PROOF TWO of the Finite Canonical Ramsey Theorem: Mileti's FIRST Proof

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PROOF TWO: 2-ary Case

PROOF TWO and PROOF THREE are due to
Joseph Mileti (2008)

He did infinite case and his interest was logic.

He showed that if $COL : \binom{[n]}{a} \rightarrow \omega$ is computable then there exists $I \subseteq [a]$ and infinite I-homog set $H \in \Pi_{2a-2}$.

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These slides are the ONLY source for this material!

What We Use

- Use R_1 and ER_1 to prove graph version.
- Use R_{a-1} and ER_{a-1} to prove a-hypergraph version.

In Proof THREE we will get rid of use of R_{a-1}.

We use the following Lemma on Recurrences in ALL of Mileti's proofs.

Lemma: Assume $0 < c < 1$, $0 < \delta \le 1/2$ and $b \in \mathrm{R}^+$. Define a sequence as follows

$$\begin{aligned} b_0 &\ge b \\ b_i &\ge c(b_{i-1})^\delta \end{aligned}$$

Then

$$b_i \ge c^{1+\delta+\delta^2+\cdots+\delta^{i-1}} b^{\delta^i} \ge c^{1/(1+\delta)} b^{\delta^i} \ge c^2 b^{\delta^i}.$$

Note: We may use this in a recurrence like

$$\begin{aligned} b_0 &\ge b \\ b_i &\ge \frac{c}{i}(b_{i-1})^\delta \end{aligned}$$

and take our value of c to be c/i. Note that this is still good for a lower bound— c/i is the smallest that coeff can go.

We refer to this as Rec Lemma.

Proof TWO of Can Ramsey Proof 2-ary case

Given $COL : \binom{[n]}{2} \rightarrow \omega$ define a sequence.
Stage 0: $X = \emptyset$, $A_0 = [n]$.
Stage s: Have $X = \{x_1, \ldots, x_{s-1}\}$,
$COL' : X \rightarrow \omega \times \{\text{homog}, \text{rain}\}$, A_{s-1} defined.
Let x_s be least elt of A_{s-1}.

Case 1: $(\exists c)[|\{y \in A_{s-1} : COL(x_s, y) = c\}| \ge \sqrt{|A_{s-1}|}]$.

$$\begin{aligned} COL'(x_s) &= (c, \text{homog}) \\ A_s &= \{y \in A_{s-1} : COL(x_s, y) = c\} \end{aligned}$$

Note: $|A_s| \ge \sqrt{|A_{s-1}|}$.

Case 2: $(\forall c)[|\{y \in A_{s-1} : COL(x_s, y) = c\}| < \sqrt{|A_{s-1}|}$.
Make all colors coming out of x_s to right diff:
Let A_s be set of all $x \in A_s$, x is LEAST with color $COL(x_s, x)$.
Formally A_s is $\{y \in A_{s-1}$:

$$COL(x_s, y) \notin \{COL(x_s, y') : x_s < y' < y \wedge y' \in A_{s-1}\}$$

$$\}$$

Now have:

$$(\forall y, y' \in A_s)[COL(x_s, y) \neq COL(x_s, y')].$$

Note: $|A_s| \geq \sqrt{|A_{s-1}|}$.

Want to make colors DIFF

Important note and convention: For the rest of Case 2 $(\forall x \in X)$ means only those x with color $(-, rain)$. Want to make the following true:

$$(\forall x \in X)(\forall y, y' \in A_s)[COL(x, y') \neq COL(x_s, y)]$$

Its OKAY if $COL(x, y) = COL(x_s, y)$.

For each $y \in A_s$ we thin out A_s so that:

- $(\forall x \in X)(\forall y' \in A_s - \{y\})[COL(x, y') \neq COL(x_s, y)]$.
- $(\forall x \in X)(\forall y' \in A_s - \{y\})[COL(x, y) \neq COL(x_s, y')]$.

BILL- SHOW AT BOARD

More to do!

$T = A_s$ (Current Version).

```
while  T ≠ ∅
   y =  least  element  of  T.
   T = T − {y}
   If  (∃x ∈ X, y' ∈ T − {y})[COL(x, y') = COL(x_s, y)]
     then  T = T − {y'},  A_s = A_s − {y'}
     (Do  this  for  all  such  x, y')
   If  (∃x ∈ X, y' ∈ T − {y})[COL(x, y) = COL(x_s, y')]
     then  T = T − {y'},  A_s = A_s − {y'}
     (Do  this  for  all  such  x, y')
```

Note: At end $|A_s| \geq \sqrt{|A_{s-1}|}/s$ (see next slide for why).

