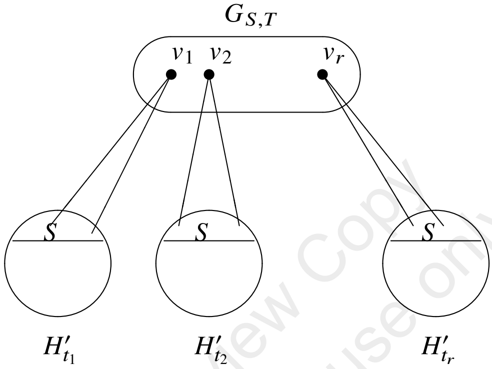

Fig. 12. 1 $G_{S,T}$ and $H'_{t_1}, H'_{t_2}, \ldots, H'_{t_r}$ for $T = \{t_1, t_2, \ldots, t_r\}$

We now show that H is 2-vertex-Ramsey for G. Let W_i be the vertex set of H'_i for $i = 1, 2, \ldots, |I|$, which is a copy of W. When we partition the vertex set of H into two parts, we have a restricted partition for each $W_i = (A_i, B_i)$. Since W has $2^{|W|}$ partitions, among those $|I| = 2^{|W|}r$ copies of H', at least r of them have the identical partition, and thus there are a set $T \in I^{(r)}$, and a partition (A, B) of W such that $(A_t, B_t) = (A, B)$ for each $t \in T$.

Let us write $T = \{t_1, t_2, \ldots, t_r\}$. As H' is 2-vertex-Ramsey for G', when the vertex set of H' is partitioned into (A, B), one of subgraph induced by A, say, contains G' as an induced subgraph. Thus we can find the copies of a fixed S of W in each H'_{t_i} that induces a subgraph isomorphic to G''. Furthermore, we consider $G_{S,T}$. If any vertex is in the same part as that of B, we have an induced G. Otherwise, one of vertex of $G_{S,T}$, say v_i, is in the same part as that of A, then the induced G' in H'_{t_i} and v_i give us a induced graph G.

We then show that $\omega(H) = \omega(G)$. It suffices to show that $\omega(H) \leq \omega(G)$ as the inverse inequality is clear. Let C be an independent set of H. Since any pair of $G_{S,T}$ are disconnected, we have that at most one $G_{S,T}$, and one of H'_i has a non-empty intersection with C. If such $G_{S,T}$ is none, then C is contained in one H'_i. As $\omega(H') = \omega(G')$, we have $|C| \leq \omega(G') \leq \omega(G)$. Otherwise, there is a $G_{S,T}$ that contains k vertices, say $v_1, \ldots, v_k$, of C. If $k = 1$, clearly $|C| \leq \omega(G)$. If $k \geq 2$, as

v_1 and v_2 has no common neighbors in H'_{t_i}, then C is contained in a $G_{S,T}$ and thus $|C| \le \omega(G)$.

For $n > 2$, we assume that there is a graph $H_{n-1} = H(n-1, G)$ that is $(n-1)$-vertex Ramsey and $\omega(H_{n-1}) = \omega(G)$. Let $H_n = H(n, G) = H(2, H_{n-1})$. Then it is easy to see that H_n is n-vertex Ramsey for G and $\omega(H_n) = \omega(G)$. □

The proof of Theorem 13.12 needs the Cartesian product of two graphs defined as follows. Let F and H be vertex disjoint graphs on vertex sets U and V, respectively, where $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$. The *Cartesian product* $F \times H$ of F and H is define as a graph on vertex set $U \times V$, in which a pair of distinct vertices (u, v) and (u', v') are adjacent if and only if $uu' \in E(F)$ and $v = v'$ or $u = u'$ and $vv' \in E(H)$. If we write the vertices of $F \times H$ as a matrix

$$\begin{matrix} (u_1, v_1) & (u_1, v_2) & \cdots & (u_1, v_n) \\ (u_2, v_1) & (u_2, v_2) & \cdots & (u_2, v_n) \\ \vdots & \vdots & & \vdots \\ (u_m, v_1) & (u_m, v_2) & \cdots & (u_m, v_n), \end{matrix}$$

then each edge is vertical or horizonal, and each column preserves the adjacency of F and each row preserves the adjacency of H.

The following is an easy property of Cartesian product of graphs.

Lemma 13.15 *Let F and H be vertex disjoint graphs. Then*

$$\omega(F \times H) = \max\{\omega(F), \omega(H)\}.$$

Theorem 13.12 (Folkman) *If $p \ge \max\{m, n\}$, then $\mathcal{F}(m, n; p) \ne \emptyset$.*

Proof. From Lemma 13.12, we may assume that $n \ge m \ge 1$. By Lemma 13.11, it suffices to construct a graph $G \in \mathcal{F}(m, n; n)$ by induction on $m + n$. As $K_n \in \mathcal{F}(1, n; n)$ and $K_n \in \mathcal{F}(2, n; n)$, we assume that $n \ge m \ge 3$ and the assertion is true for smaller value of $m+n$. Let G_1, G_2 and G_3 be vertex disjoint graphs such that $G_1 \in \mathcal{F}(m-1, n; n)$, $G_2 \in \mathcal{F}(m, n-1; \max\{m, n-1\})$, and $G_3 \in \mathcal{F}(m-1, n-1; n-1)$. From Lemma 13.11, we may assume that $\omega(G_1) = n$, $\omega(G_2) = \max\{m, n-1\}$ and $\omega(G_3) = n-1$. Let $H(k, G)$ be a graph as defined in Lemma 13.14. Write M for the number of $(n-1)$-element subsets of vertices of $G_1 \cup G_2$, and let $H_2 = H((n-1)^2 M^2, G_3)$. Write N for the number of ways partitioning edges of H_2 into two classes, and let $H_1 = H(N, G_1 \cup G_2)$.

Denote by V_1 and V_2 the vertex sets of H_1 and H_2, respectively, and let

$$\mathcal{X} = \{T \subseteq V_1 : |T| = n-1\}.$$

We then define a graph G on the vertex set $(V_1 \times V_2) \cup \mathcal{X}$, in which $V_1 \times V_2$ preserves the edges of $H_1 \times H_2$, and there is no edge between the members of $\mathcal{X}$, and each vertex (u, v) in $V_1 \times V_2$ is adjacent to each $T \in \mathcal{X}$ if $u \in T$, which means that all vertices in the u-row of $H_1 \times H_2$ are adjacent to T if $u \in T$. From the construction of G and the facts that $\omega(H_1) = n$ and $\omega(H_2) = n-1$, we have $\omega(G) = n$.

It remains to show that $G \to (m, n)$. Let (R, B) be a red-blue edge-coloring of G. For each $u \in V_1$, the edges of $H_1 \times H_2$ in the u-row, hence the edges of H_2, are colored into $(R(u), B(u))$. As there are N ways partitioning the edges of H_2, the rows of $V_1 \times V_2$, hence the vertices of H_1, are partitioned into N classes, in which two vertices u and u' of H_1 are in the same class if

$$(R(u), B(u)) = (R(u'), B(u')).$$

Since $H_1 = H(N, G_1 \cup G_2)$, which is N-vertex Ramsey for $G_1 \cup G_2$, one of classes in the above partition must contains a subset U such that $H_1[U] \cong G_1 \cup G_2$. From the definition of the partition, we know that there is a fixed coloring (R_0, B_0) of edges of H_2 such that

$$(R(u), B(u)) = (R_0, B_0) \qquad \text{for each } \ u \in U,$$

which means that the edges of H_2 in all rows that U occupies are colored in the same way. The definition of $H_1 \times H_2$ implies that, for any $v \in V_2$, the vertex set $U \times \{v\}$ induces a subgraph of G isomorphic to $G_1 \cup G_2$. If this subgraph contains a red K_m or a blue K_n, we are done. Otherwise, from the choice of $G_1 \in \mathcal{F}(m-1, n; n)$, $G_2 \in \mathcal{F}(m, n-1; \max\{m, n-1\})$, for each $v \in V_2$, both following situations must happen:

Case 1 There is a subset $S_R(v)$ of U such that $S_R(v) \times \{v\}$ induces a red K_{m-1}.

Case 2 There is a subset $T_B(v)$ of U such that $T_B(v) \times \{v\}$ induces a blue K_{n-1}, where $S_R(v) \cap T_B(v) = \emptyset$ as G_1 and G_2 are vertex disjoint.

Let $T_R(v)$ be a subset of U extended from $S_R(v)$ such that $T_R(v) \cap T_B(v) = \emptyset$ and $|T_R(v)| = |T_B(v)| = n-1$. Then both $T_R(v)$ and $T_B(v)$ are members of $\mathcal{X}$, which are vertices of G and adjacent to every vertex in the set $T_R(v) \times \{v\}$ and $T_B(v) \times \{v\}$ in $V_1 \times V_2$, respectively.

If, for some $v \in V_2$, the set $(T_R(v) \times \{v\}) \cup \{T_R(v)\}$ contains a red K_m, or the set $(T_B(v) \times \{v\}) \cup \{T_B(v)\}$ induces a blue K_n, we are done. Otherwise, for each $v \in V_2$, there is a vertex $u_1(v) \in T_R(v)$ such that the edge $\{(u_1(v), v), T_R(v)\}$ is blue, and there is a vertex $u_2(v) \in T_B(v)$ such that the edge $\{(u_2(v), v), T_B(v)\}$ is red.

As U has M subsets T with $|T| = n-1$, there are $(n-1)^2 M^2$ ordered quadruples in the form (u_1, u_2, T_1, T_2) with $u_1 \in T_1 \subseteq U$, $u_2 \in T_2 \subseteq U$ and $|T_1| = |T_2| = n-1$. Then, by considering the rows in $V_1 \times V_2$ that U occupies, we can partition V_2 hence the columns of $V_1 \times V_2$ into $(n-1)^2 M^2$ classes by putting v and v' in the same class if

$$(u_1(v), u_2(v), T_R(v), T_B(v)) = (u_1(v'), u_2(v'), T_R(v'), T_B(v')).$$

As $H_2 = H((n-1)^2 M^2, G_3)$, we thus obtain a set $V \subseteq V_2$ and a fixed quadruple (u_1, u_2, T_R, T_B) such that $H_2[V] \cong G_3$ and

$$(u_1(v), u_2(v), T_R(v), T_B(v)) = (u_1, u_2, T_R, T_B) \qquad \text{for each } \ v \in V,$$

This is to say, for $v \in V$, all $(u_1(v), v)$ are in the u_1-row, and all $(u_2(v), v)$ are in the u_2-row, and the rows that $T_R(v) \times \{v\}$ occupies are as the same as T_R does, and the rows that $T_R(v) \times \{v\}$ occupies are as the same as T_B does.

