Ramsey’s Theorem on Graphs

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1 Introduction

Imagine that you have 6 people at a party. We assume that, for every pair of them, either THEY KNOW EACH OTHER or NEITHER OF THEM KNOWS THE OTHER. So we are assuming that if $x$ knows $y$, then $y$ knows $x$.

**Claim:** Either there are at least 3 people all of whom know one another, or there are at least 3 people no two of whom know each other (or both).

**Proof of Claim:**

Let the people be $p_1, p_2, p_3, p_4, p_5, p_6$. Now consider $p_6$.

Among the other 5 people, either there are at least 3 people that $p_6$ knows, or there are at least 3 people that $p_6$ does not know.

Why is this? Well, suppose that, among the other 5 people, there are at most 2 people that $p_6$ knows, and at most 2 people that $p_6$ does not know. Then there are only 4 people other than $p_6$, which contradicts the fact that there are 5 people other than $p_6$.

Suppose that $p_6$ knows at least 3 of the others. We consider the case where $p_6$ knows $p_1, p_2, and p_3$. All the other cases are similar.

If $p_1$ knows $p_2$, then $p_1, p_2,$ and $p_6$ all know one another. HOORAY!

If $p_1$ knows $p_3$, then $p_1, p_3,$ and $p_6$ all know one another. HOORAY!

If $p_2$ knows $p_3$, then $p_2, p_3,$ and $p_6$ all know one another. HOORAY!

What if none of these scenarios holds? Then none of these three people $(p_1, p_2, p_3)$ knows either of the other 2. HOORAY!

*End of Proof of Claim*

We want to generalize this observation.

**Notation 1.1** $\mathbb{N}$ is the set of all positive integers. If $n \in \mathbb{N}$, then $[n]$ is the set $\{1, \ldots, n\}$.

**Def 1.2** A graph $G$ consists of a set $V$ of vertices and a set $E$ of edges. The edges are unordered pairs of vertices.
Note 1.3 In general, a graph can have an edge \( \{i, j\} \) with \( i = j \). Here, however, every edge of a graph is an unordered pair of distinct vertices (i.e., an unordered pair \( \{i, j\} \) with \( i \neq j \)).

Def 1.4 Let \( c \in \mathbb{N} \). Let \( G = (V, E) \) be a graph. A \( c \)-coloring of the edges of \( G \) is a function \( \text{COL} : E \to [c] \). Note that there are no restrictions on \( \text{COL} \).

Note 1.5 In the Graph Theory literature there are (at least) two kinds of coloring. We present them in this note so that if you happen to read the literature and they are using coloring in a different way than in these notes, you will not panic.

- Vertex Coloring. Usually one says that the vertices of a graph are \( c \)-colorable if there is a way to assign each vertex a color, using no more than \( c \) colors, such that no two adjacent vertices (vertices connected by an edge) are the same color. Theorems are often of the form ‘if a graph \( G \) has property BLAH BLAH then \( G \) is \( c \)-colorable’ where they mean vertex \( c \)-colorable. We will not be considering these kinds of colorings.

- Edge Colorings. Usually this is used in the context of Ramsey Theory and Ramsey-type theorems. Theorems begin with ‘for all \( c \)-coloring of a graph \( G \) BLAH BLAH happens’ We will be considering these kinds of colorings.

Def 1.6 Let \( n \in \mathbb{N} \). The complete graph on \( n \) vertices, denoted \( K_n \), is the graph

\[
V = [n] \\
E = \{\{i, j\} \mid i, j \in [n]\}
\]

Example 1.7 Let \( G \) be the complete graph on 10 vertices. Recall that the vertices are \( \{1, \ldots, 10\} \). We give a 3-coloring of the edges of \( G \):

\[
\text{COL}(\{x, y\}) = \begin{cases} 
1 & \text{if } x + y \equiv 1 \pmod{3} \\
2 & \text{if } x + y \equiv 2 \pmod{3} \\
3 & \text{if } x + y \equiv 0 \pmod{3}.
\end{cases}
\]
Let’s go back to our party! We can think of the 6 people as vertices of $K_6$. We can color edge $\{i, j\}$ RED if $i$ and $j$ know each other, and BLUE if they do not.

**Def 1.8** Let $G = (V, E)$ be a graph, and let $COL$ be a coloring of the edges of $G$. A set of edges of $G$ is *monochromatic* if they are all the same color.

