Cutting Plane Rank Lower Bound for Ramsey’s Theorem

Daniel Apon

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2. What is the relative **complexity** of proving upper bounds on Ramsey numbers?

3. **Focus on Cutting Plane proofs**
   3.1 High-dimensional geometric proofs
   3.2 IPs vs LPs
   3.3 More details soon =)
**QUESTION:**

What is the proof complexity of the propositional statement $r(k, k) \leq 4^k$?
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\[ r(k, k) \leq 4^k? \]

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2. We’d like: An exp lower bound on proof size w.r.t. formula size
3. We get: An exp lower bound on RANK w.r.t. \( k \)
Plan for the Talk

1. Intro to Cutting Plane Proofs
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   3.1 Long games $\implies$ High CP Rank
4. The Delayer’s Strategy
   4.1 Games are long!
   4.2 Proof by Io’s Method
A **CUTTING PLANE PROOF** is a series of lines where:

1. First line: List of AXIOMS (e.g. \( \vec{a} \cdot \vec{x} \leq \vec{b} \in A \) \( \vec{x} \leq \vec{b} \))
2. Final line: An arithmetically FALSE statement (e.g. \( 1 \leq 0 \))
3. In between: Anything derivable from previous lines using:
   3.1 Inequality Addition
   3.2 Scalar Multiplication
   3.3 Rounded Division

A CP derivation of a false statement from \( A \) \( \vec{x} \leq \vec{b} \) is EQUIVALENT to showing "\( A \vec{x} \leq \vec{b} \not\in \text{SAT} \)"
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A CP derivation of a false statement from $A\vec{x} \leq \vec{b}$ is EQUIVALENT to showing “$A\vec{x} \leq \vec{b} \notin SAT$”
A geometric interpretation...

1. Start at $P \overset{\text{def}}{=} \{ x \in \mathbb{R}^n : A\vec{x} \leq \vec{b} \}$ for integral $A, b$
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4. Another option: Derive $P'$ from $P$ with rounded division.
   4.1 Observe that for all $c \in \mathbb{Z}^n, \delta \in \mathbb{R}$,
   
   $$c^T y \leq \delta \text{ for all } y \in P \Rightarrow c^T x \leq \lfloor \delta \rfloor$$
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   4.1 $\exists$ integral pt inside $c^T x \leq d$ with rank $\geq s$ relative to $Ax \leq b$
      $\Rightarrow$ any CP derivation from $Ax \leq b$ has depth $\geq s$
OUR EVER-HEROIC CHAMPIONS PROVER and DELAYER will fight a BLOODY DUEL TO THE DEATH over:

“Any graph with $4^k$ vertices has either a clique of size $k$ or an independent set of size $k$.”
A Prover/Delayer Game

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This is serious action movie material.
Rules of THE GAME

1. Parameter: \( k \in \mathbb{N} \)

2. Game begins on an uncolored complete graph on \( n = 4^k \) vertices. Players \textbf{color} the edges until...

3. \textbf{PROVER} wants to force a monochromatic complete graph on \( k \) vertices

4. \textbf{DELayer} wants to \ldots delay!

5. \textbf{PROVER} will eventually win. The interesting question is \ldots how long does it take?
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   2.2 **COLOR** $(u_i, v_i)$

3. DELAYER $i^{th}$ move:
   3.1 For uncolored $(w, w') \in (C_i \setminus \{u_i, v_i\})^2$, **COLOR** them
**Goal:** Show that Long Games $\Rightarrow$ High Rank
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1. Equate colored graphs with POINTS in high-dim space
   1.1 $\{\text{BLUE, NONE, RED}\} \mapsto \{0, \frac{1}{2}, 1\}$
   1.2 $G \in \{0, \frac{1}{2}, 1\}^{\binom{|V|}{2}} \mapsto G \in [0, 1]^{\binom{|V|}{2}}$
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2. **DEFN**: The AVERAGE, $\frac{1}{2}(G_1 + G_2)$, of two graphs $G_1, G_2$ is the graph $H = (V, \frac{E_1 + E_2}{2})$
DEFN: The PROTECTION SET $S = S(G)$ for a colored graph $G$ is the set of all graph pairs $(G(u, v), G(u, v)) \in (V, E)^2$ s.t.

1. The charged part of $G$ is $C$
2. The charged part of both $G(u, v)$ and $G(u, v)$ is $C \cup \{u, v\}$
3. $G = 1/2 (G(u, v) + G(u, v))$
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Note: For fixed $(u, v)$, the two colored graphs PROVER can choose in the $i^{th}$ round average to the $(i - 1)^{th}$ round graph
A Protection Lemma

**KEY LEMMA**: Let $G$ be a colored graph with an even number of vertices and a charged part of even size. If $G$ has a protection set $S(G) \subseteq P^{(i)}$, then $G \in P^{(i+1)}$. 

Intuitively, Long Games $\Rightarrow$ High Rank

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Consider some $G$ at the start of some round $i$ in the P/D game. Note there are an even number of vertices and charged part of even size.

