Ramsey's Theorem on Graphs

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1 Introduction

Imagine that you have 6 people at a party. We assume that, for every pair of them, either THEY KNOW EACH OTHER or NEITHER OF THEM KNOWS THE OTHER. So we are assuming that if x knows y, then y knows x.
Claim: Either there are at least 3 people *all of whom know one another*, or there are at least 3 people *no two of whom know each other* (or both).
Proof of Claim:

Let the people be $p_1, p_2, p_3, p_4, p_5, p_6$. Now consider p_6.

Among the other 5 people, either there are at least 3 people that p_6 knows, or there are at least 3 people that p_6 does not know.

Why is this?

Well, suppose that, among the other 5 people, there are at most 2 people that p_6 knows, and at most 2 people that p_6 does not know. Then there are only 4 people other than p_6, which contradicts the fact that there are 5 people other than p_6.

Suppose that p_6 knows at least 3 of the others. We consider the case where p_6 knows p_1, p_2, and p_3. All the other cases are similar.

If p_1 knows p_2, then p_1, p_2, and p_6 all know one another. HOORAY!

If p_1 knows p_3, then p_1, p_3, and p_6 all know one another. HOORAY!

If p_2 knows p_3, then p_2, p_3, and p_6 all know one another. HOORAY!

What if *none* of these scenarios holds? Then none of these three people (p_1, p_2, p_3) knows either of the other 2. HOORAY!
End of Proof of Claim

We want to generalize this observation.

Notation 1.1 N is the set of all positive integers. If $n \in \mathsf{N}$, then $[n]$ is the set $\{1, \ldots, n\}$.

Def 1.2 A *graph* G consists of a set V of vertices and a set E of edges. The edges are *unordered* pairs of vertices.

Note 1.3 In general, a graph can have an edge $\{i, j\}$ with $i = j$. Here, however, every edge of a graph is an unordered pair of *distinct* vertices (i.e., an unordered pair $\{i, j\}$ with $i \neq j$).

Def 1.4 Let $c \in \mathsf{N}$. Let $G = (V, E)$ be a graph. A *c-coloring of the edges of G* is a function $COL : E \rightarrow [c]$. Note that there are no restrictions on COL.

Note 1.5 In the Graph Theory literature there are (at least) two kinds of coloring. We present them in this note so that if you happen to read the literature and they are using coloring in a different way then in these notes, you will not panic.

- Vertex Coloring. Usually one says that the vertices of a graph are c-colorable if there is a way to assign each vertex a color, using no more than c colors, such that no two adjacent vertices (vertices connected by an edge) are the same color. Theorems are often of the form 'if a graph G has property BLAH BLAH then G is c-colorable' where they mean vertex c-colorable. We **will not** be considering these kinds of colorings.

- Edge Colorings. Usually this is used in the context of Ramsey Theory and Ramsey-type theorems. Theorems begin with 'for all c-coloring of a graph G BLAH BLAH happens' We **will** be considering these kinds of colorings.

Def 1.6 Let $n \in \mathsf{N}$. The *complete graph on n vertices*, denoted K_n, is the graph

$$\begin{aligned} V = & \ [n] \\ E = & \ \{\{i, j\} \mid i, j \in [n]\} \end{aligned}$$

Example 1.7 Let G be the complete graph on 10 vertices. Recall that the vertices are $\{1, \ldots, 10\}$. We give a 3-coloring of the edges of G:

$$COL(\{x, y\}) = \begin{cases} 1 & \text{if } x + y \equiv 1 \pmod 3; \\ 2 & \text{if } x + y \equiv 2 \pmod 3; \\ 3 & \text{if } x + y \equiv 0 \pmod 3). \end{cases}$$

Lets go back to our party! We can think of the 6 people as vertices of K_6. We can color edge $\{i, j\}$ RED if i and j know each other, and BLUE if they do not.

Def 1.8 Let $G = (V, E)$ be a graph, and let COL be a coloring of the edges of G. A set of edges of G is *monochromatic* if they are all the same color.

