Infinite Canonical Ramsey's Theorem
Exposition by William Gasarch

1 Introduction

We recall Ramsey's theorem.

Convention 1.1 In this paper (1) a *coloring of a graph* is a coloring of the edges of the graph. and (2) a *coloring of a hypergraph* is a coloring of the edges of the hypergraph.

Def 1.2 Let $c \in \mathsf{N}$. Let $COL : \binom{\mathsf{N}}{2} \rightarrow [c]$. Let $V \subseteq \mathsf{N}$. The set V is *homog* if there exists a color c such that every elements of $\binom{V}{2}$ is colored c.

Convention 1.3 We write $COL(x_1, \ldots, x_a)$ rather than the more formally correct $COL(\{x_1, \ldots, x_1\})$. We do not mean to imply that $x_1 < \cdots < x_a$.

The following is Ramsey's theorem for graphs:

Theorem 1.4 *For all $c \in \mathsf{N}$, for all $COL : \binom{\mathsf{N}}{2} \rightarrow [c]$, there is an infinite homog set.*

Note that Ramsey's Theorem uses only a fixed number of colors. What if we color $\binom{\mathsf{N}}{2}$ with as many colors as we like? You may say *that's just stupid—color each edge a different color.* That is true— however, your coloring has an infinite *rainbow set* — every edge is different! This leads to the following conjecture:

Def 1.5 Let $COL : \binom{\mathsf{N}}{2} \rightarrow \omega$. Let $V \subseteq \mathsf{N}$. The set V is *rainbow* if every edge in $\binom{V}{2}$ is colored differently.

Conjecture: For every coloring of $\binom{\mathsf{N}}{2}$ there is either an infinite homog set or an infinite rainbow set.

This looks good. But alas its not true. Consider the following colorings:

$$COL(i, j) = i.$$

$$COL(i,j) = j.$$

We leave it to the reader to show that neither of these colorings has an infinite homog set, nor an infinite rainbow set. However both lead to a certain kind of homogeneity. We now define 4 types of homogeneity. Even rainbow sets can be viewed as homog in this definition.

Def 1.6 Let $COL : \binom{\mathsf{N}}{2} \rightarrow \omega$. Let $V \subseteq \mathsf{N}$.

1. V is $\emptyset$-*homog* (henceforth *homog*) if for all $x_1 < x_2 \in V$, and $y_1 < y_2 \in V$,
$$COL(x_1, x_2) = COL(y_1, y_2) \text{ iff } TRUE.$$
(This is just normal homog.)

2. V is $\{1\}$-*homog* (henceforth *min-homog*) if for all $x_1 < x_2 \in V$, and $y_1 < y_2 \in V$,
$$COL(x_1, x_2) = COL(y_1, y_2) \text{ iff } x_1 = y_1.$$

3. V is $\{2\}$-*homog* (henceforth *max-homog*) if for all $x_1 < x_2 \in V$, and $y_1 < y_2 \in V$,
$$COL(x_1, x_2) = COL(y_1, y_2) \text{ iff } x_1 = y_1.$$

4. V is $\{1,2\}$-*homog* (henceforth *rainbow*) if for all $x_1 < x_2 \in V$ and $y_1 < y_2 \in V$
$$COL(x_1, x_2) = COL(y_1, y_2) \text{ iff } x_1 = y_2 \text{ AND } x_2 = y_2.$$

We now state the Canonical Ramsey Theorem for finite graphs [**?**].

Theorem 1.7 *For all $COL : \binom{\mathsf{N}}{2} \rightarrow \omega$ there is either an infinite homog set, an infinite min-homog set, an infinite max-homog set, or an infinite rainbow set.*

We will give a proof of Theorem 1.7. We will need the hypergraph Ramsey Theorem [**?**, **?**, **?**] for the proof.

Def 1.8 Let COL be a coloring of $\binom{\mathsf{N}}{a}$ (the edges of the infinite complete a-hypergraph). Let $V \subseteq \mathsf{N}$. The set V is *homog* if there exists a color c such that every edge in $\binom{V}{a}$ is colored c.