Analysis of while Loop

Recall: only looking at $x \in X$ colored $(-, rain)$. Hence all of the $x \in X$ we consider have all DIFF colors coming out of it. Call this statement $DIFF(x)$.

Consider the statement:

$$\text{If } (\exists x \in X, y' \in T - \{y\})[COL(x, y') = COL(x_s, y)]$$

We think of x as tossing y' OUT.

CLAIM: x can only toss out ONE y'.
PROOF: If $COL(x, y') = COL(x, y'') = COL(x_s, y)$ then $DIFF(x)$ is false. Constradiction.
So it now *seems* that each $x \in X$ could toss out an element, and hence you could toss $s - 1$ elements. But NO- see next slide.

Analysis of while loop

CLAIM: If x, x' toss out y', y'' then $y' = y''$.
PROOF: Recall again that we are only looking at $x \in X$ colored $(-, rain)$. Inductively we know that
$(\forall x \neq x' \in Y)(\forall y' \neq y'' \in A_s)[COL(x, y') \neq COL(x', y'')]$.
Hence the only way that $COL(x, y') = COL(x'y'')$ is if $y' = y''$.
BOTTOMLINE: This first clause can only toss out ONE element.

Analysis of while loop

By the construction x_s has all DIFF colors coming out of. Call this statement $DIFF(x_s)$.

Consider the statement:

$$\text{If } (\exists x \in X, y' \in T - \{y\})[COL(x,y) = COL(x_s, y')]$$

If this happens we think of x as tossing y' out.

CLAIM: x can only toss out ONE y'.
PROOF: If x tosses out y' and y'' then
$COL(x,y) = COL(x_s, y') = COL(x_s, y'')$. This violates $DIFF(x_s)$.
BOTTOMLINE: Each $x \in X$ dumps at most one element per stage. Hence this second IF statement dumps at most $|X| \le s-1$ elements.
BOTTOMBOTTOMLINE: Each stage A_s declares one element IN (namely y) and declares at most s elements OUT.

So how big is A_{s+1} after all of this?

In stage i we KEEP y_i in A_s and we DUMP a set of elements $|Y_i|$ from A_s. We know $|Y_i| \le s$. Let b be the number of elements in A_s after the while loop.

We begin with the set $\{y_1, \ldots, y_b\} \cup Y_1 \cup \cdots \cup Y_b$. Hence

$$\begin{aligned} \sqrt{|A_{s-1}|} &\le b + bs = (b+1)s \\ (b+1)s &\ge \sqrt{|A_{s-1}|} \\ b &\ge \sqrt{|A_{s-1}|}/s \end{aligned}$$

To Reiterate:
Note: At end $|A_s| \ge \sqrt{|A_{s-1}|}/s$.

OKAY- What is $COL'(x_s)$?

RECAP: Have

- $(\forall y, y' \in A_s)[COL(x_s, y) \neq COL(x_s, y')]$.
- $(\forall x \in X)(\forall y, y' \in A_s)[COL(x_s, y) \neq COL(x, y')]$

$f(s)$ TBD. $t = |A_s| \geq \frac{\sqrt{|A_{s-1}|}}{s}$.
Case 2.1: $(\exists i < s)[|\{y \in A_s : COL(x_i, y) = COL(x_s, y)\}| \geq \frac{t}{f(s)}]$.

$$\begin{aligned} COL'(x_s) &= COL'(x_i) \\ A_s &= \{y \in A_s : COL(x_i, y) = COL(x_s, y)\} \end{aligned}$$

Note: $|A_s| \geq \frac{t}{f(s)}$.

OKAY- What is $COL'(x_s)$?

Case 2.2: $(\forall i < s)[|\{y \in A_s : COL(x_i, y) = COL(x_s, y)\}| < \frac{t}{f(s)}]$.

$$
\begin{aligned}
COL'(x_s) &= (\ell, \mathrm{rain})\ \ell \text{ is least-unused-rain-number} \\
A_s &= A_s - \{y : (\exists i)[COL(x_i, y) = COL(x_s, y)].
\end{aligned}
$$

Note: If (say) $COL'(x_s) = (19, \mathrm{rain})$ then the 19 has no real meaning except that its NOT $1, 2, \ldots, 18$.

Note: $|A_s| \geq t - (s-1)\frac{t}{f(s)} \geq t(1 - \frac{(s-1)}{f(s)})$.

