The Distinct Volumes Problem

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1. **Infinite Ramsey Theorem:** For any 2-coloring of the edges of $K_\omega$ there exists an infinite *monochromatic* $K_\omega$.

2. **Infinite Canonical Ramsey Theorem:** For any $\omega$-coloring of the edges of $K_\omega$ there exists an infinite *monochromatic* $K_\omega$ OR an infinite *rainbow* $K_\omega$ OR OTHER STUFF

3. **Want an “application”**. Give an infinite set of points in the plane, color pairs by the distance between.

**Result:** For any infinite set of points in the plane there is an infinite subset where all distances are distinct. (Already known by Erdős via diff proof.)

**Next Step:** Finite version: For every set of $n$ points in the plane there is a subset of size $\Omega(\log n)$ where all distances are distinct. (Much better is known.)
1. Dumped Ramsey approach! Added co-authors! Got new results!

2. What about **Area**? If there are $n$ points in $\mathbb{R}^2$ want large subset so that all areas are distinct.

3. More general question: $n$ points in $\mathbb{R}^d$ and looking for all $a$-volumes to be different. (This question seems to be new.)
The following is an **EXAMPLE** of the kind of theorems we will be talking about.

*If there are *n* points in $\mathbb{R}^2$ then there is a subset of size $\Omega(n^{1/3})$ with all distances between points **DIFF**.*
EXAMPLES with AREAS

If there are $n$ points in $\mathbb{R}^2$ then there is a subset of size $\Omega(n^{1/5})$ with all triangle areas DIFF.
EXAMPLES with AREAS

*If there are* $n$ *points in* $\mathbb{R}^2$ *then there is a subset of size* $\Omega(n^{1/5})$ *with all triangle areas DIFF.*

**FALSE:** Take $n$ points on a LINE. All triangle areas are 0.
If there are $n$ points in $\mathbb{R}^2$ then there is a subset of size $\Omega(n^{1/5})$ with all triangle areas DIFF.

**FALSE:** Take $n$ points on a LINE. All triangle areas are 0.

Two ways to modify:

1. *If there are $n$ points in $\mathbb{R}^2$, no three collinear, then there is a subset of size $\Omega(n^{1/5})$ with all triangle areas DIFF.*

2. *If there are $n$ points in $\mathbb{R}^2$, then there is a subset of size $\Omega(n^{1/5})$ with all nonzero triangle areas DIFF.*

We state theorems in **no three collinear** form.
Maximal Rainbow Sets

**Definition:** A (2)-Rainbow Set is a set of points in \( \mathbb{R}^d \) where all of the distances are distinct. Also called a dist-rainbow.

**Definition:** A 3-Rainbow Set is a set of points in \( \mathbb{R}^d \) where all nonzero areas of triangles are distinct. Also called an area-rainbow.

**Definition:** An \( a \)-Rainbow Set is a set of points in \( \mathbb{R}^d \) where all nonzero \( a \)-volumes are distinct. An \( a \)-volume is the volume enclosed by \( a \) points. Also called a vol-rainbow.

**Definition:** Let \( X \subseteq \mathbb{R}^d \). A Maximal Rainbow Set is a rainbow set \( Y \subseteq X \) such that if any more points of \( X \) are added then it STOPS being a rainbow set.

**Definition:** Let \( X \subseteq \mathbb{R}^d \). An \( a \)-Maximal Rainbow Set is a \( a \)-rainbow set \( Y \subseteq X \) such that if any more points of \( X \) are added then it STOPS being an \( a \)-rainbow set.
Lemma If there is a MAP from $X$ to $Y$ that is $\leq c$-to-1 then $|Y| \geq |X|/c$.
We will call this LEMMA.
The $d = 1$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^1$ of size $n$ there exists a dist-rainbow subset of size $\Omega(n^{1/3})$.

**Proof:** Let $M$ be a **MAXIMAL DIST-RAINBOW SET**. Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

$f$ maps an element of $X - M$ to reason $x \notin M$.

$f : X - M \rightarrow \binom{M}{2} \cup M \times \binom{M}{2}$

What is $f^{-1}(\{x_1, x_2\})$?
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$f : X - M \to \binom{M}{2} \cup M \times \binom{M}{2}$ is $\leq 2$-to-1.
The $d = 1$ Case- Cont

$f : X - M \rightarrow \binom{M}{2} \cup M \times \binom{M}{2}$ is $\leq$ 2-to-1.

