The Complexity of Grid Coloring

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Grid Coloring

Notation: If $n \in \mathbb{N}$ then $[n]$ is the set $\{1, \ldots, n\}$.

Definition

$G_{n,m}$ is the grid $[n] \times [m]$.

1. $G_{n,m}$ is c-colorable if there is a c-coloring of $G_{n,m}$ such that no rectangle has all four corners the same color.
2. $\chi(G_{n,m})$ is the least c such that $G_{n,m}$ is c-colorable.

Examples

A FAILED 2-Coloring of $G_{4,4}$

$$\begin{array}{cccc} R & B & B & R \\ B & R & R & B \\ B & B & R & R \\ R & R & R & B \end{array}$$

A 2-Coloring of $G_{4,4}$

$$\begin{array}{cccc} R & B & B & R \\ B & R & R & B \\ B & B & R & R \\ R & B & R & B \end{array}$$

EXAMPLE: A 3-Coloring of $G_{10,10}$

R	R	R	R	B	B	G	G	B	G
R	B	B	G	R	R	R	G	G	B
G	R	B	G	R	B	B	R	R	G
G	B	R	B	B	R	G	R	G	R
R	B	G	G	G	B	G	B	R	R
G	R	B	B	G	G	R	B	B	R
B	G	R	B	G	B	R	G	R	B
B	B	G	R	R	G	B	G	B	R
G	G	G	R	B	R	B	B	R	B
B	G	B	R	B	G	R	R	G	G

It is known that CANNOT 2-color $G_{10,10}$. Hence $\chi(G_{10,10}) = 3$.

Obstruction Sets

Fenner-Gasarch-Glover-Purewall [FGGP] showed:

1. For all c there exists OBS_c, a finite set of grids, such that

   $G_{n,m}$ is c-colorable iff no element of OBS_c is inside $G_{m,n}$.

2. FGGP have a proof which shows $|OBS_c| \leq 2c^2$.
3. If OBS_c is known then the set of c-colorable grids is completely characterized.

OBS-2 and OBS-3 Known

FGGP showed

$$OBS_2 = \{G_{3,7}, G_{5,5}, G_{7,3}\}$$
$$OBS_3 = \{G_{4,19}, G_{5,16}, G_{7,13}, G_{10,11}, G_{11,10}, G_{13,7}, G_{16,5}, G_{19,4}\}$$

2-colorability table. *C* for Colorable, *U* for Uncolorability.

	2	3	4	5	6	7
2	*C*	*C*	*C*	*C*	*C*	*C*
3	*C*	*C*	*C*	*C*	*C*	*U*
4	*C*	*C*	*C*	*C*	*C*	*U*
5	*C*	*C*	*C*	*U*	*U*	*U*
6	*C*	*C*	*C*	*U*	*U*	*U*
7	*C*	*U*	*U*	*U*	*U*	*U*

4-Colorability

1. FGGP did not (as of 2009) determine OBS_4.
2. FGGP had reasons to think $G_{17,17}$ is 4-colorable but they did not have a 4-coloring.
3. In 2009 Gasarch offered a prize of $289.00 for the first person to email him a 4-coloring of $G_{17,17}$.
4. Brian Hayes, Scientific American Math Editor, popularized the challenge.

Challenge Was Hard

1. Lots of people worked on it.
2. No progress.
3. Finally solved in 2012 by Bernd Steinbach and Christian Posthoff [SP]. Clever, and SAT-solver, but did not generalize.
4. They and others also found colorings that lead to $\mathrm{OBS}_4 = \{$

$$G_{5,41}, G_{6,31}, G_{7,29}, G_{9,25}, G_{18,23}, G_{11,22}, G_{13,21}, G_{17,19},$$
$$G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,18}, G_{22,11}, G_{21,13}, G_{19,17}$$

$\}$

Is Grid Coloring Hard?

We view this two ways:

1. Is there an NP-complete problem lurking here somewhere? YES!
2. Is there a Prop Statement about Grid Coloring whose resolution proof requires exp size? YES!