Theorem 1.9 *For all a, for all c, for all $COL : \binom{\mathsf{N}}{a} \rightarrow [c]$ there exists an infinite homog set.*

In Section 2 we prove lemmas that we will need. In Section 3 we give a proof of the canonical Ramsey Theorem for graphs. In Section 4 we give an "application." In Section 6 we give a proof of the 3-ary canonical Ramsey Theorem. In Section 7 we give an "application."

2 Needed Lemmas

2.1 One Dim Infinite Can Ramsey Theorem

We need the following lemma which could be called the 1-dimensional Canonical Ramsey Theorem. We leave the proof to the reader.

Def 2.1 Let $V \subseteq \mathsf{N}$ be infinite. If $COL : V \rightarrow \omega$ then (1) a *homog subset of V relative to COL* is a set that is all the same color, and (2) a *rainbow subset of V relative to COL* is a set where every element has a different color.

Lemma 2.2 *Let V be an countable set. Let $COL : V \rightarrow \omega$. Then there exists either an infinite homog set or an infinite rainbow set.*

2.2 A Premise that Yields a Rainbow Set

The next definition and lemma gives a way to get an infinite rainbow set under some conditions.

Def 2.3 Let $COL : \binom{\mathsf{N}}{2} \rightarrow \omega$. If c is a color and $v \in \mathsf{N}$ then $\deg_c(v)$ is the number of c-colored edges with an endpoint in v.

The following theorem is an infinite version of a theorem of Babai [?].

Lemma 2.4 *Let X be infinite. Let $COL : \binom{X}{2} \rightarrow \omega$. If for $x \in X$ and $c \in \omega$, $\deg_c(x) \leq 1$ then there exists an infinite rainbow set.*

Proof:

Let R be a maximal rainbow set of X. This means that that R is rainbow and

$$(\forall y \in X - R)[R \cup \{y\} \text{ is not a rainbow set}].$$

Let $y \in X - R$. Why is $y \notin R$? One of the following must occur:

1. There exists $u \in R$ and $\{a, b\} \in \binom{R}{2}$ such that $COL(u, y) = COL(a, b)$.
2. There exists $\{a, b\} \in \binom{R}{2}$ such that $COL(a, y) = COL(b, y)$. This cannot happen since then $[\exists c)[\deg_c(y) \geq 2]$.

We map $X - R$ to $R \times \binom{R}{2}$ by mapping $y \in X - R$ to $(u, \{a, b\})$ as indicated in item 1 above. This map is injective since if y_1 and y_2 both map to $(u, \{a, b\})$ then $COL(u, y_1) = COL(u, y_2$ which can't happen since $\deg_c(u) \leq 1$.

The mapping is an injection from $X - R$ to $R \times \binom{R}{2}$. If R was finite then this would be an injection from an infinite set to a finite set which is impossible. Hence R is infinite. ∎

3 Proof of Can Ramsey Theorem for Graphs

Theorem 3.1 *For all $COL : \binom{\mathsf{N}}{2} \rightarrow \omega$ there is either an infinite homog set, an infinite min-homog set, an infinite max-homog set, or an infinite rainbow set.*

Proof:

We are given $COL : \binom{\mathsf{N}}{3} \rightarrow \omega$. We use COL to obtain a $COL' : \binom{\mathsf{N}}{4} \rightarrow [7]$. We will use the (ordinary) 3-ary Ramsey theorem.

We define $COL'(x_1 < x_2 < x_3 < x_4)$ by looking at COL on all $\binom{4}{3}$ triples of $\{x_1, x_2, x_3, x_4\}$ and see how their colors compare to each other.

For each case we assume the negation of all the prior cases. In each case, we indicate what happens if this is the color of the infinite homog set.

In all the cases below we use the following notation: if we are referring to a set X and $x \in X$ then x^+ is the next element of X after x.

1. If $COL(x_1, x_2) = COL(x_1, x_3)$ then $COL'(x_1, x_2, x_3, x_4) = 1$. Assume X is an infinite homog set of color 1. Let $COL'' : X \to \omega$ be defined by $COL''(x) = COL(x, x^+)$. Note that, for all $y > x$, $COL''(x) = COL(x, y)$. Apply Lemma 2.2 to COL'' to obtain either an infinite homog (relative to COL'') subset of X which is a homog set relative to COL, or an infinite rainbow subset of X (relative to COL'') which is a min-homog set relative to COL.

