# Applications of the Erdös-Rado Canonical Ramsey Theorem to Erdös-Type Problems

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# EXAMPLES

The following are known **EXAMPLES** of the kind of theorems we will be talking about.

1. If there are $n$ points in $\mathbb{R}^2$ then there is a subset of size $\Omega(n^{1/3})$ such that all distances between points are **DIFFERENT**. (KNOWN)
2. If there are $n$ points in $\mathbb{R}^2$, no 3 collinear, then there is a subset of size $\Omega((\log\log n)^{1/186})$ such that all triangle areas are **DIFFERENT**. (OURS)

# An Erdös Problem Plus Plus

**Definition:**

1. $h_{2,d}(n)$ is the largest integer so that the following happens: For all subsets of $\mathbb{R}^d$ of size $n$ there is a subset $Y$ of size $h_{2,d}(n)$ such that all distances are **DIFFERENT**.
2. $h_{a,d}(n)$ is the largest integer so that the following happens: For all subsets of $\mathbb{R}^d$ of size $n$, no $a$ on the same $(a-1)$-hyperplane, there is a subset $Y$ of size $h_{a,d}(n)$ such that all $a$-volumes are **DIFFERENT**.
3. $h_{a,d}(\alpha)$ where $\aleph_0 \le \alpha \le 2^{\aleph_0}$ makes sense.
4. Erdös, others studied $h_{2,d}(n)$. Little was known about $h_{a,d}(n)$.

# BEST KNOWN RESULTS AND OURS

**BEST KNOWN RESULTS:**

1. $h_{2,d}(n) = \Omega(n^{1/(3d-2)})$. Torsten (1995).
2. $h_{2,2}(n) = \Omega(n^{1/3}/\log n)$. Charalambides (2012).
3. (AC) $h_{2,d}(\alpha) = \alpha$. Erdös (1950)
4. (AC) If $\alpha$ regular than $h_{a,d}(\alpha) = \alpha$.

**OUR RESULTS (FEB 2013):**

1. $h_{2,d}(n) \ge \Omega(n^{1/(6d)})$. (Uses Canonical Ramsey)
2. $h_{3,2}(n) \ge \Omega((\log\log n)^{1/186})$ (Uses Canonical Ramsey)
3. $h_{3,3}(n) \ge \Omega((\log\log n)^{1/396})$ (Uses Canonical Ramsey)

**OUR RECENT RESULTS:**
(With **David Harris** and **Douglas Ulrich**)

1. $h_{2,d}(n) \ge \Omega(n^{\frac{1}{3d}})$ (Simple Proof!)
2. $h_{2,d}(n) \ge \Omega(n^{\frac{1}{3d-3}})$ (Simple Proof PLUS hard known result)
3. $h_{a,d}(n) \ge \Omega(n^{\frac{1}{(2a-1)d}})$ (Uses Algebraic Geometry)
4. (AC) If $\alpha$ regular then $h_{a,d}(\alpha) = \alpha$ (Simple Proof)
5. (AD) If $\alpha$ regular then $h_{a,d}(\alpha) = \alpha$

**Definition** Let $COL : \binom{[n]}{2} \rightarrow \omega$. Let $V \subseteq [n]$.
$V$ is **homog** if $(\forall a < b, c < d)[COL(a,b) = COL(c,d)]$

$V$ is **min-homog** if
$(\forall a < b, c < d)[COL(a,b) = COL(c,d) \text{ iff } a = c]$

$V$ is **max-homog** if
$(\forall a < b, c < d)[COL(a,b) = COL(c,d) \text{ iff } b = d]$

$V$ is **rainbow** if
$(\forall a < b, c < d)[COL(a,b) = COL(c,d) \text{ iff } a = c \text{ and } b = d]$

**Theorem:** (Lefmann-Rodl, 1995) $(\forall k)(\exists n \le 2^{O(k^2 \log k)})$,
$(\forall COL : \binom{[n]}{2} \rightarrow \omega)$ $(\exists V, |V| = k)$, $V$ is either homog, min-homog, max-homog, or rainbow.

# Variant of Canonical Ramsey

**Definition:** The set $V$ is **weak-homog** if either
$(\forall a, b, c, d \in V)[COL(a, b) = COL(c, d)]$
$(\forall a < b, c < d \in V)[a = c \implies COL(a, b) = COL(c, d)]$
$(\forall a < b, c < d \in V)[b = d \implies COL(a, b) = COL(c, d)]$
(**Note:** only one direction.)

