Deriving the Finite Ramsey Theorem from the Infinite Ramsey Theorem

Exposition by William Gasarch

1 Finite Ramsey from Infinite Ramsey

Having proved the infinite Ramsey Theorem, we then want to prove the finite Ramsey Theorem. Can we prove the finite Ramsey Theorem *from* the infinite Ramsey Theorem? Yes, we can!

Def 1.1 $R(m)$ is the smallest n such that, for all 2-colorings of K_n, there is a homog set of size m. (Ramsey's Theorem is that $R(m)$ exists.)

Theorem 1.2 *For every $m \geq 2$, $R(m)$ exists.*

Proof: Suppose, by way of contradiction, that there is some $m \geq 2$ such that $R(m)$ does not exist. Then, for every $n \geq m$, there is some way to color K_n so that there is no monochromatic K_m (we have called this before *homogenous set of size* m). Hence there exist the following:

1. COL_1, a 2-coloring of K_m that has no monochromatic K_m
2. COL_2, a 2-coloring of K_{m+1} that has no monochromatic K_m
3. COL_3, a 2-coloring of K_{m+2} that has no monochromatic K_m

   $\vdots$

j. COL_j, a 2-coloring of K_{m+j-1} that has no monochromatic K_m

   $\vdots$

We will use these 2-colorings to form a 2-coloring COL of K_{N} that has no monochromatic K_m.

Let $e_1, e_2, e_3, \ldots$ be a list of all unordered pairs of elements of N such that every unordered pair appears exactly once. We will color e_1, then e_2, etc.

How should we color e_1? We will color it the way an infinite number of the COL_i's color it. Call that color c_1. Then how to color e_2? Well, first consider ONLY the colorings that colored e_1 with color c_1. Color e_2 the way an infinite number of those colorings color it. And so forth.

We now proceed formally:

$J_0 = \mathsf{N}$

$$COL(e_1) = \begin{cases} \text{RED} & \text{if } |\{j \in J_0 \mid COL_j(e_1) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases} \tag{1}$$

$J_1 = \{j \in J_0 \mid COL(e_1) = COL_j(e_1)\}$

Let $i \geq 2$, and assume that $e_1, \ldots, e_{i-1}$ have been colored. Assume, furthermore, that J_{i-1} is infinite and, for every $j \in J_{i-1}$,

$$\begin{aligned} COL(e_1) &= COL_j(e_1) \\ COL(e_2) &= COL_j(e_2) \\ &\vdots \\ COL(e_{i-1}) &= COL_j(e_{i-1}) \end{aligned}$$

We now color e_i:

$$COL(e_i) = \begin{cases} \text{RED} & \text{if } |\{j \in J_{i-1} \mid COL_j(e_i) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases} \tag{2}$$

$J_i = \{j \in J_{i-1} \mid COL(e_i) = COL_j(e_i)\}$.

One can show by induction that, for every i, J_i is infinite. Hence this process *never* stops.

Claim: If K_{N} is 2-colored with COL, then there is no monochromatic K_m.

Proof of Claim:

Suppose, by way of contradiction, that there is a monochromatic K_m. Let the edges between vertices in that monochromatic K_m be

$$e_{i_1}, \ldots, e_{i_M},$$

where $i_1 < i_2 < \cdots < i_M$ and $M = \binom{m}{2}$. For every $j \in J_{i_M}$, COL_j and COL agree on the colors of those edges. Choose $j \in J_{i_M}$ so that all the vertices of the monochromatic K_m are elements of the vertex set of K_{m+j-1}. Then COL_j is a 2-coloring of the edges of K_{m+j-1} that has a monochromatic K_m, in contradiction to the definition of COL_j.

End of Proof of Claim

Hence we have produced a 2-coloring of K_{N} that has no monochromatic K_m. This contradicts The Infinite Ramsey Theorem. Therefore, our initial supposition—that $R(m)$ does not exist—is false. ∎

Note that this proof does not give an upper bounds on $R(m)$.

