The Distinct Volumes Problem

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INITIAL MOTIVATION

1. **Infinite Ramsey Theorem:** For any 2-coloring of the EDGES of K_ω there exists an infinite *monochromatic* K_ω.
2. **Infinite Canonical Ramsey Theorem:** For any ω-coloring of the EDGES of K_ω there exists an infinite *monochromatic* K_ω OR an infinite *rainbow* K_ω OR OTHER STUFF
3. **Want an "application"**. Give an infinite set of points in the plane, color pairs by the distance between.

Result: For any infinite set of points in the plane there is an infinite subset where all distances are distinct. (Already known by Erdös via diff proof.)

Next Step: Finite version: For every set of n points in the plane there is a subset of size $\Omega(\log n)$ where all distances are distinct. (Much better is known.)

INITIAL MOTIVATION ABANDONED

1. Dumped Ramsey approach**!** Added co-authors**!** Got **new** results!
2. What about **Area**? If there are n points in $\mathbb{R}^2$ want large subset so that all areas are distinct.
3. More general question: n points in $\mathbb{R}^d$ and looking for all a-volumes to be different. (This question seems to be new.)

EXAMPLES with DISTANCES

The following is an **EXAMPLE** of the kind of theorems we will be talking about.
If there are n points in $\mathbb{R}^2$ then there is a subset of size $\Omega(n^{1/3})$ with all distances between points **DIFF**.

EXAMPLES with AREAS

If there are n points in $\mathbb{R}^2$ then there is a subset of size $\Omega(n^{1/5})$ with all triangle areas **DIFF**.

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FALSE: Take n points on a LINE. All triangle areas are 0.

EXAMPLES with AREAS

If there are n points in $\mathbb{R}^2$ then there is a subset of size $\Omega(n^{1/5})$ with all triangle areas **DIFF**.

FALSE: Take n points on a LINE. All triangle areas are 0.

Two ways to modify:

1. *If there are n points in $\mathbb{R}^2$, no three collinear, then there is a subset of size $\Omega(n^{1/5})$ with all triangle areas* **DIFF**.
2. *If there are n points in $\mathbb{R}^2$, then there is a subset of size $\Omega(n^{1/5})$ with all nonzero triangle areas* **DIFF**.

We state theorems in **no three collinear** form.

Maximal Rainbow Sets

Definition: A **(2)-Rainbow Set** is a set of points in $\mathbb{R}^d$ where all of the distances are distinct. Also called a **dist-rainbow.**
Definition: A **3-Rainbow Set** is a set of points in $\mathbb{R}^d$ where all nonzero areas of triangles are distinct. Also called an **area-rainbow.**
Definition: An ***a*-Rainbow Set** is a set of points in $\mathbb{R}^d$ where all nonzero a-volumes are distinct. An a-volume is the volume enclosed by a points. Also called a **vol-rainbow.**

Definition: Let $X \subseteq \mathbb{R}^d$. A **Maximal Rainbow Set** is a rainbow set $Y \subseteq X$ such that if any more points of X are added then it STOPS being a rainbow set.
Definition: Let $X \subseteq \mathbb{R}^d$. An ***a*-Maximal Rainbow Set** is a a-rainbow set $Y \subseteq X$ such that if any more points of X are added then it STOPS being an a-rainbow set.

Easy Lemma

Lemma If there is a MAP from X to Y that is $\leq c$-to-1 then $|Y| \geq |X|/c$.
We will call this LEMMA.

The $d = 1$ Case

Theorem: For all $X \subseteq \mathbb{R}^1$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/3})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

f maps an element of $X - M$ to **reason** $x \notin M$.
$f : X - M \to \binom{M}{2} \cup M \times \binom{M}{2}$
What is $f^{-1}(\{x_1, x_2\})$?

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$f : X - M \to \binom{M}{2} \cup M \times \binom{M}{2}$ is $\leq$ 2-to-1.

The $d = 1$ Case- Cont

$f : X - M \to \binom{M}{2} \cup M \times \binom{M}{2}$ is $\le$ 2-to-1.
Case 1: $|M| \ge n^{1/3}$ DONE!

Case 2: $|M| \le n^{1/3}$. So $|X - M| = \Theta(|X|)$. By LEMMA

$$\begin{aligned} |\binom{M}{2} + M \times \binom{M}{2}| &\ge 0.5|X - M| = \Omega(|X|) = \Omega(n) \\ M &\ge \Omega(n^{1/3}) \end{aligned}$$

On Circle

Theorem: For all $X \subseteq \mathbb{S}^1$ (the circle) of size n there exists a dist-rainbow subset of size $\Omega(n^{1/3})$.
Proof: Use **MAXIMAL DIST-RAINBOW SET**. Similar Proof.