**Definition:** $WER(k_1, k_2)$ is least $n$ such that for all $COL : \binom{[n]}{2} \to \omega$ either have **weak homog** set of size $k_1$ or **rainbow** set of size $k_2$.

**Theorem:** $WER(k_1, k_2) \le k_2^{O(k_1)}$.

# Easy Geom Lemma

**Lemma:** Let $p_1, \ldots, p_n \subseteq \mathbb{R}^d$. Let $COL$ be defined by $COL(i,j) = |p_i - p_j|$. Then $COL$ has no weak homog set of size $d+3$.

# POINT 1: $h_{2,d}(n) \ge \Omega(n^{1/(6d)})$ VIA CAN RAMSEY

**Theorem:** For all $d \ge 1$, $h_{2,d}(n) = \Omega(n^{1/(6d)})$.
**Proof:** Let $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$. Let $COL : \binom{[n]}{2} \to \mathbb{R}$ be defined by $COL(i,j) = |p_i - p_j|$.

$k$ is largest integer s.t. $n \ge WER(d+3, k)$.
By **VARIANT OF CANONICAL RAMSEY** $k = \Omega(n^{1/(6d)})$.

By the definition of $WER_3(d+3, k)$ there is either a weak homog set of size $d+3$ or a rainbow set of size $k$.

By **GEOMETRIC LEMMA** can't be weak homog case.
Hence there must be a rainbow set of size $k$.
THIS is the set we want!

# POINT 2: $h_{3,2}(n) \ge \Omega((\log\log n)^{1/186})$ VIA CAN RAMSEY

**Theorem:** $h_{3,2}(n) = \Omega((\log\log n)^{1/186})$.
**Proof:** Let $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^2$. Let $COL : \binom{[n]}{3} \to \mathbb{R}$ be defined by $COL(i,j,k) = AREA(p_i, p_j, p_k)$.

$k$ is largest integer s.t. $n \ge WER_3(6,k)$.
By **VARIANT OF CANONCIAL RAMSEY**
$n \ge \Omega((\log\log n)^{1/186})$.

By the definition of $WER_3(6,k)$ there is either a weak homog set of size 6 or a rainbow set of size $k$.

By **HARDER GEOMETRIC LEMMA** can't be weak homog case.
Hence there must be a rainbow set of size $k$.
THIS is the set we want!

# POINT 3: $h_{3,3}(n) \ge \Omega((\log\log n)^{1/396})$ VIA CAN RAMSEY

**Theorem:** $h_{3,3}(n) = \Omega((\log\log n)^{1/396})$.
**Proof:** Let $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^3$. Let $COL : \binom{[n]}{3} \rightarrow \mathbb{R}$ be defined by $COL(i,j,k) = AREA(p_i, p_j, p_k)$.

$k$ is largest integer s.t. $n \ge WER_3(13, k)$.
By **VARIANT OF CANONICAL RAMSEY**
$n \ge \Omega((\log\log n)^{1/396})$.

By the definition of $WER_3(13, k)$ there is either a weak homog set of size 13 or a rainbow set of size $k$.

By **HARDER GEOMETRIC LEMMA** can't be weak homog case.
Hence there must be a rainbow set of size $k$.
THIS is the set we want!

**ONWARD to NEW Results**

To prove $h_{2,d}(n) \ge \Omega(n^{\frac{1}{3d}})$ need result on spheres first.

**Definition** $h'_{2,d}(n)$ is the largest integer so that the following happens: For all subsets of $S^d$ of size $n$ there is a subset $Y$ of size $h'_{2,d}(n)$ such that all distances are **DIFFERENT**.
We prove
**Theorem** For $d \ge 1$, $h'_{2,d}(n) \ge \Omega(n^{\frac{1}{3d}})$.
Use induction on $d$.

# BASE CASE

**Base Case:** $d = 1$. $X \subseteq S^1$ (a circle). $M$ is the maximal subset of $X$ with all distances distinct. $m = |M|$.
$x \in X - M$. Either

1. $(\exists u \in M)(\exists \{u_1, u_2\} \in \binom{M}{2})[|x - u| = |u_1 - u_2|]$.
2. $(\exists \{u_1, u_2\} \in \binom{M}{2})[|x - u_1| = |x - u_2|]$.