Let $n \geq 2$. Then G *has a monochromatic* K_n if there is a set V' of n vertices (in V) such that

- there is an edge between every pair of vertices in V': $\{\{i, j\} \mid i, j \in V'\} \subseteq E$
- all the edges between vertices in V' are the same color: there is some $l \in [c]$ such that $COL(\{i, j\}) = l$ for all $i, j \in V'$

We now restate our 6-people-at-a-party theorem:

Theorem 1.9 *Every 2-coloring of the edges of K_6 has a monochromatic K_3.*

2 The Full Theorem

From the last section, we know the following:

If you want an n such that you get a monochromatic K_3 no matter how you 2-color K_n, then $n = 6$ will suffice.

What if you want to guarantee that there is a monochromatic K_4? What if you want to use 17 colors?

The following is known as *Ramsey's Theorem.* It was first proved in [3] (see also [1], [2]).

For all $c, m \geq 2$, there exists $n \geq m$ such that every c-coloring of K_n has a monochromatic K_m.

We will provide several proofs of this theorem for the $c = 2$ case. We will assume the colors are RED and BLUE (rather than the numbers 1 and 2). The general-c case (where c can be *any* integer $i \geq 2$) and other generalizations may show up on homework assignments.

3 First Proof of Ramsey's Theorem

Given m, we really want n such that every 2-coloring of K_n has a RED K_m or a BLUE K_m. However, it will be useful to let the parameter for BLUE differ from the parameter for RED.

Notation 3.1 Let $a, b \geq 2$. Let $R(a, b)$ denote the least number, if it exists, such that every 2-coloring of $K_{R(a,b)}$ has a RED K_a or a BLUE K_b. We abbreviate $R(a, a)$ by $R(a)$.

We state some easy facts.

1. For all a, b, $R(a, b) = R(b, a)$.
2. For $b \geq 2$, $R(2, b) = b$: First, we show that $R(2, b) \leq b$. Given any 2-coloring of K_b, we want a RED K_2 or a BLUE K_b. Note that a RED K_2 is just a RED edge. Hence EITHER there exists one RED edge (so you get a RED K_2) OR all the edges are BLUE (so you get a BLUE K_b). Now we prove that $R(2, b) = b$. If $b = 2$, this is obvious. If $b > 2$, then the all-BLUE coloring of K_{b-1} has neither a RED K_2 nor a BLUE K_b, hence $R(2, b) \geq b$. Combining the two inequalities ($R(2, b) \leq b$ and $R(2, b) \geq b$), we find that $R(2, b) = b$.
3. $R(3, 3) \leq 6$ (we proved this in Section 1)

We want to show that, for every $n \geq 2$, $R(n, n)$ exists. In this proof, we show something more: that for all $a, b \geq 2$, $R(a, b)$ exists. We do not really care about the case where $a \neq b$, but that case will help us get our result. This is a situation where proving more than you need is easier.

Theorem 3.2

1. *$R(2, b) = b$ (we proved this earlier)*
2. *For all $a, b \geq 3$: If $R(a-1, b)$ and $R(a, b-1)$ exist, then $R(a, b)$ exists and*
$$R(a, b) \leq R(a-1, b) + R(a, b-1)$$
3. *For all $a, b \geq 2$, $R(a, b)$ exists and $R(a, b) \leq 2^{a+b}$.*

Proof:

Since we proved part 1 earlier, we now prove parts 2 and 3.

Part 2 Assume $R(a-1,b)$ and $R(a,b-1)$ exist. Let

$$n = R(a-1,b) + R(a,b-1)$$

Let COL be a 2-coloring of K_n, and let x be a vertex. Note that there are

$$R(a-1,b) + R(a,b-1) - 1$$

edges coming out of x (edges $\{x,y\}$ for vertices y).