Let us identify G_3 with the graph induced by $\{u\} \times V$ for $u \in U$. Then the edges of G_3 in the u_1-row and in the u_2-row are colored in the same way as $u_1, u_2 \in U$. Since $G_3 \in \mathcal{F}(m-1, n-1, n-1)$, we either have a subset $W_R \subseteq V$ which induces a red K_{m-1}, or a subset $W_B \subseteq V$ which induces a blue K_{n-1}. If the former is the case, then $\{u_2\} \times W_R \cup \{T_B\}$ induces a red K_m of G, or otherwise, $\{u_1\} \times W_B \cup \{T_R\}$ induces a blue K_n of G. □

Let us have a brief analysis on the bound for the growth of Folkman number from the proofs.

Let $g(r)$ be an upper bound for the order of the graph $H = H(2, r)$ constructed in Lemma 13.14 with $g(1) = 1$. As

$$V(H) = (V \times \mathcal{X} \times I^{(r)}) \cup (W \times I).$$

Then $|W| \leq g(r-1)$ and $|I| \leq r2^{g(r-1)}$, and thus $g(r)$ is a tower of height r. For $n > 2$, let $H_{n-1} = H(n-1, G)$ and $H_n = H(n, G) = H(2, H_{n-1})$. Then the tower of the order of H_n is around the value of the order H_{n-1}. In the proof of Theorem 13.12, as N and M are big, the growth of the height of the tower in the obtained upper bound for $f(n, n; n)$ is rapid, which is somehow like that for $w(n, n)$ in the original proof of van der Waerden, see Chapter 11.

Reducing the upper bound for $f(n, n; n)$ or $f(3, n; n)$ is not easy. For $p = r(m, n) - 1$, Lin (1972) proved that in some cases $f(m, n; p) = r(m, n) + 2$. It is known that $f(3, 3; 5) = 8$ and $f(3, 3; 4) = 15$, due to Graham (1968) and Lin (1972), and Piwakowski, Radziszowski and Urbanski (1999), respectively.

It is known that $f(3, 3; 3) \leq 3 \times 10^9$, due to Spencer (1988), which improved an upper bound 10^{12} of Frankl and Rödl (1986). Chung and Graham (1999) conjectured $f(3, 3; 3) < 1000$, which was confirmed by Lu (2008) with $f(3, 3; 3) < 9697$, and by Dudek and Rödl (2008) with more computer aid. No reasonable lower bound for $f(n, n; n)$, even for $f(3, n; n)$ or $f(3, 3; 3)$, is known.

13.5 For Parameters and Coloring Types

We have seen that it is usually difficult to estimate the value of $r(G, H)$, particularly that of $r(m, n)$. Many researchers made some generalization on Ramsey numbers. For a parameter $f(F)$ of graph F, similarly to define $r(G, H)$, we can define

$$r_f(G, H) = \min\{f(F) : F \to (G, H)\}.$$

We write $r_f(G)$ for $r_f(G, G)$. Burr, Erdős and Lovasz (1976) define *chromatic Ramsey number* $r_\chi(G, H)$ by taking the parameter f as the chromatic number χ. They proved $r_\chi(K_m, K_n) = r(m, n)$ and $r_\chi(G) \geq (n-1)^2 + 1$ if $n = \chi(G)$, and conjectured that for any positive integer n, there is a graph G with $\chi(G) = n$ such that $r_\chi(G) = (n-1)^2 + 1$. This conjecture was proved to be true by Zhu (2011) in an effort to solve another conjecture.

Benedict, Chartand, and Lick (1977) defined $\mu(g; m, H)$ to be the least integer N such that any graph G of N , either $g(G) \ge m$ or $\overline{G}$ contains H as a subgraph, where $g(G)$ is a parameter of G.

Let g_1 and g_2 be graph parameters and let m and n be integers. Define the Ramsey number $r(g_1 \ge m, g_2 \ge n)$ as the least integer N such that for any graph G of order N either $g_1(G) \ge m$ or $g_2(\overline{G}) \ge n$. We call $r(g_1 \ge m, g_2 \ge n)$ as the Ramsey number on g_1 and g_2, or the *mixed Ramsey number* on g_1 and g_2 if g_1 and g_2 are different. In this new notation, we have

$$r(m, n) = r(\omega \ge m, \omega \ge n) = r(\alpha \ge m, \alpha \ge n).$$

Let $g_H(G)$ be the indicator that H is a subgraph of G. Then $r(g_H = 1, g_F = 1)$ is $r(H, F)$.

The most natural question is that for which pair g_1 and g_2, the defined Ramsey number $r(g_1 \ge m, g_2 \ge n)$ exists. The answer is not always positive. For example, $r(\chi \ge 2, \alpha \ge 2)$ does not exist since the empty graph N_n of order n satisfies that $\chi(N_n) = 1$ and $\alpha(\overline{N_n}) = 1$. However, for many pairs of parameters, the existence of the Ramsey number on these parameters can be easily verified.

Theorem 13.13 *Let g_1 and g_2 be graph parameters. Then for any positive integers m and n, $r(g_1 \ge m; g_2 \ge n)$ exists if and only if*

$$\lim_{k\to\infty}\left(\min_{|V(G)|=k}(g_1(G) + g_2(G))\right) = \infty. \qquad (13.1)$$

Proof. Suppose that for any positive integers m and n, $r(g_1 \ge m; g_2 \ge n)$ exists. Then for any integer $M > 0$, $K(M) = r(f_1 \ge M; f_2 \ge M)$ exists. So for any graph G of order $k \ge K(M)$, either $g_1(G) \ge M$ or $g_2(\overline{G}) \ge M$. Therefore,

$$\min_{|V(G)|=k}(g_1(G) + g_2(G)) \ge M + 1$$

for $k \ge K(M)$, so (13.1) holds.

Conversely, suppose (13.1) holds. Then for any positive integers m and n, there exists $K = K(m, n)$ such that if $k \ge K$,

$$\min_{|V(G)|=k}(g_1(G) + g_2(G)) \ge m + n.$$

So for any graph G of order $k \ge K$, $g_1(G) + g_2(\overline{G}) \ge m + n$, hence $g_1(G) \ge m$ or $g_2(G) \ge n$. Minimizing such K shows the existence of $r(g_{\ge} m; g_2 \ge n)$. □

The following easy result has a similar form as $r(K_m, T_n) = (m-1)(n-1) + 1$ of Chvátal in Chapter 1.

Theorem 13.14 *Let m and n be positive integers. Then*

$$r(\chi \geq m, \chi \geq n) = (m-1)(n-1)+1,$$

where χ signifies the chromatic number of a graph.

Proof. The assertion is obviously true if one of m and n is one. Assume that $m, n \geq 2$. Let $G = (n-1)K_{m-1}$. Then $\chi(G) = m-1$ and $\chi(\overline{G}) = n-1$, yielding that

$$r(\chi \geq m, \chi \geq n) \geq (m-1)(n-1)+1.$$

On the other hand, if G is a graph of order $N = (m-1)(n-1)+1$ and $\chi(G) \leq m-1$, then by the fact that $\chi(G)\alpha(G) \geq N$ we have

$$\alpha(G) \geq \left\lceil \frac{N}{\chi(G)} \right\rceil = \left\lceil \frac{(m-1)(n-1)+1}{m-1} \right\rceil = n.$$

Therefore $\chi(\overline{G}) \geq \omega(\overline{G}) = \alpha(G) \geq n$, proving

$$r(\chi \geq m, \chi \geq n) \leq (m-1)(n-1)+1,$$

and the equality follows. □

Let G' be a graph obtained by deleting a vertex from G. Similarly as that for proving

$$r(F, H) \leq r(F', H) + r(F, H'), \tag{13.2}$$

we have the following bound.

Theorem 13.15 *For a fixed graph F, define $g_F(G)$ be the number of subgraphs isomorphic to F with different vertex sets in graph G. Then for any graphs F and H of orders at least two and any positive integers m and n,*

$$r(g_F \geq m, g_H \geq n) \leq r(g_{F'} \geq m, g_H \geq n) + r(g_F \geq m, g_{H'} \geq n).$$

Let us turn to some other generalizations of Ramsey number. An obvious one is as follows. Let $\mathcal{F}_i$ is a family of graphs for $i = 1, 2$. Define the *class Ramsey number* $r(\mathcal{F}_1, \mathcal{F}_2)$ as the minimum integer N such that any edge coloring of K_N by red and blue contains some red $F_1 \in \mathcal{F}_1$ or some blue $F_2 \in \mathcal{F}_2$. Let $\mathcal{F}(n, s)$ denote the family of connected graphs with n vertices and s edges for $n-1 \leq s \leq \binom{n}{2}$. We then get a definition of $r(n, s; m, t)$ as $r(\mathcal{F}(n, s), \mathcal{F}(m, t))$. General results on class Ramsey numbers are difficult since it covers the classic Ramsey numbers $r(m, n)$.

In traditional definitions of graph Ramsey numbers, we asked the minimum number N such that any edge coloring of K_N contains some *monochromatic* graphs. If we change the coloring types either in K_N or in subgraphs that are contained, we may have some other definitions of Ramsey numbers. We introduce some of them without details of discussion.

Local k-coloring

Let G be a graph and let $k \geq 0$ be an integer. A subset $S \subseteq V(G)$ is said to be a k-independent set if the subgraph induced by S has a maximum degree at most k. Correspondingly, an edge-coloring of G by any number of colors is called a local k-coloring if each vertex of G is adjacent to at most k distinct colors.

There are two ways to define Ramsey numbers with k-independence. One way is to ask what is the minimum integer N such that any local edge k-coloring of K_N contains a monochromatic G. See, for example, Gyárfás, Lehel, Nešetřil, Rödl, Schelp, and Tuza (1987), and Caro and Tuza (1993). Other way is to ask what is the minimum integer N such that any edge coloring of K_N in m colors contains a local k-colored G.

Zero-sum coloring

Most recent combinatorial research on zero-sum problems is related to a result of Erdős, Ginzburg and Ziv (1961) as follows.

Theorem 13.16 *Let $m \geq k \geq 2$ be integers such that $k|m$. Then for any sequence of integers $\{a_1, a_2, \ldots, a_{m+k-1}\}$, there exists a subset I of indices of $\{1, 2, \ldots, m+k-1\}$ such that $|I| = m$ and $\sum_{i \in I} a_i = 0 \pmod{k}$.*

Let Z_k be the additive group modulo k. A Z_k-coloring of edges of graph G is a function $f : E(G) \to Z_k$. We say that G is zero-sum, relative to f, if $\sum_{e \in E(G)} f(e) = 0 \pmod{k}$.

The zero-sum Ramsey number is to ask the minimum integer N such that any Z_k-coloring of edges of K_N contains a zero sum G. For this topic, see a survey by Caro (1996).