**Case 1:** $|M| \geq n^{1/3}$ DONE!

**Case 2:** $|M| \leq n^{1/3}$. So $|X - M| = \Theta(|X|)$. By LEMMA

\[
\left| \binom{M}{2} + M \times \binom{M}{2} \right| \geq 0.5|X - M| = \Omega(|X|) = \Omega(n)
\]

$M \geq \Omega(n^{1/3})$
Theorem: For all $X \subseteq S^1$ (the circle) of size $n$ there exists a dist-rainbow subset of size $\Omega(n^{1/3})$.

Proof: Use MAXIMAL DIST-RAINBOW SET. Similar Proof.
Better is known: In 1975 Komlos, Sulyok, Szemeredi showed:

**Theorem:** For all $X \subseteq S^1$ or $\mathbb{R}^1$ of size $n$ there exists a dist-rainbow subset of size $\Omega(n^{1/2})$.

This is optimal in $S^1$ and $\mathbb{R}^1$

**Theorem:** If $X = \{1, \ldots, n\}$ then the largest dist-rainbow subset is of size $\leq (1 + o(1))n^{1/2}$. 
The $d = 2$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^2$ of size $n$ there exists a dist-rainbow subset of size $\Omega(n^{1/6})$.

**Proof:** Let $M$ be a **MAXIMAL DIST-RAINBOW SET**. Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.
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What is $f^{-1}(\{x_1, x_2\})$? Lies on LINE.
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What is $f^{-1}(x_1, \{x_2, x_3\})$?
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What is $f^{-1}(x_1, \{x_2, x_3\})$? Lies on CIRCLE.
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All INVERSE IMG’s lie on LINES or CIRCLES.
The $d = 2$ Case - Cont

$f : X - M \rightarrow (\binom{M}{2}) \cup M \times (\binom{M}{2})$
All INVERSE IMG’s lie on LINES or CIRCLES. $\delta$ TBD.
Cases 1 and 2 induct into line and circle case.

Case 1: $(\exists x_1, x_2)[(f^{-1}(\{x_1, x_2\})| \geq n^\delta]$
$\geq n^\delta$ points on a line, so rainbow set size $\geq \Omega(n^{\delta/3})$.

Case 2: $(\exists x_1, x_2, x_3)[|f^{-1}(\{x_1, x_2, x_3\})| \geq n^\delta]$
$\geq n^\delta$ points on a circle, so rainbow set size $\geq \Omega(n^{\delta/3})$.

Case 3: $|M| \geq n^{1/6}$ DONE!

Case 4: Map is $\leq n^\delta$-to-1 AND $|X - M| = \Theta(|X|)$. By LEMMA

$|\binom{M}{2} \cup M \times \binom{M}{2}| \geq \frac{n}{n^\delta} = n^{1-\delta}$
$|M| \geq \Omega(n^{(1-\delta)/3})$

Set $\delta/3 = (1 - \delta)/3$. $\delta = 1/2$. Get $\Omega(n^{1/6})$. 
On Sphere

**Theorem:** For all $X \subseteq S^2$ (surface of sphere) of size $n$ there exists a dist-rainbow subset of size $\Omega(n^{1/6})$.

**Proof:** Use **MAXIMAL DIST-RAINBOW SET**. Similar Proof.

**Note:** Better is known: Charalambides showed $\Omega(n^{1/3})$. 
General $d$ Case

**Theorem:**
For all $X \subseteq \mathbb{R}^d$ of size $n \exists$ dist-rainbow subset of size $\Omega(n^{1/3d})$.
For all $X \subseteq \mathbb{S}^d$ of size $n \exists$ dist-rainbow subset of size $\Omega(n^{1/3d})$.

**Proof:** Use **MAXIMAL DIST-RAINBOW SET** and induction.
Need result on $\mathbb{S}^d$ and $\mathbb{R}^d$ to get result for $\mathbb{S}^{d+1}$ and $\mathbb{R}^{d+1}$.

**Note:** Better is known. In 1995 Thiele showed $\Omega(n^{1/(3d-2)})$. But WE improved that!
General $d$ Case - Much Better

**Theorem:** For all $d \geq 2$, for all $X \subseteq \mathbb{R}^d$ of size $n$ there exists a dist-rainbow subset of size $\Omega(n^{1/(3d-3)}(\log n)^{1/3 - 2/(3d-3)})$.