Better is known

Better is known: In 1975 Komlos, Sulyok, Szemeredi showed:
Theorem: For all $X \subseteq \mathbb{S}^1$ or $\mathbb{R}^1$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/2})$.

This is optimal in $\mathbb{S}^1$ and $\mathbb{R}^1$
Theorem: If $X = \{1, \ldots, n\}$ then the largest dist-rainbow subset is of size $\leq (1 + o(1))n^{1/2}$.

The $d = 2$ Case

Theorem: For all $X \subseteq \mathbb{R}^2$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/6})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
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All INVERSE IMG's lie on LINES or CIRCLES.

The $d = 2$ Case- Cont

$f : X - M \rightarrow \binom{M}{2} \cup M \times \binom{M}{2}$

All INVERSE IMG's lie on LINES or CIRCLES. δ TBD.

Cases 1 and 2 induct into line and circle case.

Case 1: $(\exists x_1, x_2)[(f^{-1}(\{x_1, x_2\})| \geq n^{\delta}]$.

$\geq n^{\delta}$ points on a line, so rainbow set size $\geq \Omega(n^{\delta/3})$.

Case 2: $(\exists x_1, x_2, x_3)[|f^{-1}(\{x_1, x_2, x_3\})| \geq n^{\delta}]$.

$\geq n^{\delta}$ points on a circle, so rainbow set size $\geq \Omega(n^{\delta/3})$.

Case 3: $|M| \geq n^{1/6}$ DONE!

Case 4: Map is $\leq n^{\delta}$-to-1 AND $|X - M| = \Theta(|X|)$. By LEMMA

$$\begin{aligned} |\binom{M}{2} \cup M \times \binom{M}{2}| &\geq n/n^{\delta} = n^{1-\delta} \\ |M| &\geq \Omega(n^{(1-\delta)/3}) \end{aligned}$$

Set $\delta/3 = (1-\delta)/3$. $\delta = 1/2$. Get $\Omega(n^{1/6})$.

On Sphere

Theorem: For all $X \subseteq \mathbb{S}^2$ (surface of sphere) of size n there exists a dist-rainbow subset of size $\Omega(n^{1/6})$.
Proof: Use **MAXIMAL DIST-RAINBOW SET**. Similar Proof.

Note: Better is known: Charalambides showed $\Omega(n^{1/3})$.

General d Case

Theorem:
For all $X \subseteq \mathbb{R}^d$ of size n $\exists$ dist-rainbow subset of size $\Omega(n^{1/3d})$.
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Proof: Use **MAXIMAL DIST-RAINBOW SET** and induction.
Need result on $\mathbb{S}^d$ and $\mathbb{R}^d$ to get result for $\mathbb{S}^{d+1}$ and $\mathbb{R}^{d+1}$.

Note: Better is known. In 1995 Thiele showed $\Omega(n^{1/(3d-2)})$. But WE improved that!

General d Case- Much Better

Theorem: For all $d \geq 2$, for all $X \subseteq \mathbb{R}^d$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/(3d-3)}(\log n)^{\frac{1}{3}-\frac{2}{3d-3}})$.
Proof: Use **VARIANT ON MAX DIST-RAINBOW SET**

d	$n^{1/3d}$	$n^{1/(3d-3)}(\log n)^{\frac{1}{3}-\frac{2}{3d-3}}$
1	$n^{1/3}$	$--$
2	$n^{1/6}$	$n^{1/3}(\log n)^{-1/3}$
3	$n^{1/9}$	$n^{1/6}(\log n)^{0}$
4	$n^{1/12}$	$n^{1/9}(\log n)^{1/12}$
5	$n^{1/15}$	$n^{1/12}(\log n)^{1/6}$
6	$n^{1/18}$	$n^{1/15}(\log n)^{1/5}$

Can we do better? Best we can hope for is roughly $n^{1/d}$.

Area-$d = 2$ Case

Theorem: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.
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Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[AREA(x, x_1, x_2) = AREA(x, x_1, x_3)]$.
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What is $f^{-1}(\{x_1, x_2\}, \{x_1, x_3\})$? SEE NEXT SLIDE FOR GEOM LEMMA.

Lemma On Area

Lemma: Let L_1 and L_2 be lines in R^2.

$$\{p : AREA(L_1, p) = AREA(L_2, p)\}$$

is a line.