2. If $COL(x_1, x_3) = COL(x_2, x_3)$ then $COL'(x_1, x_2, x_3) = 2$. Assume X is an infinite homog set of color 2. By reasoning similar to Case 1, there is either an infinite homog (rel to COL) subset of X, or an infinite max-homog (rel to COL) subset of X.

3. If $COL(x_1, x_2) = COL(x_2, x_3)$ then $COL'(x_1, x_2, x_3) = 3$. Assume X is an infinite homog set of color 3. Note that $COL(x_1, x_2) = COL(x_2, x_3) = \cdots$. We call this color RED. For all $x < y$, $COL(x, y) = COL(y, y^+) = RED$, so X is homog.

4. If none of the above occur then $COL'(x_1, x_2, x_3) = 4$. Assume X is an infinite homog set of color 4. Restrict COL to X. The reader can easily show that, for all $x \in X$, for all colors c, $\deg_c(x) \le 1$. Hence, by Lemma 2.4, there is an infinite rainbow set.

▮

4 An Application to a Points-in-the-Plane Problem

Theorem 3.1 was about colorings whose co-domain was ω. We never used any property of ω so the theorems hold with any co-domain. We use this fact freely.

We leave the following lemmas to the reader.

Lemma 4.1 *Let P be a countable set of points in R^2. Let $COL : \binom{P}{2} \to \mathsf{R}^+$ be defined by $COL(x, y) = |x - y|$ (the distance between x and y). Then*

1. *There is no infinite homog set.*

2. *There is no infinite min-homog set.*

3. *There is no infinite max-homog set.*

Theorem 4.2 *Let P be a countable set of points in R^2. There exists a countable subset X of P such that all pairs of points in X have different distances.*

Proof: Let $COL : \binom{P}{2} \rightarrow \mathsf{R}^+$ be defined by $COL(x,y) = |x-y|$ (the distance between x and y). By Theorem 3.1 and Lemma 4.1 there is an infinite rainbow set X. This is the desired X. ∎

Note 4.3

1. Lemma 4.1 and Theorem 4.2 both hold if you replace R^2 with R^d for any natural $d \geq 1$.
2. The problem has not been studied for other metrics on R^d and for other metric spaces. An analog of Lemma 4.1 is all you need to obtain theorems.

5 Infinite Can Ramsey for 3-Hypergraphs

Look at the following 8 colorings of $\binom{\mathsf{N}}{3}$:

- $COL(i,j,k) = RED$.
- $COL(i,j,k) = 2^i$.
- $COL(i,j,k) = 2^j$.
- $COL(i,j,k) = 2^k$.
- $COL(i,j,k) = 2^i 3^j$.
- $COL(i,j,k) = 2^i 5^k$.
- $COL(i,j,k) = 3^j 5^k$.
- $COL(i,j,k) = 2^i 3^j 5^k$.

Each of these leads to a diff kind of homog. We define them and show that you MUST get an infinite subset that has one of these types of homog.

Def 5.1 Let $COL : \binom{\mathsf{N}}{3} \to \omega$. Let $V \subseteq \mathsf{N}$ and $I \subseteq \{1,2,3\}$. V is *I-homog* if, for all $x_1 < x_2 < x_3$, $y_1 < y_2 < y_3 \in V$

$$COL(x_1, x_2, x_3) = COL(y_1, y_2, y_3) \text{ iff } (\forall i \in I)[x_i = y_i].$$

Examples:

1. If $I = \emptyset$ then all of the triples are the same color, usually called homog.
2. If $I = \{1\}$ then the coloring depends exactly on the first coordinate. That is, two triples are equal iff they agree on their min element.
3. If $I = \{1,3\}$ then the coloring depends on the first AND last coordinate. So $(89, 100, 1000)$ and $(89, 103, 1000)$ would be the same color. More than that— if two triples are the same color then they HAVE TO agree on both the first and last coordinate.
4. If $I = \{1,2,3\}$ then this is just a rainbow set.
5. One could also have defined I-homog for colorings of pairs. If we had done that then homog would be $\emptyset$-homog, min-homog would be 1-homog, max-homog would be 2-homog, rainbow would be $\{1,2\}$-homog,

Notation 5.2 Formally we should use things like $\{1,3\}$-homog. We will instead use things like $(1,3)$-homog.