**Proof:** Use **VARIANT ON MAX DIST-RAINBOW SET**

<table>
<thead>
<tr>
<th>$d$</th>
<th>$n^{1/3d}$</th>
<th>$n^{1/(3d-3)}(\log n)^{1/3 - 2/(3d-3)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$n^{1/3}$</td>
<td>$-$</td>
</tr>
<tr>
<td>2</td>
<td>$n^{1/6}$</td>
<td>$n^{1/3}(\log n)^{-1/3}$</td>
</tr>
<tr>
<td>3</td>
<td>$n^{1/9}$</td>
<td>$n^{1/6}(\log n)^0$</td>
</tr>
<tr>
<td>4</td>
<td>$n^{1/12}$</td>
<td>$n^{1/9}(\log n)^{1/12}$</td>
</tr>
<tr>
<td>5</td>
<td>$n^{1/15}$</td>
<td>$n^{1/12}(\log n)^{1/6}$</td>
</tr>
<tr>
<td>6</td>
<td>$n^{1/18}$</td>
<td>$n^{1/15}(\log n)^{1/5}$</td>
</tr>
</tbody>
</table>

Can we do better? Best we can hope for is roughly $n^{1/d}$. 
Area-$d = 2$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^2$ of size $n$, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

**Proof:** Let $M$ be a **MAXIMAL AREA-RAINBOW SET**.

Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

$\Rightarrow (\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]$.

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What is $f^{-1}([\{x_1, x_2\}, \{x_1, x_3\}]$? SEE NEXT SLIDE FOR GEOM LEMMA.
Lemma: Let $L_1$ and $L_2$ be lines in $\mathbb{R}^2$. 

\[ \{ p : \text{AREA}(L_1, p) = \text{AREA}(L_2, p) \} \]

is a line.

Sketch: $\text{AREA}(L_1, p) = \text{AREA}(L_2, p)$ iff 

\[ |L_1| \times |L_1 - p| = |L_2| \times |L_2 - p| \text{ iff } \frac{|L_1 - p|}{|L_2 - p|} = \frac{|L_1|}{|L_2|}. \] 

This is a line.
(Reboot) Area-\(d = 2\) Case

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\[\begin{align*}
&\text{\(\exists x_1, x_2, x_3 \in M\)} [\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]. \\
&\text{\(\exists x_1, x_2, x_3, x_4 \in M\)} [\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]. \\
&\text{\(\exists x_1, x_2, x_3, x_4, x_5 \in M\)} [\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)].
\end{align*}\]

\(f\) maps an element of \(X - M\) to reason \(x \notin M\).

\[f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3} .\]

Recall that

What is \(f^{-1}(\{x_1, x_2\}, \{x_1, x_3\})\)? By Lemma all points on it are on a line- so \(\leq 2\) points. **FINITE.**

What is \(f^{-1}(\{x_1, x_2\}, \{x_3, x_4\})\)? By Lemma all points on it are on a line- so \(\leq 2\) points. **FINITE.**

What is \(f^{-1}(\{x_1, x_2\}, \{x_3, x_4, x_5\})\)?
(Reboot) Area-$d = 2$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^2$ of size $n$, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

**Proof:** Let $M$ be a **MAXIMAL AREA-RAINBOW SET**. Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)]$.

$f$ maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$. Recall that

What is $f^{-1}(\{x_1, x_2\}, \{x_1, x_3\})$? By Lemma all points on it are on a line- so $\leq 2$ points. **FINITE**.

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4\})$? By Lemma all points on it are on a line- so $\leq 2$ points. **FINITE**.

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4, x_5\})$? By Lemma all points on it are on a line- so $\leq 2$ points. **FINITE**.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$ **FINITE-to-1**.
Area $d = 2$ Case- Cont

$f : X - M \rightarrow (M \binom{2}{2}) \times (M \binom{2}{2}) \cup (M \binom{2}{2}) \times (M \binom{3}{3})$ is FINITE-to-1.

