Homework 1 SOLUTIONS. Was due Morally Tue Feb 5, 2013
(The symbol before gasarch is a tilde.)

1. (10 points) What is your name? Write it clearly. Staple your HW.
   When is the midterm (give Date and Time)? If you cannot make it
   in that day/time see me ASAP. Join the Piazza group for the course.
   The codename is cmssc858. Look at the link on the class webpage
   about projects. Come see me about a project. READ the note on the
   class webpage that say THIS YOU SHOULD READ that you haven’t
   already read.

2. (20 points) Recall that the $a$-ary infinite Ramsey Theorem dealt with
   colorings of $\binom{\mathbb{N}}{a}$. We have only dealt with $a \geq 2$.
   
   (a) Formulate the 1-ary infinite Ramsey Theorem, for $c$ colors, and
   prove it.

   (b) Formulate the $\omega$-ary infinite Ramsey Theorem. (Extra Credit-
   prove or disprove it.)

SOLUTION TO PROBLEM 2

The key to this problem was to DEFINE homog sets.

1) Given $COL : (\mathbb{N})^1 \rightarrow \{0, 1\}$, a homog set is a set of numbers that are all
   colored the same. Hence the statement is:
   
   For all $COL : (\mathbb{N})^1 \rightarrow \{0, 1\}$ there is an infinite subset $A \subseteq \mathbb{N}$ such that all
   the elements of $A$ are colored the same.

   OR, if you defined homog you could just say
   
   For all $COL : (\mathbb{N})^1 \rightarrow \{0, 1\}$ there is an infinite homog subset $A \subseteq \mathbb{N}$.

2) Given $COL : (\mathbb{N})^\omega \rightarrow \{0, 1\}$, a homog set is an infinite set $A$ such that all
   infinite subsets of $A$ are colored the same. Hence the statement is:
   
   For all $COL : (\mathbb{N})^\omega \rightarrow \{0, 1\}$ there is an infinite subset $A \subseteq \mathbb{N}$ such that all
   subsets of $A$ are colored the same.

   OR, if you defined homog you could just say
   
   For all $COL : (\mathbb{N})^\omega \rightarrow \{0, 1\}$ there is an infinite homog subset $A \subseteq \mathbb{N}$.
3. (40 points) State and prove (rigorously) the $c$-color $a$-ary Ramsey Theorem. Your statement should start out for all $a \geq 1$, for all $c \geq 1$, . . . .

The proof should be by induction on $a$ with the base case being $a = 1$.

Omitted- very similar to what we did in class.

4. (40 points) Show (rigorously) that there exists a computable 2-coloring of $\binom{\mathbb{N}}{2}$ with no c.e.-in-$HALT$ homog set. (HINT- the proof is very similar to the one you saw in class. Instead of looking at $W_{e,s}$ you look at $W^{\text{HALT}}_{e,s}$.) (NOTE- I ALLOW THE FOLLOWING TECHNICAL ASSUMPTION: if $W^{\text{HALT}}_e$ is a c.e.-in-HALT set then it can only change its mind finitely often on any one number. Formally: For every $x$ there is an $s_0 \in \mathbb{N}$ such that one of the two holds:

\[
\begin{align*}
(1) & \quad (\forall s \geq s_0)[x \in W^{\text{HALT}}_{e,s}] \\
(2) & \quad (\forall s \geq s_0)[x /\in W^{\text{HALT}}_{e,s}].
\end{align*}
\]

SOLUTION TO PROBLEM 4

The construction is similar to the one I did in class: just replace $W_{e,s}$ with $W^{Ks}_{e,s}$. But the proof that it works needs some serious changes.

I do the proof as though its the proof I did in class and then say where it differs.

We show that each requirement is eventually satisfied.

For pedagogy we first look at $R_1$.

If $W^K_1$ is finite then $R_1$ is satisfied.

Assume $W^K_1$ is infinite. We show that $R^K_1$ is satisfied. Let $x < y$ be the least two elements in $W^K_1$. Let $s_0$ be the least number such that $x, y \in W^{Ks_0}_{1,s_0}$.

NO NO NO!!!- It could be that for some later $s \geq s_0$ we have $x, y \not\in W^{Ks}_{1,s}$. ALSO it is possible that for some later $s \geq s_0$ some SMALLER values $x', y'$ are in $W^{Ks}_{1,s}$ and they will be the ones whose edges to $s$ get colored.

It is ESSENTIAL to take $x_0$ such that

- $x, y \in W^{Ks_0}_{1,s_0}$
• $(\forall s \geq s_0)[x, y \in W_{1,s}^K]$.
• $(\forall s \geq s_0)[0, \ldots, x-1, x+1, x+2, \ldots, y-1 \notin W_{1,s}^K]$.

NOW we have that, for ALL $s \geq s_0$:

$COL(x, s) = RED$
$COL(y, s) = BLUE$

Since $W_1^K$ is infinite there is SOME $s \geq s_0$ with $s \in W_{e,s}^K$. Hence $x, y, s \in W_1^K$ and show that $W_1^K$ is NOT homogenous.

Can we show $R_2$ is satisfied the same way? Yes but with a caveat—we won’t use the least two elements of $W_2^K$. We’ll use the least two elements of $W_2^K$ that are bigger than the least two elements of $W_1^K$.

We now do this rigorously and more generally.

**Claim:** For all $e$, $R_e$ is satisfied:

**Proof:** Fix $e$. If $W_e^K$ is finite then $R_e$ is satisfied.

Assume $W_e^K$ is infinite. We show that $R_e$ is satisfied. Let $x_1 < x_2 < \cdots < x_{2e}$ be the first (numerically) $2e$ elements of $W_e^K$. Let $s_0$ be the least number such that

• $x_1, \ldots, x_e \in W_{1,s_0}^{K,e}$
• $(\forall s \geq s_0)[x_1, \ldots, x_e \in W_{1,s}^{K,e}]$.
• $(\forall s \geq s_0)(\forall z \in [x_{2e}] - \{x_1, \ldots, x_{2e}\}[z \notin W_{1,s}^{K,e}]$.

**KEY:** for all $s \geq s_0$, during stage $s$, the requirements $R_1, \ldots, R_{e-1}$ may define $COL(x, s)$ for some of the $x \in \{x_1, \ldots, x_{2e}\}$. But they will NOT define $COL(x, s)$ for ALL of those $x$. Why? Because $R_i$ only defines $COL(x, s)$ for at most TWO of those $x$’s, and there are $e-1$ such $i$, so at most $2e-2$ of those $x$’s have $COL(x, s)$ defined. Hence there will exist $x, y$ such that $R_e$ gets to define $COL(x, s)$ and $COL(y, s)$. Furthermore, they will always be the SAME $x, y$ since the $R_i$ with $i < e$ have already made up their minds about the $x$ in $\{x_1, \ldots, x_{2e}\}$.

**UPSHOT:** There exists $x, y \in W_e^K$ such that, for all $s \geq s_0$,

$COL(x, s) = RED$
$COL(y, s) = BLUE$
Since $W^K_e$ is infinite there is SOME $s \geq s_0$ with $s \in W^K_e$. Hence $x, y, s \in W^K_e$ and show that $W_e$ is NOT homogenous.