Think about: Is there a proof that gives an upper bound on $R(m)$?

2 Proof of Large Ramsey Theorem

In all of the theorems presented in the course so far, the labels on the vertices did *not* matter. In this section, the labels *do* matter.

Def 2.1 A finite set $F \subseteq \mathsf{N}$ is called *large* if the size of F is at least as large as the smallest element of F.

Example 2.2

1. The set $\{1, 2, 10\}$ is large: It has 3 elements, the smallest element is 1, and $3 \geq 1$.
2. The set $\{5, 10, 12, 17, 20\}$ is large: It has 5 elements, the smallest element is 5, and $5 \geq 5$.
3. The set $\{20, 30, 40, 50, 60, 70, 80, 90, 100\}$ is not large: It has 9 elements, the smallest element is 20, and $9 < 20$.
4. The set $\{5, 30, 40, 50, 60, 70, 80, 90, 100\}$ is large: It has 9 elements, the smallest element is 5, and $9 \geq 5$.
5. The set $\{101, \ldots, 190\}$ is not large: It has 90 elements, the smallest element is 101, and $90 < 101$.

We will be considering monochromatic K_m's where the underlying set of vertices is a large set. We need a definition to identify the underlying set.

Def 2.3 Let COL be a 2-coloring of K_n. A set A of vertices is *homogeneous* if there exists a color c such that, for all $x, y \in A$ with $x \neq y$, $COL(\{x, y\}) = c$. In other words, all of the edges between elements of A are the same color. One could also say that there is a monochromatic $K_{|A|}$.

Let COL be a 2-coloring of K_n. Recall that the vertex set of K_n is $\{1, 2, \ldots, n\}$. Consider the set $\{1, 2\}$. It is clearly both homogeneous and large (using our definition of large). Hence the statement

"for every $n \geq 2$, every 2-coloring of K_n has a large homogeneous set"

is true but trivial.

What if we used $V = \{m, m+1, \ldots, m+n\}$ as our vertex set? Then a large homogeneous set would have to have size at least m.

Notation 2.4 K_n^m is the graph with vertex set $\{m, m+1, \ldots, m+n\}$ and edge set consisting of all unordered pairs of vertices. The superscript (m) indicates that we are labeling our vertices starting with m, and the subscript (n) is one less than the number of vertices.

Note 2.5 The vertex set of K_n^m (namely, $\{m, m+1, \ldots, m+n\}$) has $n+1$ elements. Hence if K_n^m has a large homogeneous set, then $n+1 \geq m$ (equivalently, $n \geq m-1$). We could have chosen to use K_n^m to denote the graph with vertex set $\{m+1, \ldots, m+n\}$, so that the smallest vertex is $m+1$ and the number of vertices is n, but the set we have designated as K_n^m will better serve our purposes.

Notation 2.6 $LR(m)$ is the least n, if it exists, such that every 2-coloring of K_n^m has a large homogeneous set.

We first prove a theorem about infinite graphs and large homogeneous sets.

Theorem 2.7 *If COL is any 2-coloring of K_{N}, then, for every $m \geq 2$, there is a large homogeneous set whose smallest element is at least as large as m.*

Proof: Let COL be any 2-coloring of K_{N}. By The Infinite Ramsey Theorem there exist an infinite set of vertices,

$$v_1 < v_2 < v_3 < \cdots,$$

and a color c such that, for all i, j, $COL(\{v_i, v_j\}) = c$. (This could be called an infinite homogeneous set.) Let i be such that $v_i \geq m$. The set

$$\{v_i, \ldots, v_{i+v_i-1}\}$$

is a homogeneous set that contains v_i elements and whose smallest element is v_i. Since $v_i \geq v_i$, it is a large set; hence it is a large homogeneous set. ∎

Theorem 2.8 *For every $m \geq 2$, $LR(m)$ exists.*

Proof: This proof is similar to our proof of the finite Ramsey Theorem *from* the infinite Ramsey Theorem.

