Constructions in Computable Ramsey Theory (An Exposition)

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Notation:

1. $M_1, M_2, \ldots$ is a standard list of Turing Machines.
2. Note that from $e$ we can extract the code for $M_e$.
3. $M_{e,s}(x)$ means that we run $M_e$ for $s$ steps.
4. $W_e$ is the domain of $M_e$, that is,

$$W_e = \{ x \mid (\exists s)[M_{e,s}(x) \downarrow] \}.$$ 

Note that $W_1, W_2, \ldots$ is a list of ALL c.e. sets.
5. 

$$W_{e,s} = \{ x \mid M_{e,s}(x) \downarrow \}.$$
There is a Comp Coloring with no Inf c.e. Homog Set

**Theorem**

There exists computable \( \text{COL} : \left( \mathbb{N}^2 \right) \rightarrow [2] \) such that there is NO infinite c.e. homog set.
We construct $COL : (\mathbb{N}^2) \to [2]$ to satisfy:

$$R_e : W_e \text{ infinite } \implies \text{we NOT a homog set}.$$ 

**CONSTRUCTION OF COLORING**

*Stage 0*: $COL$ is not defined on anything.

*Stage s*: We will define $COL(0, s), \, COL(1, s), \ldots, \, COL(s - 1, s)$. For all $0 \leq e \leq s$ do the following, starting with $e = 0$:

If $(\exists x, y \leq s - 1)[x, y \in W_e, s \land COL(x, s), COL(y, s) \text{ undefined}]$ then define take LEAST such $x, y$ and do: (1) $COL(x, s) = RED$, (2) $COL(y, s) = BLUE$. (Note that IF $s \in W_e$ then $R_e$ would be satisfied.)

After all this, for all $(x, s)$ not yet colored, $COL(x, s) = RED$.

**END OF CONSTRUCTION**
There is a Comp Coloring with no Inf c.e.-in-HALT Homog Set

Theorem

There exists computable COL : \( \binom{\mathbb{N}}{2} \rightarrow [2] \) such that there is NO infinite c.e-in-HALT homog set.

This is on HW1.
Theorem

For every computable coloring \( \text{COL} : \binom{\mathbb{N}}{2} \rightarrow [2] \) there is an infinite \( \Pi_2 \) homog set.
Given computable $COL : (\mathbb{N}^2) \to [2]$.

**CONSTRUCTION of** $x_1, x_2, \ldots$ **and** $c_1, c_2, \ldots$.

$x_1 = x$ and $c_1 = RED$ (We are guessing. Might change later)

$s \geq 2$, assume $x_1, \ldots, x_{s-1}$ and $c_1, \ldots, c_{s-1}$ are defined.

Ask $K ((\exists x \geq x_{s-1})(\forall 1 \leq i \leq s - 1)[COL(x_i, x) = c_i])$?

**YES:** Find least such $x$.

- $x_i = x$
- $c_i = RED$ (Guessing.)
Construction of Inf $\Pi_2$ Homog Set: NO Case

**NO:** Ask $K$:

- $(\exists x \geq x_{s-1})(\forall 1 \leq i \leq s - 2)[COL(x_i, x) = c_i])$?
- ...
- $(\exists x \geq x_{s-1})(\forall 1 \leq i \leq 1)[COL(x_i, x) = c_i])$?

Let $i_0$ be largest such that

$(\exists x \geq x_{s-1})(\forall 1 \leq i \leq i_0)[COL(x_i, x) = c_i])$?

1. Change color of $c_{i+1}$.
2. Wipe out $x_{i+2}, \ldots, x_{s-1}$.
3. Find $x \geq x_{s-1}$ such that $(\forall 1 \leq i \leq i_0)[COL(x_i, x) = c_i]$.
4. $x_{i+2} = x$. $c_{i+2} = RED$ (Guessing)

**END OF CONSTRUCTION of $x_1, x_2 \ldots$ and $c_1, c_2, \ldots$**
Getting the Inf $\Pi_2$ Homog Set

$X = \{x_1, x_2, \ldots\}$. $R$ is the set of red elts of $X$

$\overline{X} \in \Sigma_2$ (so $X \in \Pi_2$).

$\overline{X} = \{x \mid (\exists x)[ \text{at stage } s \text{ of the construction } x \text{ was tossed out }\} \}.

\overline{R} \in \Sigma_2$ (so $R \in \Pi_2$).

\overline{R} = \overline{X} \cup \{x \mid (\exists x)[ \text{at stage } s \text{ of the construction } x \text{ was turned BLUE}\} \}.

1. If $R$ is infinite then $R$ is inf homog set in $\Pi_2$.
2. If $R$ is finite then $B = M - R$ is inf homog set in $\Pi_2$. 

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