Let $n \geq 2$. Then $G$ has a monochromatic $K_n$ if there is a set $V'$ of $n$ vertices (in $V$) such that

- there is an edge between every pair of vertices in $V'$: $\{\{i, j\} \mid i, j \in V'\} \subseteq E$
- all the edges between vertices in $V'$ are the same color: there is some $l \in [c]$ such that $COL(\{i, j\}) = l$ for all $i, j \in V'$

We now restate our 6-people-at-a-party theorem:

**Theorem 1.9** Every 2-coloring of the edges of $K_6$ has a monochromatic $K_3$.

## 2 The Full Theorem

From the last section, we know the following:

*If you want an $n$ such that you get a monochromatic $K_3$ no matter how you 2-color $K_n$, then $n = 6$ will suffice.*

What if you want to guarantee that there is a monochromatic $K_4$? What if you want to use 17 colors?

The following is known as *Ramsey’s Theorem*. It was first proved in [3] (see also [1], [2]).

**For all $c, m \geq 2$, there exists $n \geq m$ such that every $c$-coloring of $K_n$ has a monochromatic $K_m$.**

We will provide several proofs of this theorem for the $c = 2$ case. We will assume the colors are RED and BLUE (rather than the numbers 1 and 2). The general-$c$ case (where $c$ can be *any* integer $i \geq 2$) and other generalizations may show up on homework assignments.
3 First Proof of Ramsey’s Theorem

Given $m$, we really want $n$ such that every 2-coloring of $K_n$ has a RED $K_m$ or a BLUE $K_m$. However, it will be useful to let the parameter for BLUE differ from the parameter for RED.

**Notation 3.1** Let $a, b \geq 2$. Let $R(a, b)$ denote the least number, if it exists, such that every 2-coloring of $K_{R(a, b)}$ has a RED $K_a$ or a BLUE $K_b$. We abbreviate $R(a, a)$ by $R(a)$.

We state some easy facts.

1. For all $a, b$, $R(a, b) = R(b, a)$.

2. For $b \geq 2$, $R(2, b) = b$: First, we show that $R(2, b) \leq b$. Given any 2-coloring of $K_b$, we want a RED $K_2$ or a BLUE $K_b$. Note that a RED $K_2$ is just a RED edge. Hence EITHER there exists one RED edge (so you get a RED $K_2$) OR all the edges are BLUE (so you get a BLUE $K_b$). Now we prove that $R(2, b) = b$. If $b = 2$, this is obvious. If $b > 2$, then the all-BLUE coloring of $K_{b-1}$ has neither a RED $K_2$ nor a BLUE $K_b$, hence $R(2, b) \geq b$. Combining the two inequalities ($R(2, b) \leq b$ and $R(2, b) \geq b$), we find that $R(2, b) = b$.

3. $R(3, 3) \leq 6$ (we proved this in Section 1)

We want to show that, for every $n \geq 2$, $R(n, n)$ exists. In this proof, we show something more: that for all $a, b \geq 2$, $R(a, b)$ exists. We do not really care about the case where $a \neq b$, but that case will help us get our result. This is a situation where proving more than you need is easier.

**Theorem 3.2**

1. $R(2, b) = b$ (we proved this earlier)

2. For all $a, b \geq 3$: If $R(a - 1, b)$ and $R(a, b - 1)$ exist, then $R(a, b)$ exists and

   $$R(a, b) \leq R(a - 1, b) + R(a, b - 1)$$

3. For all $a, b \geq 2$, $R(a, b)$ exists and $R(a, b) \leq 2^{a+b}$. 
Proof:

Since we proved part 1 earlier, we now prove parts 2 and 3.

**Part 2** Assume $R(a-1, b)$ and $R(a, b-1)$ exist. Let

$$n = R(a - 1, b) + R(a, b - 1)$$

Let $COL$ be a 2-coloring of $K_n$, and let $x$ be a vertex. Note that there are

$$R(a - 1, b) + R(a, b - 1) - 1$$

edges coming out of $x$ (edges $\{x, y\}$ for vertices $y$).