5.1 A Premise that Yields a Rainbow Set

The next definition and lemma gives a way to get an infinite rainbow set under some conditions.

Def 5.3 Let $COL : \binom{X}{3} \to \omega$. Let c be a color and let $x_1, x_2 \in X$.

1. $\deg_c(x_1)$ is the number of $\{y, z\} \in \binom{X}{2}$ such that $COL(x_1, y, z) = c$.
2. $\deg_c^<(x_1)$ is the number of $\{y, z\} \in \binom{X}{2}$ such that $y, z < x_1$ and $COL(x_1, y, z) = c$.
3. $\deg_c(x_1, x_2)$ is the number of $z \in \binom{X}{1}$ such that $COL(x_1, x_2, z) = c$.

We show two ways to go from $(\forall c)(\forall x,y)[\deg_c(x,y) \le 1]$ leads to a rainbow set.

A Maximal Set Arg- The Ulrich Hypergraph Ramsey Approach

Lemma 5.4 *Let X be infinite. Let $COL : \binom{X}{3} \rightarrow \omega$. Assume both of the following hold:*

- *For all $x \in X$ and $c \in \omega$ $\deg_c^{<}(x) \le 1$.*
- *For all $\{x,y\} \in \binom{X}{2}$ and $c \in \omega$ $\deg_c(x,y) \le 1$.*

Then there exists an infinite rainbow set.

Proof:

Let R be a maximal rainbow subset of X. This means that R is rainbow and

$$(\forall y \in X - R)[R \cup \{y\} \text{ is not a rainbow set}].$$

Assume R is finite. Throw out of X all of the elements of X that are less than any element or R. Rename the new set X.

Let $y \in X - R$. Why is $y \notin R$? One of the following must occur:

1. There exists $\{u_1, u_2\} \in \binom{R}{2}$ and $\{a,b,c\} \in \binom{R}{3}$ such that $COL(y, u_1, u_2) = COL(a,b,c)$.
2. There exists $\{a_1, b_2\}, \{a_2, b_2\} \in \binom{R}{2}$ such that $COL(y, a_1, a_2) = COL(y, b_1, b_2)$. This cannot happen:

   (1) If $\{a_1, a_2\} \cap \{b_1, b_2\} = \emptyset$ then we get $(\exists c)[\deg_c^{<}(y) \ge 2]$ which violates the premise.

   (2) If (say) $a_1 = b_1$ then $(\exists c)[\deg_c(y, a_1) \ge 2]$, which violates the premise.

We map $X-R$ to $\binom{R}{2} \times \binom{R}{3}$ by mapping $y \in X-R$ to $(\{u_1,u_2\}, \{a,b,c\})$ as indicated in item 1 above. This map is injective since if y_1 and y_2 both map to $(\{u_1,u_2\}, \{a,b,c\})$ then $COL(y_1, u_1, u_2) = COL(y_2, u_1, u_2)$ which can't happen since $(\forall c)[\deg_c(u_1, u_2) \le 1]$.

This is an injection from $X - R$ to $\binom{R}{2} \times \binom{R}{3}$. Since R is finite we have an injection from an infinite set to a finite set which is impossible. Hence R is infinite. ▮

Lemma 5.4 *seems* useful. However, when we do a proof similar to that of Theorem 3.1 we will have as the last case that there is an infinite set X such that $(\forall c)(\forall x, y \in X)[\deg_c(x, y) \leq 1]$. To apply Lemma 5.4 we do indeed need that condition; however, we also need $(\forall c)(\forall x \in X[\deg_c^<(x) \leq 1]$.