Rainbow coloring

We call an edge coloring of G to be a *rainbow* or anti-Ramsey if each pair of edges have distinct colors. Weakening this condition, an edge coloring of G is called a (p, q)-coloring if every subset of p vertices spans at least q colors. Erdős (1981) asked what is the minimum number $f(n, p, q)$ of colors such that there exists a (p, q)-coloring of K_n. To avoid the trivial cases, we assume that $2 \leq q \leq \binom{p}{2}$. Clearly $f(n, p, \binom{p}{2}) = \binom{n}{2}$. Observe that $(p, 2)$-coloring are equivalent to coloring without monochromatic K_p. For more information, see Erdős and Gyárfás (1997).

Some researchers ask some kind types of Ramsey numbers involving rainbow subgraphs, among which some are as follows.

(1) Given graph G of order m and integer $n \geq m$, what is the minimum number k of colors such that each k-coloring of edges of K_n yields a rainbow G?

(2) Given graphs G and H, and integer k, what is the minimum integer N such that each k-coloring of edges of K_N contains a monochromatic G or a rainbow H?

(3) Given graphs G and H, what is the minimum integer N such that each edge-coloring of K_N contains a monochromatic G or a rainbow H? Here, no constraint is placed on the number of colors. Note that this number exists if and only if G is a star or H is a forest, see Jamison, Jiang and Ling (2003).

Regular coloring

A graph G is called regular by k if $\Delta(G) - \delta(G) \le k$. Recall that many Ramsey graphs correspond to the Ramsey numbers $r(G, H)$ we met are regular or regular by 1. We may ask what is the minimum integer N such that if a graph G of order N is regular by 1 then it contains a copy of K_m or an independent set of order n. We believe that this N is very close to $r(m, n)$.

13.6 Exercises

1. Find size Ramsey numbers of C_4 and path of lenght 3.

2. By computing the constant C in the proof of Theorem 13.4.

3. Prove the lower bounds in Theorem 13.8 and in Theorem 13.10, respectively.

4. Let $f(n, p, q)$ be defined in Section 13.5. Show that $f(16, 3, 2) = 3$ and $f(17, 3, 2) = 4$. (Hint: $r_3(3) = 17$)

5. Let B_1 and B_2 be bipartite graphs. Show that the Folkman number $f(B_1, B_2; 2)$ is between $br(B_1, B_2)$ and $2br(B_1, B_2)$.

6.* Prove that $\hat{r}(sK_{1,m}, tK_{1,n}) = (s + t - 1)(m + n - 1)$. (Hint: Burr and Erdős, Faudree, Rousseau and Schelp (1978). They conjectured that for $F_1 = \cup_{i=1}^{s} K_{n_i}$ and $F_2 = \cup_{i=1}^{t} K_{m_i}$, $\hat{r}(F_1, F_2) = \sum_{k=2}^{s+t} \ell_k$, where $\ell_k = \max\{n_i + m_j - 1, i + j = k\}$.)

7.* (Conlon, 2008) obtained that for all large n,

$$br(K_{n,n}) \le (1 + o(1))2^{n+1} \log_2 n.$$

What expression of the small term o(1) can we have?

References

H. L. Abbott, Lower bounds for some Ramsey numbers, *Discrete Math.* **2** (1972), 289–293.

H. Abbott and D. Hanson, A problem of Schur and its generalizations, *Acta Arith.* **20** (1972), 175–187.

D. Achlioptas and A. Naor, The two possible values of the chromatic number of a random graph, *Ann. of Math.* **162** (2005), 1335–1351.

M. Aigner and G. Ziegler, *Proofs from THE BOOK, 4th Edition*, Springer, Berlin, 2012.

M. Ajtai, P. Erdős, J. Komlós and E. Szemerédi, On Turán theorem for sparse graphs, *Combinatorica* **1** (1981), 313–317.

M. Ajtai, P. Erdős, J. Komlós and E. Szemerédi, A dense infinite Siden sequence, *European J. Combin.* **2** (1981), 1–11.

M. Ajtai, J. Komlós and E. Szemerédi, A note on Ramsey numbers, *J. Combin. Theory Ser. A* **29** (1980), 354–360.

M. Ajtai, J. Komlós and E. Szemerédi, A dense infinite Sidon sequence, *European J. Combin.* **2** (1981), 1–11.

P. Allen, G. Brightwell and J. Skokan, Ramsey–goodness and otherwise, *Combinatorica* **33** (2013), 125–160.

N. Alon, The maximal number of Hamiltonian paths in tournaments, *Combinatorica* **10** (1990), 319–324.

N. Alon, Subdivided graphs have linear Ramsey numbers, *J. Graph Theory* **18** (1994), 343–347.

N. Alon, Explicit Ramsey graphs and orthonormal labelings, *Electron. J. Combin.* **1** (1994b), Research Paper 12, approx. 8 pp.

N. Alon, L. Babai and H. Suzuki, Multilinear polynomials and Frankl-Ray-Chaudhuri-Wilson type intersection theorems, *J. Combin. Theory Ser. A* **58** (1991), 165–180.

N. Alon, E. Fischer, M. Krivelevich and M. Szegedy, Efficient testing of large graphs, *Combinatorica* **20** (2000), 451–476.

N. Alon and M. Krivelevich, Constructive bounds for a Ramsey-type problem, *Graphs Combin.* **13** (1997), 217–225.

N. Alon, M. Krivelevich and B. Sudakov, Turán Numbers of Bipartite Graphs and Related Ramsey-Type Questions, *Combin. Probab. Comput.* **12** (2003), 477–494.

N. Alon and A. Orlitsky, Repeated communication and Ramsey Graphs, *IEEE Trans. Inform. Theory* **41** (1995), 1276–1289.

Y. Li, Q. Lin, *Elementary Methods of Graph Ramsey Theory*, Applied Mathematical Sciences 211, https://doi.org/10.1007/978-3-031-12762-5

N. Alon and P. Pudlák, Constructive lower bounds for off-diagonal Ramsey numbers, *Israel J. Math.* **122** (2001) 243–251.

N. Alon and V. Rödl, Sharp bounds for some multicolor Ramsey numbers, *Combinatorica* **25** (2005), 125–141.

N. Alon, L. Rónyai, and T. Szabó, Norm-graphs: Variations and applications, *J. Combin. Theory Ser. B* **76** (1999), 280–290.

N. Alon and J. Spencer, *The Probabilistic Method*, 4th ed., Wiley–Interscience, New York, 2016.

V. Angeltveit and B. D. McKay, $R(5, 5) \leq 48$, *J. Graph Theory* **89** (2018), 5–13.

N. Ankeny, The least quadratic non-residue, *Ann. of Math.* **55** (1952), 65–72.

M. Axenovich, Z. Füredi , and D. Mubayi, On generalized Ramsey theory: The bipartite case, *J. Combin. Theory Ser. B* **79** (2000), 66–86.

R. Baker, G. Harman and J. Pintz, The difference between consecutive primes, II, *Proc. Lond. Math. Soc.* **83** (2001), 532–562.

J. Beck, On size Ramsey numbers of paths, trees and circuits I, *J. Graph Theory* **7** (1983), 115–129.

J. Beck, On size Ramsey numbers of paths, trees and circuits II, in: *Mathematics of Ramsey Theory, J. Nesětřil and V. Rödl Eds.*, Springer-Verlag, Berlin, 1990, pp. 34–45.

C. T. Benson, Minimal regular graphs of girths eight and twelve, *Canad. J. Math.* **18** (1966), 1091–1094.

E. R. Berlekamp, A construction for partitions which avoid long arithmetic progressions, *Canad. Math. Bull.* **11** (1968), 409–414.

E. R. Berlekamp, On subsets with intersections of even cardinality, *Canad. Math. Bull.* **12** (1969), 471–477.

A. Bishnoi, F. Ihringer and V. Pepe, A construction for clique-free pseudorandom graphs, https://arxiv.org/abs/1905.04677.

T. Bloom, A quantitative improvement for Roth's theorem on arithmetic progressions, *J. Lond. Math. Soc.* **93** (2016), 643–663.

T. Bloom and O. Sisask, Logarithmic bounds for Roth's theorem via almost-periodicity, *Discrete Anal.* **4** (2019), 20 pp.

T. Bohman, The triangle-free process, *Adv. Math.* **221** (2009), 1653–1677.

T. Bohman and R. Holzman, A nontrivial lower bound on the Shannon capacities of the complements of odd cycles, *IEEE Trans. Inform. Theory* **49** (2003), 721–722.

T. Bohman and P. Keevash, The early evolution of the H-free process, *Invent. Math.* **181** (2010), 291–336.

T. Bohman and P. Keevash, Dynamic concentration of the triangle-free process, *Random Structures Algorithms* **58** (2021), no. 2, 221–293.

B. Bollobás, On generalized graphs, *Acta Math. Hungar.* **16** (1965), 447–452.

B. Bollobás, *Extremal Graph Theory*, Academic Press, New York-London, 1978.

B. Bollobás, The diameter of random graphs, *Trans. Amer. Math. Soc.* **267** (1981), 41–52.

B. Bollobás, *Combinatorics*, Cambridge University Press, London, 1986.

B. Bollobás, The chromatic numbers of random graphs, *Combinatorica* **8** (1988), 49–55.

B. Bollobás, *Graph Theory: An Introductory Course, 3rd Edition*, Springer-Verlag, Berlin, 1994.

B. Bollobás, *Modern Graph Theory*, Springer–Verlag, New York, 1998.

B. Bollobás, *Random Graphs*, 2nd ed., Vol. 73 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2001.

B. Bollobás, S. E. Eldridge, Packing of graphs and applications to computational complexity, *J. Combin. Theory Ser. B* **25** (1978), 105–124.

B. Bollobás and P. Erdős, On a Ramsey-Turán type problem, *J. Combin. Theory Ser. B* **21** (1976), 166–168.

B. Bollobás and P. Erdős, Cliques in random graphs, *Math. Proc. Cambridge Philos. Soc.* **80** (1976), 419–427.

B. Bollobás, O. Riordan, J. Spencer and G. Tusnády, The degree sequence of a scale-free random graph process, *Random Structures Algorithms* **18** (2001), 279–290.

B. Bollobás and A. Thomason, Graphs which contain all small graphs, *European J. Combin.* **2** (1981), 13–15.

J. Bondy and P. Erdős, Ramsey numbers for cycles in graphs, *J. Combin. Theory Ser. B* **11** (1973), 46–54.

J. Bondy and M. Simonovits, Cycles of even length in graphs, *J. Combin. Theory Ser. B* **16** (1974), 97–105.

K. Borsuk, Drei Sätze über die n-dimensionale euklidische sphäre, *Fund. Math.* **20** (1933), 177–190.

J. Böttcher, P. Heinig and A. Taraz, Embedding into bipartite graphs, *SIAM J. Discrete Math.* **24** (2010), 1215–1233.

J. Bourgain, On triples in arithmetic progression, *Geom. Funct. Anal.* **9** (1999), 968–984.

J. Bourgain, Roth's theorem on progressions revisited, *J. Anal. Math.* **104** (2008), 155–206.

W. G. Brown, On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.* **9** (1966), 281–285.

T. Brown, B. Landman and A. Robertson, Bounds on some van der Waerden numbers, *J. Combin. Theory Ser. A* **115** (2008), 1304–1309.