THERE IS AN NP-COMPLETE PROBLEM LURKING!

Grid Coloring Hard!-NP stuff

1. Let $c, N, M \in \mathbb{N}$. A partial mapping χ of $N \times M$ to $\{1, \ldots, c\}$ is a *extendable to a c-coloring* if there is an extension of χ to a total mapping which is a c-coloring of $N \times M$.
2. $$GCE = \{(N, M, c, \chi) \mid \chi \text{ is extendable}\}.$$

   GCE is NP-complete!

GCE is NP-complete

$\phi(x_1, \ldots, x_n) = C_1 \wedge \cdots \wedge C_m$ is a 3-CNF formula. We determine N, M, c and a partial c-coloring χ of $N \times M$ such that

$$\phi \in \text{3-SAT iff } (N, M, c, \chi) \in GCE$$

Forcing a Color to Only Appear Once in Main Grid

G								
G								
R		G						
G								
G								
G								
G	G	G	G	G	G	G	G	G

G can only appear once in the main grid, where it is, but what about R? (The double lines are not part of the construction. They are there to separate the main grid from the rest.)

Forcing a Color to Only Appear Once in Main Grid

R	G								
R	G								
R	R		G						
R	G								
R	G								
R	G								
R	G	G	G	G	G	G	G	G	G
R	G	R	R	R	R	R	R	R	R

G can only appear once in the main grid, where it is. *R* cannot appear anywhere in the main grid.

Using Variables

D means that the color is some *distinct*, unique color.

	D	D	D	D	D	D	D	D	D	D	D
$\overline{x}_1$		D	D	D	D	D	D	D	D	T	F
x_1		D	D	D	D	D	D	T	F	T	F
$\overline{x}_1$		D	D	D	D	T	F	T	F	D	D
x_1		D	D	T	F	T	F	D	D	D	D
$\overline{x}_1$		T	F	T	F	D	D	D	D	D	D
x_1		T	F	D	D	D	D	D	D	D	D

The labeled $x_1, \overline{x}_1$ are not part of the grid. They are visual aids.

Coding a Clause

$C_1 = L_1 \vee L_2 \vee L_3$. Where L_1, L_2, L_3 are literals (vars or their negations).

	$\cdots$	D	T	T
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
L_1	$\cdots$		D	F
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
L_2	$\cdots$			
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
L_3	$\cdots$		F	D
	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The L_1, L_2, L_3 are not part of the grid. They are visual aids.

Coding a Clause—More Readable

$C_1 = L_1 \vee L_2 \vee L_3$.

	D	T	T
L_1		D	F
L_2			
L_3		F	D

One can show that

- If put any of TTT, TTF, TFT, FTT, FFT, FTF, TFF in first column then can extend to full coloring.
- If put FFF in first column then cannot extend to a full coloring.

Example: (F,F,T)

$C_1 = L_1 \vee L_2 \vee L_3$.

	D	T	T
L_1	F	D	F
L_2	F		$*$
L_3	T	F	D

The * is forced to be T.

Example: (F,F,T)

$C_1 = L_1 \vee L_2 \vee L_3$.

	D	T	T
L_1	F	D	F
L_2	F	$*$	T
L_3	T	F	D

The * is forced to be F.

Example: (F,F,T)

$C_1 = L_1 \vee L_2 \vee L_3$.

	D	T	T
L_1	F	D	F
L_2	F	F	T
L_3	T	F	D

Other Assignments

1. We did (F, F, T).
2. (F, T, F), (T, F, F) are similar.
3. (F, T, T), (T, F, T), (T, T, F), (T, T, T) are easier.

Cannot Use (F,F,F)

$C_1 = L_1 \vee L_2 \vee L_3$. Want that (F, F, F) CANNOT be extended to a coloring.

	D	T	T
L_1	F	D	F
L_2	F	$*$	$*$
L_3	F	F	D

The *'s are forced to be T.