Let NUM-RED-EDGES be the number of red edges coming out of x, and let NUM-BLUE-EDGES be the number of blue edges coming out of x. Note that

$$\text{NUM-RED-EDGES} + \text{NUM-BLUE-EDGES} = R(a-1,b) + R(a,b-1) - 1$$

Hence either

$$\text{NUM-RED-EDGES} \geq R(a-1,b)$$

or

$$\text{NUM-BLUE-EDGES} \geq R(a,b-1)$$

To see this, suppose, by way of contradiction, that both inequalities are false. Then

$$\begin{array}{rcl} \text{NUM-RED-EDGES} & + & \text{NUM-BLUE-EDGES} \\ \leq R(a-1,b)-1 & + & R(a,b-1)-1 \\ = R(a-1,b) & + & R(a,b-1)-2 \\ < R(a-1,b) & + & R(a,b-1)-1 \end{array}$$

There are two cases:

1. *Case 1:* NUM-RED-EDGES $\geq R(a-1,b)$. Let

   $$U = \{y \mid COL(\{x,y\}) = \text{RED}\}$$

   U is of size NUM-RED-EDGES $\geq R(a-1,b)$. Consider the restriction of the coloring COL to the edges between vertices in U. Since

   $$|U| \geq R(a-1,b),$$

   this coloring has a RED K_{a-1} or a BLUE K_b. Within Case 1, there are two cases:

(a) There is a RED K_{a-1}. Recall that all of the edges in

$$\{\{x, u\} \mid u \in U\}$$

are RED, hence all the edges between elements of the set $U \cup \{x\}$ are RED, so they form a RED K_a and WE ARE DONE.

(b) There is a BLUE K_b. Then we are DONE.

2. *Case 2:* NUM-BLUE-EDGES $\geq R(a, b-1)$. Similar to Case 1.

Part 3 To show that $R(a, b)$ exists and $R(a, b) \leq 2^{a+b}$, we use induction on $n = a + b$. Since $a, b \geq 2$, the smallest value of $a + b$ is 4. Thus $n \geq 4$.

Base Case: $n = 4$. Since $a + b = 4$ and $a, b \geq 2$, we must have $a = b = 2$. From part 1, we know that $R(2, 2)$ exists and $R(2, 2) = 2$. Note that

$$R(2, 2) = 2 \leq 2^{2+2} = 16$$

Induction Hypothesis: For all $a, b \geq 2$ such that $a + b = n$, $R(a, b)$ exists and $R(a, b) \leq 2^{a+b}$.

Inductive Step: Let a, b be such that $a, b \geq 2$ and $a + b = n + 1$.
There are three cases:

1. *Case 1:* $a = 2$. By part 1, $R(2, b)$ exists and $R(2, b) = b$. Since $b \geq 2$, we have

$$b \leq 2^b \leq 4 \cdot 2^b = 2^2 \cdot 2^b = 2^{2+b}$$

Hence $R(2, b) \leq 2^{2+b}$.

2. *Case 2:* $b = 2$. Follows from Case 1 and $R(a, b) = R(b, a)$.

3. *Case 3:* $a, b \geq 3$. Since $a, b \geq 3$, we have $a - 1 \geq 2$ and $b - 1 \geq 2$. Also, $a + b = n + 1$, so $(a - 1) + b = n$ and $a + (b - 1) = n$. By the induction hypothesis, $R(a - 1, b)$ and $R(a, b - 1)$ exist; moreover,

$$R(a-1, b) \leq 2^{(a-1)+b} = 2^{a+b-1}$$

$$R(a, b-1) \leq 2^{a+(b-1)} = 2^{a+b-1}$$

From part 3, $R(a, b)$ exists and

$$R(a, b) \leq R(a-1, b) + R(a, b-1)$$

Hence

$$R(a,b) \le R(a-1,b) + R(a,b-1) \le 2^{a+b-1} + 2^{a+b-1} = 2 \cdot 2^{a+b-1} = 2^{a+b}$$

▮

Corollary 3.3 *For every $m \ge 2$, $R(m)$ exists and $R(m) \le 2^{2m}$.*

4 Second Proof of Ramsey's Theorem

We now present a proof that does not use $R(a,b)$. It also gives a mildly better bound on $R(m)$ than the one given in Corollary 3.3.