Recurrence for $|A_s|$

Case 1 yields: $|A_s| \ge \frac{t}{f(s)}$
Case 2 yields: $|A_s| \ge t(1 - \frac{s-1}{f(s)})$

Take $f(s) = 1 + (s-1) = s$ to obtain that in both cases get:
$|A_s| \ge \frac{t}{s} \ge \frac{\sqrt{|A_{s-1}|}}{s^2}$
Let $a_s = |A_s|$.
$a_0 = n$
$a_s \ge \frac{\sqrt{a_{s-1}}}{s^2}$
By Rec Lemma with $b = n$, $c = \frac{1}{s^2}$, $\delta = 1/2$, $i = s$ we get

$$a_s \ge \frac{n^{1/2^s}}{s^2}$$

Will later see how far we need to go.

How far out do we need to go?

We determine r later.
Have $X = \{x_1, x_2, \ldots, x_r\}$, $COL' : X \rightarrow \omega \times \{\mathrm{homog}, \mathrm{rain}\}$.
Case 1: There are $r/2$ colors of the form $(-, \mathrm{homog})$.
Case 1a: There are $\sqrt{r/2}$ that are the same. HOMOG.
Case 1b: There are $\sqrt{r/2}$ that are the different. MIN-HOMOG

Case 2: There are $r/2$ colors of the form $(-, \mathrm{rain})$.
Case 1a: There are $\sqrt{r/2}$ that are the same. MAX-HOMOG
Case 1b: There are $\sqrt{r/2}$ that are the different. RAINBOW

Need $r = 2k^2$.

Estimate n

Need: $a_r \geq 2$ where $r = 2k^2$.
Have: $a_s \geq \frac{n^{1/2^s}}{s^2}$
Let $s = 2k^2$. Need

$$\frac{n^{1/2^s}}{s^2} \geq 1$$

$$n^{1/2^s} \geq s^2$$

$$n \geq s^{2^{s+1}}$$

Suffice to take $n = 2^{2^{2s}} = \Gamma_2(4k^2)$

UPSHOT: $ER_2(k) \leq \Gamma_2(4k^2)$.

PROS and CONS

1. GOOD-Proof reminsicent of Ramsey Proof.
2. BAD-Proof complicated(?).
3. GOOD- $ER_2(k) \le \Gamma_2(4k^2)$. (We've seen worse).

PROOF TWO: 3-ary CASE

JUST LIKE 2-ary case!

Will use R_2 and ER_2.

Theorem: For all k there exists n such that for all $COL : \binom{[n]}{3} \rightarrow \omega$ there exists $I \subseteq [3]$ and an I-homog set of size k.

Proof in the Style of Ramsey

Given $COL : \binom{[n]}{3} \rightarrow \omega$ define a sequence.
Stage 1 $a_1 = 1$, $X = \{x_1\}$, $A_1 = [n] - X$.
Stage s: Have $X = \{x_1, \ldots, x_{s-1}\}$,
$COL' : \binom{X}{a-1} \rightarrow \omega \times \{\mathrm{homog}, \mathrm{rain}\}$, and A_{s-1}.
Let $A_s^0 = A_{s-1}$ and x_s be least element of A_{s-1}.
For all $0 \le L \le s-1$ we define $COL'(x_L, x_s)$ and thin out A,
Form $A_{s,0}$, $A_{s,1}$, ..., $A_{s,s}$.
Assume have $A_{s,L-1}$ and $COL'(x_1, x_s)$, ..., $COL'(x_{L-1}, x_s)$.
Notation: We denote $A_{s,L}$ by A_L throughout.

Case 1: $(\exists c)[|\{x \in A_{L-1} : COL(x_L, x_s, x) = c\}| \geq \sqrt{|A_{L-1}|}$.

$$\begin{aligned} COL'(x_L, x_s) &= (c, \mathrm{homog}) \\ A_L &= \{x \in A_{L-1} : COL(x_L, x_s, x) = c\} \end{aligned}$$

Note: $|A_L| \geq \sqrt{|A_{L-1}|}$.

Case 2

Case 2: $(\forall c)[|\{x \in A_{L-1} : COL(x_L, x_s, x) = c\}| < \sqrt{|A_{L-1}|}$.
Make all colors coming out of (x_L, x_s) to the right different:

Let A_L be the set of all $x \in A_{L-1}$ such that x is the LEAST number with the color $COL(x_L, x_s, x)$.
Formally A_L is $\{x \in A_{L-1}$:

$$COL(x_L, x_s, x) \notin \{COL(x_L, x_s, y) : x_s < y < x \wedge y \in A_{L-1}\}\}$$

$$\}$$

Now have

$$(\forall y, y' \in A_L)[COL(x_L, x_s, y) \neq COL(x_L, x_s, y')].$$

Note: $|A_L| \geq \sqrt{|A_{L-1}|}$.

