Some Nice Problems: An Exposition by William Gasarch

A Nice Problem

Given n try to find a set $A \subseteq \{1, \ldots, n\}$ such that ALL of the differences of elements of A are DISTINCT.

Try to make A as big as possible.

Students Break into small groups and try to do this.
VOTE:

1. There is a all-diff-dist set of size roughly $n/3$.
2. There is a all-diff-dist set of size roughl $n^{1/4}$.
3. There is a sumfree set of size roughly $\log n$.

An Approach

Let a be a number to be determined.

Pick a RANDOM subset A $\{1, \ldots, n\}$ of size a.

What is the probability that all of the diffs in A are distinct?

We hope the prob is strictly greater than 0.

KEY: If the prob is strictly greater than 0 then there must be SOME set of a elements where all of the diffs are distinct.

Determining the Prob

If you pick a RANDOM $A \subseteq \{1, \ldots, n\}$ of size a what is the probability that all of the diffs in A are distinct?

Determining the Prob

If you pick a RANDOM $A \subseteq \{1, \ldots, n\}$ of size a what is the probability that all of the diffs in A are distinct?

WRONG QUESTION!

Determining the Prob

If you pick a RANDOM $A \subseteq \{1, \ldots, n\}$ of size a what is the probability that all of the diffs in A are distinct?

WRONG QUESTION!

If you pick a RANDOM $A \subseteq \{1, \ldots, n\}$ of size a what is the probability that all of the diffs in A are NOT distinct?

We hope that it is NOT 1.

Determining the Prob

If you pick a RANDOM $A \subseteq \{1, \ldots, n\}$ of size a what is the probability that all of the diffs in A are NOT distinct?

Determining the Prob

If you pick a RANDOM $A \subseteq \{1, \ldots, n\}$ of size a what is the probability that all of the diffs in A are NOT distinct?

WRONG QUESTION!

Determining the Prob

If you pick a RANDOM $A \subseteq \{1, \ldots, n\}$ of size a what is the probability that all of the diffs in A are NOT distinct?
WRONG QUESTION!

We only need to show that the prob is LESS THAN 1.

Review a Little Bit of Combinatorics

The number of ways to CHOOSE x elements out of y elements is

$$\binom{x}{y} = \frac{x!}{y!(x-y)!}.$$

Determining the Prob I

If a RAND $A \subseteq \{1, \ldots, n\}$, size a, want bound on prob all of the diffs in A are NOT distinct. Numb of ways to choose a elements out of $\{1, \ldots, n\}$ is $\binom{n}{a}$.
Two ways to create a set with a diff repeated:
Way One:

- Pick $x < y$. There are $\binom{n}{2} \le n^2$ ways to do that.
- Pick diff d such that $x + d \ne y$, $x + d \le n$, $y + d \le n$. Can do $\le n$ ways. Put $x, y, x + d, y + d$ into A.
- Pick $a - 4$ more elements out of the $n - 4$ left.

Number of ways to do this is $\le n^3 \times \binom{n-4}{a-4}$.
Way Two: Pick $x < y$. Let $d = y - x$ (so we do NOT pick d). Put $x, y = x + d, y + d$ into A. Pick $a - 3$ more elements out of the $n - 3$ left.
Number of ways to do this is $\le n^2 \times \binom{n-3}{a-3}$.

Determining the Prob II

If you pick a RANDOM $A \subseteq \{1, \ldots, n\}$ of size a then a bound on the probability that all of the diffs in A are NOT distinct is

$$\frac{n^3 \times \binom{n-4}{a-4} + n^2 \times \binom{n-3}{a-3}}{\binom{n}{a}} = \frac{n^3 \times \binom{n-4}{a-4}}{\binom{n}{a}} + \frac{n^2 \times \binom{n-3}{a-3}}{\binom{n}{a}}$$

$$= \frac{n^3 a(a-1)(a-2)(a-3)}{n(n-1)(n-2)(n-3)} + \frac{n^2 a(a-1)(a-2)}{n(n-1)(n-2)}$$

$$\leq \frac{32a^4}{n} \text{ Need some Elem Algebra and uses } n \geq 5.$$

ANSWER

RECAP: If pick a RANDOM $A \subseteq \{1, \ldots, m\}$ then the prob that there IS a repeated difference is $\le \frac{32a^4}{n}$.
So WANT

$$\frac{32a^4}{n} < 1$$

Take

$$a = \frac{n^{1/4}}{33}.$$

UPSHOT: For all $n \ge 5$ there exists a set of size $\frac{n^{1/4}}{33}$ of $\{1, \ldots, n\}$ where all of the differences are distinct.

