POLY VDW THEOREM (Exposition)

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Recall:

**Theorem:** \((\forall k, c)(\exists W = W(k, c))\) such that for all 
\(COL : [W] \rightarrow [c] (\exists a, d)\) such that 

\[
COL(a) = COL(a + d) = COL(a + 2d) = \cdots = COL(a + (k - 1)d).
\]

Recall:
\(W(1, c) = 1\)
\(W(2, c) = c + 1\) (this is PHP)
\(W(k, 1) = k\)

Let \(W(k, c)\) mean both the number and the STATEMENT.
We proved

Recall:

\[ W(2, 32) \implies W(3, 2) \]
\[ W(2, 10^{10}) \implies W(3, 3) \text{ (might not be big enough.)} \]
\[ W(2, (10!^{10!})) \implies W(3, 4) \text{ (might not be big enough.)} \]

\[ W(3, BLAH) \implies W(4, 2). \]
\[ W(3, BLAHBLAH) \implies W(4, 3). \]
Order PAIRS of naturals (think \((k, c)\)) via

\[(2, 2) \leq (2, 3) \leq (2, 4) \leq \cdots \leq (3, 2) \leq (3, 3) \leq (3, 4) \cdots\]

\[(4, 2) \leq (4, 3) \leq \cdots (5, 2) \leq (5, 3) \leq \cdots \leq (6, 2) \cdots\]

Formal proof of VDW is an induction on this ordering.

Induction on an \(\omega^2\) ordering.
Order PAIRS of naturals (think \((k, c)\)) via

\[(2, 2) \leq (2, 3) \leq (2, 4) \leq \cdots \leq (3, 2) \leq (3, 3) \leq (3, 4) \cdots\]

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Formal proof of VDW is an induction on this ordering.

Induction on an \(\omega^2\) ordering.

(WARNING: Not a good example of induction for CMSC 250)
So Whats Really Going On?

Order PAIRS of naturals (think \((k, c)\)) via

\[(2, 2) \leq (2, 3) \leq (2, 4) \leq \cdots \leq (3, 2) \leq (3, 3) \leq (3, 4) \cdots\]

\[(4, 2) \leq (4, 3) \leq \cdots (5, 2) \leq (5, 3) \leq \cdots \leq (6, 2) \cdots\]

Formal proof of VDW is an induction on this ordering.

Induction on an \(\omega^2\) ordering.

(WARNING: Not a good example of induction for CMSC 250)

(WEIRDNESS: Several HS students saw this as their FIRST proof by induction and went on to live productive lives.)
Why $a, a + d, a + 2d, a + 3d, \ldots, a + (k - 1)d$?
Replace $d, 2d, 3d, \ldots, (k - 1)d$ by some other function of $d$?
Is this true:

**Theorem:** $(\forall p_1, \ldots, p_k \in \mathbb{Z}[x])(\forall c)(\exists W)$ for all $COL : [W] \rightarrow [c]$
$(\exists a, d)$

$$COL(a) = COL(a + p_1(d)) = COL(a + p_2(d)) = \cdots = COL(a + p_k(d)).$$

VOTE!
FALSE for a DUMB reason:

1. \( k = 1 \)
2. \( p_1(x) = 1 \)
3. \( c = 2 \).

NEED \( W \) such that for all \( \text{COL} : [W] \rightarrow [2] \) there exists \( a, d \) such that

\[
\text{COL}(a) = \text{COL}(a + 1)
\]

Take \( RBRBRB \cdots \).
**Definition:** $Z^*[x]$ are all polynomials with coefficients in $Z$ and zero constant term.

**Theorem:** $(\forall p_1, \ldots, p_k \in Z^*[x])(\forall c)(\exists W)$ for all $COL : [W] \rightarrow [c]$ $(\exists a, d)$

$$COL(a) = COL(a + p_1(d)) = COL(a + p_2(d)) = \cdots = COL(a + p_k(d)).$$
RESTATE IT:

**Theorem:** For all finite $S \subseteq \mathbb{Z}^*[x]$ $(\forall c)(\exists W)$ for all $COL : [W] \rightarrow [c]$ $(\exists a, d)$

$$\{a\} \cup \{a + p(d) \mid p \in S\}$$ all the same color.