Want to make colors DIFF

Important Note and Convention: For the rest of Case 2 $(\forall Z \in \binom{X}{2})$ means all such Z with $COL'(Z) = (-, \mathrm{rain})$.
Want to make the following true

$$(\forall Z \in \binom{X}{2})(\forall y, y' \in A_s)[COL(Z, y') \neq COL(x_L, x_s, y)]$$

Its OKAY if $COL(Z, y) = COL(x_L, x_s, y)$.

For each $y \in A_L$ we thin out A_L so that:

- $(\forall Z \in \binom{X}{2})(\forall y' \in A_L - \{y\})[COL(Z, y') \neq COL(x_L, x_s, y)]$.
- $(\forall Z \in \binom{X}{2})(\forall y' \in A_L - \{y\})[COL(Z, y) \neq COL(x_L, x_s, y')]$.

BILL- SHOW AT BOARD

More to do!

Use C for COL for space

$T = A_L$ (elements to process)

```
while  T ≠ ∅
   y = least element of T.
   T = T − {y}  (but y stays in A_L)
   If (∃Z ∈ (X choose 2), y' ∈ T)[C(x_L, x_s, y) = C(Z, y')] then
      T = T − {y'},        A_L = A_L − {y'}
   If (∃Z ∈ (X choose 2), y' ∈ T)[C(x_L, x_s, y') = C(Z, y)] then
      T = T − {y'},        A_L = A_L − {y'}
```

Note: At end $|A_L| \ge \sqrt{|A_{L-1}|/\binom{s-1}{2}} \ge 2\sqrt{|A_{L-1}|}/s^2$

Note: At end $(\forall Z \in \binom{X}{2}, y' \in A_L))[COL(x_L, x_s, y) \ne COL(Z, y')]$.

OKAY- What is $COL'(x_L, x_s)$?

RECAP:

- $(\forall y, y' \in A_L)[COL(x_L, x_s, y) \neq COL(x_L, x_s, y')]$.
- $(\forall y, y' \in A_L)(\forall Z \in \binom{X}{2})[COL(x_L, x_s, y) \neq COL(Z, y')]$.

$f(s)$ TBD. Let $t = |A_L| \geq \frac{2\sqrt{|A_{L-1}|}}{s^2}$.

Case 2.1:

$(\exists Z \in \binom{X}{2})[|\{y \in A_L : COL(x_L, x_s, y) = COL(Z, y)\}| \geq \frac{t}{f(s)}]$.

$$\begin{aligned} COL'(x_L, x_s) &= COL(x_i, x_s) \\ A_L &= \{y \in A_L : COL(x_L, x_s, y) = COL(Z, y)\} \end{aligned}$$

Note: This will be a color of the form $(-, \mathrm{rain})$.

Note: $|A_L| \geq \frac{t}{f(s)}$.

OKAY- What is $COL'(x_L, x_s)$

Case 2.2:
$(\forall Z \in \binom{X}{2})[|\{y \in A_L : COL(x_L, x_s, y) = COL(Z, y)\}| < \frac{t}{f(s)}]$.

$$\begin{aligned} COL'(x_L, x_s) &= (\ell, \mathrm{rain}) \ \ell \text{ is least not-used-for-}\mathrm{rain}\text{ color.} \\ A_L &= A_{L+1} - \{y : (\exists Z \in \binom{X}{2})[COL(Z, y) = COL(x_L, x_s, y)]. \end{aligned}$$

Note: $|A_L| \ge t - \binom{s-1}{2}\frac{t}{f(s)} \ge t(1 - \binom{s-1}{2}\frac{1}{f(s)})$

Picking $f(s)$

Case 1 yields: $|A_L| \ge \frac{t}{f(s)}$.

Case 2 yields: $|A_L| \ge t(1 - \binom{s-1}{2}\frac{1}{f(s)})$

Take $f(s) = 1 + \binom{s-1}{2} \le s^2/2$ to obtain that in both cases get:

$$|A_L| \ge \frac{t}{f(s)} \ge \frac{2\sqrt{|A_{L-1}|}}{s^2}\frac{2}{s^2} \ge \frac{\sqrt{|A_{L-1}|}}{s^4}.$$

We do this process s times.