Let $\text{NUM-RED-EDGES}$ be the number of red edges coming out of $x$, and let $\text{NUM-BLUE-EDGES}$ be the number of blue edges coming out of $x$. Note that

$$\text{NUM-RED-EDGES} + \text{NUM-BLUE-EDGES} = R(a - 1, b) + R(a, b - 1) - 1$$

Hence either

$$\text{NUM-RED-EDGES} \geq R(a - 1, b)$$

or

$$\text{NUM-BLUE-EDGES} \geq R(a, b - 1)$$

To see this, suppose, by way of contradiction, that both inequalities are false. Then

$$\text{NUM-RED-EDGES} + \text{NUM-BLUE-EDGES} \leq R(a - 1, b) - 1 + R(a, b - 1) - 1$$

$$= R(a - 1, b) + R(a, b - 1) - 2$$

$$< R(a - 1, b) + R(a, b - 1) - 1$$

There are two cases:

1. **Case 1:** $\text{NUM-RED-EDGES} \geq R(a - 1, b)$. Let

   $$U = \{ y \mid COL(\{x, y\}) = \text{RED} \}$$

   $U$ is of size $\text{NUM-RED-EDGES} \geq R(a-1, b)$. Consider the restriction of the coloring $COL$ to the edges between vertices in $U$. Since

   $$|U| \geq R(a - 1, b),$$

   this coloring has a RED $K_{a-1}$ or a BLUE $K_b$. Within Case 1, there are two cases:
(a) There is a RED $K_{a-1}$. Recall that all of the edges in
\[ \{x, u\} | u \in U \] are RED, hence all the edges between elements of the set $U \cup \{x\}$ are RED, so they form a RED $K_a$ and WE ARE DONE.

(b) There is a BLUE $K_b$. Then we are DONE.

2. **Case 2:** NUM-BLUE-EDGES \( \geq R(a, b-1) \). Similar to Case 1.

**Part 3** To show that $R(a, b)$ exists and $R(a, b) \leq 2a+b$, we use induction on $n = a + b$. Since $a, b \geq 2$, the smallest value of $a + b$ is 4. Thus $n \geq 4$.

**Base Case:** $n = 4$. Since $a + b = 4$ and $a, b \geq 2$, we must have $a = b = 2$. From part 1, we know that $R(2, 2)$ exists and $R(2, 2) = 2$. Note that

\[ R(2, 2) = 2 \leq 2^{2+2} = 16 \]

**Induction Hypothesis:** For all $a, b \geq 2$ such that $a + b = n$, $R(a, b)$ exists and $R(a, b) \leq 2^{a+b}$.

**Inductive Step:** Let $a, b$ be such that $a, b \geq 2$ and $a + b = n + 1$.

There are three cases:

1. **Case 1:** $a = 2$. By part 1, $R(2, b)$ exists and $R(2, b) = b$. Since $b \geq 2$, we have

\[ b \leq 2^b \leq 4 \cdot 2^b = 2^2 \cdot 2^b = 2^{2+b} \]

Hence $R(2, b) \leq 2^{2+b}$.

2. **Case 2:** $b = 2$. Follows from Case 1 and $R(a, b) = R(b, a)$.

3. **Case 3:** $a, b \geq 3$. Since $a, b \geq 3$, we have $a - 1 \geq 2$ and $b - 1 \geq 2$. Also, $a + b = n + 1$, so $(a - 1) + b = n$ and $a + (b - 1) = n$. By the induction hypothesis, $R(a - 1, b)$ and $R(a, b - 1)$ exist; moreover,

\[ R(a - 1, b) \leq 2^{(a-1)+b} = 2^{a+b-1} \]
\[ R(a, b - 1) \leq 2^{a+(b-1)} = 2^{a+b-1} \]

From part 3, $R(a, b)$ exists and

\[ R(a, b) \leq R(a - 1, b) + R(a, b - 1) \]
Hence

\[ R(a, b) \leq R(a - 1, b) + R(a, b - 1) \leq 2^{a+b-1} + 2^{a+b-1} = 2 \cdot 2^{a+b-1} = 2^{a+b} \]

\[ \blacksquare \]

**Corollary 3.3** For every \( m \geq 2 \), \( R(m) \) exists and \( R(m) \leq 2^{2m} \).

### 4 Second Proof of Ramsey’s Theorem

We now present a proof that does not use \( R(a, b) \). It also gives a mildly better bound on \( R(m) \) than the one given in Corollary 3.3.

