Cutting Plane Rank Lower Bound for Ramsey's Theorem

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Proof Complexity and Ramsey's Theorem

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2. What is the relative complexity of proving upper bounds on Ramsey numbers?
3. Focus on Cutting Plane proofs
   3.1 High-dimensional geometric proofs
   3.2 IPs vs LPs
   3.3 More details soon =)

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3. We get: An exp lower bound on RANK w.r.t. k

Plan for the Talk

1. Intro to Cutting Plane Proofs

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4. The Delayer's Strategy
   4.1 Games are long!
   4.2 Proof by Io's Method

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A CP derivation of a false statement from $A\vec{x} \leq \vec{b}$ is EQUIVALENT to showing "$A\vec{x} \leq \vec{b} \notin SAT$"

More on Rounded Division

A geometric interpretation...

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3. Add/mult gives linear comb. of earlier inequalities
4. Another option: Derive P' from P with rounded division.
   4.1 Observe that for all $c \in \mathbb{Z}^n, \delta \in \mathbb{R}$,

$$c^T y \leq \delta \text{ for all } y \in P \Rightarrow c^T x \leq \lfloor \delta \rfloor$$

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   4.1 $\exists$ integral pt inside $c^T x \leq d$ with rank $\geq s$ relative to $Ax \leq b$ $\Rightarrow$ any CP derivation from $Ax \leq b$ has depth $\geq s$

A Prover/Delayer Game

OUR EVER-HEROIC CHAMPIONS PROVER and DELAYER will fight a BLOODY DUEL TO THE DEATH over:

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This is serious action movie material.

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5. PROVER will eventually win. The interesting question is HOW LONG does it take?

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3. DELAYER i^{th} move:
   3.1 For uncolored $(w, w') \in (C_i \backslash \{u_i, v_i\})^2$, COLOR them

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    1.1 $\{$BLUE, NONE, RED$\} \mapsto \{0, \frac{1}{2}, 1\}$
    1.2 $G \in \{0, \frac{1}{2}, 1\}^{\binom{|V|}{2}} \mapsto G \in [0, 1]^{\binom{|V|}{2}}$

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2. DEFN: The AVERAGE, $\frac{1}{2}(G_1 + G_2)$, of two graphs G_1, G_2 is the graph $H = (V, \frac{E_1+E_2}{2})$

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Note: For fixed (u,v), the two colored graphs PROVER can choose in the i^{th} round average to the $(i-1)^{\mathrm{th}}$ round graph

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Intuitively, Long Games $\Rightarrow$ High Rank

Prot Lemma Proof

Consider some G at the start of some round i in the P/D game. Note there are an even number of vertices and charged part of even size.

1. $\underline{G \in P^{(i)}}$:
   1.1 By constr: For $u, v \notin C_i$, G is the average of $(G(u,v), G(u,v)) \in S(G)$.
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   2.3 Then, $G \in P^{(i)} \Rightarrow a'^T G \leq b' < q(b+1) \Rightarrow b < a^T G < b+1$.
   By constr: $a^T G = b + \frac{1}{2}$.

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Claim: This implies $\exists (u,v) \in \mathcal{U}^2$ s.t.

$$a[u,v] + \sum_{w \in \mathcal{C}} a[u,w] + \sum_{w \in \mathcal{C}} a[v,w] \equiv 1 \pmod 2.$$

Proof: Formal proof is a bit lengthy. High-level idea: Suppose not, then can show some fixed part of G has both even and odd size

Prot Lemma Proof

Fix (u, v) as implied by prev.

Look at sum over three groups of edges:

1. **(A)**: all edges between two charged vertices,
2. **(B)**: edges enumerated in defn of (u, v) (those induced in one round of P/D),
3. **(C)**: rest of the edges in G.

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3. Therefore, the numbers $a^T G(u,v), a^T G(u,v)$ are integral and (from before) less than $b + 1$.
4. Therefore, they are at most b.
5. As G is their average, $a^T G \leq b$, contradicting the assumption $G \notin P^{(i+1)}$. □

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3. We need the existence of a certain graph with extremal Ramsey properties for DELAYER to use!

The Magic Graph, H

CLAIM: There is a complete graph H, all edges colored either red or blue, s.t.

1. there is no monochromatic clique of size k,
2. above holds even if the colors of diagonal edges are toggled arbitrarily,
3. for any diagonal pair of vertices $\{2m-1, 2m\}$ and any vertex $a < 2m-1$, the color of $(a, 2m-1)$ and $(a, 2m)$ are DIFF

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2. We want to count the prob that k-size subsets have both a BLUE and RED edge that are not between diagonal pairs.
3. DEFN: K_0 – family of sets of k vertices with no diagonal pair
4. DEFN: K_1 – family of sets of k vertices where (only) the LEAST two vertices are diagonal

Io's Method

Fix $n = 2^{k/2}$. Then,

$$\Pr[H \text{ has a monochromatic } k\text{−clique}]$$

$$\leq |K_0| \frac{2}{2^{\binom{k}{2}}} + |K_1| \frac{2}{2^{\binom{k}{2}-1}}$$

$$\leq \frac{2}{2^{\binom{k}{2}}} \left[2^k \binom{n/2}{k} + 2^{k-1} \binom{n/2}{k-1} \right] < 1.$$

Therefore, some such H exists! □

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THEREFORE: Ramsey's theorem has CP Rank at least $e = \Omega(2^k)$.

Thanks for listening!