**Case 1:** $|M| \geq n^{1/5}$ DONE!

**Case 2:** $|M| \leq n^{1/5}$. Then $|X - M| = \Theta(|X|)$. Since MAP is finite-to-1, by LEMMA

$$| (M \binom{2}{2}) \times (M \binom{2}{2}) \cup (M \binom{2}{2}) \times (M \binom{3}{3})| \geq \Omega(|X - M|) = \Omega(|X|) = \Omega(n)$$

$$|M| \geq \Omega(n^{1/5})$$
Theorem: For all $X \subseteq \mathbb{R}^3$ of size $n$, no four on a plane, there exists Vol-rainbow set of size $\Omega(n^\delta)$. ($\delta$ TBD)
Similar. Left for the reader.
KEY to These Proofs

1. Used **MAXIMAL a-RAINBOW SET** $M$.
2. Used Map $f$ from $x \in X - M$ to the reason $x$ is NOT in $M$.
3. Looked at **INVERSE IMAGES** of that map.
4. Either:
   - All INVERSE IMG’s are small, so use LEMMA.
   - OR
   - Some INVERSE IMG’s are large subsets of $\mathbb{R}^d$ or $S^d$, so induct.
Area-$d = 3$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^3$ of size $n$, no three colinear, there exists Area-rainbow set of size $\Omega(n^\delta)$. ($\delta$ TBD)

**Proof:** Let $M$ be a **MAXIMAL AREA-RAINBOW SET**.
Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

1. $(\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]$.
2. $(\exists x_1, x_2, x_3, x_4 \in M)[\text{A}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]$.
3. $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)]$.

$f$ maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$.

What is $f^{-1}(\{{\{x_1, x_2}\}, \{x_1, x_3\}})$?
Area-\( d = 3 \) Case

**Theorem:** For all \( X \subseteq \mathbb{R}^3 \) of size \( n \), no three colinear, there exists Area-rainbow set of size \( \Omega(n^\delta) \). (\( \delta \) TBD)

**Proof:** Let \( M \) be a **MAXIMAL AREA-RAINBOW SET**. Let \( x \in X - M \). WHY IS \( x \) NOT IN \( M \)!? Either

- (\( \exists x_1, x_2, x_3 \in M \))[\( \text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3) \)].
- (\( \exists x_1, x_2, x_3, x_4 \in M \))[\( \text{A}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4) \)].
- (\( \exists x_1, x_2, x_3, x_4, x_5 \in M \))[\( \text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5) \)].

\( f \) maps an element of \( X - M \) to reason \( x \notin M \).

\( f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3} \).

What is \( f^{-1}(\{\{x_1, x_2\}, \{x_1, x_3\}\}) \)? THIS IS HARD!
Area-\(d = 3\) Case

**Theorem:** For all \(X \subseteq \mathbb{R}^3\) of size \(n\), no three colinear, there exists Area-rainbow set of size \(\Omega(n^\delta)\). (\(\delta\) TBD)

**Proof:** Let \(M\) be a **MAXIMAL AREA-RAINBOW SET**.

Let \(x \in X - M\). WHY IS \(x\) NOT IN \(M\)? Either

- \((\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]\).
- \((\exists x_1, x_2, x_3, x_4 \in M)[\text{A}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]\).
- \((\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)]\).

\(f\) maps an element of \(X - M\) to reason \(x \notin M\).

\(f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}\).

What is \(f^{-1}(\{\{x_1, x_2\}, \{x_1, x_3\}\})\)? THIS IS HARD!

What is \(f^{-1}(\{x_1, x_2\}, \{x_3, x_4\}\)?
Theorem: For all \( X \subseteq \mathbb{R}^3 \) of size \( n \), no three colinear, there exists an area-rainbow set of size \( \Omega(n^\delta) \). (\( \delta \) TBD)

**Proof:** Let \( M \) be a **MAXIMAL AREA-RAINBOW SET**.

Let \( x \in X - M \). Why is \( x \) not in \( M \)? Either

- \((\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]\).
- \((\exists x_1, x_2, x_3, x_4 \in M)[\text{A}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]\).
- \((\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)]\).

\( f \) maps an element of \( X - M \) to reason \( x \notin M \).

\( f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3} \).

What is \( f^{-1}(\{\{x_1, x_2\}, \{x_1, x_3\}\}) \)? THIS IS HARD!

What is \( f^{-1}(\{x_1, x_2\}, \{x_3, x_4\}) \)? THIS IS HARD!
Area-$d = 3$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^3$ of size $n$, no three colinear, there exists Area-rainbow set of size $\Omega(n^\delta)$. ($\delta$ TBD)

**Proof:** Let $M$ be a **MAXIMAL AREA-RAINBOW SET.** Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

- $(\exists x_1, x_2, x_3 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\text{A}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{AREA}(x, x_1, x_2) = \text{AREA}(x_3, x_4, x_5)]$.