Map $X - M$ to $M \times \binom{M}{2} \cup \binom{M}{2}$. Map is $\le$ 2-to-1.

$$|X - M| \le 2\left|M \times \binom{M}{2} \cup \binom{M}{2}\right|.$$

$|M| = \Omega(n^{1/3})$.

# INDUCTION STEP

$X \subseteq S^d$. $M$ a a maximal subset of $X$.
$x \in X - M$. Either

1. $(\exists u \in M)(\exists \{u_1, u_2\} \in \binom{M}{2})[|x - u| = |u_1 - u_2|]$.
2. $(\exists \{u_1, u_2\} \in \binom{M}{2})[|x - u_1| = |x - u_2|]$.

Map $X - M$ to $M \times \binom{M}{2} \cup \binom{M}{2}$. Two cases based on param $\delta$.
**Case 1:** $(\forall B \in \text{co-domain})[|\text{map}^{-1}(B)| \leq n^\delta]$. Map is $\leq n^\delta$-to-1.
$|X - M| \leq n^\delta \left| M \times \binom{M}{2} \cup \binom{M}{2} \right|$. Hence $m \geq \Omega(n^{\frac{1-\delta}{3}})$.
**Case 2:** $(\exists B \in \text{co-domain})[|\text{map}^{-1}(B)| \geq n^\delta]$.
**KEY:** $\text{map}^{-1}(B) \subseteq S^{d-1}$. By IH have set of size $\Omega(n^{\delta/3(d-1)})$.

Take $\delta = \frac{d-1}{d}$ to obtain $\Omega(n^{1/3d})$ in both cases.

# BETTER AUX RESULT: $h'_{2,d}(n) \ge \Omega(n^{\frac{1}{3d-3}})$

**Lemma** (Charalambides)

1. $h'_{2,d}(n) \ge \Omega(n^{1/3})$.
2. $h_{2,d}(n) \ge \Omega(n^{1/3})$.

**Theorem** For $d \ge 2$, $h'_{2,d}(n) \ge \Omega(n^{\frac{1}{3d-3}})$.
Only change is the **BASE CASE**.
Start at $d = 2$. Use Charalambides result that $h'_{2,d}(n) \ge \Omega(n^{1/3})$.

# NEW RESULT: $h_{2,d}(n) \ge \Omega(n^{\frac{1}{3d-3}})$ via MAXIMAL SETS

**Theorem** For $d \ge 2$, $h_{2,d}(n) \ge \Omega(n^{\frac{1}{3d-3}})$.
Induction on $d$.
**Base Case:** Use Charalambides result that $h_{2,d}(n) \ge \Omega(n^{1/3})$.
**Induction Step:** Similar to that in lower bound for $h'_{2,d}(n)$.

# NOTES ON THE PROOF

**I) Contrast:**

- $h'_{a,d}(n)$ Induction Step reduces $S^d$ to $S^{d-1}$.
- $h_{a,d}(n)$ Induction Step reduces $R^d$ to $R^{d-1}$ OR $S^{d-1}$.

**II) KEY**: In prove that $h_{2,d}(n) \ge \Omega(n^{\frac{1}{3d-3}})$ we need that inverse image of map was $S^{d-1}$ or $R^{d-1}$.

**III) Two views of result:**

- $h_{2,d}(n) \ge \Omega(n^{\frac{1}{3d}})$ via self contained elementary techniques.
- $h_{2,d}(n) \ge \Omega(n^{\frac{1}{3d-3}})$ via using hard known result.