Theorem 4.1 *For every $m \ge 2$, $R(m)$ exists and $R(m) \le 2^{2m-2}$.*

Proof:

Let COL be a 2-coloring of $K_{2^{2m-2}}$. We define a sequence of vertices,

$$x_1, x_2, \ldots, x_{2m-1},$$

and a sequence of sets of vertices,

$$V_0, V_1, V_2, \ldots, V_{2m-1},$$

that are based on COL.

Here is the intuition: Vertex $x_1 = 1$ has $2^{2m-2} - 1$ edges coming out of it. Some are RED, and some are BLUE. Hence there are at least 2^{2m-3} RED edges coming out of x_1, or there are at least 2^{2m-3} BLUE edges coming out of x_1. To see this, suppose, by way of contradiction, that it is false, and let N.E. be the total number of edges coming out of x_1. Then

$$\text{N.E.} \le (2^{2m-3} - 1) + (2^{2m-3} - 1) = (2 \cdot 2^{2m-3}) - 2 = 2^{2m-2} - 2 < 2^{2m-2} - 1$$

Let c_1 be a color such that x_1 has at least 2^{2m-3} edges coming out of it that are colored c_1. Let V_1 be the set of vertices v such that $COL(\{v, x_1\}) = c_1$. Then keep iterating this process.

We now describe it formally.

$$\begin{aligned}
V_0 &= [2^{2m-2}] \\
x_1 &= 1 \\
c_1 &= \begin{cases} \text{RED} & \text{if } |\{v \in V_0 \mid COL(\{v, x_1\}) = \text{RED}\}| \geq 2^{2m-3} \\ \text{BLUE} & \text{otherwise} \end{cases} \\
V_1 &= \{v \in V_0 \mid COL(\{v, x_1\}) = c_1\} \text{ (note that } |V_1| \geq 2^{2m-3})
\end{aligned}$$

Let $i \geq 2$, and assume that V_{i-1} is defined. We define x_i, c_i, and V_i:

$$\begin{aligned}
x_i &= \text{ the least number in } V_{i-1} \\
c_i &= \begin{cases} \text{RED} & \text{if } |\{v \in V_{i-1} \mid COL(\{v, x_i\}) = \text{RED}\}| \geq 2^{(2m-2)-i}; \\ \text{BLUE} & \text{otherwise.} \end{cases} \\
V_i &= \{v \in V_{i-1} \mid COL(\{v, x_i\}) = c_i\} \text{ (note that } |V_i| \geq 2^{(2m-2)-i})
\end{aligned}$$

How long can this sequence go on for? Well, x_i can be defined if V_{i-1} is nonempty. Note that

$$|V_{2m-2}| \geq 2^{(2m-2)-(2m-2)} = 2^0 = 1$$

Thus if $i-1 = 2m-2$ (equivalently, $i = 2m-1$), then $V_{i-1} = V_{2m-2} \neq \emptyset$, but there is no guarantee that V_i $(= V_{2m-1})$ is nonempty. Hence we can define

$$x_1, \ldots, x_{2m-1}$$

Consider the colors

$$c_1, c_2, \ldots, c_{2m-2}$$

Each of these is either RED or BLUE. Hence there must be at least $m-1$ of them that are the same color. Let $i_1, \ldots, i_{m-1}$ be such that $i_1 < \cdots < i_{m-1}$ and

$$c_{i_1} = c_{i_2} = \cdots = c_{i_{m-1}}$$

Denote this color by c, and consider the m vertices

$$x_{i_1}, x_{i_2}, \cdots, x_{i_{m-1}}, x_{i_{m-1}+1}$$

To see why we have listed m vertices but only $m-1$ colors, picture the following scenario: You are building a fence row, and you want (say) 7 sections of fence. To do that, you need 8 fence posts to hold it up. Now think of the fence posts as vertices, and the sections of fence as edges between successive vertices, and recall that every edge has a color associated with it.