Lemma 5.5 *Let X be an infinite set and $COL : \binom{X}{3} \to \omega$. Assume that $(\forall c)(\forall x, y \in X)[\deg_c(x, y) \leq 1]$. Then there exists and infinite set $X' \subseteq X$ such that $(\forall c)(\forall x \in X')[\deg_c^<(x) \leq 1]$.*

Proof: Let $COL' : \binom{X}{5} \to [4]$ be defined by
$COL'(x_1 < x_2 < x_3 < x_4 < x_5) =$

- 1 if $COL(x_1, x_2, x_5) = COL(x_3, x_4, x_5)$.
- 2 if $COL(x_1, x_3, x_5) = COL(x_2, x_4, x_5)$.
- 3 if $COL(x_1, x_4, x_5) = COL(x_2, x_3, x_5)$.
- 4 otherwise.

Apply the 5-ary Ramsey Theorem to obtain an infinite homog set. We claim that this set cannot be color 1,2, or 3.

Assume, by way of contradiction, that the infinite homog set is of color 1. Let the homog set be

$$H = \{x_1 < x_2 < x_3 < \cdots\}.$$

$COL(x_1, x_2, x_6) = COL(x_4, x_5, x_6)$.
$COL(x_2, x_3, x_6) = COL(x_4, x_5, x_6)$.
so $COL(x_1, x_2, x_6) = COL(x_2, x_3, x_6)$. Hence $(\exists c)0\deg_c(x_1, x_6) \geq 2]$. This is a contradiction.

Similar proofs hold for colors 2 and 3.

So we now have that an infinite homog set of color 4. It is easy to set that, $(\forall c)(\forall x)[\deg_c^<(x) \leq 1]$. ∎

Lemma 5.6 *Let X be an infinite set and $COL : \binom{X}{3} \to \omega$. Assume that $(\forall c)(\forall x, y \in X)[\deg_c(x, y) \leq 1]$. Then there exists an infinite rainbow set.*

Proof: By Lemma 5.5 there is an infinite $X' \subseteq X$ such that $(\forall c)(\forall x \in X')[\deg_c^<(x) \le 1]$. By Lemma 5.4 there is an infinite rainbow subset of X'. ∎

The Zbarsky Non-Naive Approach

We now use the assumption $(\forall c)(\forall x, y)[\deg_c(x, y) \le 1]$ to obtain a rainbow set in a completely different way.

Def 5.7 Let $W, X, V \subseteq \mathsf{N}$. Let $COL : \binom{X}{3} \to \omega$. $W < X$ means $(\forall w \in W)(\forall x \in X)[w < x]$.

Thought experiment: Let X be an infinite subset. Let $COL : \binom{X}{3} \to \omega$ be such that $(\forall c)(\forall x, y \in X)[\deg_c(x, y) \le 1]$. We want to construct a infinite rainbow set $W \subseteq X$. Say that so far we have a finite nice set $W' \subseteq X$. We want to find $x \in X - W'$ such that $W' \cup \{x\}$ is rainbow. Our fear: what if for every $x \in X - W$, there exists $a_1, b_1, a_2, b_2 \in W$ such that $COL(a_1, b_1, x) = COL(a_2, b_2, x)$. We now give some definitions and a lemma which will tell how to avoid this. The intuition: We build W up carefully so that when we put an element into it we are not just concerned with *is* $W \cup \{x\}$ *rainbow ?* but also with *are there an infinite number of* x' *such that* $W \cup \{x, x'\}$ *is rainbow?*

Def 5.8 Let $W, X \subseteq \mathsf{N}$ and $COL : \binom{W \cup X}{3} \to \omega$. Assume $W < X$ (so W is finite). Let $x \in X$.

1. x is W-*stupid* if

$$(\exists a_1, b_1, c_1, a_2, b_2 \in W)[COL(a_1, b_1, c_1) = COL(a_2, b_2, x)].$$

Note that if W is finite there can only be a finite number of W-stupid numbers.

2. x is W-*naively bad* (henceforth n.b.) if

$$(\exists a_1, b_1, a_2, b_2 \in W)[COL(a_1, b_1, x) = COL(a_2, b_2, x)].$$

(Note that if $(\forall c)(\forall x, y \in X)[\deg_c(x, y) \le 1]$, which will be our case, then all of the a_1, b_1, a_2, b_2 are distinct.)