**Theorem 4.1** For every \( m \geq 2 \), \( R(m) \) exists and \( R(m) \leq 2^{2m-2} \).

**Proof:**

Let \( COL \) be a 2-coloring of \( K_{2^{2m-2}} \). We define a sequence of vertices,

\[ x_1, x_2, \ldots, x_{2m-1}, \]

and a sequence of sets of vertices,

\[ V_0, V_1, V_2, \ldots, V_{2m-1}, \]

that are based on \( COL \).

Here is the intuition: Vertex \( x_1 = 1 \) has \( 2^{2m-2} - 1 \) edges coming out of it. Some are RED, and some are BLUE. Hence there are at least \( 2^{2m-3} \) RED edges coming out of \( x_1 \), or there are at least \( 2^{2m-3} \) BLUE edges coming out of \( x_1 \). To see this, suppose, by way of contradiction, that it is false, and let \( N.E. \) be the total number of edges coming out of \( x_1 \). Then

\[ N.E. \leq (2^{2m-3} - 1) + (2^{2m-3} - 1) = (2 \cdot 2^{2m-3}) - 2 = 2^{2m-2} - 2 \leq 2^{2m-2} - 1 \]

Let \( c_1 \) be a color such that \( x_1 \) has at least \( 2^{2m-3} \) edges coming out of it that are colored \( c_1 \). Let \( V_1 \) be the set of vertices \( v \) such that \( COL(\{v, x_1\}) = c_1 \). Then keep iterating this process.

We now describe it formally.
\[ V_0 = [2^{2m-2}] \]
\[ x_1 = 1 \]

\[ c_1 = \begin{cases} 
    \text{RED} & \text{if } |\{v \in V_0 \mid COL(\{v, x_1\}) = \text{RED}\}| \geq 2^{2m-3} \\
    \text{BLUE} & \text{otherwise}
\end{cases} \]

\[ V_1 = \{ v \in V_0 \mid COL(\{v, x_1\}) = c_1 \} \] (note that \(|V_1| \geq 2^{2m-3}\))

Let \( i \geq 2 \), and assume that \( V_{i-1} \) is defined. We define \( x_i, c_i, \) and \( V_i \):

\[ x_i = \text{the least number in } V_{i-1} \]

\[ c_i = \begin{cases} 
    \text{RED} & \text{if } |\{v \in V_{i-1} \mid COL(\{v, x_i\}) = \text{RED}\}| \geq 2^{(2m-2)-i} \\
    \text{BLUE} & \text{otherwise.}
\end{cases} \]

\[ V_i = \{ v \in V_{i-1} \mid COL(\{v, x_i\}) = c_i \} \] (note that \(|V_i| \geq 2^{(2m-2)-i}\))

How long can this sequence go on for? Well, \( x_i \) can be defined if \( V_{i-1} \) is nonempty. Note that

\[ |V_{2m-2}| \geq 2^{(2m-2)-(2m-2)} = 2^0 = 1 \]

Thus if \( i-1 = 2m-2 \) (equivalently, \( i = 2m-1 \)), then \( V_{i-1} = V_{2m-2} \neq \emptyset \), but there is no guarantee that \( V_i (= V_{2m-1}) \) is nonempty. Hence we can define

\[ x_1, \ldots, x_{2m-1} \]

Consider the colors

\[ c_1, c_2, \ldots, c_{2m-2} \]

Each of these is either RED or BLUE. Hence there must be at least \( m-1 \) of them that are the same color. Let \( i_1, \ldots, i_{m-1} \) be such that \( i_1 < \cdots < i_{m-1} \) and

\[ c_{i_1} = c_{i_2} = \cdots = c_{i_{m-1}} \]

Denote this color by \( c \), and consider the \( m \) vertices

\[ x_{i_1}, x_{i_2}, \cdots, x_{i_{m-1}}, x_{i_{m-1}+1} \]
To see why we have listed $m$ vertices but only $m - 1$ colors, picture the following scenario: You are building a fence row, and you want (say) 7 sections of fence. To do that, you need 8 fence posts to hold it up. Now think of the fence posts as vertices, and the sections of fence as edges between successive vertices, and recall that every edge has a color associated with it.

