Ramsey's Theorem and the Canonical Ramsey's Theorem for the Infinite Complete Graph and the Infinite Compleete Hypergraph

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1 Introduction

Notation 1.1 K_{N} is the graph (V,E) where

$$\begin{aligned} V &= \mathsf{N} \\ E &= \binom{\mathsf{N}}{2} \end{aligned}$$

Notation 1.2 K_{N}^a is the hypergraph (V,E) where

$$\begin{aligned} V &= \mathsf{N} \\ E &= \binom{\mathsf{N}}{a} \end{aligned}$$

Convention 1.3 In this paper (1) a *coloring of a graph* is a coloring of the edges of the graph. and (2) a *coloring of a hypergraph* is a coloring of the edges of the hypergraph.

Def 1.4 Let $c \in \mathsf{N}$. Let COL be a c-coloring of the edges of K_{N}. Let $V' \subseteq V$. The set V' is *homogenous* if there exists a color c such that EVERY edge between vertices in V' is colored c.

Def 1.5 Let $c \in \mathsf{N}$. Let COL be a c-coloring of the edges of K_{N}^a. Let $V' \subseteq V$. The set V' is *homogenous* if there exists a color c such that EVERY edge that uses only vertices of V' is colored c.

2 Ramsey's Theorem for the Infinite Complete Graphs

The following is Ramsey's Theorem for K_{N}.

Theorem 2.1 *For every 2-coloring of the edges of* K_{N} *there is an infinite homogenous set.*

Proof:

Let COL be a 2-coloring of K_{N}. We define an infinite sequence of vertices,

$$x_1, x_2, \dots,$$

and an infinite sequence of sets of vertices,

$$V_0, V_1, V_2, \dots,$$

that are based on COL.

Here is the intuition: Vertex $x_1 = 1$ has an infinite number of edges coming out of it. Some are RED, and some are BLUE. Hence there are an infinite number of RED edges coming out of x_1, or there are an infinite number of BLUE edges coming out of x_1 (or both). Let c_1 be a color such that x_1 has an infinite number of edges coming out of it that are colored c_1. Let V_1 be the set of vertices v such that $COL(v, x_1) = c_1$. Then keep iterating this process.

We now describe it formally.

$$\begin{aligned} V_0 &= \mathsf{N} \\ x_1 &= 1 \\ \\ c_1 &= \text{RED if } |\{v \in V_0 \mid COL(v, x_1) = \text{RED}\}| \text{ is infinite} \\ &= \text{BLUE otherwise} \\ V_1 &= \{v \in V_0 \mid COL(v, x_1) = c_1\} \text{ (note that } |V_1| \text{ is infinite)} \end{aligned}$$

Let $i \geq 2$, and assume that V_{i-1} is defined. We define x_i, c_i, and V_i:

$$\begin{aligned} x_i &= \text{the least number in } V_{i-1} \\ \\ c_i &= \text{RED if } |\{v \in V_{i-1} \mid COL(v, x_i) = \text{RED}\}| \text{ is infinite} \\ &= \text{BLUE otherwise} \\ V_i &= \{v \in V_{i-1} \mid COL(v, x_i) = c_i\} \text{ (note that } |V_i| \text{ is infinite)} \end{aligned}$$

How long can this sequence go on for? Well, x_i can be defined if V_{i-1} is nonempty. We an show by induction that, for every i, V_i is infinite. Hence the sequence

$$x_1, x_2, \ldots,$$

is infinite.

Consider the infinite sequence

$$c_1, c_2, \ldots$$

Each of the colors in this sequence is either RED or BLUE. Hence there must be an infinite sequence $i_1, i_2, \ldots$ such that $i_1 < i_2 < \cdots$ and

$$c_{i_1} = c_{i_2} = \cdots$$

Denote this color by c, and consider the vertices

$$H = \{x_{i_1}, x_{i_2}, \cdots\}$$

We leave it to the reader to show that H is homogenous. ∎

Exercise 1 Show that, for all $c \geq 3$, for every c-coloring of the edges of K_{N}, thre is a an infinite homogenous set.