Whats Really Going on?

$$\begin{aligned} b_0 &= b = a_{s-1} \\ b_L &\geq \frac{\sqrt{b_{L-1}}}{s^4} \end{aligned}$$

By Rec Lemma with $c = 1/s^4$, $\delta = 1/2$, $i = L$ we get

$$b_L \geq \frac{m^{1/2^L}}{s^8}.$$

In stage s do this for s times. Hence

$$a_s \geq b_s \geq \frac{a_{s-1}^{1/2^s}}{s^8}.$$

Bound on A_s

Let $a_s = |A_s|$.

$$
\begin{aligned}
a_0 &= n \\
a_s &\ge \frac{a_{s-1}^{1/2^s}}{s^8}
\end{aligned}
$$

By Rec Lemma with $b_i = a_i$, $c = 1/s^8$, $\delta = 1/2^s$, $i = s$, we get

$$
a_s \ge \frac{n^{1/2^{s^2}}}{s^{16}}
$$

We later see how far we need to go.

Now CASES

We determine r later

Have $X = \{x_1, x_2, \ldots, x_r\}$, $COL' : \binom{X}{2} \rightarrow \omega \times \{\mathrm{homog}, \mathrm{rain}\}$.

- Some of the colors are of form $(-, \mathrm{homog})$,
- Some of the colors are of form $(-, \mathrm{rain})$,

We would like to have a subset that has colors of the same type.

What to do?

Now CASES

We determine r later
Have $X = \{x_1, x_2, \ldots, x_r\}$, $COL' : \binom{X}{2} \rightarrow \omega \times \{\text{homog}, \text{rain}\}$.

- Some of the colors are of form $(-, \text{homog})$,
- Some of the colors are of form $(-, \text{rain})$,

We would like to have a subset that has colors of the same type.

What to do?
Use RAMSEY'S THEOREM ON PAIRS JUST 2 COLORS!

$COL''(x, y) = \Pi_2(COL'(x, y))$.

Let $r = R_2(m)$. Let H be the homog set of size m rel to COL''.
We determine m later.

Homog of color homog

Case 1: All pairs in H colored homog (real colors). Have

$$(\forall x < y < z_1 < z_2)[COL(x,y,z_1) = COL(x,y,z_2)].$$

$$COL'''(x,y) = \Pi_1(COL'(x,y)) = COL(x,y,-)$$

Get an I-homog set where $I \subseteq [2]$.

$$COL(y_1,y_2,y_3) = COL(z_1,z_2,z_3) \text{ iff}$$

$$COL'''(y_1,y_2) = COL'''(z_1,z_2)(\text{def of } COL''' \text{ iff})$$

$$(\forall i \in I)[y_i = z_i](\text{def of } I\text{-homog})$$

Get I-homog set.

Homog of color rain

Case 2: All pairs in H colored rain.
Have

$$(\forall x < y < z_1 < z_2)[COL(x,y,z_1) \ne COL(x,y,z_2)].$$

$COL'''(x,y) = \Pi_1(COL'(x,y))$
Get an I-homog set where $I \subseteq [2]$.

$$COL(y_1,y_2,y_3) = COL(z_1,z_2,z_3) \text{ iff}$$

$$y_3 = z_3 \wedge COL'''(y_1,y_2) = COL'''(z_1,z_2) \text{ (from the construction}$$

$$\text{iff } y_3 = z_3 \wedge (\forall i \in I)[y_i = z_i] \text{ (def of } I\text{-homog)).}$$

Get $I \cup \{3\}$-homog.

Estimate n

NEED: $m = ER_2(k) = k^2$ for COL'''.
NEED $r = R_2(m)$ Note that $r \le$

$$\Gamma_1(2ER_2(k)) \le \Gamma_1(2\Gamma_2(4k^2)) \le \Gamma_1(\Gamma_2(8k^2)) \le \Gamma_3(8k^2)$$

Note $r^2 \le \Gamma_3(16k^2)$.
Need construction to run r steps. Need n such that

$$\frac{n^{1/2^{r^2}}}{r^{16}} \ge 1$$

$$n \ge r^{16 \times 2^{r^2}}$$

Suffices to take

$$n = 2^{2^{r^2}} = \Gamma_2(r^2) \le \Gamma_2(\Gamma_3(16k^2)) \le \Gamma_5(16k^2).$$

So

PROS and CONS

1. GOOD-Proof reminsicent of Ramsey Proof.
2. GOOD-Seemed to be able to avoid alot of cases.
3. BAD-Proof complicated(?).
4. GOOD?- $ER_3(k) \le \Gamma_5(16k^2)$.

