The Infinite Ramsey Theorem (An Exposition)

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Notation

Notation

1. K_{N} is the graph (V, E) where

$$V = \mathsf{N}$$
$$E = \binom{\mathsf{N}}{2}$$

2. K_{N}^{a} is the hypergraph (V, E) where

$$V = \mathsf{N}$$
$$E = \binom{\mathsf{N}}{a}$$

3. A *coloring of a graph* is a coloring of the edges of the graph. A *coloring of a hypergraph* is a coloring of the edges of the hypergraph.

Notation

Definition

1. Let $c \in \mathbb{N}$. Let COL be a c-coloring of the edges of $K_{\mathbb{N}}$. Let $V' \subseteq V$. The set V' is *homogenous* if there exists a color c such that EVERY edge between vertices in V' is colored c.
2. Let $c \in \mathbb{N}$. Let COL be a c-coloring of the edges of $K_{\mathbb{N}}^{a}$. Let $V' \subseteq V$. The set V' is *homogenous* if there exists a color c such that EVERY edge that uses only vertices of V' is colored c.

Ramsey's Theorem For Graphs

Theorem: For every 2-coloring of the edges of K_{N} there is an infinite homogenous set.
Proof: *COL* is a 2-coloring of K_{N}. We define an infinite sequence of vertices,

$$x_1, x_2, \ldots,$$

and an infinite sequence of sets of vertices,

$$V_0, V_1, V_2, \ldots,$$

that are based on *COL*. See next slide

Ramsey's Theorem For Graphs-CONSTRUCTION

$$
\begin{aligned}
V_0 &= \mathrm{N} \\
x_1 &= 1
\end{aligned}
$$

$$
\begin{aligned}
c_1 &= \text{RED} \;\text{ if } |\{v \in V_0 \mid COL(v, x_1) = \text{RED}\}| \text{ is infinite} \\
&= \text{BLUE} \;\text{ otherwise} \\
V_1 &= \{v \in V_0 \mid COL(v, x_1) = c_1\} \text{ (note that } |V_1| \text{ is infinite)}
\end{aligned}
$$

Let $i \geq 2$. Assume V_{i-1} is defined. We define x_i, c_i, and V_i:

$$
x_i = \text{the least number in } V_{i-1}
$$

$$
\begin{aligned}
c_i &= \text{RED} \;\text{ if } |\{v \in V_{i-1} \mid COL(v, x_i) = \text{RED}\}| \text{ is infinite} \\
&= \text{BLUE} \;\text{ otherwise} \\
V_i &= \{v \in V_{i-1} \mid COL(v, x_i) = c_i\} \text{ (note that } |V_i| \text{ is infinite)}
\end{aligned}
$$

Ramsey's Theorem- the sequence

$$c_1, c_2, \ldots$$

There is an infinite sequence $i_1, i_2, \ldots$ such that $i_1 < i_2 < \cdots$ and

$$c_{i_1} = c_{i_2} = \cdots = c$$

$$H = \{x_{i_1}, x_{i_2}, \cdots\}$$

We leave it to the reader to show that H is homogenous.
End of Proof