GENERAL UPSHOT

We proved an object existed by showing that the Prob that it exists is NONZERO!
We use this technique in a more sophisticated way.

SUM SET PROBLEM

Definition: A set of numbers A is *sum free* if there is NO $x, y, z \in A$ such that $x + y = z$.

Example: Let $y_1, \ldots, y_m \in (1/3, 2/3)$ (so they are all between 1/3 and 2/3). Note that $y_i + y_j > 2/3$, hence $y_i + y_j \notin \{y_1, \ldots, y_m\}$.

ANOTHER EXAMPLE

Def: $\mathrm{frac}(x)$ is the fractional part of x. E.g., $\mathrm{frac}(1.414) = .414$.

Lemma: If y_1, y_2, y_3 are such that $\mathrm{frac}(y_1), \mathrm{frac}(y_2), \mathrm{frac}(y_3) \in (1/3, 2/3)$ then $y_1 + y_2 \neq y_3$.

Proof: Let $y_1 = z_1 + \epsilon_1$, $y_2 = z_2 + \epsilon_2$, $y_3 = z_3 + \epsilon_3$ where ϵ's are fracpart.

If $y_1 + y_2 = y_3$ then

$$z_1 + z_2 + \epsilon_1 + \epsilon_2 = z_3 + \epsilon_3$$

Since $\epsilon_1, \epsilon_2, \epsilon_3 \in (1/3, 2/3)$ we have that

$$z_1 + z_2 = z_3$$

$$\epsilon_1 + \epsilon_2 = \epsilon_3.$$

Impossible since $\epsilon_1, \epsilon_2, \epsilon_3 \in (1/3, 2/3)$.

Example: Let $A = \{y_1, \ldots, y_m\}$ all have fractional part in $(1/3, 2/3)$. A is sum free by above Lemma.

QUESTION

Given $x_1, \ldots, x_n \in R$ does there exist a LARGE sum-free subset? How Large?

QUESTION

Given $x_1, \ldots, x_n \in R$ does there exist a LARGE sum-free subset? How Large?

- STUDENTS - WORK ON THIS IN GROUPS.

QUESTION

Given $x_1, \ldots, x_n \in R$ does there exist a LARGE sum-free subset? How Large?

- STUDENTS - WORK ON THIS IN GROUPS.

VOTE:

1. There is a sumfree set of size $n/3$.
2. There is a sumfree set of size $\sqrt{n}$.
3. There is a sumfree set of size $\log n$.

SUM SET PROBLEM

Theorem If A is a set of n real numbers there is a sum-free subset of size $n/3$.
Proof:
For $a \in \mathsf{R}$ let

$$B_a = \{x \in A : \mathrm{frac}(ax) \in (1/3, 2/3)\}.$$

Claim: For all a, B_a is sum-free.
Proof of Claim: $x, y, z \in B_a$. If $x + y = z$ then $ax + ay = az$. But $\mathrm{frac}(ax), \mathrm{frac}(ay), \mathrm{frac}(az) \in (1/3, 2/3)$. Can't happen by above Lemma.

HOW BIG IS B_a?

SO we need an a such that B_a is LARGE.
What is the EXPECTED VALUE of $|B_a|$?
Let $x \in A$.

$$\Pr_{a \in \mathsf{R}}(ax \in (1/3, 2/3)) = 1/3$$

.

$$E(|B_a|) = \sum_{x \in A} \Pr_{a \in \mathsf{R}}(ax \in (1/3, 2/3)) = \sum_{x \in A} 1/3 = n/3.$$

So THERE EXISTS an a such that $|B_a| \geq n/3$.
What is a? I DON"T KNOW AND I DON"T CARE!
End of Proof

Notation:

1. A Graph is (V, E) where V is the set of vertices and E is a set of pairs of vertices. Easy to draw!
2. An Ind Set in a graph (V, E) is a set $V' \subseteq V$ such that there are NO edges between elements of V'.
3. If (V, E) is a graph and $v \in V$ then the degree of v, denoted d_v, is the number of edges coming out of it.

DO EXAMPLES ON BOARD

Turan's Theorem

Theorem If $G = (V, E)$ is a graph, $|V| = n$, and $|E| = e$, then G has an ind set of size at least

$$\frac{n}{\frac{2e}{n} + 1}.$$

We proof this using Probability, but first need a lemma.