Cannot Use (F,F,F)

	D	T	T
L_1	F	D	F
L_2	F	T	T
L_3	F	F	D

There is a mono rectangle of T's. NOT a valid coloring!

Put it all Together

Do the above for all variables and all clauses to obtain the result that GRID EXT is NP-complete!

Big Example

$$(x_1 \vee x_2 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee x_3 \vee x_4) \wedge (\overline{x}_1 \vee \overline{x}_3 \vee \overline{x}_4)$$

												C_1	C_1	C_2	C_2	C_3	C_3
	D	D	D	D	D	D	D	D	D	D	D	T	T	T	T	T	T
$\overline{x}_4$		D	D	D	D	D	D	D	D	T	F	D	D	D	D	D	F
x_4		D	D	D	D	D	D	D	D	T	F	D	D	D	F	D	D
$\overline{x}_3$		D	D	D	D	D	D	T	F	D	D	D	D	D	D	D	D
x_3		D	D	D	D	T	F	T	F	D	D	D	D			D	D
$\overline{x}_3$		D	D	D	D	T	F	D	D	D	D	D	F	D	D		
$\overline{x}_2$		D	D	T	F	D	D	D	D	D	D	D	D	F	D	D	D
x_2		D	D	T	F	D	D	D	D	D	D			D	D	D	D
$\overline{x}_1$		T	F	D	D	D	D	D	D	D	D	D	D	D	D	F	D
x_1		T	F	D	D	D	D	D	D	D	D	F	D	D	D	D	D

Does this Explain why the Challenge was Hard?

1. MAYBE NOT: *GCE* is Fixed Parameter Tractable: For fixed c GCE_c is in time $O(N^2M^2 + 2^{O(c^4)})$. But for $c = 4$ this is huge!
2. MAYBE NOT: Our result says nothing about the case where the grid is originally all blank.

KEY to $O(N^2M^2 + 2^{O(c^4)})$ Result

Lemma Let χ be a partial c-coloring of $G_{n,m}$. Let U be the uncolored grid points. Let $|U| = u$. There is an algorithm that will determine if χ can be extended to a full c-coloring that runs in time $O(cnm2^{2u}) = 2^{O(nm)}$.
Sketch: For $S \subseteq U$ and $1 \le i \le c$ let

$$f(S,i) = \begin{cases} YES & \text{if } \chi \text{ can be extended to } S \text{ using colors } \{1,\ldots,i\}; \\ NO & \text{if not.} \end{cases}$$

For $S \subseteq U$ and $1 \le i \le c$ use Dynamic Programming to compute $f(S,i)$. $f(U,c)$ is your answer.
End of Sketch

Computing $f(S, i)$

Assume that $(\forall S', |S'| < |S|)(\forall 1 \le i \le c)[f(S', i)$ is known].

1. For all 1-colorable $T \subseteq S$ do the following
   1.1 If $f(S - T, i) = NO$ then $f(S, i) = NO$ and STOP.
   1.2 If $f(S - T, i - 1) = YES$ then determine if coloring T with i works. If yes then $f(S, i) = YES$ and STOP. Note that this takes $O(nm)$.
2. We know that for all 1-colorable $T \subseteq S$ $f(S - T, i) = YES$ and either
   (1) $f(S - T, i - 1) = NO$ or
   (2) $f(S - T, i - 1) = YES$ and coloring T with i bad.
   In all cases $f(S, i) = NO$.

Open Questions

1. Improve Fixed Parameter Tractable algorithm.
2. NPC results for mono squares? Other shapes?
3. Show that

$$\{(n, m, c) : G_{n,m} \text{ is } c\text{-colorable } \}$$

is hard.

- If n, m in unary then sparse set, not NPC—New framework for hardness needed.
- If n, m binary then not in NP. Could try to prove *NEXP*-complete. But we the difficulty of the problem is not with the grid being large, but with the number-of-possibilities being large.

Part II of Talk—Lower Bounds on Tree Resolution

YOU SAY YOU WANT A RESOLUTION!