**Claim:** The $m$ vertices listed above form a monochromatic $K_m$.

**Proof of Claim:**

First, consider vertex $x_{i_1}$. The vertices

$$x_{i_2}, \ldots, x_{i_{m-1}}, x_{i_{m-1}+1}$$

are elements of $V_{i_1}$, hence the edges

$$\{x_{i_1}, x_{i_2}\}, \ldots, \{x_{i_1}, x_{i_{m-1}}\}, \{x_{i_1}, x_{i_{m-1}+1}\}$$

are colored with $c_{i_1} (= c)$.

Then consider each of the remaining vertices in turn, starting with vertex $x_{i_2}$. For example, the vertices

$$x_{i_3}, \ldots, x_{i_{m-1}}, x_{i_{m-1}+1}$$

are elements of $V_{i_2}$, hence the edges

$$\{x_{i_2}, x_{i_3}\}, \ldots, \{x_{i_2}, x_{i_{m-1}}\}, \{x_{i_2}, x_{i_{m-1}+1}\}$$

are colored with $c_{i_2} (= c)$.

*End of Proof of Claim*  

## 5 Proof of the Infinite Ramsey Theorem

We now consider infinite graphs.

**Notation 5.1** $K_N$ is the graph $(V, E)$ where

$$V = N$$

$$E = \{\{x, y\} \mid x, y \in N\}$$
**Def 5.2** Let $G = (V, E)$ be a graph with $V = \mathbb{N}$, and let $\text{COL}$ be a coloring of the edges of $G$. A set of edges of $G$ is *monochromatic* if they are all the same color (this is the same as for a finite graph).

$G$ has a monochromatic $K_N$ if there is an infinite set $V'$ of vertices (in $V$) such that

- there is an edge between every pair of vertices in $V'$
- all the edges between vertices in $V'$ are the same color

**Theorem 5.3** Every 2-coloring of the edges of $K_N$ has a monochromatic $K_N$. 

**Proof:**

(Note: this proof is similar to the proof of Theorem 4.1.)

Let $\text{COL}$ be a 2-coloring of $K_N$. We define an infinite sequence of vertices, $x_1, x_2, \ldots$, and an infinite sequence of sets of vertices, $V_0, V_1, V_2, \ldots$, that are based on $\text{COL}$.

Here is the intuition: Vertex $x_1 = 1$ has an infinite number of edges coming out of it. Some are RED, and some are BLUE. Hence there are an infinite number of RED edges coming out of $x_1$, or there are an infinite number of BLUE edges coming out of $x_1$ (or both). Let $c_1$ be a color such that $x_1$ has an infinite number of edges coming out of it that are colored $c_1$. Let $V_1$ be the set of vertices $v$ such that $\text{COL}(\{v, x_1\}) = c_1$. Then keep iterating this process.

We now describe it formally.

\[
\begin{align*}
V_0 &= V \\
x_1 &= 1 \\
c_1 &= \begin{cases} 
\text{RED} & \text{if } |\{v \in V_0 \mid \text{COL}(\{v, x_1\}) = \text{RED}\}| \text{ is infinite} \\
\text{BLUE} & \text{otherwise}
\end{cases} \\
V_1 &= \{v \in V_0 \mid \text{COL}(\{v, x_1\}) = c_1\} \text{ (note that } |V_1| \text{ is infinite)}
\end{align*}
\]
Let $i \geq 2$, and assume that $V_{i-1}$ is defined. We define $x_i$, $c_i$, and $V_i$:

- $x_i =$ the least number in $V_{i-1}$
- $c_i =$ \{ RED if $|\{v \in V_{i-1} \mid \text{COL}(\{v, x_i\}) = \text{RED}\}|$ is infinite
  
- BLUE otherwise

- $V_i =$ \{ $v \in V_{i-1} \mid \text{COL}(\{v, x_i\}) = c_i$ \} (note that $|V_i|$ is infinite)

How long can this sequence go on for? Well, $x_i$ can be defined if $V_{i-1}$ is nonempty. We an show by induction that, for every $i$, $V_i$ is infinite. Hence the sequence

$x_1, x_2, \ldots,$

is infinite.

Consider the infinite sequence

$c_1, c_2, \ldots$

Each of the colors in this sequence is either RED or BLUE. Hence there must be an infinite sequence $i_1, i_2, \ldots$ such that $i_1 < i_2 < \cdots$ and

$c_{i_1} = c_{i_2} = \cdots$

Denote this color by $c$, and consider the vertices

$x_{i_1}, x_{i_2}, \cdots$

Using an argument similar to the one we used in the proof of Theorem 4.1 (to show that we had a monochromatic $K_m$), we can show that these vertices form a monochromatic $K_N$.