M. Bucić, S. Letzter and B. Sudakov, 3-color bipartite Ramsey number of cycles and paths, *J. Graph Theory* **92** (2019), no. 4, 445–459.

J. P. Burling and S. W. Reyner, Some Lower Bounds of the Ramsey Numbers $n(k,k)$, *J. Combin. Theory Ser. B* **13** (1972), 168–169.

S. Burr, Ramsey numbers involving graphs with long suspended paths, *J. Lond. Math. Soc.* **24** (1981), 405–413.

S. Burr, Multicolor Ramsey numbers involving graphs with long suspended paths, *Discrete Math.* **40** (1982), 11–20.

S. Burr and P. Erdős, On the magnitude of generalized Ramsey numbers for graphs, in Infinite and Finite Sets I (Colloq., Keszthely, 1973), Colloq. Math. Soc. Janos Bolyai 10, North–Holland, Amsterdam (1975), 214–240.

S. A. Burr and P. Erdős, Generalizations of a Ramsey-theoretic result of Chvátal, *J. Graph Theory* **7** (1983), 39–51.

S. A. Burr and P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp, Ramsey-minimal graphs for multiple copies, *Nederl. Akad. Wetensch. Indag. Math.* **40** (1978), 187–195.

S. Burr, P. Erdős and L. Lovász, On graphs of Ramsey type, *Ars Combin.* **1** (1976), 167–190.

S. A. Burr and J. A. Roberts, On Ramsey numbers for stars, *Util. Math.* **4** (1973), 217–220.

T. Cai, T. Liang and A. Rakhlin, On dectection and structural reconstruction of small–world random networks, *IEEE Trans. Network Sci. Eng.* **4** (3) (2017), 165–176.

Y. Caro, New results on the independence number, *Technical Report*, Tel–Aviv University, 1979.

Y. Caro, Zero-sum problems-A survey, *Discrete Math.* **152** (1996), 93–113.

Y. Caro and C. Rousseau, Asymptotic bounds for bipartite Ramsey numbers, *Electron. J.*

Combin. **8** (2001), no. 1, Research Paper 17, 5 pp.

Y. Caro and Z. Tuza, On k-local and k-mean colorings of graphs and hypergraphs, *Quart. J. Math. Oxford* **44** (1993), 358–398.

P. Catlin, Hajós' graph-coloring conjecture: variations and counterexamples, *J. Combin. Theory, Ser. B* **26** (1979), 268–274.

G. Chartrand and S. Schuster, On the existence of specified cycles in complementary graphs, *Bull. Amer. Math. Soc.* **77** (1971), 995–998.

X. Chen, Q. Lin and C. You, Ramsey numbers of large books, arXiv:2103.05814v3, 2021.

G. Chen and R. Schelp, Graphs with linearly bounded Ramsey numbers, *J. Combin. Theory Ser. B* **57** (1993), 138–149.

G. Chen, X. Yu and Y. Zhao, Improved bounds on the Ramsey number of fans, *European J. Combin.* **96** (2021), 103347.

H. Chernoff, A measure of the asymptotic efficiency for tests of a hypothesis based on the sum of observations, *Ann. Math. Statis.* **23** (1952), 493–509.

V. Chvátal, V. Rödl, E. Szemerédi and T. Trotter, The Ramsey Number of a graph with bounded maximum degree, *J. Combin. Theory Ser. B* **34** (1983), 239–243.

F. R. Chung, On the Ramsey number $N(3, 3, 3, 3; 2)$, *Discrete Math.* **5** (1973), 317–321.

F. R. Chung, A note on constructive methods for Ramsey numbers, *J. Graph Theory* **5** (1981), 109–113.

F. R. Chung, Regularity lemmas for hypergraphs and quasi-randomness, *Random Structures Algorithms* **2** (1991), 241–252.

F. Chung, P. Erdős and R. Graham, On sparse sets hitting linear forms, in: *Number Theory for the Millennium, I*, Urbana, IL, 2000, A.K. Peters, Natick, MA, 2002, pp. 257–272.

F. R. Chung and R. L. Graham, On multicolor Ramsey numbers for bipartite graphs, *J. Combin. Theory Ser. B* **18** (1975), 164–169.

F. R. Chung and R. L. Graham, Erdős on Graphs: His Legacy of Unsolved Problems, AK Peters, Massachusetts, 1999.

F. R. Chung, R. L. Graham, Sparse quasi-random graphs, *Combinatorica* **22** (2002), 217–244.

F. R. Chung, R. L. Graham and R. Wilson, Quasi-random graphs, *Combinatorica* **9** (1989), 345–362.

F. R. Chung and L. Lu, Complex Graphs and Networks, American Mathematical Society, U.S.A., 2004.

V. Chvátal, Tree-complete graph Ramsey numbers, *J. Graph Theory* **1** (1977), 93.

V. Chvátal and F. Harary, Generalized Ramsey Theory for Graphs II, Small diagonal numbers, *Proc. Amer. Math. Soc.* **32** (1972), 389–394.

V. Chvátal and F. Harary, Generalized Ramsey theory for graphs. III, Small off-diagonal numbers, *Pacific J. Math.* **41** (1972), 335–345.

V. Chvátal, V. Rödl, E. Szemerédi and T. Trotter, The Ramsey Number of a graph with bounded maximum degree, *J. Combin. Theory Ser. B* **34** (1983), 239–243.

B. Codenotti, P. Pudlák and G. Resta, Some structural properties of low rank matrices related to computational complexity, *Theoret. Comput. Sci.* **235** (2000), 89–107.

D. Conlon, A new upper bound for the bipartite Ramsey problem, *J. Graph Theory* **58** (2008), 351–356.

D. Conlon, A new upper bound for diagonal Ramsey numbers, *Ann. of Math.* **170** (2009), 941–960.

D. Conlon, A sequence of triangle-free pseudorandom graphs, *Combin. Probab. Comput.* **26**

(2017), 195–200.

D. Conlon, Graphs with few paths of prescribed length between any two vertices, *Bull. Lond. Math. Soc.* **51** (2019), 1015–1021.

D. Conlon, The Ramsey number of books, *Adv. Combin.* 2019: **3**, 12pp.

D. Conlon, Lectures on graph Ramsey theory,
http://www.its.caltech.edu/ dconlon/Ramsey-course.html.

D. Conlon, A. Ferber, Lower bounds for multicolor Ramsey numbers, *Adv. Math.* **378** (2021), Paper No. 107528.

D. Conlon and J. Fox, Bounds for graph regularity and removal lemmas, *Geom. Funct. Anal.* **22** (2012), 1191–1256.

D. Conlon, J. Fox, C. Lee and B. Sudakov, Ramsey numbers of cubes versus cliques, *Combinatorica* **36** (2016), 37–70.

D. Conlon, J. Fox and B. Sudakov, Hypergraph Ramsey numbers, *J. Amer. Math. Soc.* **23** (2010), no. 1, 247–266.

D. Conlon, J. Fox and B. Sudakov, Short proofs of some extremal results, *Combin. Probab. Comput.* **23** (2014), 8–28.

D. Conlon, J. Fox and B. Sudakov, Short proofs of some extremal results II, *J. Combin. Theory Ser. B* **121** (2016), 173–196.

D. Conlon, J. Fox and B. Sudakov, Recent developments in graph Ramsey theory, in Surveys in combinatorics 2015, London Math. Soc. Lecture Note Ser., vol. 424, Cambridge Univ. Press, Cambridge, 2015, pp. 49–118.

D. Conlon, J. Fox and Y. Wigderson, Ramsey numbers of books and quasirandomness, *Combinatorica*, to appear.

E. Davies, M. Jenssen and B. Roberts, Multicolour Ramsey numbers of paths and even cycles, *European J. Combin.* 63 (2017), 124–133.

A. N. Day and J. R. Johnson, Multicolor Ramsey numbers of odd cycles, *J. Combin. Theory Ser. B* **124** (2017), 56–63.

D. Jr. Dellamonica, The size-Ramsey number of trees, *Random Structures Algorithms* **40** (2012), no. 1, 49–73.

W. Deuber, A generalization of Ramsey's number, in: *Infinite and Finite Sets (to Paul Erdős on His 60th Birthday 1973) I, Hajnal, Rado, Sos Eds.*, Colloq. Math. Soc. Janos Bolyai, North-Holland, Amsterdam/London, 1975.

M. Deza and P. Frankl, Every large set of equidistant $(0, +1, -1)$-vectors forms a sunflower, *Combinatorica* **1** (1981), 225–231.

M. Deza, P. Frankl and N. Singhi, On functions of strength t, *Combinatorica* **3** (1983), 331–339.

G. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, *J. Lond. Math. Soc.* **27** (1952), 85–92.

R. Diestel, *Graph Theory, 2nd Edition*, Springer, Berlin, 2010.

L. Dong, Y. Li and Q. Lin, Ramsey numbers involving graphs with large degrees, *Appl. Math. Lett.* **22** (2009), 1577–1580.

S. Dorogovtsev, J. Mendes, Evolution of networks, *Adv. Phy.* **51** (4) (2002), 1079.

A. Dudek and P. PraLat, An alternative proof of the linearity of the size-Ramsey number of paths, *Combin. Probab. Comput.* **24** (2015), 551–555.

A. Dudek and P. PraLat, On some multicolor Ramsey properties of random graphs, *SIAM J. Discrete Math.* **31** (2017), no. 3, 2079–2092.

A. Dudek and V. Rödl, On the Folkman number $f(2, 3, 4)$, *Experiment. Math.* **17** (2008), 63–67.

N. Eaton, Ramsey numbers for sparse graphs, *Discrete Math.* **185** (1998), 63–75.

C. S. Edwards, Some extremal properties of bipartite subgraphs, *Canad. J. Math.* **3** (1973), 475–485.

C. S. Edwards, An improved lower bound for the number of edges in a largest bipartite subgraph, Proceedings of Second Czechoslovak Symposium on Graph Theory, Prague, 1975, pp. 167–181.

P. Erdős, On sequences of integers no one of which divides the product of two others and on some related problems, *Mitt. Forsch. Inst. Math. Mech. Univ. Tomsk* **2** (1938), 74–82.

P. Erdős, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.* **53** (1947), 292–294.

P. Erdős, Graph theory and probability, *Canad. J. Math.* **11** (1959), 34–38.

P. Erdős, Graph theory and probability II, *Canad. J. Math.* **13** (1961), 346–352.

P. Erdős, On circuits and subgraphs of chromatic graphs, *Mathematika* **9** (1962), 170–175.

P. Erdős, Extremal problems in graph theory, in: *Theory of Graphs and Its Applications, M. Fiedler ed.*, Proc. Symp. Smolenice, 1963, Academic Press, New York-London, 1965.