Suppose, by way of contradiction, that there is some $m \geq 2$ such that $LR(m)$ does not exist. Then, for every $n \geq m-1$, there is some way to color K_n^m so that there is no large homogeneous set. Hence there exist the following:

1. COL_1, a 2-coloring of K_{m-1}^m that has no large homogeneous set
2. COL_2, a 2-coloring of K_m^m that has no large homogeneous set
3. COL_3, a 2-coloring of K_{m+1}^m that has no large homogeneous set

$\vdots$

j. COL_j, a 2-coloring of K_{m+j-2}^m that has no large homogeneous set

$\vdots$

We will use these 2-colorings to form a 2-coloring COL of K_{N} that has no large homogeneous set whose smallest element is at least as large as m.

Let $e_1, e_2, e_3, \ldots$ be a list of all unordered pairs of elements of N such that every unordered pair appears exactly once. We will color e_1, then e_2, etc.

How should we color e_1? We will color it the way an infinite number of the COL_i's color it. Call that color c_1. Then how to color e_2? Well, first consider ONLY the colorings that colored e_1 with color c_1. Color e_2 the way an infinite number of those colorings color it. And so forth.

We now proceed formally:

$J_0 = \mathsf{N}$

$$COL(e_1) = \begin{cases} \text{RED} & \text{if } |\{j\ inJ_0\ midCOL_j(e_1) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases} \quad (3)$$

$J_1 = \{j \in J_0 \mid COL(e_1) = COL_j(e_1)\}$

Let $i \geq 2$, and assume that $e_1, \ldots, e_{i-1}$ have been colored. Assume, furthermore, that J_{i-1} is infinite and, for every $j \in J_{i-1}$,

$$\begin{aligned} COL(e_1) = &\ COL_j(e_1) \\ COL(e_2) = &\ COL_j(e_2) \\ &\vdots \\ COL(e_{i-1}) = &\ COL_j(e_{i-1}) \end{aligned}$$

We now color e_i:

$$COL(e_i) = \begin{cases} \text{RED} & \text{if } |\{j \in J_{i-1} \mid COL_j(e_i) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases} \tag{4}$$

$J_i = \{j \in J_{i-1} \mid COL(e_i) = COL_j(e_i)\}$

One can show by induction that, for every i, J_i is infinite. Hence this process *never* stops.

Claim: If K_{N} is 2-colored with COL, then there is no large homogeneous set whose smallest element is at least as large as m.

Proof of Claim:

Suppose, by way of contradiction, that there is a large homogeneous set whose smallest element is at least as large as m. Without loss of generality, we can assume that the size of the large homogeneous set is equal to its smallest element. Let the vertices of that large homogeneous set be $v_1, v_2, \ldots v_{v_1}$, where $m \le v_1 < v_2 < \cdots < v_{v_1}$, and let the edges between those vertices be

$$e_{i_1}, \ldots, e_{i_M},$$

where $i_1 < i_2 < \cdots < i_M$ and $M = \binom{v_1}{2}$. For every $j \in J_{i_M}$, COL_j and COL agree on the colors of those edges. Choose $j \in J_{i_M}$ so that all the vertices of the large homogeneous set are elements of the vertex set of K^m_{m+j-2}. Then COL_j is a 2-coloring of the edges of K^m_{m+j-2} that has a large homogeneous set, in contradiction to the definition of COL_j.

End of Proof of Claim

Hence we have produced a 2-coloring of K_{N} that has no large homogeneous set whose smallest element is at least as large as m. This contradicts The Infinite Ramsey Theorem. Therefore, our initial supposition—that $LR(m)$ does not exist—is false. ∎

Note that this proof does not give an upper bounds on $LR(m)$.

Think about: Is there a proof that gives an upper bound on $LR(m)$?

References

[1] R. Graham, A. Rothchild, and J. Spencer. *Ramsey Theory*. Wiley, 1990.

[2] B. Landmann and A. Robertson. *Ramsey Theory over the Integers*. AMS, 2003.

[3] F. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, 30:264–286, 1930. Series 2.