Claim: The m vertices listed above form a monochromatic K_m.
Proof of Claim:
First, consider vertex x_{i_1}. The vertices

$$x_{i_2}, \ldots, x_{i_{m-1}}, x_{i_{m-1}+1}$$

are elements of V_{i_1}, hence the edges

$$\{x_{i_1}, x_{i_2}\}, \ldots \{x_{i_1}, x_{i_{m-1}}\}, \{x_{i_1}, x_{i_{m-1}+1}\}$$

are colored with c_{i_1} $(= c)$.
Then consider each of the remaining vertices in turn, starting with vertex x_{i_2}. For example, the vertices

$$x_{i_3}, \ldots, x_{i_{m-1}}, x_{i_{m-1}+1}$$

are elements of V_{i_2}, hence the edges

$$\{x_{i_2}, x_{i_3}\}, \ldots \{x_{i_2}, x_{i_{m-1}}\}, \{x_{i_2}, x_{i_{m-1}+1}\}$$

are colored with c_{i_2} $(= c)$.
End of Proof of Claim ∎

5 Proof of the Infinite Ramsey Theorem

We now consider infinite graphs.

Notation 5.1 K_{N} is the graph (V, E) where

$$\begin{aligned} V &= \mathsf{N} \\ E &= \{\{x, y\} \mid x, y \in \mathsf{N}\} \end{aligned}$$

Def 5.2 Let $G = (V, E)$ be a graph with $V = \mathsf{N}$, and let COL be a coloring of the edges of G. A set of edges of G is *monochromatic* if they are all the same color (this is the same as for a finite graph).

G has a monochromatic K_{N} if there is an infinite set V' of vertices (in V) such that

- there is an edge between every pair of vertices in V'
- all the edges between vertices in V' are the same color

Theorem 5.3 *Every 2-coloring of the edges of K_{N} has a monochromatic K_{N}.*

Proof:

(Note: this proof is similar to the proof of Theorem 4.1.)

Let COL be a 2-coloring of K_{N}. We define an infinite sequence of vertices,

$$x_1, x_2, \ldots,$$

and an infinite sequence of sets of vertices,

$$V_0, V_1, V_2, \ldots,$$

that are based on COL.

Here is the intuition: Vertex $x_1 = 1$ has an infinite number of edges coming out of it. Some are RED, and some are BLUE. Hence there are an infinite number of RED edges coming out of x_1, or there are an infinite number of BLUE edges coming out of x_1 (or both). Let c_1 be a color such that x_1 has an infinite number of edges coming out of it that are colored c_1. Let V_1 be the set of vertices v such that $COL(\{v, x_1\}) = c_1$. Then keep iterating this process.

We now describe it formally.

$$\begin{aligned} V_0 &= \mathsf{N} \\ x_1 &= 1 \end{aligned}$$

$$c_1 = \begin{cases} \text{RED} & \text{if } |\{v \in V_0 \mid COL(\{v, x_1\}) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases}$$

$$V_1 = \{v \in V_0 \mid COL(\{v, x_1\}) = c_1\} \text{ (note that } |V_1| \text{ is infinite)}$$

Let $i \geq 2$, and assume that V_{i-1} is defined. We define x_i, c_i, and V_i:

$$x_i = \quad \text{the least number in } V_{i-1}$$

$$c_i = \begin{cases} \text{RED} & \text{if } |\{v \in V_{i-1} \mid COL(\{v, x_i\}) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases}$$

$$V_i = \quad \{v \in V_{i-1} \mid COL(\{v, x_i\}) = c_i\} \text{ (note that } |V_i| \text{ is infinite)}$$

How long can this sequence go on for? Well, x_i can be defined if V_{i-1} is nonempty. We an show by induction that, for every i, V_i is infinite. Hence the sequence

$$x_1, x_2, \ldots,$$

is infinite.