3. x is (W, X)-*sneaky bad* if

$$(\forall^\infty y \in X)[y \text{ is } (W \cup \{x\})\text{-n.b.}].$$

4. W is X-nice if both of the following hold.
   - W is rainbow.
   - There are no W-n.b. numbers in X.

Lemma 5.9 *Let $W, X \subseteq \mathsf{N}$. Let $COL : \binom{W \cup X}{3} \rightarrow \omega$ be such that*

- $\forall x, y \in W \cup X)(\forall c)[\deg_c(x, y) \leq 1]$.
- *$W < X$ (so W is finite). Assume W is X-nice.*

Then there exists $x \in X$ and infinite $X' \subseteq X$ such that $W \cup \{x\}$ is X'-nice.

Proof: We seek an element x and a subset X'. We will do a construction to find x, X' and later prove that it works. The construction will look infinite, but must eventually stop with a proper x, X'.

Let $x \in X$. Since y is not W-n.b. but y is $(W \cup \{x\})$-n.b. there must be $a, b, a' \in W$ such that $COL(a, b, y) = COL(a', x, y)$. Since $(\forall c)(\forall z_1, z_2)[\deg_c(z_1, z_1) \leq 1]$ we know that a, b, a', x are all distinct. Let $f_x(y) = (a, b, a')$ where $a < b$. Since W is finite there must be some (a, b, a') such that an infinite number of y's map to it.

We do the following construction. We carry it out for an infinite number of steps but later will only use a finite number of those steps.

CONSTRUCTION

Stage 0: Let X_0 be X with the stupid elements removed.

Stage $s + 1$: We can assume that X_s is defined. Let x_s is the least element of X_s. Since W is nice we know that $W \cup \{x_s\}$ is rainbow. We just ask

Is $W \cup \{x_1\}$ X_s-nice? If so then GREAT, we are doe If not then we have

$$(\forall^\infty y \in X_s)[y \text{ is } W \cup \{x_s\}\text{-naively bad}].$$

First remove from X_s the finite number of y's for which this is true. Rename X_s with just X_s. We have

$$(\forall y \in X_s)[y \text{ is } W \cup \{x_s\}\text{-naively bad}].$$

Let $y \in X_s$. Since y is $W \cup \{x_s\}$-naively bad

$$(\exists a, b, a' \in W)[COL(a, b, x_s) = COL(a', x_s, y)].$$

(We will later call these (a_s, b_s, a'_s).)

Since $(\forall c)(\forall z_1, z_2)[\deg_c(z_1, z_1) \le 1]$ we know that a, b, a', x are all distinct. Let $f_{x_s}(y) = (a, b, a')$ where $a < b$. Since W is finite there must be some (a, b, a') such that an infinite number of y's map to it.

Let

$$X_{s+1} = \{y \mid f_{x_s}(y) = (a, b, a')\}.$$

Note that X_{s+1} has no W-stupid elements since X_s didn't.

END OF CONSTRUCTION

The number of possible $(a_s, b_s, a'_s) \in W \times W \times W$ is finite since W is finite. Hence there must be an $s < t$ such that $(a_s, b_s, a'_s) = (a_t, b_t, a'_t) = (a, b, a')$. Let $y \in X_t \supseteq X_s$.

Since $y \in X_s$, $f_{x_s}(y) = (a_s, b_s, a'_s) = (a, b, a')$. Hence

$$COL(a, b, y) = COL(a', x_s, y)$$

Since $y \in X_t$, $f_{x_t}(y) = (a_t, b_t, a'_t) = (a, b, a')$. Hence

$$COL(a, b, y) = COL(a', x_t, y)$$

Hence

$$COL(a', x_s, y) = COL(a', x_t, y)$$

so $\deg_c(a', y) \ge 2$. This violates the premise of the lemma. Contradiction.

So we now know that

$$(\exists x \in X)(\exists^\infty y \in X)[y \text{ is not } (W \cup \{x\})\text{-n.b.}]$$

Let x be that x. Let X' be the infinite set of those y's.