Resolution

Definition

Let $\varphi = C_1 \wedge \cdots \wedge C_L$ be a CNF formula. A Resolution Proof of $\varphi \notin SAT$ is a sequence of clauses such that on each line you have either

1. One of the C's in φ (called an AXIOM).
2. $A \vee B$ if $A \vee x$ and $B \vee \neg x$ were on prior lines. Variable that is resolved on is x.
3. The last line has the empty clause.

Example

$\varphi = x_1 \wedge x_2 \wedge (\neg x_1 \vee \neg x_2)$

1. x_1 (AXIOM)
2. $\neg x_1 \vee \neg x_2$ (AXIOM)
3. $\neg x_2$ (From lines 1,2, resolve on x_1.)
4. x_2 (AXIOM)
5. $\emptyset$ (From lines 3,4, resolve on x_2.)

Resolution is Complete

Definition

Let $\varphi = C_1 \wedge \cdots \wedge C_L$ be a CNF formula on n variables.

1. If exists a Res Proof of $\varphi \notin SAT$ then $\varphi \notin SAT$.
   Proof: Any assignment that satisfies φ satisfies any node of the Res Proof including the last node $\emptyset$.
2. If $\varphi \notin SAT$ then exists a Res Proof of $\varphi \notin \mathrm{SAT}$ of size $2^{O(n)}$.
   Proof: Form a Decision Tree that has at every node on level i the variable x_i. Right$=T$ and Left$=F$. A leaf is the first clause that is false with that assignment. Turn Decision Tree upside down! View nodes as which var to resolve on! This will be Res Proof! (It will even be Tree Res Proof.)

Another Example

The AND of the following:

1. For $i, j \in \{1, \ldots, 5\}$

$$x_{ij1} \vee x_{ij2}.$$

Another Example

The AND of the following:

1. For $i, j \in \{1, \ldots, 5\}$

$$x_{ij1} \vee x_{ij2}.$$

   Interpretation: (i, j) is colored either 1 or 2.

Another Example

The AND of the following:

1. For $i, j \in \{1, \ldots, 5\}$
$$x_{ij1} \vee x_{ij2}.$$
Interpretation: (i, j) is colored either 1 or 2.
2. For $i, j, i', j' \in \{1, \ldots, 5\}$
$$\neg x_{ij1} \vee \neg x_{i'j1} \vee \neg x_{ij'1} \vee \neg x_{i'j'1}$$

Another Example

The AND of the following:

1. For $i, j \in \{1, \ldots, 5\}$
$$x_{ij1} \vee x_{ij2}.$$
Interpretation: (i, j) is colored either 1 or 2.
2. For $i, j, i', j' \in \{1, \ldots, 5\}$
$$\neg x_{ij1} \vee \neg x_{i'j1} \vee \neg x_{ij'1} \vee \neg x_{i'j'1}$$
Interpretation: There is no mono 1-rectangle.

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$$\neg x_{ij1} \vee \neg x_{i'j1} \vee \neg x_{ij'1} \vee \neg x_{i'j'1}$$
Interpretation: There is no mono 1-rectangle.
3. For $i, j, i', j' \in \{1, \ldots, 5\}$
$$\neg x_{ij2} \vee \neg x_{i'j2} \vee \neg x_{ij'2} \vee \neg x_{i'j'2}$$

Another Example

The AND of the following:

1. For $i, j \in \{1, \ldots, 5\}$
$$x_{ij1} \vee x_{ij2}.$$
Interpretation: (i,j) is colored either 1 or 2.
2. For $i, j, i', j' \in \{1, \ldots, 5\}$
$$\neg x_{ij1} \vee \neg x_{i'j1} \vee \neg x_{ij'1} \vee \neg x_{i'j'1}$$
Interpretation: There is no mono 1-rectangle.
3. For $i, j, i', j' \in \{1, \ldots, 5\}$
$$\neg x_{ij2} \vee \neg x_{i'j2} \vee \neg x_{ij'2} \vee \neg x_{i'j'2}$$
Interpretation: There is no mono 2-rectangle.