P. Erdős, Some recent results on extremal problems in graph theory, in "Theory of Graphs" (Internat. Sympos., Rome, 1966), pp. 117–130, Gordon 6 Breach, New York, 1967.

P. Erdős, On the graph theorem of Turán, *Mat. Lapok* **21** (1970), 249–251.

P. Erdős, Problems and results on finite and infinite graphs, in: Recent Advances in Graph Theory, Proc. Second Czechoslovak Sympos., Prague, 1974, Academia, Prague, 1975, pp. 183–192.

P. Erdős, Solved and unsolved problems in combinatorics and combinatorial number theory, *Congr. Numer.* **32** (1981), 49–62.

P. Erdős (1981), Problems and results in graph theory. In *The Theory and Applications of Graphs: Kalamazoo, MI, 1980*, Wiley, pp. 331–341.

P. Erdős, Some new problems and results in graph theory and other branches of combinatorial mathematics. Combinatorics and graph theory (Calcutta, 1980), pp. 9–17, Lecture Notes in Math., 885, Springer, Berlin-New York, 1981.

P. Erdős, On some problems in graph theory, combinatorial analysis and combinatorial number theory, in: B. Bololobás (Ed.), Graph Theory and Combinatorics, Cambridge, 1983, Academic Press, London, New York, 1984, pp. 1–17.

P. Erdős and S. Fajtlowicz, On the conjecture of Hajós, *Combinatorica* **1** (1981), 141–143.

P. Erdős, R. Faudree, C. Rousseau and R. Schelp, The size Ramsey number, Period. Math. Hungar. **9** (1978), 145–161.

P. Erdős, R. Faudree, C. Rousseau and R. Schelp, On cycle-complete Ramsey numbers, *J. Graph Theory* **2** (1978), 53–64.

P. Erdős, R. Faudree, C. Rousseau and R. Schelp, Ramsey numbers for brooms, *Congressus Numerantium* **35** (1982), 283–293.

P. Erdős, R. Faudree, C. Rousseau, and R. Schelp, A Ramsey problem of Harary on graphs with prescribed size, *Discrete Math.* **67** (1987), 227–233.

P. Erdős and T. Gallai, Maximal paths and circuits in graphs, *Acta Math. Hungar.* **10** (1959), 337–356.

P. Erdős, A. Ginzburg and A. Ziv, Theorem in additive number theory, *Bull. Res. Council Israel Sect. F* **10F** (1961), 41–43.

P. Erdős and A. Gyárfás, A variant of the classical Ramsey problem, *Combinatorica* **17** (1997), 459–467.

P. Erdős and A. Hajnal, Research problem 2-5, *J. Combin. Theory* **2** (1967), 104.

P. Erdős and A. Hajnal, On Ramsey like theorems, Problems and results, Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972), pp. 123–140, Inst. Math. Appl., Southend-on-Sea, 1972.

P. Erdős and A. Hajnal, Ramsey-type theorems, *Discrete Appl. Math.* **25** (1989), 37–52.

P. Erdős, A. Hajnal and L. Pósa, Strong embeddings of graphs into colored graphs, in: *Infinite and Finite Sets (to Paul Erdős on His 60th Birthday 1973) II, A. Hajnal, R. Rado and V. Sos Eds.*, Colloq. Math. Soc. Janos Bolyai, North-Holland, Amsterdam/London, 1975.

P. Erdős, A. Hajnal and R. Rado, Partition relations for cardinal numbers, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 93–196.

P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford* **12** (1961), 313–320.

P. Erdős and L. Lovász, Problems and results on 3–chromatic hypergraph and some related questions, in: *Infinite and Finite Sets (to Paul Erdős on His 60th Birthday 1973) II, Hajnal, Rado, Sos Eds.*, Colloq. Math. Soc. Janos Bolyai, North–Holland, Amsterdam/London, 1975.

P. Erdős, R. McEliece and H. Taylor, Ramsey bounds for graph products, *Pacific J. Math.* **37** (1971), 45–46.

P. Erdős and R. Rado, Combinatorial theorems on classifications of subsets of a given set, *Proc. Lond. Math. Soc.* **2** (1952), 417–439.

P. Erdős and A. Rényi, On the evolution of random graphs, *Magyar Tud. Akad. Mat. Kutato Int. Kozl.* **5** (1960), 17-61.

P. Erdős and A. Rényi, On a problem in the theory of graphs, *Magyar Tud. Akad. Mat. Kutato Int. Kozl.* **7** (1962), 623–641.

P. Erdős, A. Rényi, and V. T. Sós, On a problem of graph theory, *Studia Sci. Math. Hungar.* **1** (1966), 215–235.

P. Erdős and C. C. Rousseau, The size Ramsey number of a complete bipartite graph, *Discrete Math.* **113** (1993), no. 1-3, 259–262.

P. Erdős and Sós, Mentioned in: P. Erdős, Extremal problems in graph theory, Theory of graphs and its applications, in: *Proc. Symposium held in Smolenica in June 1963*.

P. Erdős and M. Simonovits, A limit theorem in graph theory, *Studia Sci. Math. Hungar.* **1** (1966), 51–57.

P. Erdős and J. H. Spencer, Probabilistic Methods in Combinatorics, Academic Press, 1974.

P. Erdős and J. H. Spencer, Lopsided Lovász lemma and Latin transversals, *Discrete Appl. Math.* **30** (1991), 151–154.

P. Erdős and A. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.* **5** (1946), 1087–1091.

P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compos. Math.* **2** (1935), 464–470.

P. Erdős and P. Turán, On some sequences of integers, *J. Lond. Math. Soc.* **11** (1936), 261–264.

G. Exoo, A Lower Bound for $R(5,5)$, *J. Graph Theory* **13** (1989), 97–98.

G. Exoo, A lower bound for Schur numbers and multicolor Ramsey numbers, *Electron. J. Combin.* **1** (1994), # R8.

N. Fagle, A. Pentland and D. Lazer, Infering friendship network structure by using mobile phone data, *Proc. Natl. Acad. Sci. USA*, 106 (36) (2009), 15274–15278.

G. Fan, The Erdős-Sós conjecture for spiders of large size, *Discrete math.* **313** (2013), 2513–2517.

R. Faudree, C. Rousseau and J. Sheehan, Strongly regular graphs and finite Ramsey theory, *Linear Algebra Appl.* **46** (1982), 221–241.

R. Faudree and R. Schelp, All Ramsey numbers for cycles in graphs, *Discrete Math.* **8** (1974), 313–329.

R. Faudree and R. Schelp, Path-path Ramsey-type numbers for the complete bipartite graph, *J. Combin. Theory Ser. B* **19** (1975), 161–173.

R. Faudree and M. Simonovits, On a class of degenerate extremal graph problems, *Combinatorica* **3** (1983), 83–93.

S. Fettes, R. L. Kramer and S. P. Radziszowski, An upper bound of 62 on the classical Ramsey number $r(3,3,3,3)$ *Ars Combin.* **72** (2004), 41–63.

A. Figaj and T. Luczak, The Ramsey numbres for a triple of long even cycles, *J. Combin. Theory Ser. B* **97** (2007), 584–596.

R. Fisher, An examination of the different possible solutions of a problem in incomplete blocks, *Annals of Eugenics (London)* **10** (1940), 53–75.

G. Fiz Pontiveros, S. Griffiths and R. Morris, The triangle-free process and $R(3,k)$, *Mem. Amer. Math. Soc.* **263** (2020), no. 1274, v+125 pp.

J. Folkman, Graphs with monochromatic complete subgraphs in every edge coloring, *SIAM J. Appl. Math.* **18** (1970), 19–24.

J. Folkman, Notes on the Ramsey numbers $N(3,3,3,3)$, *J. Combin. Theory Ser. A* **16** (1974), 371–379.

J. Fox, X. He and Y. Wigderson, Ramsey goodness of books revisited, arXiv:2109.09205v1, 2021.

J. Fox and L. M. Lovász, A tight lower bound for Szemerédi's regularity lemma, *Combinatorica* **37** (2017), 911–951.

J. Fox and B. Sudakov, Dependent random choice, *Random Structures Algorithms* **38** (2011), 68–99.

P. Frankl, A constructive lower bound for Ramsey numbers, *Ars Combin.* **3** (1977), 297–302.

P. Frankl and V. Rödl, The uniformity lemma for hypergraphs, *Graphs Combin.* **8** (1992), 309–312.

P. Frankl and V. Rödl, Large triangle-free subgraphs in graphs without K_4, *Graphs Combin.* **2** (1986), 136–144.

P. Frankl and R. Wilson, Intersection theorems with geometric consequences, *Combinatorica* **1** (1981), 357–368.

H. Fredricksen and M. Sweet, Symmetric sum-free partitions and lower bounds for Schur numbers, *Electron. J. Combin.* **7** (2000), #R32.

J. Friedman and N. Pippenger, Expanding graphs contain all small trees, *Combinatorica* **7** (1987), 71–76.

A. Frieze and M. Karoński, *Introduction to random graphs*, Cambridge University Press, Cambridge, 2016.

J. Fukuyama, Improved bound on sets including no sunflower with three petals, arXiv:1809.10318, 2018.

Z. Füredi, Graphs without quadrilaterals, *J. Combin. Theory Ser. B* **34** (1983), 187–190.

Z. Füredi, On a Turán type problem of Erdős, *Combinatorica* **11** (1991), 75–79.

Z. Füredi (a), An upper bound on Zarankiewicz' problem, *Combin. Probab. Comput.* **5** (1996), 29–33.

Z. Füredi (b), New asymptotics for bipartite Turán numbers, *J. Combin. Theory Ser. A* **75** (1996), 141–144.

Z. Füredi (c), On the number of edges of quadrilateral-free graphs, *J. Combin. Theory Ser. B*

68 (1996), 1–6.

Z. Füredi, A. Naor and J. Verstraëte, On the Turán number for the hexagon, *Adv. Math.* **203** (2006), 476–496.

Z. Füredi and M. Simonovits, The history of degenerate (bipartite) extremal graph problems, Erdős centennial, Bolyai Soc. Math. Stud., vol. 25, János Bolyai Math. Soc., Budapest, 2013, 169–264.

L. Geréncser and A. Gyárfás, On Ramsey-type problems, *Ann. Univ. Sci. Budapest, Eőtvős Sect. Math.* **10** (1967), 167–170.

W. Goddard and D. Kleitmen, An upper bound for the Ramsey number $r(K_3, G)$, *Discrete Math.* **125** (1994), 177–182.

C. Godsil and G. Royle, Algebraic graph theory, Springer–Verlag, New York, 2001.

R. Goodman, On sets of a acquaintances and strangers at any party, *Amer. Math. Monthly* **66** (1959), 778–783.

W. T. Gowers, Lower bounds of tower type for Szemerédi's uniformity lemma, *Geom. Funct. Anal.* **7** (1997), 322–337.

W. T. Gowers, A new proof of Szemerédi's theorem for arithmetic progressions of length four, *Geom. Funct. Anal.* **8** (1998), 529–551.