# $h_{3,d}(n)$ ATTEMPT

**Theorem Attempt:** For all $d \ge 2$, $h_{3,d}(n) \ge$ LET'S FIND OUT!
**Base Case:** $d = 2$. $X \subseteq \mathbb{R}^2$, no 3 collinear. $M$ is the maximal subset of $X$ with all areas diff. $m = |M|$.
$x \in X - M$. Either
$(\exists \{u_1, u_2\} \in \binom{M}{2})(\exists \{u_3, u_4\} \in \binom{M}{2})$

$$AREA(x, u_1, u_2) = AREA(x, u_3, u_4).$$

$(\exists \{u_1, u_2\} \in \binom{M}{2})(\exists \{u_3, u_4, u_5\} \in \binom{M}{3})$

$$AREA(x, u_1, u_2) = AREA(u_3, u_4, u_5).$$

Map $X - M$ to $\binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$.
Need **Nice** Inverse Images. **DO NOT HAVE THAT!**

# Definition of $h_{a,d,r}(n)$

**Definition:** Let $1 \le a \le d+1$. Let $r \in \mathrm{N}$. $h_{a,d,r}$ is the largest integer so that the following happens: For all varieties $V$ of dimension $d$ and degree $r$ (in complex proj space), for all subsets of $V$ of size $n$, no $a$ points in the same $(a-1)$-hyperplane, there is a subset $Y$ of size $h_{2,d,r}(n)$ such that all $a$-volumes are **DIFFERENT**.

# Theorem about $h_{a,d}(n)$

**Theorem** Let $1 \le a \le d+1$. Let $r \in \mathrm{N}$. $h_{a,d,r}(n) \ge \Omega(n^{\frac{1}{(2a-1)d}})$. (The constant depends on $a, d, r$.)

**Comments on the Proof**

1. Proof uses Algebraic Geometry in Proj Space over C.
2. Cannot define Volume in Proj space!
3. Can define $VOL(a,b,c) \ne VOL(d,e,f)$ via difference of determinents (a homog poly) being 0.
4. Proof uses Maximal subsets.

**Corollary** Let $1 \le a \le d+1$. Let $r \in \mathrm{N}$. $h_{a,d}(n) \ge \Omega(n^{\frac{1}{(2a-1)d}})$. (The constant depends on $a, d$.)

# $h_{a,d}(\alpha)$ Under AC

**Theorem:** (AC) $\aleph_0 \le \alpha \le 2^{\aleph_0}$, $\alpha$ regular, then $h_{a,d}(\alpha) = \alpha$. We do $h_{3,2}$ case.
$X \subseteq \mathbb{R}^2$, no 3 collinear. $M$ is a maximal subset of $X$. $m = |M|$.
$x \in X - M$. Either
$(\exists \{u_1, u_2\} \in \binom{M}{2})(\exists \{u_3, u_4\} \in \binom{M}{2})$

$$AREA(x, u_1, u_2) = AREA(x, u_3, u_4)$$

$(\exists \{u_1, u_2\} \in \binom{M}{2})(\exists \{u_3, u_4, u_5\} \in \binom{M}{3})$

$$AREA(x, u_1, u_2) = AREA(u_3, u_4, u_5)$$

Map $X - M$ to $\binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$. Assume $|M| < \alpha$.

# $h_{a,d}(\alpha) = \alpha$ Cases of Proof

**Case 1:** $(\forall B \in \text{co-domain})[|\text{map}^{-1}(B)| < \alpha]$. Contradicts $\alpha$ regularity.
**Case 2:** $(\exists B \in \text{co-domain})[|\text{map}^{-1}(B)| = \alpha]$.
**KEY:** Using Determinant Def of AREA, any such $B$ is alg variety.
Let $B_1$ be one such $B$. Can show $B_1 \subset X$.
Repeat procedure on $B_1$. If get Case 1—DONE. If not get alg variety $B_2 \subset B_1 \subset X$,
If process does not stop then have

$$X \supset B_1 \supset B_2 \supset B_3 \cdots$$

Contradicts Hilbert Basis Theorem.

# $h_{a,d}(\alpha)$ Under AD

**Theorem:** (AD+DC) If $\aleph_0 \le \alpha \le 2^{\aleph_0}$ and $\alpha$ is regular then for all $1 \le a \le d+1$, $h_{a,d}(\alpha) = \alpha$.
Proof omitted for space.

# Open Questions

1. Get better lower bounds and **ANY** non-trivial upper bounds on $h_{a,d}(n)$.
2. What is $h_{a,d}(\alpha)$ for $\alpha$ singular? What axioms will be needed to prove results (e.g., AC, AD, DC)?
3. (DC) Assume $\alpha = 2^{\aleph_0}$ is regular. We have $AC \rightarrow h_{a,d}(\alpha) = \alpha$. We have $AD \rightarrow h_{a,d}(\alpha) = \alpha$. What if we have neither AC or AD?