Another Example

The AND of the following:

1. For $i, j \in \{1, \ldots, 5\}$

$$x_{ij1} \vee x_{ij2}.$$

   Interpretation: (i, j) is colored either 1 or 2.

2. For $i, j, i', j' \in \{1, \ldots, 5\}$

$$\neg x_{ij1} \vee \neg x_{i'j1} \vee \neg x_{ij'1} \vee \neg x_{i'j'1}$$

   Interpretation: There is no mono 1-rectangle.

3. For $i, j, i', j' \in \{1, \ldots, 5\}$

$$\neg x_{ij2} \vee \neg x_{i'j2} \vee \neg x_{ij'2} \vee \neg x_{i'j'2}$$

   Interpretation: There is no mono 2-rectangle.

We interpret this statement as saying

There is a 2-coloring of $G_{5,5}$.

This statement is known to be false.

GRID(n,m,c)

Definition

Let $n, m, c \in$ N. $GRID(n, m, c)$ is the AND of the following:

1. For $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$,

$$x_{ij1} \vee x_{ij2} \vee \cdots \vee x_{ijc}$$

Interpretation: (i, j) is colored either 1 or $\cdots$ or c.

2. For $i, i' \in \{1, \ldots, n\}$, $j, j' \in \{1, \ldots, m\}$, $k \in \{1, \ldots, c\}$,

$$\neg x_{ijk} \vee \neg x_{i'jk} \vee \neg x_{ij'k} \vee \neg x_{i'j'k}$$

Interpretation: There is no mono rectangle.

We interpret this statement as saying

There is a c-coloring of $G_{n,m}$.

NOTE: $GRID(n, m, c)$ has nmc VARS and $O(cn^2m^2)$ CLAUSES.

GRID(n,m,c)—How to View Assignments

Given an assignment:

1. For all $i \in [n]$ and $j \in [m]$ let k be the LEAST number such that $x_{ijk} = T$. View this as saying that (i,j) is colored k.
2. If there is NO such number then (i,j) is not colored and this assignment makes $GRID(n,m,c)$ FALSE.

Hence we view assignments as attempted colorings of the grid where some points are not colored.

Two Ways to Invalidate GRID(n,m,c)

1. There is a mono rectangle.
2. There is some point that is not colored: there is some i, j such that all x_{ijk} are FALSE.

Tree Resolution Proofs

Definition

A Tree Res Proof is a Res Proof where the underlying graph is a tree. Note that if you remove the bottom node that is labeled $\emptyset$ then the Tree Res Proof is cut into two disjoint parts.

Known: If $\varphi \notin SAT$ and φ has v variables then there is a Tree Res Proof of φ of size $2^{O(v)}$.

Our Goal

Assume that there is no c-coloring of $G_{n,m}$.

1. $GRID(n, m, c)$ has a size $2^{O(cnm)}$ Tree Res Proof.
2. We show $2^{\Omega(c)}$ size is REQUIRED. THIS IS OUR POINT!
3. The lower bound is IND of n, m.

Interesting Examples

1. Fenner et al [FGGP] showed that $G_{2c^2-c,,2c}$ is not c-colorable. Hence

$$GRID(2c^2 - c, 2c)$$

   has $O(c^3)$ vars, $O(c^6)$ clauses but $2^{\Omega(c)}$ Tree Res proof.

2. Easy to show G_{c^3,c^3} is not c-colorable.

$$GRID(c^3, c^3, c)$$

   has $O(c^7)$ vars, $O(c^{13})$ clauses and $2^{\Omega(c)}$ Tree Res proof.

These are poly-in-c formulas that require $2^{\Omega(c)}$ Tree Res proofs.