Lemma

Lemma If $G = (V, E)$ is a graph. Then

$$\sum_{v \in V} deg(v) = 2e.$$

Proof: Try to count the edges by summing the degrees at each vertex. This counts every edge TWICE.

Proof of Turan's Theorem

Theorem If $G = (V, E)$ is a graph, $|V| = n$, and $|E| = e$, then G has an ind set of size

$$\geq \frac{n}{\frac{2e}{n} + 1}.$$

Proof: Take the graph and RANDOMLY permute the vertices.

(DO EXAMPLE ON BOARD.) The set of vertices that have NO edges coming out on the right form an Ind Set. Call this set I.

How Big is I?

How big is I

How Big is I?

How big is I

WRONG QUESTION!

How Big is I?

How big is I

WRONG QUESTION!

What is the EXPECTED VALUE of the size of I.

(NOTE- we permuted the vertices RANDOMLY)

What is Prob $v \in I$

Let $v \in V$. What is prob that $v \in I$ DRAW PICTURE OF v. v has degree d_v. How many ways can v and its vertices be laid out: $(d_v + 1)!$. In how many of them is v on the right? $d_v!$.

$$\Pr(v \in I) = \frac{d_v!}{(d_v + 1)!} = \frac{1}{d_v + 1}.$$

Hence

$$E(|I|) = \sum_{v \in V} \frac{1}{d_v + 1}.$$

How Big is this Sum?

Need to find lower bound on

$$\sum_{v \in V} \frac{1}{d_v + 1}.$$

Rephrase

NEW PROBLEM:
Minimize

$$\sum_{v\in V} \frac{1}{x_v+1}$$

relative to the constraint:

$$\sum_{v\in V} x_v = 2e.$$

KNOWN: This sum is minimized when all of the x are $\frac{2e}{|V|} = \frac{2e}{n}$.
So the min the sum can be is

$$\sum_{v\in V} \frac{1}{\frac{2e}{n}+1} = \frac{n}{\frac{2e}{n}+1}.$$

DONE!

$$\sum_{v \in V} \frac{1}{x_v + 1} \geq \sum_{v \in V} \frac{1}{\frac{2e}{n} + 1} = \frac{n}{\frac{2e}{n} + 1}.$$

END OF THIS TALK

TAKEAWAY: There are TWO ways (probably more) to show that an object exists using probability.

1. Show that the probability that it exists is NONZERO. Hence there must be some set of random choices that makes it exist. We did this for the distinct-sums problem.
2. You want to show that an object of a size $\geq s$ exists. Show that if you do a probabilistic experiment then you (a) always get the object of the type you want, and (b) the expected size is $\geq s$. Hence again SOME set of random choices produces an object of size $\geq s$.

Lower Bound on Ramsey Numbers

PROBLEM: We want to find a coloring of $\binom{[n]}{2}$ without a k-homog set for some $n = f(k)$.

Lower Bound on Ramsey Numbers

PROBLEM: We want to find a coloring of $\binom{[n]}{2}$ without a k-homog set for some $n = f(k)$.
WRONG!- I want to just show that such exists!

Pick a coloring at Random!

Number of colorings: $2^{\binom{n}{2}}$.
Number of colorings: that HAVE a homog set of size k is bounded by

$$\binom{n}{k} \times 2 \times 2^{\binom{n}{2}-\binom{k}{2}}$$

Prob that a random 2-coloring HAS a homog set is bounded by

$$\frac{\binom{n}{k} \times 2 \times 2^{\binom{n}{2}-\binom{k}{2}}}{2^{\binom{n}{2}}} \le \frac{\binom{n}{k} \times 2}{2^{\binom{k}{2}}} \le \frac{n^k}{k!2^{k^2/2}}$$

Want n large and $\frac{n^k}{k!2^{k^2/2}} < 1$. Take $n = 2^{k/2}$.
UPSHOT: $R(k) \ge 2^{k/2}$.
SUMMARY OF WHAT WE KNOW: $2^{k/2} \le R(k) \le 2^{2k-1}$.

Better Upper Bounds

Definition: $R(a,b)$ is the least n such that for all 2-colorings of K_n there is either a RED K_a or a BLUE K_b.

Easy Theorem: $R(2,b)=b$ and $R(a,2)=a$.

Theorem: For all $a,b \geq 3$ $R(a,b) \leq R(a-1,b)+R(a,b-1)$
Proof: BILL DOES ON ON BOARD

Better Upper Bounds

Definition: $R(a,b)$ is the least n such that for all 2-colorings of K_n there is either a RED K_a or a BLUE K_b.