PROOF TWO: a-ary Case

REALLY JUST LIKE 3-ary case! (I mostly replaced 3 with a).
Will use R_{a-1} and ER_{a-1}.
Theorem: For all k there exists n such that for all $COL : \binom{[n]}{a} \rightarrow \omega$ there exists $I \subseteq [a]$ and an I-homog set of size k.

Proof in the Style of Ramsey

Given $COL : \binom{[n]}{a} \rightarrow \omega$ define a sequence.
Stage $a-2$ $(\forall 1 \le i \le a-2)[x_i = i]$. $X = \{x_1, \ldots, x_{a-1}\}$.
$A_{a-1} = [n] - X$.
Stage s: Have $X = \{x_1, \ldots, x_{s-1}\}$,
$COL' : \binom{X}{a-1} \rightarrow \omega \times \{\text{homog}, \text{rain}\}$, and A_{s-1}.
Let $A_s^0 = A_{s-1}$ and x_s be least element of A_{s-1}.
For all $X_L \in \binom{X}{a-2}$ we define $COL'(X_L, x_s)$ and thin out A,
Form A_s^0, A_s^1, ..., $A_s^{\binom{s}{a-2}}$
Assume have A_s^{L-1} and $COL'(X_1, x_s)$, ..., $COL'(X_{L-1}, x_s)$ defined.
Notation: We denote A_s^L by A_L throughout.

Case 1

Case 1: $(\exists c)[|\{x \in A_{L-1} : COL(X_L, x_s, x) = c\}| \geq \sqrt{|A_{L-1}|}$.

$$\begin{aligned} COL'(X_L, x_s) &= (c, \text{homog}). \\ A_L &= \{x \in A_{L-1} : COL(X_L, x_s, x) = c\} \end{aligned}$$

Note: $|A_L| \geq \sqrt{|A_{L-1}|}$.

Can Ramsey Proof

Case 2: $(\forall c)[|\{x \in A_{L-1} : COL(X_L, x_s, x) = c\}| < \sqrt{|A_{L-1}|}$.
Make all colors coming out of (X_L, x_s) to the right different:

Let A_L be the set of all $x \in A_{L-1}$ such that x is the LEAST number with the color $COL(X_L, x_s, x)$.
Formally $A_L = \{x \in A_{L-1}$:

$$COL(X_L, x_s, x) \notin \{COL(X_L, x_s, y) : x_s < y < x \wedge y \in A_{L-1}\}$$

$$\}$$

Now have

$$(\forall y, y' \in A_L)[COL(X_L, x_s, y) \neq COL(X_L, x_s, y')].$$

Note: $|A_L| \geq \sqrt{|A_{L-1}|}$.

Want to make colors DIFF

Important Note and Convention: For the rest of Case 2 we only care about $Z \in \binom{X}{a-1}$ such that $COL'(Z) = (-, \mathrm{rain})$.
Want to make the following true

$$(\forall Z \in \binom{X}{a-1})(\forall y, y' \in A_s)[COL(Z, y') \neq COL(X_L, x_s, y)]$$

Its OKAY if $COL(Z, y) = COL(X_L, y)$.

For each $y \in A_L$ we thin out A_L so that:

- $(\forall Z \in \binom{X}{a-1})(\forall y' \in A_L - \{y\})[COL(Z, y') \neq COL(X_L, x_s, y)]$.
- $(\forall Z \in \binom{X}{a-1})(\forall y' \in A_L - \{y\})[COL(Z, y) \neq COL(X_L, x_s, y')]$.

BILL- SHOW AT BOARD

More to do!

Use C for COL for space

$T = A_L$ (elements to process)

```
while  T ≠ ∅
    y = least element of T.
    T = T − {y} (but y stays in A_L)
    If (∃Z ∈ (X choose a−1), y' ∈ T)[C(X_L, x_s, y) = C(Z, y')] then
      T = T − {y'}  A_L = A_L − {y'}
    If (∃Z ∈ (X choose a−1)y' ∈ T)[C(X_L, x_s, y') = C(Z, y)] then
      T = T − {y'}  A_L = A_L − {y'}
```

Can show that for each $y \in T$ that is considered:

1) There is at most ONE Z such that there is a $y' \in T$ such that $C(X_L, x_s, y) = C(Z, y')$.