$f$ maps an element of $X - M$ to reason $x \notin M$.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$.

What is $f^{-1}({\{x_1, x_2\}, \{x_1, x_3\}})$? THIS IS HARD!

What is $f^{-1}({\{x_1, x_2\}, \{x_3, x_4\}})$? THIS IS HARD!

What is $f^{-1}({\{x_1, x_2\}, \{x_3, x_4, x_5\}})$?
Area-$d = 3$ Case

**Theorem:** For all $X \subseteq \mathbb{R}^3$ of size $n$, no three colinear, there exists an area-rainbow set of size $\Omega(n^\delta)$. ($\delta$ TBD)

**Proof:** Let $M$ be a **MAXIMAL AREA-RAINBOW SET**.
Let $x \in X - M$. WHY IS $x$ NOT IN $M$!? Either

1. $(\exists x_1, x_2, x_3 \in M)[\text{Area}(x, x_1, x_2) = \text{Area}(x, x_1, x_3)]$.
2. $(\exists x_1, x_2, x_3, x_4 \in M)[A(x, x_1, x_2) = \text{Area}(x, x_3, x_4)]$.
3. $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\text{Area}(x, x_1, x_2) = \text{Area}(x_3, x_4, x_5)]$.

$f$ maps an element of $X - M$ to reason $x \notin M$.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$.

What is $f^{-1}(\{\{x_1, x_2\}, \{x_1, x_3\}\})$? THIS IS HARD!

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4\})$? THIS IS HARD!

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4, x_5\})$? THIS IS HARD!

What to do?
WHAT CHANGED?

Why is this proof harder?

**KEY** statement about prior proof:

1. If INVERSE IMG’s are all finite so $M$ is large.
2. If INVERSE IMG’s are subsets of $\mathbb{R}^d$ or $S^d$ then induct.

**KEY:** We cared about $X \subseteq \mathbb{R}^d$ but had to work with $S^d$ as well. NOW we will have to work with more complicated objects.
What Do Inverse Images Look Like?

\[
\{ x : \text{AREA}(x, x_1, x_2) = \text{AREA}(x, x_3, x_4) \} = \\
\{ x : |\text{DET}(x, x_1, x_2)| = |\text{DET}(x, x_3, x_4)| \}.
\]

**Definition:** (Informally) An **Algebraic Variety in** \( \mathbb{R}^d \) **is a set of points in** \( \mathbb{R}^d \) **that satisfy a polynomial equation in** \( d \) **variables.**
**General Theorem**

**Theorem** Let \(2 \leq a \leq d + 1\). Let \(r \in \mathbb{N}\). For all varieties \(V\) of dim \(d\) and degree \(r\) for all sets of \(n\) points on \(V\) there exists an \(a\)-rainbow set of size \(\Omega(n^{1/(2a-1)d})\).

**Corollary** Let \(2 \leq a \leq d + 1\). For all \(X \subseteq \mathbb{R}^d\) of size \(n\) there exists an \(a\)-rainbow set of size \(\Omega(n^{1/(2a-1)d})\).

**Corollary** For all \(X \subseteq \mathbb{R}^d\) of size \(n\) there exists a 2-rainbow set (dist. distances) of size \(\Omega(n^{1/3d})\).

**Corollary** For all \(X \subseteq \mathbb{R}^d\) of size \(n\) there is a 3-rainbow set (dist. areas) of size \(\Omega(n^{1/5d})\).

**Corollary** For all \(X \subseteq \mathbb{R}^d\) of size \(n\) there is a 4-rainbow set (dist. volumes) of size \(\Omega(n^{1/7d})\).

**Comments on the Proof**

1. Proof uses Algebraic Geometry in Proj Space over \(\mathbb{C}\).
2. Proof uses Maximal subsets in same way as easier proofs.
3. Proof is by induction on \(d\).
Open Questions

1. Better Particular Results: e.g., want
   for all $X \subseteq \mathbb{R}^2$ of size $n$, there exists a rainbow set of size
   $\Omega(n^{1/2})$.

2. General Better Results: e.g., want
   Let $1 \leq a \leq d + 1$. For all $X \subseteq \mathbb{R}^d$ of size $n$ there exists a
   rainbow set of size $\Omega(n^{1/ad})$.

3. Get easier proofs of general theorem.

4. Find any nontrivial limits on what we can do. (Trivial: $n^{1/d}$).

5. Algorithmic aspects.