R. Graham, On edgewise 2-colored graphs with monochromatic triangles and containing no complete hexagon, *J. Combin. Theory* **4** (1968), 300.

R. L. Graham and V. Rödl, Numbers in Ramsey theory, Surveys in combinatorics 1987 (New Cross, 1987), London Math. Soc. Lecture Note Ser., vol. 123, Cambridge Univ. Press, Cambridge, 1987, pp. 111–153.

R. L. Graham, V. Rödl, and A. Ruciński, On graphs with linear Ramsey numbers, *J. Graph Theory* **35** (2000), 176–192.

R. L. Graham, V. Rödl, and A. Ruciński, On bipartite graphs with linear Ramsey numbers, *Combinatorica* **21** (2001), 199–209.

R. L. Graham, B. L. Rothschild and J. H. Spencer, Ramsey theory (2nd ed.). John Wiley & Sons, Inc., New York, 1990. xii+196 pp.

J. Graver and J. Yackel, Some graph theoretic results associated with Ramsey's theorem, *J. Combin. Theory* **4** (1968), 125–175.

B. Green, New lower bounds for van der Waerden numbers, Preprint (2021), arXiv:2102.01543.

R.E. Greenwood and A. M. Gleason, Combinatorial relations and chromatic graphs, *Canad. J. Math.* **7** (1955), 1–7.

B. Green and T. Tao, The primes contains arbitrarily long arithmetic progression, *Ann. of Math.* **167** (2008), 481–547.

J. Griggs, An upper bound on the Ramsey number $R(3, n)$, *J. Combin. Theory Ser. A* **35** (1983), 145–153.

G. Grimmett and C. McDiarmid, On coloring random graphs, *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 313–324.

C. Grinstead and S. Roberts, On the Ramsey numbers $R(3, 8)$ and $R(3, 9)$, *J. Combin. Theory Ser. B* **33** (1982), 27–51.

H. Guo and L. Warnke, On the power of random greedy algorithms, Preprint 2021, arXiv:2104.07854v1.

A. Gyárfás and J. Lehel, A Ramsey-type problem in directed and bipartite graphs, *Pereodica Math. Hung.* **3** (1973), no. 3-4, 299–304.

A. Gyárfás, J. Lehel, J. Nešetřil, V. Rödl, R. Schelp and Z. Tuza, Local k-colorings of graphs and hypergraphs, *J. Combin. Theory B* **43** (1987), 127–139.

A. Gyárfás, M. Ruszinkó, G. Sárközy and E. Szemerédi, Three-color Ramsey numbers for paths, *Combinatorica* **27** (2007), 35–69.

A. Hales and R. Jewett, Regularity and position games, *Trans. Amer. Math. Soc.* **106** (1963), 222–229.

P. E. Haxell and Y. Kohayakawa, The size-Ramsey number of trees, *Israel J. Math.* **89** (1995), 261–274.

P. E. Haxell, Y. Kohayahawa and T. Luczak, The induced size-Ramsey number of cycles, *Combin. Probab. Comput.* **4** (1995), 217–239.

D. R. Heath-Brown, Integer sets containing no arithmetic progressions, *J. Lond. Math. Soc.* **35** (1987), 385–394.

P. Herwig, M. Heule, P. van Lambalgen and H. van Maaren, A new method to construct lower bounds for van der Waerden numbers, *Electron. J. Combin.* **14** (2007), # R6.

R. Jamison, T. Jiang and A. Ling, Constrained Ramsey numbers of graphs, *J. Graph Theory* **42** (2003), 1–16.

S. Janson, T. Luczak and A. Ruciński, *Random Graphs*, Wiley-Interscience, New York, 2000.

R. Javadi, F. Khoeini, G. R. Omidi and A. Pokrovskiy, On the size-Ramsey number of cycles, *Combin. Probab. Comput.* **28** (2019), no. 6, 871–880.

M. Jenssen and J. Skokan, Exact ramsey numbers of odd cycles via nonlinear optimisation, *Adv. Math.* 376 (2021), 107444, 46 pp.

J. Kahn and G. Kalai, A counterexample to Borsuk's conjecture, *Bull. Amer. Math. Soc.* **29** (1993), 60–62.

J. G. Kalbfleisch, Construction of special edge-chromatic graphs, *Canad. Math. Bull.* **8** (1965), 575–584.

G. Katona, A simple proof of the Erdős-Ko-Rado theorem, *J. Combin. Theory Ser. B* **13** (1972), 183–184.

X. Ke, The size Ramsey number of trees with bounded degree, *Random Structures Algorithms* **4** (1993), 85–97.

P. Keevash, E. Long and J. Skokan, Cycle-complete Ramsey numbers, *Int. Math. Res. Not. IMRN* (2021), no. 1, 277–302.

J. Kim, The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$, *Random Structures Algorithms* **7** (1995), 173–207.

V. Klee and D. Larman, Diameters of random graphs, *Canad. J. Math.* **33** (1981), 618–640.

C. Knierim and P. Su, Improved bounds on the multicolor Ramsey numbers of paths and even cycles, *Electron. J. Combin.* **26** (2019), no. 1, Paper No. 1.26, 17 pp.

Y. Kohayakawa, Szemerédi's regularity lemma for sparse graphs, Foundations of computational mathematics (Rio de Janeiro, 1997), Springer, Berlin, 1997, 216–230.

Y. Kohayakawa and V. Rödl, Szemerédi's regularity lemma and quasi-randomness, Recent advances in algorithms and combinatorics, CMS Books Math./Ouvrages Math. SMC, vol. 11, Springer, New York, 2003, 289–351.

Y. Kohayakawa, V. Rödl, M. Schacht and E. Szemeredi, Sparse partition universal graphs for graphs of bounded degree, *Adv. Math.* **226** (2011), 5041–5065

J. Kollár, L. Rónyai, and T. Szabó, Norm-graphs and bipartite Turán numbers, *Combinatorica* **16** (1996), 399–406.

J. Komlós and M. Simonovits, *Szemerédi's regularity lemma and its applications in graph theory*, in: Combinatorics, Paul Erdős is eighty, vol. 2 (Miklós, Sós, and Szőnyi eds.), Bolyai Math. Soc., Budapest (1996), 295–352.

Y. Kohayakawa, H. Prömel and V. Rödl, Induced Ramsey numbers, *Combinatorica* **18** (1998), 373–404.

Y. Kohayakawa, M. Simonovits and J. Skokan, The 3-colored Ramsey number of odd cycles, *Electron. Notes Discrete Math.* **19** (2005), 397–402.

G. Kossinets and D. Watts, Empirical analysis of evolving social network, *Science* **311** (2006) 5757, 88–90.

A. Kostochka, A bound of the cardinality of families not containing Δ-systems, In *The Mathematics of Paul Erdős II*, pp. 229–235, Springer, 1997.

A. V. Kostochka, P. Pudlák and V. Rödl, Some constructive bounds on Ramsey numbers, *J. Combin. Theory Ser. B* **100** (2010), 439–445.

A. V. Kostochka and V. Rödl, On graphs with small Ramsey numbers, *J. Graph Theory* **37** (2001), 198–204.

A. V. Kostochka and V. Rödl, On graphs with small Ramsey numbers II, *Combinatorica* **24** (2004), 389–401.

T. Kövári, T. Sós, and P. Turán, On a problem of K. Zarankiewicz, *Colloq. Math.* **3** (1954), 50–57.

M. Krivelevich, Bounding Ramsey numbers through large deviation inequalities, *Random Structures Algorithms* **7** (1995), 145–155.

M. Krivelevich and B. Sudakov, Pseudo–random graphs, *Bolyai Soc. Math. Stud.* **15** (2006), 199–262.

F. Lazebnik and V. A. Ustimenko, Explicit construction of graphs with an arbitrary large girth and of large size, *Discrete Appl. Math.* **60** (1995), 275–284.

F. Lazebnik, V. A. Ustimenko and A. J. Woldar, A new series of dense graphs of high girth, *Bull. Amer. Math. Soc.* **32** (1995), 73–79.

F. Lazebnik and A. Woldar, New lower bounds on the multicolor Ramsey numbers $r_k(C_4)$, *J. Combin. Theory Ser. B* **79** (2000), 172–176.

C. Lee, Ramsey numbers of degenerate graphs, *Ann. of Math.* **185** (2017), 791–829.

H. Lefmann, A note on Ramsey numbers, *Studia Sci. Math. Hung.* **22** (1987), 445–446.

S. Letzter, Path Ramsey number for random graphs, *Combin. Probab. Comput.* **25** (2016), no. 4, 612–622.

Y. Li and C.C. Rousseau, Fan-complete graph Ramsey numbers, *J. Graph Theory* **23** (1996), 413–420.

Y. Li and C. C. Rousseau, On book-complete graph Ramsey numbers, *J. Combin. Theory Ser. B* **68** (1996), 36–44.

Y. Li and C. C. Rousseau, On the Ramsey number $r(H+\bar{K}_n, K_n)$, *Discrete Math.* **170** (1997), 265–267.

Y. Li, C. C. Rousseau and L. Šoltés, Ramsey linear families and generalized subdivided graphs, *Discrete Math.* **170** (1997), 269–275.

Y. Li, C. C. Rousseau and W. Zang, Asymptotic upper bounds for Ramsey functions, *Graphs Combin.* **17** (2001), 123–128.

Y. Li, C. Rousseau and W. Zang, An upper bound for Ramsey numbers, *Appl. Math. Lett.* **17** (2004), 663–665.

Y. Li, Y. Shang and Y. Yang, Clustering coefficients of large networks, *Inform. Sci.* **382/383** (2017), 350–358.

Y. Li and J. Shu, A lower bound for off-diagonal van der Waerden numbers, *Adv. Appl. Math.* **44** (2010), 243–247.

Y. Li and W. Zang, Ramsey number $r(C_{2m+1}, K_n)$ with large n, *Adv. Math. (China)* **30** (2001), 286–287.

Y. Li and W. Zang, Ramsey numbers involving large dense graphs and bipartite Turán numbers, *J. Combin. Theory Ser. B* **87** (2003), 280–288.

Y. Li and W. Zang, The independence number of graphs with a forbidden cycle and Ramsey numbers, *J. Comb. Optim.* **7** (2003), 353–359.

Q. Lin and Y. Li, On Ramsey numbers of fans, *Discrete Applied Math.* **157** (2009), 191–194.

Q. Lin, Y. Li and L. Dong, Ramsey goodness and generalized stars, *European J. Combin.* **31** (2010), 1228–1234.

Q. Lin and Y. Li, Multicolor Bipartite Ramsey Number of C_4 and large $K_{n,n}$, *J. Graph Theory* **67** (2011), 47–54.

Q. Lin and Y. Li, A Folkman Linear Family, *SIAM J. Discrete Math.* **29** (2015), 1988–1998.

Q. Lin and Y. Li, Sparse multipartite graphs as partition universal for graphs with bounded degree, *J. Comb. Optim.* **35** (2018), 724–739.