6 Finite Ramsey from Infinite Ramsey

Picture the following scenario: Our first lecture on the Ramsey Theorem began by proving Theorem 5.3. This is not absurd: The proof we gave of the infinite Ramsey Theorem does not need some of the details that are needed in the proof we gave of the finite Ramsey Theorem.

Having proved the infinite Ramsey Theorem, we then want to prove the finite Ramsey Theorem. Can we prove the finite Ramsey Theorem from the infinite Ramsey Theorem? Yes, we can!
Theorem 6.1 For every $m \geq 2$, $R(m)$ exists.

Proof: Suppose, by way of contradiction, that there is some $m \geq 2$ such that $R(m)$ does not exist. Then, for every $n \geq m$, there is some way to color $K_n$ so that there is no monochromatic $K_m$. Hence there exist the following:

1. $COL_1$, a 2-coloring of $K_m$ that has no monochromatic $K_m$
2. $COL_2$, a 2-coloring of $K_{m+1}$ that has no monochromatic $K_m$
3. $COL_3$, a 2-coloring of $K_{m+2}$ that has no monochromatic $K_m$

... 

j. $COL_j$, a 2-coloring of $K_{m+j-1}$ that has no monochromatic $K_m$

We will use these 2-colorings to form a 2-coloring $COL$ of $K_N$ that has no monochromatic $K_m$.

Let $e_1, e_2, e_3, \ldots$ be a list of all unordered pairs of elements of $N$ such that every unordered pair appears exactly once. We will color $e_1$, then $e_2$, etc.

How should we color $e_1$? We will color it the way an infinite number of the $COL_i$’s color it. Call that color $c_1$. Then how to color $e_2$? Well, first consider ONLY the colorings that colored $e_1$ with color $c_1$. Color $e_2$ the way an infinite number of those colorings color it. And so forth.

We now proceed formally:

$$J_0 = N$$

$$COL(e_1) = \begin{cases} \text{RED} & \text{if } |\{j \in J_0 \mid COL_j(e_1) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases}$$

$$J_1 = \{j \in J_0 \mid COL(e_1) = COL_j(e_1)\}$$

Let $i \geq 2$, and assume that $e_1, \ldots, e_{i-1}$ have been colored. Assume, furthermore, that $J_{i-1}$ is infinite and, for every $j \in J_{i-1}$,
\[ \begin{align*}
\text{COL}(e_1) &= \text{COL}_j(e_1) \\
\text{COL}(e_2) &= \text{COL}_j(e_2) \\
&\quad \vdots \\
\text{COL}(e_{i-1}) &= \text{COL}_j(e_{i-1})
\end{align*} \]

We now color \( e_i \):

\[ \text{COL}(e_i) = \begin{cases} 
\text{RED} & \text{if } |\{j \in J_{i-1} | \text{COL}_j(e_i) = \text{RED}\}| \text{ is infinite} \\
\text{BLUE} & \text{otherwise}
\end{cases} \]

\[ J_i = \{ j \in J_{i-1} | \text{COL}(e_i) = \text{COL}_j(e_i) \} \]

One can show by induction that, for every \( i \), \( J_i \) is infinite. Hence this process never stops.

**Claim:** If \( K_N \) is 2-colored with \( \text{COL} \), then there is no monochromatic \( K_m \).

**Proof of Claim:**

Suppose, by way of contradiction, that there is a monochromatic \( K_m \). Let the edges between vertices in that monochromatic \( K_m \) be

\( e_{i_1}, \ldots, e_{i_M} \),

where \( i_1 < i_2 < \cdots < i_M \) and \( M = \binom{m}{2} \). For every \( j \in J_{i_M} \), \( \text{COL}_j \) and \( \text{COL} \) agree on the colors of those edges. Choose \( j \in J_{i_M} \) so that all the vertices of the monochromatic \( K_m \) are elements of the vertex set of \( K_{m+j-1} \). Then \( \text{COL}_j \) is a 2-coloring of the edges of \( K_{m+j-1} \) that has a monochromatic \( K_m \), in contradiction to the definition of \( \text{COL}_j \).

**End of Proof of Claim**

Hence we have produced a 2-coloring of \( K_N \) that has no monochromatic \( K_m \). This contradicts Theorem 5.3. Therefore, our initial supposition—that \( R(m) \) does not exist—is false.