■

Lemma 5.10 *Let X be infinite. Let $COL : \binom{X}{3} \to \omega$. Assume that for $x, y \in X$ and $c \in \omega$ $\deg_c(x, y) \le 1$. Then there is an infinite rainbow $W \subseteq X$.*

Proof: We construct a set W in stages. At every stage W is rainbow so the final W is rainbow.

CONSTRUCTION

$W_0 = \emptyset$. $X_0 = X$.

Stage s: Assume inductively that W_{s-1} and X_{s-1}) exist, X_{s-1} is infinite, W_{s-1} is X_{s-1}-nice, and X_s has no W_s-stupid elements. Then, by Lemma 5.9, there exists $x \in X_{s-1}$ and $X' \subseteq X_{s-1}$, X' infinite, such that $W_{s-1} \cup \{x\}$ is X'-nice. Let

$$W_s = W_{s-1} \cup \{x\}$$

$$X_s = X' - \text{ the stupid } W_s\text{-elements} .$$

END OF CONSTRUCTION

Let $W = \bigcup_{s=1}^{\infty} W_s$. ∎

6 Proof of Can Ramsey Theorem for 3-Hypergraphs

Theorem 6.1 *For all $COL : \binom{\mathsf{N}}{3} \to \omega$ there is an $I \subseteq [3]$ and an infinite I-homog set.*

Proof:

We are given $COL : \binom{\mathsf{N}}{3} \to \omega$. We use COL to obtain $COL' : \binom{\mathsf{N}}{4} \to [8]$. We will use the 4-ary Ramsey theorem.

We define $COL'(x_1 < x_2 < x_3 < x_4)$ by looking at COL on all $\binom{4}{3}$ triples of $\{x_1, x_2, x_3, x_4\}$ and see how their colors compare to each other.

For each case we assume the negation of all the prior cases. In each case, we indicate what happens if this is the color of the infinite homog set.

In all the cases below we use the following notation: if we are referring to a set X and $x \in X$ then x^+ is the next element of X after x.

1. If $COL(x_1, x_2, x_3) = COL(x_1, x_2, x_4)$ then $COL'(x_1, x_2, x_3, x_4) = 1$. Assume X is an infinite homog set with color 1. Note that for all x, y, z, z' we have $COL(x, y, z) = COL(x, y, z')$. Let $COL'' : \binom{X}{2} \to \omega$ be defined by $COL''(x, y) = COL(x, y, y^+)$. Note that, for all $x < y < z$, $COL''(x, y) = COL(x, y, z)$. We use this fact freely. Note that COL'' is a coloring of pairs- WE CAN APPLY CAN RAMSEY ON GRAPHS TO IT! One of the following must occur:
   (a) There is an infinite homog set H (relative to COL''). It is easy to see that H is homog for COL.

(b) There is an infinite min-homog set H (relative to COL''). We will show that H is 1-homog relative to COL. Let $x, y, z, x', y', z' \in H$.

$$COL(x,y,z) = COL(x',y',z') \text{ iff } COL''(x,y) = COL''(x',y')$$

This is by the definition of COL'' which ultimately goes back to X being 1-homog for COL.

$$COL''(x,y) = COL''(x',y') \text{ iff } x = x'$$

This is because H is 1-homog for COL''.

(c) There is an infinite max-homog set H (relative to COL''). H is 2-homog by a proof similar to that of case b.

(d) There is an infinite rainbow set H (relative to COL''). We show that H is (1,2)-homog rel to COL.
Let $x, y, z, x', y', z' \in H$.

$$COL(x,y,z) = COL(x',y',z') \text{ iff } COL''(x,y) = COL''(x',y')$$

This is by the definition of COL'' which ultimately goes back to X being homog for COL with color 1.

$$COL''(x,y) = COL''(x',y') \text{ iff } x = x' \text{ and } y = y'$$

This is because H is rainbow for COL''.