Consider the infinite sequence

$$c_1, c_2, \ldots$$

Each of the colors in this sequence is either RED or BLUE. Hence there must be an infinite sequence $i_1, i_2, \ldots$ such that $i_1 < i_2 < \cdots$ and

$$c_{i_1} = c_{i_2} = \cdots$$

Denote this color by c, and consider the vertices

$$x_{i_1}, x_{i_2}, \cdots$$

Using an argument similar to the one we used in the proof of Theorem 4.1 (to show that we had a monochromatic K_m), we can show that these vertices form a monochromatic K_{N}. ▮

6 Finite Ramsey from Infinite Ramsey

Picture the following scenario: Our first lecture on the Ramsey Theorem *began* by proving Theorem 5.3. This is not absurd: The proof we gave of the infinite Ramsey Theorem does not need some of the details that are needed in the proof we gave of the finite Ramsey Theorem.

Having proved the infinite Ramsey Theorem, we then want to prove the finite Ramsey Theorem. Can we prove the finite Ramsey Theorem *from* the infinite Ramsey Theorem? Yes, we can!

Theorem 6.1 *For every $m \geq 2$, $R(m)$ exists.*

Proof: Suppose, by way of contradiction, that there is some $m \geq 2$ such that $R(m)$ does not exist. Then, for every $n \geq m$, there is some way to color K_n so that there is no monochromatic K_m. Hence there exist the following:

1. COL_1, a 2-coloring of K_m that has no monochromatic K_m
2. COL_2, a 2-coloring of K_{m+1} that has no monochromatic K_m
3. COL_3, a 2-coloring of K_{m+2} that has no monochromatic K_m

$\vdots$

j. COL_j, a 2-coloring of K_{m+j-1} that has no monochromatic K_m

$\vdots$

We will use these 2-colorings to form a 2-coloring COL of K_{N} that has no monochromatic K_m.

Let $e_1, e_2, e_3, \ldots$ be a list of all unordered pairs of elements of N such that every unordered pair appears exactly once. We will color e_1, then e_2, etc.

How should we color e_1? We will color it the way an infinite number of the COL_i's color it. Call that color c_1. Then how to color e_2? Well, first consider ONLY the colorings that colored e_1 with color c_1. Color e_2 the way an infinite number of those colorings color it. And so forth.

We now proceed formally:

$$J_0 = \mathsf{N}$$

$$COL(e_1) = \begin{cases} \text{RED} & \text{if } |\{j \in J_0 \mid COL_j(e_1) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases}$$

$$J_1 = \{j \in J_0 \mid COL(e_1) = COL_j(e_1)\}$$

Let $i \geq 2$, and assume that $e_1, \ldots, e_{i-1}$ have been colored. Assume, furthermore, that J_{i-1} is infinite and, for every $j \in J_{i-1}$,

$$\begin{aligned}
COL(e_1) = &\ COL_j(e_1)\\
COL(e_2) = &\ COL_j(e_2)\\
&\vdots\\
COL(e_{i-1}) = &\ COL_j(e_{i-1})
\end{aligned}$$

We now color e_i:

$$COL(e_i) = \begin{cases} \text{RED} & \text{if } |\{j \in J_{i-1} \mid COL_j(e_i) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases}$$

$$J_i = \{j \in J_{i-1} \mid COL(e_i) = COL_j(e_i)\}$$

One can show by induction that, for every i, J_i is infinite. Hence this process *never* stops.

Claim: If K_{N} is 2-colored with COL, then there is no monochromatic K_m.

Proof of Claim:

Suppose, by way of contradiction, that there is a monochromatic K_m. Let the edges between vertices in that monochromatic K_m be

$$e_{i_1}, \ldots, e_{i_M},$$

where $i_1 < i_2 < \cdots < i_M$ and $M = \binom{m}{2}$. For every $j \in J_{i_M}$, COL_j and COL agree on the colors of those edges. Choose $j \in J_{i_M}$ so that all the vertices of the monochromatic K_m are elements of the vertex set of K_{m+j-1}. Then COL_j is a 2-coloring of the edges of K_{m+j-1} that has a monochromatic K_m, in contradiction to the definition of COL_j.