Note that two of our proofs of the finite Ramsey Theorem (the proofs of Theorems 3.2 and 4.1) give upper bounds on \( R(m) \), but that our proof of the finite Ramsey Theorem from the infinite Ramsey Theorem gives no upper bound on \( R(m) \).
7 Proof of Large Ramsey Theorem

In all of the theorems presented earlier, the labels on the vertices did not matter. In this section, the labels do matter.

Def 7.1 A finite set $F \subseteq \mathbb{N}$ is called large if the size of $F$ is at least as large as the smallest element of $F$.

Example 7.2

1. The set $\{1, 2, 10\}$ is large: It has 3 elements, the smallest element is 1, and $3 \geq 1$.

2. The set $\{5, 10, 12, 17, 20\}$ is large: It has 5 elements, the smallest element is 5, and $5 \geq 5$.

3. The set $\{20, 30, 40, 50, 60, 70, 80, 90, 100\}$ is not large: It has 9 elements, the smallest element is 20, and $9 < 20$.

4. The set $\{5, 30, 40, 50, 60, 70, 80, 90, 100\}$ is large: It has 9 elements, the smallest element is 5, and $9 \geq 5$.

5. The set $\{101, \ldots, 190\}$ is not large: It has 90 elements, the smallest element is 101, and $90 < 101$.

We will be considering monochromatic $K_m$’s where the underlying set of vertices is a large set. We need a definition to identify the underlying set.

Def 7.3 Let $COL$ be a 2-coloring of $K_n$. A set $A$ of vertices is homogeneous if there exists a color $c$ such that, for all $x, y \in A$ with $x \neq y$, $COL(\{x, y\}) = c$. In other words, all of the edges between elements of $A$ are the same color. One could also say that there is a monochromatic $K_{|A|}$.

Let $COL$ be a 2-coloring of $K_n$. Recall that the vertex set of $K_n$ is $\{1, 2, \ldots, n\}$. Consider the set $\{1, 2\}$. It is clearly both homogeneous and large (using our definition of large). Hence the statement

“for every $n \geq 2$, every 2-coloring of $K_n$ has a large homogeneous set”
is true but trivial.

What if we used $V = \{m, m + 1, \ldots, m + n\}$ as our vertex set? Then a large homogeneous set would have to have size at least $m$.

**Notation 7.4** $K_n^m$ is the graph with vertex set $\{m, m + 1, \ldots, m + n\}$ and edge set consisting of all unordered pairs of vertices. The superscript $(m)$ indicates that we are labeling our vertices starting with $m$, and the subscript $(n)$ is one less than the number of vertices.

**Note 7.5** The vertex set of $K_n^m$ (namely, $\{m, m + 1, \ldots, m + n\}$) has $n + 1$ elements. Hence if $K_n^m$ has a large homogeneous set, then $n + 1 \geq m$ (equivalently, $n \geq m - 1$). We could have chosen to use $K_n^m$ to denote the graph with vertex set $\{m + 1, \ldots, m + n\}$, so that the smallest vertex is $m + 1$ and the number of vertices is $n$, but the set we have designated as $K_n^m$ will better serve our purposes.

**Notation 7.6** $LR(m)$ is the least $n$, if it exists, such that every 2-coloring of $K_n^m$ has a large homogeneous set.

We first prove a theorem about infinite graphs and large homogeneous sets.

**Theorem 7.7** If $COL$ is any 2-coloring of $K_N$, then, for every $m \geq 2$, there is a large homogeneous set whose smallest element is at least as large as $m$.

**Proof:** Let $COL$ be any 2-coloring of $K_N$. By Theorem 5.3, there exist an infinite set of vertices, $v_1 < v_2 < v_3 < \cdots$, and a color $c$ such that, for all $i, j$, $COL(\{v_i, v_j\}) = c$. (This could be called an infinite homogeneous set.) Let $i$ be such that $v_i \geq m$. The set

$$\{v_i, \ldots, v_i+v_i-1\}$$

is a homogeneous set that contains $v_i$ elements and whose smallest element is $v_i$. Since $v_i \geq v_i$, it is a large set; hence it is a large homogeneous set. 

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Theorem 7.8 For every $m \geq 2$, $LR(m)$ exists.