2) For each $Z \in \binom{X}{a-1}$ there is at most one $y' \in T$ such that $C(X_L, x_s, y') = C(Z, y)$.

Analysis

Begin with $T = A_L$. Every iteration we

- Ensure one elements stays in A_L.
- Remove at most $\binom{s}{a-1} + 1 \le s^{a-1}$ elements of A_L.

$c_0 = \sqrt{A_{L-1}}$ (initial size of A_L)
$c_i = c_{i-1} - s^{a-1}$.
Can show $c_i = c_0 - is^{a-1}$.
New $|A_L| \ge$ Numb of iterations $\ge c_0/s^{a-1} \ge \sqrt{|A_{L-1}|}/s^{a-1}$.

Also: At end
$(\forall Z \in \binom{X}{a-1}, y' \in A_L))[COL(X_L, x_s, y) \neq COL(Z, y')]$.

OKAY- What is $COL'(X_L, x_s)$?

RECAP:

- $(\forall y, y' \in A_L)[COL(X_L, x_s, y) \ne COL(X_L, x_s, y')]$
- $(\forall y, y' \in A_L)(\forall Z \in \binom{X}{a-1})[COL(X_L, x_s, x) \ne COL(Z, y')]$

$f(s)$ TBD. Let $t = |A_L| \ge \frac{\sqrt{|A_{L-1}|}}{s^{a-1}}$

Case 2.1:

$(\exists Z \in \binom{X}{a-1})[|\{y : COL(X_L, x_s, y) = COL(Z, y)\}| \ge \frac{t}{f(s)}]$.

$$\begin{aligned} COL'(X_L, x_s) &= COL(X_i, x_s) \\ A_L &= \{y \in A_L : COL(X_L, x_s, y) = COL(Z, y)\} \end{aligned}$$

Note: This will be a color of the form $(-, \mathrm{rain})$.

Note: $|A_L| \ge \frac{t}{f(s)}$.

OKAY- What is $COL'(X_L, x_s)$

Case 2.2:
$(\forall Z \in \binom{X}{a-1})[|\{y \in A_L : COL(X_L, x_s, y) = COL(Z, y)\}| < \frac{t}{f(s)}]$.

$$\begin{aligned} COL'(X_L, x_s) &= (\ell, \mathrm{rain}) \text{ } (\ell \text{ is least not-used-for-}\mathrm{rain} \text{ color.}) \\ A_L &= A_L - \{y : (\exists Z \in \tbinom{X}{a-1})[COL(X_L, x_s, y) = COL(Z, y)]. \end{aligned}$$

Note: $|A_L| \geq t - \binom{s-1}{a-1}\frac{t}{f(s)} \geq t(1 - \binom{s-1}{a-1}\frac{1}{f(s)})$

Picking $f(s)$

Case 1 yields $|A_L| \geq \frac{t}{f(s)}$.

Case 2 yields $|A_L| \geq t(1 - \binom{s-1}{a-1}\frac{1}{f(s)})$

Take $f(s) = 1 + \binom{s-1}{a-1} \leq s^a/a!$. Both cases yield:

$$|A_L| \geq \frac{t}{f(s)} \geq \frac{\sqrt{|A_{L-1}|}}{s^{a-1}}\frac{a!}{s^a} \geq \frac{\sqrt{|A_{L-1}|}}{s^{2a}}.$$

(We could have kept the $a!$ and have denom s^{2a-1} but what we do is simpler and does not lose much.)

We do this process $\binom{s-1}{a-1} \leq s^{a-1}$ times.

Whats Really Going on?

$$\begin{aligned} b_0 &= b = a_{s-1} \\ b_L &\ge \frac{\sqrt{b_{L-1}}}{s^{2(a-1)}} \end{aligned}$$

By Rec Lemma with $\delta = 1/2$, $c = s^{2a-2}$, $i = L$ we get

$$b_L \ge \frac{b^{1/2^L}}{s^{4a-4}}.$$

In stage s do this for $\le s^{a-1}$ times. Hence

$$a_s \ge b_{s^{a-1}} \ge \frac{a_{s-1}^{1/2^{s^{a-1}}}}{s^{4a-4}}.$$

Bound on A_s

Let $a_s = |A_s|$.

$$\begin{aligned} a_0 &= n \\ a_s &\geq \frac{a_{s-1}^{1/2^{s^{a-1}}}}{s^{4a-4}}. \end{aligned}$$

by Rec Lemma with $\delta = 1/2^{s^{a-1}}$, $c = 1/s^{4a-4}$, $b = a_0 = n$, $i = s$, we get

$$a_s \geq \frac{n^{1/2^{s^a}}}{s^{8a-8}}$$

We later see how far we need to go.