The Prover-Delayer Game

(Due to Pudlak and Impagliazzo [PI].) Parameters of the game: $p \in R^+$,

$$\varphi = C_1 \wedge \cdots \wedge C_L \notin SAT.$$

Do the following until a clause is proven false:

1. PROVER picks a variable x that was not already picked.
2. DEL either
   2.1 Sets x to F or T, OR
   2.2 Defers to PROVER who then sets x to T or F while DEL gets a point.

At end if DEL has $\geq p$ pts then he WINS; else PROVER WINS.

Convention

We assume that PROVER and DEL play perfectly.

1. *PROVER wins* means *PROVER has a winning strategy*.
2. *DEL wins* means *DEL has a winning strategy*.

Prover-Delayer Game and Tree Res Proofs

Lemma

Let $p \in \mathrm{R}^{+}$, $\varphi \notin SAT$. If φ has a Tree Res Proof of size $< 2^p$ then PROVER wins.

Prover-Delayer Game and Tree Res Proofs

Lemma

Let $p \in \mathrm{R}^+$, $\varphi \notin SAT$. If φ has a Tree Res Proof of size $< 2^p$ then PROVER wins.

Proof.

PROVER Strategy:

1. Initially T is res tree of size $< 2^p$ and DEL has 0 pts.
2. PROVER picks x, the LAST var resolved on.
3. If DEL sets x then DEL gets no pts.
4. If DEL defers then PROVER sets T or F—whichever yields a smaller tree. NOTE: One of the trees will be of size $< 2^{p-1}$. DEL gets 1 point.
5. Repeat: after ith stage will always have T of size $< 2^{p-i}$, and DEL has $\leq i$ pts. □

Contrapositive is Awesome!

Recall:

Lemma

Let $p \in \mathrm{R}^+$, $\varphi \notin SAT$. If φ has a Tree Res Proof of size $< 2^p$ then PROVER wins.

Contrapositive is Awesome!

Recall:

Lemma

Let $p \in \mathrm{R}^+$, $\varphi \notin SAT$. If φ has a Tree Res Proof of size $< 2^p$ then PROVER wins.

Contrapositive:

Lemma

Let $p \in \mathrm{R}^+$, $\varphi \notin SAT$. If DEL wins then EVERY Tree Res Proof for φ has size $\geq 2^p$.

PLAN: Get AWESOME strategy for DEL when $\varphi = GRID(n, m, c)$.

GRID(n,m,c) Requires Exp Tree Res Proofs

Theorem

Let n, m, c be such that $G_{n,m}$ is not c-colorable. Let $c \geq 2$. Any tree resolution proof of $GRID(n, m, c) \notin SAT$ requires size $2^{0.5c}$.

PROOF: Parameters: $p = 0.5c$, $\varphi = GRID(n, m, c)$.

Del Strategy

Assume x_{ijk} was chosen by the PROVER.

1. If setting $x_{ijk} = T$ creates a mono rect (of color k) then DEL DOES NOT let this happen— he sets x_{ijk} to F.
2. If none of the x_{ij*} are T and $\geq \frac{c}{2}$ of the x_{ij*} are F via PROVER then DEL sets x_{ijk} to T.
3. In all other cases the DEL defers to the PROVER.

Case 1: Prover Set c/2 Vars to F

Game ends when there is some i, j such that

$$x_{ij1} = x_{ij2} = \cdots = x_{ijc} = F.$$

Who set those variables to F?

Case 1: At least $\frac{c}{2}$ set F by Prover. Then DEL gets at least

$$0.5c \text{ pts.}$$

Case 2: Del Set c/2 Vars to F

$x_{ij1} = x_{ij2} = \cdots = x_{ijc} = F$. Who set those vars to F?
Case 2: At least $\frac{c}{2}$ set F by DEL. Assume they are $x_{ij1}, x_{ij2}, \ldots, x_{ijc/2}$.

- x_{ij1} set to F by DEL. Why? There exists i', j' such that $x_{i'j1}$, $x_{ij'1}$, $x_{i'j'1}$ all set to T. (Do not know by who.)
- x_{ij2} set to F by DEL. Why? There exists i'', j'' such that $x_{i''j2}$, $x_{ij''2}$, $x_{i''j''2}$ all set to T. (Do not know by who.)
- etc.