End of Proof of Claim

Hence we have produced a 2-coloring of K_{N} that has no monochromatic K_m. This contradicts Theorem 5.3. Therefore, our initial supposition—that $R(m)$ does not exist—is false. ▮

Note that two of our proofs of the finite Ramsey Theorem (the proofs of Theorems 3.2 and 4.1) give upper bounds on $R(m)$, but that our proof of the finite Ramsey Theorem *from* the infinite Ramsey Theorem (the proof of Theorem 6.1) gives no upper bound on $R(m)$.

7 Proof of Large Ramsey Theorem

In all of the theorems presented earlier, the labels on the vertices did *not* matter. In this section, the labels *do* matter.

Def 7.1 A finite set $F \subseteq \mathsf{N}$ is called *large* if the size of F is at least as large as the smallest element of F.

Example 7.2

1. The set $\{1, 2, 10\}$ is large: It has 3 elements, the smallest element is 1, and $3 \geq 1$.
2. The set $\{5, 10, 12, 17, 20\}$ is large: It has 5 elements, the smallest element is 5, and $5 \geq 5$.
3. The set $\{20, 30, 40, 50, 60, 70, 80, 90, 100\}$ is not large: It has 9 elements, the smallest element is 20, and $9 < 20$.
4. The set $\{5, 30, 40, 50, 60, 70, 80, 90, 100\}$ is large: It has 9 elements, the smallest element is 5, and $9 \geq 5$.
5. The set $\{101, \ldots, 190\}$ is not large: It has 90 elements, the smallest element is 101, and $90 < 101$.

We will be considering monochromatic K_m's where the underlying set of vertices is a large set. We need a definition to identify the underlying set.

Def 7.3 Let COL be a 2-coloring of K_n. A set A of vertices is *homogeneous* if there exists a color c such that, for all $x, y \in A$ with $x \neq y$, $COL(\{x, y\}) = c$. In other words, all of the edges between elements of A are the same color. One could also say that there is a monochromatic $K_{|A|}$.

Let COL be a 2-coloring of K_n. Recall that the vertex set of K_n is $\{1, 2, \ldots, n\}$. Consider the set $\{1, 2\}$. It is clearly both homogeneous and large (using our definition of large). Hence the statement

"for every $n \geq 2$, every 2-coloring of K_n has a large homogeneous set" is true but trivial.

What if we used $V = \{m, m+1, \ldots, m+n\}$ as our vertex set? Then a large homogeneous set would have to have size at least m.

Notation 7.4 K_n^m is the graph with vertex set $\{m, m+1, \ldots, m+n\}$ and edge set consisting of all unordered pairs of vertices. The superscript (m) indicates that we are labeling our vertices starting with m, and the subscript (n) is one less than the number of vertices.

Note 7.5 The vertex set of K_n^m (namely, $\{m, m+1, \ldots, m+n\}$) has $n+1$ elements. Hence if K_n^m has a large homogeneous set, then $n+1 \geq m$ (equivalently, $n \geq m-1$). We could have chosen to use K_n^m to denote the graph with vertex set $\{m+1, \ldots, m+n\}$, so that the smallest vertex is $m+1$ and the number of vertices is n, but the set we have designated as K_n^m will better serve our purposes.

Notation 7.6 $LR(m)$ is the least n, if it exists, such that every 2-coloring of K_n^m has a large homogeneous set.

We first prove a theorem about infinite graphs and large homogeneous sets.

Theorem 7.7 *If COL is any 2-coloring of K_{N}, then, for every $m \geq 2$, there is a large homogeneous set whose smallest element is at least as large as m.*

Proof: Let COL be any 2-coloring of K_{N}. By Theorem 5.3, there exist an infinite set of vertices,

$$v_1 < v_2 < v_3 < \cdots,$$

and a color c such that, for all i, j, $COL(\{v_i, v_j\}) = c$. (This could be called an infinite homogeneous set.) Let i be such that $v_i \geq m$. The set

$$\{v_i, \ldots, v_{i+v_i-1}\}$$

is a homogeneous set that contains v_i elements and whose smallest element is v_i. Since $v_i \geq v_i$, it is a large set; hence it is a large homogeneous set. ∎

Theorem 7.8 *For every $m \geq 2$, $LR(m)$ exists.*

Proof: This proof is similar to our proof of the finite Ramsey Theorem *from* the infinite Ramsey Theorem (the proof of Theorem 6.1).