Proof: This proof is similar to our proof of the finite Ramsey Theorem from the infinite Ramsey Theorem (the proof of Theorem 6.1).

Suppose, by way of contradiction, that there is some $m \geq 2$ such that $LR(m)$ does not exist. Then, for every $n \geq m - 1$, there is some way to color $K_n^m$ so that there is no large homogeneous set. Hence there exist the following:

1. $COL_1$, a 2-coloring of $K_{m-1}^m$ that has no large homogeneous set
2. $COL_2$, a 2-coloring of $K_m^m$ that has no large homogeneous set
3. $COL_3$, a 2-coloring of $K_{m+1}^m$ that has no large homogeneous set
   ...
   j. $COL_j$, a 2-coloring of $K_{m+j-2}^m$ that has no large homogeneous set
   ...

We will use these 2-colorings to form a 2-coloring $COL$ of $K_N$ that has no large homogeneous set whose smallest element is at least as large as $m$.

Let $e_1, e_2, e_3, \ldots$ be a list of all unordered pairs of elements of $\mathbb{N}$ such that every unordered pair appears exactly once. We will color $e_1$, then $e_2$, etc.

How should we color $e_1$? We will color it the way an infinite number of the $COL_i$’s color it. Call that color $c_1$. Then how to color $e_2$? Well, first consider ONLY the colorings that colored $e_1$ with color $c_1$. Color $e_2$ the way an infinite number of those colorings color it. And so forth.

We now proceed formally:

$$J_0 = \mathbb{N}$$

$$COL(e_1) = \begin{cases} \text{RED} & \text{if } |\{j \in J_0 \mid COL_j(e_1) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases}$$

$$J_1 = \{j \in J_0 \mid COL(e_1) = COL_j(e_1)\}$$

Let $i \geq 2$, and assume that $e_1, \ldots, e_{i-1}$ have been colored. Assume, furthermore, that $J_{i-1}$ is infinite and, for every $j \in J_{i-1}$,
\[ \begin{align*}
\text{COL}(e_1) &= \text{COL}_j(e_1) \\
\text{COL}(e_2) &= \text{COL}_j(e_2) \\
&\vdots \\
\text{COL}(e_{i-1}) &= \text{COL}_j(e_{i-1})
\end{align*} \]

We now color \( e_i \):

\[ \text{COL}(e_i) = \begin{cases} 
\text{RED} & \text{if } |\{ j \in J_{i-1} \mid \text{COL}_j(e_i) = \text{RED} \}| \text{ is infinite} \\
\text{BLUE} & \text{otherwise}
\end{cases} \]

\[ J_i = \{ j \in J_{i-1} \mid \text{COL}(e_i) = \text{COL}_j(e_i) \} \]

One can show by induction that, for every \( i \), \( J_i \) is infinite. Hence this process \textit{never} stops.

\textit{Claim:} If \( K_N \) is 2-colored with \( \text{COL} \), then there is no large homogeneous set whose smallest element is at least as large as \( m \).

\textit{Proof of Claim:}
Suppose, by way of contradiction, that there is a large homogeneous set whose smallest element is at least as large as \( m \). Without loss of generality, we can assume that the size of the large homogeneous set is equal to its smallest element. Let the vertices of that large homogeneous set be \( v_1, v_2, \ldots v_{v_1} \), where \( m \leq v_1 < v_2 < \cdots < v_{v_1} \), and let the edges between those vertices be \( e_{i_1}, \ldots, e_{i_M} \),

where \( i_1 < i_2 < \cdots < i_M \) and \( M = \binom{v_1}{2} \). For every \( j \in J_{i_M} \), \( \text{COL}_j \) and \( \text{COL} \) agree on the colors of those edges. Choose \( j \in J_{i_M} \) so that all the vertices of the large homogeneous set are elements of the vertex set of \( K_{m+j-2} \). Then \( \text{COL}_j \) is a 2-coloring of the edges of \( K_{m+j-2} \) that has a large homogeneous set, in contradiction to the definition of \( \text{COL}_j \).

\textit{End of Proof of Claim}

Hence we have produced a 2-coloring of \( K_N \) that has no large homogeneous set whose smallest element is at least as large as \( m \). This contradicts Theorem 7.7. Therefore, our initial supposition—that \( LR(m) \) does not exist—is false. \( \blacksquare \)
References