For every k such that x_{ijk} is set to F by DEL there exists THREE vars of form x_{**k} set to T.
KEY: All these 3-sets are DISJOINT, so at least $3c/2$ vars set T (by who?).

Case 2a: Prover Set 3c/2 Vars to T

KEY: At least $3c/2$ vars set T (by who?).
Case 2a: PROVER set $\geq \frac{3c}{4}$ to T. DEL gets at least

$$0.75c = 0.75c \text{ pts.}$$

Case 2b: Del Set 3c/2 Vars To T

Case 2b: DEL set $\geq \frac{3c}{4}$ to T.
DEL set x_{ijk} to T:

- At time there are $c/2$ k' such that PROVER set $x_{ijk'}$ to F.
- DEL will NEVER set an x_{ij*} to T again! NEVER!!

Every x_{ijk} set T by DEL implies that $c/2$ vars set F by PROVER, and these sets of $c/2$ vars are disjoint.
UPSHOT: PROVER had set $\frac{3c}{4} \times \frac{c}{2}$ to F. DEL gets at least

$$0.375c^2 = 0.375c^2 \text{ pts.}$$

Final Analysis

- Case 1: DEL gets at least $0.5c$ pts.
- Case 2a: DEL gets at least $0.75c$ pts.
- Case 2b: DEL gets at least $0.375c^2$ pts.

UPSHOT: For $c \geq 2$ DEL gets at least $0.5c$ pts.
PUNCHLINE: By Lemma any Tree Res Proof has size $\geq 2^{0.5c}$.

Optimize

1. In construction use cutoff of $c/2$ for when DEL sets x_{ijk} to T. Choose fraction CAREFULLY!
2. In analysis we twice do a half-half cutoff. Choose fractions CAREFULLY!
3. Use asymmetric PROVER-DEL game (next slide) and choose a, b CAREFULLY!

Theorem

Let n, m, c be such that $G_{n,m}$ is not c-colorable. Let $c \geq 9288$. Any tree resolution proof of $GRID(n, m, c) \notin SAT$ requires size $2^{0.836c}$.

Asymmetric Prover-Delayer Game

(Due to Beyersdorr, Galesi, Lauria [BGL].) Parameters of the game: $a, b \in (1, \infty)$, with $\frac{1}{a} + \frac{1}{b} = 1$, $p \in R^+$,

$$\varphi = C_1 \wedge \cdots \wedge C_L \notin SAT.$$

Do the following until a clause is proven false:

1. PROVER picks a variable x that was not already picked.
2. DEL either
    2.1 Sets x to F or T, OR
    2.2 Defers to PROVER.
        2.2.1 If PROVER sets $x = F$ then DEL gets $\lg a$ pts.
        2.2.2 If PROVER sets $x = T$ then DEL gets $\lg b$ pts.

At end if DEL has $\geq p$ pts then he WINS; else PROVER WINS.

Other Shapes

What is special about rectangles? NOTHING!

Definition

(Informally) Let S be a set of at least 2 grid points. Let $GRID(n, m, c; S)$ be the prop statement that there is a c-coloring of $G_{n,m}$ with no mono configuration that is "like S".

Theorem

(Informally) Let S be a set of at least 2 grid points. Let n, m, c be such that $GRID(n, m, c; S) \notin SAT$. Any tree resolution proof of $GRID(n, m, c; S) \notin SAT$ requires size $2^{\Omega(c)}$.

Open Questions

1. Want matching upper bounds for Tree Res Proofs of $GRID(n, m, c) \notin SAT$.
2. Want lower bounds on Gen Res Proofs of $GRID(n, m, c) \notin SAT$.
3. Want lower bounds on in other proof systems $GRID(n, m, c) \notin SAT$.
4. Upper and lower bounds for $GRID(n, m, c; S)$ for various S in various proof systems.

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