Q. Lin, Y. Li and J. Shen, Lower bounds for $r_2(K_1 + G)$ and $r_3(K_1 + G)$ from Paley graph and generalization, *European J. Combin.* **40** (2014), 65–72.

Q. Lin and X. Liu, Ramsey numbers involving large books, *SIAM J. Discrete Math.* **35** (2021), 23–34.

S. Lin, On Ramsey numbers and K_r-coloring of graphs, *J. Combin. Theory Ser. B* **12** (1972), 82–92.

M. Liu and Y. Li, Ramsey numbers and bipartite Ramsey numbers via quasi-random graphs, *Discrete Math.* **344** (2021), no. 1, Paper No. 112162, 5 pp.

M. Liu, Y. Li and Q. Lin, Class Ramsey numbers of odd cycles in many colors, *Appl. Math. Comput.* **363** (2019), 124613, 4 pp.

L. Lovász, On the ratio of optimal and integral fractional covers, *Discrete Math.* **13** (1975), 383–390.

L. Lovász, Flats in matroids and geometric graphs, in: *Combinatorial Survey, Proc. 6th British Combin. Conf., Egham 1977, P. Cameron ed.*, Academic Press, New York-London, 1977.

L. Lovász, On the Shannon capacity of a graph, *IEEE Trans. Inform. Theory* **25** (1979), 1–7.

L. Lovász, *Combinatorial Problems and Exercise, 2nd Edition*, Akadémiai Kiadó, Budapest, 1993.

L. Lu, Explicit construction of small Folkman graphs, *SIAM J. Discrete Math.* **21** (2008), 1053–1060.

L. Lu and L. Székely, Using Lovász local lemma in the space of random injections, *Electron. J. Combin.* **14** (2007), no. 1, 63.

D. Lubell, A short proof of Sperner's lemma, *J. Combin. Theory* **1** (1966), 299.

T. Luczak, The chromatic number of random graphs, *Combinatorica* **11** (1991), 45–54.

T. Luczak, A note on the sharp concentration of the chromatic number of random graphs, *Combinatorica* **11** (1997), 295–297.

T. Luczak, $R(C_n, C_n, C_n) \leq (4 + o(1))n$, *J. Combin. Theory Ser. B* **75** (1999), 174–187.

T. Luczak, V. Rödl, On induced Ramsey numbers for graphs with bounded maximum degree, *J. Combin. Theory Ser. B* 66 (1996), 324–333.

J. Ma and T. Yang, Upper bounds on the extremal number of the 4-cycle, arXiv:2107.11601v1, 2021.

K. A. Mathew and P. R. J. Östergård, New lower bounds for the Shannon capacity of odd cycles, *Des. Codes Cryptogr.* 84 (2017), 13–22.

R. Mathon, Lower bounds for Ramsey numbers and association schemes, *J. Combin. Theory Ser. B* **42** (1987), 122–127.

D. W. Matula, On the complete subgraphs of a random graph, in: *Proc. Second Chapel Hill*

Conference on Combinatory Mathematics and its Applications, University of North Carolina, Chapel Hill, North Caroline, 1970.

D. W. Matula, The employee party problem, *Notices Amer. Math. Soc.* **19** (1972), A–328.

D. W. Matula, The largest clique size in a random graph, *Tech. Rep.*, Dept. Comput. Sci., Southern Methodist University, Dallas, 1976.

B. McKay and S. Radziszowski, $R(4,5) = 25$, *J. Graph Theory* **19** (1995), 309–322.

B. McKay and K. Zhang, The value of the Ramsey number $R(3,8)$, *J. Graph Theory* **16** (1992), 99–105.

K. E. Mellinger and D. Mubayi, Constructions of bipartite graphs from finite geometries, *J. Graph Theory* **49** (2005), 1–10.

L. D. Meshalkin, A generalization of Sperner's theorem on the number of subsets of a finite set, *Teor. Veroyatn Primen.* **8** (1963), 219–220.

H. Montgomery, *Topics in Multiplicative Number Theory*, Lecture Notes in Mathematics, 227, Springer-Verlag, 1971.

G. Moshkovitz and A. Shapira, A short proof of Gowers' lower bound for the regularity lemma, *Combinatorica* **36** (2016), 187–194.

D. Mubayi and A. Suk, Off-diagonal hypergraph Ramsey numbers, *J. Combin. Theory Ser. B* **125** (2017), 168–177.

D. Mubayi and A. Suk, Constructions in Ramsey theory, *J. Lond. Math. Soc.* **(2) 97** (2018), 247–257.

D. Mubayi and A. Suk, New lower bounds for hypergraph Ramsey numbers, *Bull. Lond. Math. Soc.* **50** (2018), 189–201.

D. Mubayi and J. Verstraëte, A note on pseudorandom Ramsey graphs, arXiv:1909.01461v2, 2019.

D. Mubayi and J. Williford, On the independence number of the Erdős-Rényi and projective norm graphs and a related hypergraph, *J. Graph Theory* **56** (2007), 113–127.

J. Mycielski, Sur le coloriage des graphs, *Colloq. Math.* **3** (1955), 161–162.

Z. Nagy, A certain constructive estimate of the Ramsey number (Hungarian), *Mat. Lapok*, **23** (1972), 301–302.

J. Nešetřil and V. Rödl, The Ramsey property for graphs with forbidden complete graphs, *J. Combin. Theory Ser. B* **20** (1976), 243–249.

J. Nešetřil and V. Rödl, Simple proof of the existence of restricted Ramsey graphs by means of a partite construction, *Combinatorica* **1** (1981), 199–202.

J. Nešetřil and M. Rosenfeld, I. Schur, C. E. Shannon and Ramsey numbers, a short story. Combinatorics, graph theory, algorithms and applications. *Discrete Math.* **229** (2001), no. 1–3, 185–195.

V. Nikiforov, The cycle-complete graph Ramsey numbers, *Combin. Probab. Comput.* 14 (2005), 349–370.

V. Nikiforov and C. C. Rousseau, Book Ramsey numbers. I, *Random Structures Algorithms* **27** (2005), 379–400.

V. Nikiforov and C. C. Rousseau, A note on Ramsey numbers for books, *J. Graph Theory* **49** (2005), 168–176.

V. Nikiforov and C. Rousseau, Ramsey goodness and beyond, *Combinatorica* **29** (2009), 227–262.

V. Nikiforov, C. C. Rousseau and R. H. Schelp, Book Ramsey numbers and quasi-randomness, *Combin. Probab. Comput.* **14** (2005), 851–860.

A. Nilli, On the second eigenvalue of a graph, *Discrete Math.* **91** (1991), 207–210.

A. Nilli, Tight estimates for eigenvalues of regular graphs, *Electron. J. Combin.* **11** (2004), no. 1, Note 9, 4 pp.

D. Osthus and A. Taraz, Random maximal H-free graphs, *Random Structures Algorithms* **18** (2001), 61–82.

T. Parsons, Ramsey graphs and block designs. I, *Trans. Amer. Math. Soc.* **209** (1975), 33–44.

R. Pastor-Satorras and A. Vespignani, Epidemic spreading in scale-free networks, *Phys. Rew. Lett.* **86** (2001), 3200–3203.

K. Piwakowski, S. Radziszowski and S. Urbanski, Computation of the Folkman number $F_e(3,3;5)$, *J. Graph Theory* **32** (1999), 41–49.

A. Pokrovskiy, Partitioning edge-coloured complete graphs into monochromatic cycles and paths, *J. Combin. Theory Ser. B* **106** (2014), 70–97.

L. Pósa, Hamiltonian circuits in random graphs, *Discrete Math.* **14** (1976) , 359–364.

K. Prachar, Primzahlverteilung, Springer-Verlag, Berlin and New York, 1957. MR 19, 393.

S. Radziszowski, Small Ramsey numbers, *Electron. J. Combin.* **1** (1994), a dynamic survey.

F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* **30** (1929), 264–286.

D. Ray-Chaudhuri and R. Wilson, On t-designs, *Osaka J. Math.* **12** (1975), 737–744.

L. Reiman, Über ein problem von K. Zarankiewicz, *Acta Math. Hungar.* **9** (1958), 269-278.

V. Rödl, The dimension of a graph and generalized Ramsey theorems, Master's thesis, Charles University, 1973.

V. Rödl, On a packing and covering problem, *European J. Combin.* **5** (1985), 69–78.

V. Rödl and E. Szemerrédi, On size Ramsey numbers of graphs with bounded degree, *Combinatorica* **20** (2000), 257–262.

V. Rosta, On a Ramsey type problem of J. A. Bondy and P. Erdős, I & II, *J. Combin. Theory Ser. B* **15** (1973), 94–120.

K. F. Roth, On certain sets of integers, *J. Lond. Math. Soc.* **28** (1953), 104–109.

K. F. Roth, On certain sets of integers, II, *J. Lond. Math. Soc.* **29** (1954), 20–26.

C. Rousseau and J. Sheehan, On Ramsey numbers for books, *J. Graph Theory* **2** (1978), 77–87.

A. Ruciński and N. Wormald, Random graph processes with degree restrictions, *Combin. Probab. Comput.* **1** (1992), 169–180.

A. Sah, Diagonal Ramsey via effective quasirandomness, arXiv:2005.09251, 2020.

A. Sanchez-Flores, An Improved Bound for Ramsey Number $N(3,3,3,3;2)$, *Discrete Math.* **140** (1995), 281–286.

T. Sanders, On certain other sets of integers, *J. Anal. Math.* **116** (2012), 53–82.

T. Sanders, On Roth's theorem on progressions, *Ann. Math.* **174** (2011), 619–636.

G. N. Sárközy, On the multi-colored Ramsey numbers of paths and even cycles, *Electron. J. Combin.* **23** (3) (2016), P3–53.

N. Sauer and J. H. Spencer, Edge disjoint placement of graphs, *J. Combin. Theory Ser. B* **25** (1978) 295–302.

I. Schiermeyer, All cycle–complete graph Ramsey numbers $r(C_m, K_6)$, *J. Graph Theory* **44** (2003), 251–260.

T. Schoen (a), A subexponential upper bound for van der Waerden numbers $W(3,k)$, *Electron. J. Combin.* **28** (2021), # p2.34.

T. Schoen (b), Improved bound in Roth's theorem on arithmetic progressions, *Adv. Math.* **386** (2021), Paper No. 107801, 20 pp.

O. Schramm, Illuminating sets of constant width, *Mathematika*, **35** (1988), 180–199.

I. Schur, Über die Kongruenz $x^m + y^m = z^m \pmod{p}$, *Jahresber. Deutsch. Math.-Verein.* **25** (1916), 114–117.

J. Seidel, A survey of two-graphs, in: Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973), vol I, Atti dei Convegni Lincei, No. 17, Accad. Naz. Lincei, Rome, 1976, 481–511.