**Notation:** $PVDW(S)$ means that Poly VDW theorem holds for the set $S \subseteq \mathbb{Z}^*[x]$. Note that

$$PVDW(x, 2x, 3x) \implies (\forall c)[VDW(3, c)].$$
Definition: A finite set $S \subseteq \mathbb{Z}^*[x]$ is of type $(n_e, n_{e-1}, \ldots, n_1)$ if

- the number of different leading coefficients of polynomials of degree $e$ is $\leq n_e$.
- the number of different leading coefficients of polynomials of degree $e - 1$ is $\leq n_{e-1}$.

\[ \vdots \]
- the number of different leading coefficients of polynomials of degree 1 is $\leq n_1$.

BILL DO EXAMPLES ON BOARD.
Definition: Let \((n_e, n_{e-1}, \ldots, n_1) \in \mathbb{N}^e\). \(PVDW(n_e, \ldots, n_1)\) means that \(PVDW(S)\) holds for all \(S\) of type \((n_e, n_{e-1}, \ldots, n_1)\).

VDW’s theorem is \(PVDW(1) \land PVDW(2) \land \cdots\).

We showed

\[
\left( \bigwedge_{i \in \mathbb{N}} PVDW(i) \right) \implies PVDW(1, 0).
\]
**Definition:** Let \((n_e, n_{e-1}, \ldots, n_1) \in (\omega \cup \mathbb{N})^e\). \(PVDW(n_e, \ldots, n_1)\) means that, for all \((m_e, \ldots, m_1) \leq (n_e, \ldots, n_1)\) (component wise) \(PVDW(S)\) holds for all \(S\) of type \((m_e, m_{e-1}, \ldots, m_1)\).

We showed

\[
PVDW(\omega) \implies PVDW(1, 0).
\]
**Theorem:** $(\forall c)(\exists W)$ for all $COL : [W] \rightarrow [c]$ $(\exists a, d)$

$$COL(a) = COL(a + d^2).$$

Proof by proving Lemma:

**Lemma:** $(\forall c)(\forall r)(\exists U)$ for all $COL : [U] \rightarrow [c]$ EITHER

- $(\exists a, d)[COL(a) = COL(a + d^2)]$, OR
- $(\exists a, d_1, d_2, \ldots, d_r)[COL(a), COL(a + d_i^2)]$ all colored DIFFERENTLY.

BILL- REDO OR NOT IN CLASS.
Theorem: \((\forall c)(\forall k)(\forall A \in Z)(\forall B \subseteq Z, \ B \text{ finite})(\exists W)\) for all \(COL : [W] \rightarrow [c] (\exists a, d)\)

all elements of \(\{a\} \cup \{a + d^2 + id : i \in B\}\) are the same color.

Proof by proving Lemma:

Lemma: \((\forall c)(\forall k)(\forall A \in Z)(\forall B \subseteq Z, \ B \text{ finite})(\exists U)\) for all \(COL : [U] \rightarrow [c] \) EITHER

\(\exists a, d\) all elements of \(\{a\} \cup \{a + d^2 + id : i \in B\}\) are the same color, OR

\((\exists a, d_1, d_2, \ldots, d_r)\)

\((\forall 1 \leq j \leq r\) the elements of \(\{a + d_j^2 + id_j : i \in B\}\) are the same color. We call the \(j\)th one the \(j\)th BUBBLE.

\(\forall\) All the bubbles are colored differently and all are a different color than \(a\).

BILL- DO IN CLASS AND DO BASE CASE
Theorem: \((\forall c))(\exists W)\) for all \(COL : [W] \rightarrow [c] (\exists a, d)\)

all elements of \(\{a\} \cup \{a + d, a + d^2\}\) are the same color.

Proof by proving Lemma:

Lemma: \((\forall c)(\exists U)\) for all \(COL : [U] \rightarrow [c]\) EITHER

\(\exists a, d\) all elements of \(\{a\} \cup \{a + d, a + d^2\}\) are the same color, OR

\(\exists a, d_1, d_2, \ldots, d_r\)

\(\forall 1 \leq j \leq r\) the elements of \(\{a + d_j, a + d_j^2\}\) are the same color. We call the \(j\)th one the \(j\)th BUBBLE.

All the bubbles are colored differently and all are a different color than \(a\).

BILL- DO IN CLASS AND DO BASE CASE