Now CASES

We determine r later

Have $X = \{x_1, x_2, \ldots, x_r\}$, $COL' : \binom{X}{a} \rightarrow \omega \times \{\mathrm{homog}, \mathrm{rain}\}$.

- Some of the colors are of form $(-, \mathrm{homog})$,
- Some of the colors are of form $(-, \mathrm{rain})$,

We would like to have a subset that has colors of the same type.

What to do?

Now CASES

We determine r later

Have $X = \{x_1, x_2, \ldots, x_r\}$, $COL' : \binom{X}{a} \rightarrow \omega \times \{\text{homog}, \text{rain}\}$.

- Some of the colors are of form $(-, \text{homog})$,
- Some of the colors are of form $(-, \text{rain})$,

We would like to have a subset that has colors of the same type.

What to do?

Use RAMSEY'S THEOREM ON $(a-1)$-tuples JUST 2 COLORS!

$COL''(W) = \Pi_2(COL'(W))$.

Let $r = R_{a-1}(m)$. Let H be the homog set of size m rel to COL''.

We determine m later.

Homog of color homog

Case 1: Color is homog (real colors). Have

$$(\forall Y \in \binom{H}{a-1}, z_1, z_2)[COL(Y, z_1) = COL(Y, z_2)].$$

$$COL'''(Y) = \Pi_1(COL'(Y)) = COL(Y, -)$$

Get an I-homog set where $I \subseteq [a-1]$.

$$COL(y_1, \ldots, y_a) = COL(z_1, \ldots, z_a) \text{ iff}$$

$$COL'''(y_1, \ldots, y_{a-1}) = COL'''(z_1, \ldots, z_{a-1}) (\text{def of } COL''' \text{ iff}$$

$$(\forall i \in I)[y_i = z_i] (\text{def of } I\text{-homog})$$

So get I-homog set.

Case 2: Color is rain.
Have

$$(\forall Y \in \binom{H}{a-1}, z_1, z_2)[COL(Y, z_1) \ne COL(Y, z_2)].$$

$COL'''(Y) = \Pi_1(COL'(Y))$
Get an I-homog set where $I \subseteq [a-1]$.

$$COL(y_1, \ldots, y_a) = COL(z_1, \ldots, z_a) \text{ iff}$$

$$y_a = z_a \wedge COL'''(y_1, \ldots, y_{a-1}) = COL'''(z_1, \ldots, z_{a-1}) \text{ (from const.}$$

$$\text{iff } y_a = z_a \wedge (\forall i \in I)[y_i = z_i] \text{ (def of } I\text{-homog)).}$$

Get $I \cup \{a\}$-homog set.
Need $m = ER_{a-1}(k)$ for COL'''.

Estimate n

NEED $m = ER_{a-1}(k)$ for COL'''.
NEED $r = R_{a-1}(m)$.

$$r = R_{a-1}(ER_{a-1}(k)) \le \Gamma_{a-2}(ER_{a-1}(k)).$$

Need construction to run r steps. Need n such that

$$\frac{n^{1/2^r}}{r^{8a-8}} \ge 1$$

$$n \ge r^{8a \times 2^r}$$

Suffices to take $n = 2^{2^{2ar}} = \Gamma_2(2ar)$

$$n = \le \Gamma_2(2ar) = \Gamma_2(2a\Gamma_{a-2}(ER_{a-1}(k)) \le \Gamma_a(ER_{a-1}(2ak).$$

So

$$ER_a(k) \le \Gamma_a(ER_{a-1}(2ak)).$$

SOLVE REC

$$
\begin{aligned}
ER_1(k) &\le \Gamma_0(k^2) \\
ER_a(k) &\le \Gamma_a(ER_{a-1}(2ak))
\end{aligned}
$$

Can show
$ER_a(k) \le \Gamma_{f(a)}(4ak^2)$ where $f(a) = \frac{a^2+a-2}{2}$.

PROS and CONS

1. GOOD-Proof reminiscent of Ramsey Proof.
2. GOOD-Seemed to be able to avoid alot of cases.
3. BAD-Proof complicated(?).
4. GOOD?- $ER_a(k) \le \Gamma_{f(a)}(4ak^2)$ where $f(a) = \frac{a^2+a-2}{2}$. An improvement!