E. Shamir and J. H. Spencer, Sharp concentration of the chromatic number in random graph $G_{n,p}$, *Combinatorica* **7** (1987), 121–130.

C. Shannon, The zero-error capacity of a noisy channel, *IRE Trans. Inform. Theory* **IT-2** (1956), 8–19.

J. Shao, C. He and H. Shan, The existence of even cycles in Wenger's graphs, *Acta Math. Appl. Sin. Engl. Ser.* **24** (2008), 281–288.

J. Shearer, A note on the independence number of triangle-free graphs, *Discrete Math.* **46** (1983), 83–87.

J. Shearer, Lower bounds for small diagonal Ramsey numbers, *J. Combin. Theory Ser. A* **42** (1986), 213–216.

J. Shearer, The independence number of dense graphs with large odd girth, *Electron. J. Combin.* **2** (1995), #N2.

J. Shearer, On the independence number of sparse graphs, *Random Structures Algorithms* **7** (1995), 269–271.

S. Shelah, Primitive recursive bounds for van der Waerden numbers, *J. Amer. Math. Soc.* **1** (1988), 683–697.

L. Shen, Q. Lin and Q. Liu, Bipartite Ramsey numbers for bipartite graphs of small bandwidth, *Electron. J. Combin.* **25** (2018), no. 2, Paper No. 2.16, 12 pp.

A. Sidorenko, The Ramsey number of an N-edge graph versus a triangle is at most $2N + 1$, *J. Combin. Theory Ser. B* **58** (1993), 185–196.

M. Simonovits, A method for solving extremal problems in graph theory, stability problems. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 279–319. Academic Press, New York, 1968.

M. Simonovits and V. Sós, Ramsey-Turán theory, *Discrete Math.* **229** (2001), 293–340.

R. R. Singleton, On minimal graphs of maximum even girth, *J. Combin. Theory* **1** (1966), 306–332.

J. H. Spencer, Ramsey's theorem-a new lower bound, *J. Combin. Theory Ser. A* **18** (1975), 108–115.

J. H. Spencer, Asymptotic lower bound for Ramsey functions, *Discrete Math.* **20** (1977), 69–76.

J. H. Spencer, Three hundred million points suffice, *J. Combin. Theory Ser. A* **49** (1988), 210–217, an erratum in Vol. 50 (1989), 323.

J. H. Spencer, *Ten Lectures on the Probabilistic Method, 2nd Edition*, SIAM, Philadelphia, 1994.

E. Sperner, Ein Satz über Untermengen einer endlichen Menge, *Math. Z.* **27** (1928), 544–548.

B. Sudkov, A note on odd cycle-complete graph Ramsey numbers, *Electron. J. Combin.* **9** (2002), #N1.

B. Sudakov, A few remarks on the Ramsey–Turán-type problems, *J. Combin. Theory Ser. B* **88** (2003), 99–106.

B. Sudakov, Large K_r-free subgraphs in K_s-free graphs and some other Ramsey-type problems, *Random Structures Algorithms* **26** (2005), 253–265.

B. Sudakov, Ramsey Numbers and the Size of Graphs, *SIAM Discrete Math.* **21** (2008), 980–986.

B. Sudakov, A Conjecture of Erdős on graph Ramsey numbers, *Adv. Math.* **227** (2011), 601–609.

B. Sudakov, T. Szabó and V. Vu, A generalization of Turán's theorem, *J. Graph Theory* **49** (2005), 187–195.

T. Szabó, An application of Lovász' local lemma-a new lower bound for the van der Waerden numbers, *Random Structures Algorithms* **1** (1990), 343–360.

T. Szabó, On the spectrum of projective norm–graphs, *Inform. Process. Lett.* **86** (2) (2003), 71–74.

T. Szele, Kombinatorikai vizsgálatok az irányitott teljes gráffal, Kapcsolatban, Mat. Fiz. Lapok **50** (1943), 223–256.

E. Szemerédi, On graphs containing no complete subgraph with 4 vertices, *Mat. Lapok* **23** (1972), 113–116.

E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, *Acta Arith.* **27** (1975), 199–245.

E. Szemerédi, *Regular partitions of graphs*, in: Problémes Combinatories et théorie des graphs, Colloque Inter. CNRS, Univ. Orsay, Orsay, 1976, J. Bermond, J. Fournier, M. Las Vergnas, and D. Scotteau, Eds. (1978), 399–402.

E. Szemerédi, Integer sets containing no arithmetic progressions, *Acta Math. Hung.* **56** (1990), 155–158.

A. Thomason, On finite Ramsey numbers, *European J. Combin.* **3** (1982), 263–273.

A. Thomason, An upper bound for some Ramsey numbers, *J. Graph Theory* **12** (1988), 509–517.

A. Thomason, Pseudo-random graphs, in: *Proceedings of Random Graphs, Poznań 1985, M. Karoński, Eds. Ann. Discrete Math.* **33** (1987), 307–331.

P. Turán, Eine Extremalaufgabe aus der Graphentheorie [Hungarian], *Math Fiz. Lapok* **48** (1941), 436–452.

B. L. van der Waerden, Beweis einer Baudetschen Vermutung, *Nieuw Arch. Wiskd.* **15** (1927), 257–271.

B. L. van der Waerden, How the proof of Baudet's conjecture was found, in: *Studies in Pure Mathematics (Presented to Richard Rado)*, Academic Press, New York-London, 1971.

K. Walker, Dichromatic graphs and Ramsey numbers, *J. Combin. Theory* **5** (1968), 238–243.

H. Wan, Upper bounds for Ramsey numbers $R(3, \ldots, 3)$ and Schur numbers, *J. Graph Theory* **26** (1997), 119–122.

A. Weil, Sur les courbes algébriques et les variéstés qui sèn déduisent. Actualités Sci. Ind., 1041 (1948), Herman, Paris.

V. K. Wei, A lower bound on the stability number of a simple graph, *Bell Laboratories Technical Memorandum*, No. 81-11217–9, 1981.

V. K. Wei, A lower bound on the stability number of a simple graph, Technical memorandum TM 81–11217–9, Bell Laboratories, Berkley Heights, NJ, 1981.

R. Wenger, Extremal graphs with no C^4's, C^6's, or C^{10}'s, *J. Combin. Theory Ser. B* **52** (1991), 113–116.

D. B. West, *Introduction to Graph Theory, 2nd Edition*, Prentice-Hall, Englewood Cliffs, NJ, 2001.

Y. Wigderson, An improved lower bound on multicolor Ramsey numbers, *Proc. Amer. Math. Soc.* **149** (2021), no. 6, 2371–2374.

G. Wolfovitz, Lower bounds for the size of random maximal H-free graphs, *Electronic J. Combin.* **16** (2009), R4.

M. Woźniak, On the Erdős Sós conjecture, *J. Graph Theory* **21** (1996), 229–234.

J. Yackel, Inequalities and asymptotic bounds for Ramsey numbers, *J. Combin. Theory* **13** (1972), 56-68.

K. Yamamoto, Logarithmic order of free distributive lattices, *J. Math. Soc. Japan* **6** (1954), 343–353.

S. Yongqi, Y. Yuansheng, X. Feng and L. Bingxi, New lower bounds on the multicolor Ramsey numbers $R_r(C_{2m})$, *Graphs Combin.* **22** (2006), 283–288.

K. Zarankiewicz, Problem P 101, *Colloq. Math.* **2** (1951), 301.

Y. Zhang, H. Broersma and Y. Chen, A note on Ramsey numbers for fans, *Bull. Aust. Math. Soc.* **92** (2015), 19–23.

X. Zhang, Y. Chen and T. C. Edwin Cheng, Some values of Ramsey numbers for C_4 versus stars, *Finite Fields Appl.* **45** (2017), 73–85.

X. Zhang, Y. Chen and T. C. Edwin Cheng, Polarity graphs and Ramsey numbers for C_4 versus stars, *Discrete Math.* **340** (2017), 655–660.

X. Zhu, The fractional version of Hedetniemi's conjecture, *European J. Combin.* **32** (2011), 1168–1175.

Glossary

Throughout this book, we use the standard notation of graphs. The entries of the glossary are divided into two lists. Entries such as $\beta(G)$ and $r(G)$ that have fixed letters as part of their representation occur in the first list, in alphabetic order (phonetically for Greek characters). Entries of the second list correspond to the meanings of the first entries.

a.a.s.	asymptotically almost surely, 86
$\mathcal{A}(F)$	automorphism group of F, 77
B_n	book graph, 7
$B_n^{(m)}$	general book graph, 72
$B(n, p)$	binomial distribution, 67
$br(B_1, B_2)$	bipartite Ramsey number of B_1 and B_2, 307
C_n	a cycle on n vertices, 13
$\Delta(G)$	the maximum degree of G, 3
$diam(G)$	the greatest distance between two vertices of G, 98
ER_q	Erdős-Rényi graph, 163
$ex(n, H)$	Turán number of graph H, 149
F_n	Friendship graph or a k-fan, 168
$F(q)$ or F_q	the field of order q, 21
$\gamma(G)$	the largest r such that G contains a subdivision of K_r, 83
$G + H$	join graph of G and H, 7
$G_{q,t}$	projective norm graph, 177
$\chi(G)$	chromatic number of G, 83
K_N	complete graph of order N, 1
$K_n^{(r)}$	complete r-uniform hypergraph of order n, 14
$K_{n_1,\ldots,n_{k-1}}$	complete $(k-1)$-partite graph, 149
$K_{1,n}$	a star of n edges, 12
$[N]$	$\{1, 2, \ldots, N\}$, 26
$\omega(G)$	clique number of G, 3
P_q	Paley graph of order q, 35

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P_{1+n}	a path of order $1+n$, 10
$\hat{r}(G_1, G_2)$	size Ramsey number, 295
$r_k(G_1, \ldots, G_k)$	Ramsey number of $G_1, \ldots, G_k$, 1
$r_k(G)$	k-color Ramsey number of G, 2
$r_\chi(G, H)$	chromatic Ramsey number of G and H, 317
$r_k(n)$	k-color Ramsey number of K_n, 2
$r_k^{(r)}(n_1, \ldots, n_k)$	k-color hypergraph Ramsey number, 14
$r(G)$	Ramsey number $r(G, G)$, 2
$r_{ind}(H)$	induced Ramsey number of H, 304
$r(\ell, \ldots, n)$	classical Ramsey number, 3
$r(n)$	Ramsey number $r(K_n, K_n)$, 2
$\Theta(G)$	Shannon capacity of G, 195
t-AP	an arithmetic progression of t terms, 248
$t_{k-1}(n)$	the edge number of Turán graph $T_{k-1}(n)$, 149
T_n	a tree of order n, 12
$V^{(r)}$	the family of all r-subsets of V, 14
$w_k(t)$	k-color van der Waerden number, 248
$w(m, n)$	off-diagonal van der Waerden number, 252
$z(m, n; s, t)$	Zarankiewicz number, 157

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