The Infinite Ramsey Theorem (An Exposition)

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Notation

1. $K_N$ is the graph $(V, E)$ where

   \[ V = N \]
   \[ E = \binom{N}{2} \]

2. $K^a_N$ is the hypergraph $(V, E)$ where

   \[ V = N \]
   \[ E = \binom{N}{a} \]

3. A \textit{coloring of a graph} is a coloring of the edges of the graph. A \textit{coloring of a hypergraph} is a coloring of the edges of the hypergraph.
Definition

1. Let $c \in \mathbb{N}$. Let $COL$ be a $c$-coloring of the edges of $K_N$. Let $V' \subseteq V$. The set $V'$ is *homogenous* if there exists a color $c$ such that EVERY edge between vertices in $V'$ is colored $c$.

2. Let $c \in \mathbb{N}$. Let $COL$ be a $c$-coloring of the edges of $K_N^a$. Let $V' \subseteq V$. The set $V'$ is *homogenous* if there exists a color $c$ such that EVERY edge that uses only vertices of $V'$ is colored $c$. 

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**Theorem:** For every 2-coloring of the edges of $K_N$ there is an infinite homogenous set.

**Proof:** $COL$ is a 2-coloring of $K_N$. We define an infinite sequence of vertices,

$$x_1, x_2, \ldots,$$

and an infinite sequence of sets of vertices,

$$V_0, V_1, V_2, \ldots,$$

that are based on $COL$. See next slide
Ramsey’s Theorem For Graphs—CONSTRUCTION

\( V_0 = \mathbb{N} \)
\( x_1 = 1 \)

\( c_1 = \begin{cases} \text{RED} & \text{if } |\{v \in V_0 \mid \text{COL}(v, x_1) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases} \)
\( V_1 = \{v \in V_0 \mid \text{COL}(v, x_1) = c_1\} \) (note that \( |V_1| \) is infinite)

Let \( i \geq 2 \). Assume \( V_{i-1} \) is defined. We define \( x_i, c_i, \) and \( V_i \):

\( x_i = \) the least number in \( V_{i-1} \)

\( c_i = \begin{cases} \text{RED} & \text{if } |\{v \in V_{i-1} \mid \text{COL}(v, x_i) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases} \)
\( V_i = \{v \in V_{i-1} \mid \text{COL}(v, x_i) = c_i\} \) (note that \( |V_i| \) is infinite)
Ramsey’s Theorem- the sequence

\(c_1, c_2, \ldots\)

There is an infinite sequence \(i_1, i_2, \ldots\) such that \(i_1 < i_2 < \cdots\) and

\[c_{i_1} = c_{i_2} = \cdots = c\]

\[H = \{x_{i_1}, x_{i_2}, \cdots\}\]

We leave it to the reader to show that \(H\) is homogenous.

End of Proof