Sketch: $AREA(L_1, p) = AREA(L_2, p)$ iff

$|L_1| \times |L_1 - p| = |L_2| \times |L_2 - p|$ iff $\frac{|L_1 - p|}{|L_2 - p} = \frac{|L_1}{|L_2|}$. This is a line.

(Reboot) Area-$d = 2$ Case

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What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4\})$? By Lemma all points on it are on a line- so ≤ 2 points. FINITE.

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4, x_5\})$? By Lemma all points on it are on a line- so ≤ 2 points. FINITE.

$f : X - M \to \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$ FINITE-to-1.

Area $d = 2$ Case- Cont

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$ is FINITE-to-1.
Case 1: $|M| \geq n^{1/5}$ DONE!

Case 2: $|M| \leq n^{1/5}$. Then $|X - M| = \Theta(|X|)$. Since MAP is finite-to-1, by LEMMA

$$\begin{aligned} |\binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}| &\geq \Omega(|X - M|) = \Omega(|X|) = \Omega(n) \\ |M| &\geq \Omega(n^{1/5}) \end{aligned}$$

Volume $d = 3$

Theorem: For all $X \subseteq \mathbb{R}^3$ of size n, no four on a plane, there exists Vol-rainbow set of size $\Omega(n^\delta)$. (δ TBD)
Similar. Left for the reader.

KEY to These Proofs

1. Used **MAXIMAL a-RAINBOW SET** M.
2. Used Map f from $x \in X - M$ to the reason x is NOT in M.
3. Looked at **INVERSE IMAGES** of that map.
4. Either:
   All INVERSE IMG's are small, so use LEMMA.
   OR
   Some INVERSE IMG's are large subsets of $\mathbb{R}^d$ or $\mathbb{S}^d$, so induct.

Area-$d = 3$ Case

Theorem: For all $X \subseteq \mathbb{R}^3$ of size n, no three colinear, there exists Area-rainbow set of size $\Omega(n^\delta)$. (δ TBD)
Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[AREA(x, x_1, x_2) = AREA(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[A(x, x_1, x_2) = AREA(x, x_3, x_4)]$.
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f maps an element of $X - M$ to **reason** $x \notin M$.
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What to do?

WHAT CHANGED?

Why is this proof harder?
KEY statement about prior proof:

1. If INVERSE IMG's are all finite so M is large.
2. If INVERSE IMG's are subsets of $\mathbb{R}^d$ or $\mathbb{S}^d$ then induct.

KEY: We cared about $X \subseteq \mathbb{R}^d$ but had to work with $\mathbb{S}^d$ as well.
NOW we will have to work with more complicated objects.

What Do Inverse Images Look Like?

$$\{x : AREA(x, x_1, x_2) = AREA(x, x_3, x_4)\} =$$

$$\{x : |DET(x, x_1, x_2)| = |DET(x, x_3, x_4)|\}.$$

Definition: (Informally) An **Algebraic Variety in $\mathbb{R}^d$** is a set of points in $\mathbb{R}^d$ that satisfy a polynomial equation in d variables.

General Theorem

Theorem Let $2 \le a \le d+1$. Let $r \in \mathbb{N}$. For all varieties V of dim d and degree r for all sets of n points on V there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

Corollary Let $2 \le a \le d+1$. For all $X \subseteq \mathbb{R}^d$ of size n there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there exists a 2-rainbow set (dist. distances) of size $\Omega(n^{1/3d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there is a 3-rainbow set (dist. areas) of size $\Omega(n^{1/5d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there is a 4-rainbow set (dist. volumes) of size $\Omega(n^{1/7d})$.

Comments on the Proof

1. Proof uses Algebraic Geometry in Proj Space over $\mathbb{C}$.
2. Proof uses Maximal subsets in same way as easier proofs.
3. Proof is by induction on d.

Open Questions

1. Better Particular Results: e.g., want
   for all $X \subseteq \mathbb{R}^2$ *of size* n*, there exists a rainbow set of size* $\Omega(n^{1/2})$*.*
2. General Better Results: e.g., want
   Let $1 \le a \le d+1$*. For all* $X \subseteq \mathbb{R}^d$ *of size* n *there exists a rainbow set of size* $\Omega(n^{1/ad})$*.*
3. Get easier proofs of general theorem.
4. Find any nontrivial limits on what we can do. (Trivial: $n^{1/d}$).
5. Algorithmic aspects.