1. $G \in P^{(i)}$:
   
   1.1 By constr: For $u, v \notin C_i$, $G$ is the average of $(G(u, v), G(u, v)) \in S(G)$. By assmp, $S(G) \subseteq P^{(i)}$, so $G \in P^{(i)}$. 
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   2.2 Let $a'^T x \leq b'$ have rank $i$ s.t. for some $q, r \in \mathbb{Z}, 0 < r < q$,
      2.2.1 $a'[u,v] = qa[u,v]$
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      2.2.1 $a'[u,v] = qa[u,v]$
      2.2.2 $b' = qb + r$
   2.3 Then, $G \in P^{(i)} \Rightarrow a'^T G \leq b' < q(b + 1) \Rightarrow b < a^T G < b + 1$. By constr: $a^T G = b + \frac{1}{2}$. 
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Prot Lemma Proof

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Since $a \in \mathbb{Z}^m$, then

$$\sum_{(u,v) \in \mathcal{U}^2} a[u,v] + \sum_{u \in \mathcal{U}, w \in C} a[u,w] \equiv 1 \pmod{2},$$

else $a^T G$ would be integral.
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Claim: This implies \( \exists (u, v) \in \mathcal{U}^2 \) s.t.

\[
a[u, v] + \sum_{w \in \mathcal{C}} a[u, w] + \sum_{w \in \mathcal{C}} a[v, w] \equiv 1 \pmod{2}.
\]

Proof: Formal proof is a bit lengthy. High-level idea: Suppose not, then can show some fixed part of \( G \) has both even and odd size.
Fix \((u, v)\) as implied by prev.

Look at sum over three groups of edges:

1. \((A)\): all edges between two charged vertices,
2. \((B)\): edges enumerated in defn of \((u, v)\) (those induced in one round of P/D),
3. \((C)\): rest of the edges in \(G\).
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3. Therefore, the numbers \(a^T G(u, v), a^T G(u, v)\) are integral and (from before) less than \(b + 1\).
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4. Therefore, they are at most \(b\).

5. As \(G\) is their average, \(a^T G \leq b\), contradicting the assumption \(G \notin P^{(i+1)}\). \(\square\)
A Delayer Strategy to Force Long Games

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1. **DEFN**: A diagonal pair of vertices is any pair \(\{2m - 1, 2m\}\) for \(m \in [2^{k/2-1}]\).

2. A diagonal edge is an edge between a diagonal pair of vertices.
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3. We need the existence of a certain graph with extremal Ramsey properties for DELAYER to use!
CLAIM: There is a complete graph $H$, all edges colored either red or blue, s.t.

1. there is no monochromatic clique of size $k$,
2. above holds even if the colors of diagonal edges are toggled arbitrarily,
3. for any diagonal pair of vertices \( \{2m - 1, 2m\} \) and any vertex \( a < 2m - 1 \), the color of \((a, 2m - 1)\) and \((a, 2m)\) are DIFF
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1. For all $i \in [2^{k/2-1}]$ and $v < 2i - 1$, color $(v, 2i - 1)$ uniformly at random; set $(v, 2i)$ to the opposite
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2. We want to count the prob that $k$-size subsets have both a BLUE and RED edge that are not between diagonal pairs.
Io’s Method

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2. We want to count the prob that $k$-size subsets have both a BLUE and RED edge that are not between diagonal pairs.

3. **DEFN:** $K_0$ – family of sets of $k$ vertices with no diagonal pair

4. **DEFN:** $K_1$ – family of sets of $k$ vertices where (only) the LEAST two vertices are diagonal
Fix \( n = 2^{k/2} \). Then,

\[
\Pr[H \text{ has a monochromatic } k-\text{clique}] \\
\leq |K_0| \frac{2}{2^\binom{k}{2}} + |K_1| \frac{2}{2^\binom{k}{2} - 1}
\]

\[
\leq \frac{2}{2^\binom{k}{2}} \left[ 2^k \binom{n/2}{k} + 2^{k-1} \binom{n/2}{k - 1} \right] < 1.
\]

Therefore, some such \( H \) exists! \( \square \)
Putting It All Together

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1. In each round, map the new charged vertices \((2i - 1, 2i)\) onto vertices \((2i - 1, 2i)\) of \(H\).
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2. Let PROVER color these diagonal edges however he wants. (NO ONE CARES WHAT YOU DO, PROVER)

Therefore: The P/D game continues for \(e^{\text{def}} = 2^k / 2 - 1\) rounds.

Therefore: \(G_{e - 1} \subseteq P_1\), \(G_{e - 2} \subseteq P_2\), \[2120x2073\], \(G_0 \subseteq P_e\).

Therefore: Ramsey's theorem has CP Rank at least \(e = \Omega(2^k)\).
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3. Color the remaining edges according to \(H\). \(\square\)
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**THEREFORE:** The P/D game continues for \(e \overset{\text{def}}{=} 2^{k/2-1}\) rounds.
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3. Color the remaining edges according to \(H\). □

**THEREFORE:** The P/D game continues for \(e \equiv \frac{2^k}{2^k - 1}\) rounds.

**THEREFORE:** \(G_e \subseteq P_0\). So, by the Prot Lemma, \(G_{e-1} \subseteq P_1\), \(G_{e-2} \subseteq P_2\), \ldots, \(G_0 \subseteq P_e\).
AWESOME DELAYER STRATEGY:

1. In each round, map the new charged vertices \((2i - 1, 2i)\) onto vertices \((2i - 1, 2i)\) of \(H\).

2. Let PROVER color these diagonal edges however he wants. (NO ONE CARES WHAT YOU DO, PROVER)

3. Color the remaining edges according to \(H\).


\[
\text{THEREFORE: The P/D game continues for } e \overset{\text{def}}{=} 2^{k / 2 - 1} \text{ rounds.}
\]

\[
\text{THEREFORE: } G_e \subseteq P_0. \text{ So, by the Prot Lemma, } \\
G_{e-1} \subseteq P_1, G_{e-2} \subseteq P_2, \cdots, G_0 \subseteq P_e.
\]

\[
\text{THEREFORE: Ramsey’s theorem has CP Rank at least } e = \Omega(2^k).
\]
Thanks for listening!