2. If $COL(x_1,x_2,x_3) = COL(x_1,x_3,x_4)$ then $COL'(x_1,x_2,x_3,x_4) = 2$. Assume X is an infinite homog set with color 2. We first show that if two elements of $\binom{X}{3}$ have the same first element then they are the same color. (This does not show that X is 1-homog since the converse need not hold. For example, X could be $\emptyset$-homog.) Assume (1) $x < y_1 < z_1$, (2) $x < y_2 < z_2$, and (3) $x, y_1, y_2, z_1, z_2 \in X$. We need $COL(x,y_1,z_1) = COL(x,y_2,z_2)$. Let $w > \max\{z_1, z_2\}$.

$$COL(x,y_1,z_1) = COL(x,z_1,z_1^+) = \cdots = COL(x,w,w^+)$$

$$COL(x, y_2, z_2) = COL(x, z_2, z_2^+) = \cdots = COL(x, w, w^+)$$

We define another auxiliary coloring $COL'' : X \to \omega$. Let $COL''(x) = COL(x, x^+, x^{++})$. Note that, for all $x < y < z$, $COL(x, y, z) = COL''(x)$. Apply Lemma 2.2 to COL'' to obtain either (1) an infinite homog (relative to COL'') which is $\emptyset$-homog rel to COL, or (2) an infinite rainbow (relative to COL'') which is a 1-homog set relative to COL.

3. If $COL(x_1, x_2, x_3) = COL(x_2, x_3, x_4)$ then $COL'(x_1, x_2, x_3, x_4) = 3$. Assume X is an infinite homog set of with color 3. We show that X is $\emptyset$-homog (all triples are the same color). Note that all triples of the form $(x, x^+, x^{++}\}$ have the same COL. Denote that color RED. Assume (1) $x < y < z$, (2) $x' < y' < z'$, and (3) $x, y, z, x', y', z' \in X$. We need $COL(x, y, z) = COL(x', y', z')$. Note that

$$COL(x, y, z) = COL(y, z, z^+) = COL(z, z^+, z^{++}) = RED.$$

By the same reasoning $COL(x', y', z') = RED$.

4. If $COL(x_1, x_2, x_4) = COL(x_1, x_3, x_4)$ then $COL'(x_1, x_2, x_3, x_4) = 4$. Assume X is an infinite homog set with color 4. First we omit every other element of X to obtain X'. We do this so that every two elements of X' have an element inbetween them in X. We let x^+ be the next element in X (not X'). Let $COL'' : \binom{X'}{2} \to \omega$ be defined by $COL'(x, z) = COL(x, x^+, z)$. Note that, for all $x, y, z \in X'$, $COL', (x, z) = COL(x, y, z)$. Note that COL' is a coloring of pairs- WE CAN APPLY CAN RAMSEY ON GRAPHS TO IT! The rest of the proof is similar to the proof of Case 1 so we just say what kind of homog set is obtained. One of the following must occur:

   (a) There is an infinite homog set H (relative to COL''). H is homog rel to COL.

   (b) There is an infinite min-homog set (relative to COL''). H is a 1-homog set rel to COL.

   (c) There is an infinite max-homog set (relative to COL''). H is a 3-homog set rel to COL.

(d) There is an infinite rainbow set (relative to COL''). H is a (1,3)-homog set rel to COL.

5. If $COL(x_1, x_2, x_4) = COL(x_2, x_3, x_4)$ then $COL'(x_1, x_2, x_3, x_4) = 5$. This is similar to the Case 2. We either get an infinite $\emptyset$-homog set or an infinite 2-homog set.

6. If $COL(x_1, x_3, x_4) = COL(x_2, x_3, x_4)$ then $COL'(x_1, x_2, x_3, x_4) = 6$. This is similar to Case 1. We either obtain a $\emptyset$-homog set or a 2-homog set or a 3-homog set or a $(1, 3)$-homog set.

7. If none of the above occur then $COL'(x_1, x_2, x_3, x_4) = 7$. Assume X is an infinite homog set of this color. Note that, for all colors c, for all $x, y \in X$, $\deg_c(x, y) \leq 1$. By Lemma 5.6 or 5.10 there is an infinite rainbow subset of X.

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7 Another Application to a Points-in-the-Plane Problem