Suppose, by way of contradiction, that there is some $m \geq 2$ such that $LR(m)$ does not exist. Then, for every $n \geq m-1$, there is some way to color K_n^m so that there is no large homogeneous set. Hence there exist the following:

1. COL_1, a 2-coloring of K_{m-1}^m that has no large homogeneous set

2. COL_2, a 2-coloring of K_m^m that has no large homogeneous set

3. COL_3, a 2-coloring of K_{m+1}^m that has no large homogeneous set

$\vdots$

j. COL_j, a 2-coloring of K_{m+j-2}^m that has no large homogeneous set

$\vdots$

We will use these 2-colorings to form a 2-coloring COL of K_{N} that has no large homogeneous set whose smallest element is at least as large as m.

Let $e_1, e_2, e_3, \ldots$ be a list of all unordered pairs of elements of N such that every unordered pair appears exactly once. We will color e_1, then e_2, etc.

How should we color e_1? We will color it the way an infinite number of the COL_i's color it. Call that color c_1. Then how to color e_2? Well, first consider ONLY the colorings that colored e_1 with color c_1. Color e_2 the way an infinite number of those colorings color it. And so forth.

We now proceed formally:

$$J_0 = \mathsf{N}$$

$$COL(e_1) = \begin{cases} \text{RED} & \text{if } |\{j \in J_0 \mid COL_j(e_1) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases}$$

$$J_1 = \{j \in J_0 \mid COL(e_1) = COL_j(e_1)\}$$

Let $i \geq 2$, and assume that $e_1, \ldots, e_{i-1}$ have been colored. Assume, furthermore, that J_{i-1} is infinite and, for every $j \in J_{i-1}$,

$$\begin{aligned}
COL(e_1) = &\ COL_j(e_1) \\
COL(e_2) = &\ COL_j(e_2) \\
&\vdots \\
COL(e_{i-1}) = &\ COL_j(e_{i-1})
\end{aligned}$$

We now color e_i:

$$COL(e_i) = \begin{cases} \text{RED} & \text{if } |\{j \in J_{i-1} \mid COL_j(e_i) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases}$$

$$J_i = \{j \in J_{i-1} \mid COL(e_i) = COL_j(e_i)\}$$

One can show by induction that, for every i, J_i is infinite. Hence this process *never* stops.

Claim: If K_{N} is 2-colored with COL, then there is no large homogeneous set whose smallest element is at least as large as m.

Proof of Claim:

Suppose, by way of contradiction, that there is a large homogeneous set whose smallest element is at least as large as m. Without loss of generality, we can assume that the size of the large homogeneous set is equal to its smallest element. Let the vertices of that large homogeneous set be $v_1, v_2, \ldots v_{v_1}$, where $m \le v_1 < v_2 < \cdots < v_{v_1}$, and let the edges between those vertices be

$$e_{i_1}, \ldots, e_{i_M},$$

where $i_1 < i_2 < \cdots < i_M$ and $M = \binom{v_1}{2}$. For every $j \in J_{i_M}$, COL_j and COL agree on the colors of those edges. Choose $j \in J_{i_M}$ so that all the vertices of the large homogeneous set are elements of the vertex set of K^m_{m+j-2}. Then COL_j is a 2-coloring of the edges of K^m_{m+j-2} that has a large homogeneous set, in contradiction to the definition of COL_j.

End of Proof of Claim

Hence we have produced a 2-coloring of K_{N} that has no large homogeneous set whose smallest element is at least as large as m. This contradicts Theorem 7.7. Therefore, our initial supposition—that $LR(m)$ does not exist—is false. ∎

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