Easy Theorem: $R(2,b) = b$ and $R(a,2) = a$.

Theorem: For all $a, b \geq 3$ $R(a,b) \leq R(a-1,b) + R(a,b-1)$
Proof: BILL DOES ON ON BOARD
We can use this to get bounds on $R(a,b)$: DO IN GROUPS

Better Upper Bounds

Definition: $R(a,b)$ is the least n such that for all 2-colorings of K_n there is either a RED K_a or a BLUE K_b.

Easy Theorem: $R(2,b) = b$ and $R(a,2) = a$.

Theorem: For all $a,b \ge 3$ $R(a,b) \le R(a-1,b) + R(a,b-1)$
Proof: BILL DOES ON ON BOARD
We can use this to get bounds on $R(a,b)$: DO IN GROUPS

Theorem: For all $a,b \ge 3$ $R(a,b) \le \binom{a+b-2}{a-1}$
Proof: BILL DOES ON ON BOARD UNLESS GROUPS GOT IT.

Slight Improvement

Theorem: For all $a, b \geq 3$, if $R(a-1, b)$ and $R(a, b-1)$ are both EVEN then $R(a, b) \leq R(a-1, b) + R(a, b-1) - 1$

PROVE IN GROUPS- THEN BILL WILL DO IT IF NEEDS TO.

Slight Improvement

Theorem: For all $a, b \geq 3$, if $R(a-1, b)$ and $R(a, b-1)$ are both EVEN then $R(a, b) \leq R(a-1, b) + R(a, b-1) - 1$

PROVE IN GROUPS- THEN BILL WILL DO IT IF NEEDS TO.

Actual Numbers:
$R(3,3) \leq R(2,3) + R(3,2) \leq 3 + 3 = 6$.
$R(3,4) \leq R(2,4) + R(3,3) \leq 4 + 6 = 10$. BOTH EVEN:
$R(3,4) \leq 9$.
$R(3,5) \leq R(2,5) + R(3,4) \leq 5 + 9 = 14$
$R(4,4) \leq R(3,4) + R(4,3) \leq 9 + 9 = 18$
$R(4,5) \leq R(3,5) + R(4,4) \leq 14 + 18 \leq 32$. BOTH EVEN:
$R(3,5) \leq 31$.
$R(5,5) \leq R(4,5) + R(5,4) \leq 31 + 31 \leq 62$.
NEED MATCHING LOWER BOUNDS.

$R(3,3) \ge 6$: We need a coloring of K_5 with NO mono K_3.
Vertices are $\{0,1,2,3,4\}$.
$COL(a,b) =$ RED if $a-b \equiv SQ \pmod 5$, BLUE OW.

- $-1 \equiv SQ \pmod 5$: $a-b \equiv SQ$ iff $b-a \equiv SQ$. COL is sym.
- Squares mod 5: 1,4.
- If there is a RED triangle then $a-b$, $b-c$, $c-a$ all SQ's. SUM is 0. So

  $$x^2 + y^2 + z^2 \equiv 0 \pmod 5$$

  Can show this is impossible.
- If there is a BLUE triangle then $a-b$, $b-c$, $c-a$ all non-SQ's. Product of nonsq's is a sq. So $2(a-b)$, $2(b-c)$, $2(c-a)$ all squares. SUM to zero- same proof.

UPSHOT: $R(3,3) = 6$.

$R(4,4) = 18.$

Vertices are $\{0, \ldots, 16\}$.
Use
$COL(a,b) =$ RED if $a - b \equiv SQ \pmod{17}$, BLUE OW.
Same idea as above, but more cases.

$R(3,5) = 14$.

Vertices are $\{0, \ldots, 12\}$.
Use
$COL(a,b) =$ RED if $a - b \equiv CUBE \pmod{13}$, BLUE OW.
Same idea as above, but more cases.

$R(3,4) = 9$.
Subgraph of the above graph.

Can we extend this? Are there patterns?

Good news: $R(4,5) = 25$.

Can we extend this? Are there patterns?

Good news: $R(4,5)=25$.

Bad news: THATS IT. No other $R(a,b)$ are known using NICE methods.
($R(5,5)$- there will be some news on that after the midterm.)

Can we extend this? Are there patterns?

Good news: $R(4,5)=25$.

Bad news: THATS IT. No other $R(a,b)$ are known using NICE methods.
($R(5,5)$- there will be some news on that after the midterm.)

The Law of Small Numbers: Patterns that you see for small values vanish when the numbers get to large to compute.