Deriving the Finite Ramsey Theorem from the Infinite Ramsey Theorem

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1 Finite Ramsey from Infinite Ramsey

Having proved the infinite Ramsey Theorem, we then want to prove the finite Ramsey Theorem. Can we prove the finite Ramsey Theorem from the infinite Ramsey Theorem? Yes, we can!

**Def 1.1** $R(m)$ is the smallest $n$ such that, for all 2-colorings of $K_n$, there is a homog set of size $m$. (Ramsey’s Theorem is that $R(m)$ exists.)

**Theorem 1.2** For every $m \geq 2$, $R(m)$ exists.

**Proof:** Suppose, by way of contradiction, that there is some $m \geq 2$ such that $R(m)$ does not exist. Then, for every $n \geq m$, there is some way to color $K_n$ so that there is no monochromatic $K_m$ (we have called this before homogenous set of size $m$). Hence there exist the following:

1. $COL_1$, a 2-coloring of $K_m$ that has no monochromatic $K_m$
2. $COL_2$, a 2-coloring of $K_{m+1}$ that has no monochromatic $K_m$
3. $COL_3$, a 2-coloring of $K_{m+2}$ that has no monochromatic $K_m$
   
   ...

   $j$. $COL_j$, a 2-coloring of $K_{m+j-1}$ that has no monochromatic $K_m$
   
   ...

We will use these 2-colorings to form a 2-coloring $COL$ of $K_N$ that has no monochromatic $K_m$.

Let $e_1, e_2, e_3, \ldots$ be a list of all unordered pairs of elements of $N$ such that every unordered pair appears exactly once. We will color $e_1$, then $e_2$, etc.

How should we color $e_1$? We will color it the way an infinite number of the $COL_i$’s color it. Call that color $c_1$. Then how to color $e_2$? Well, first
consider ONLY the colorings that colored \( e_1 \) with color \( c_1 \). Color \( e_2 \) the way an infinite number of those colorings color it. And so forth.

We now proceed formally:

\[
J_0 = \mathbb{N}
\]

\[
COL(e_1) = \begin{cases} 
    \text{RED} & \text{if } |\{ j \in J_0 \mid COL_j(e_1) = \text{RED}\}| \text{ is infinite} \\
    \text{BLUE} & \text{otherwise}
\end{cases}
\] (1)

\[
J_1 = \{ j \in J_0 \mid COL(e_1) = COL_j(e_1) \}
\]

Let \( i \geq 2 \), and assume that \( e_1, \ldots, e_{i-1} \) have been colored. Assume, furthermore, that \( J_{i-1} \) is infinite and, for every \( j \in J_{i-1} \),

\[
COL(e_1) = COL_j(e_1) \\
COL(e_2) = COL_j(e_2) \\
\vdots \\
COL(e_{i-1}) = COL_j(e_{i-1})
\]

We now color \( e_i \):

\[
COL(e_i) = \begin{cases} 
    \text{RED} & \text{if } |\{ j \in J_{i-1} \mid COL_j(e_i) = \text{RED}\}| \text{ is infinite} \\
    \text{BLUE} & \text{otherwise}
\end{cases}
\] (2)

\[
J_i = \{ j \in J_{i-1} \mid COL(e_i) = COL_j(e_i) \}
\]

One can show by induction that, for every \( i \), \( J_i \) is infinite. Hence this process never stops.

**Claim:** If \( K_N \) is 2-colored with \( COL \), then there is no monochromatic \( K_m \).

**Proof of Claim:**

Suppose, by way of contradiction, that there is a monochromatic \( K_m \). Let the edges between vertices in that monochromatic \( K_m \) be

\[
e_{i_1}, \ldots, e_{i_M},
\]

where \( i_1 < i_2 < \cdots < i_M \) and \( M = \binom{m}{2} \). For every \( j \in J_{i_M} \), \( COL_j \) and \( COL \) agree on the colors of those edges. Choose \( j \in J_{i_M} \) so that all the vertices of the monochromatic \( K_m \) are elements of the vertex set of \( K_{m+j-1} \). Then \( COL_j \) is a 2-coloring of the edges of \( K_{m+j-1} \) that has a monochromatic \( K_m \), in contradiction to the definition of \( COL_j \).
Hence we have produced a 2-coloring of $K_N$ that has no monochromatic $K_m$. This contradicts The Infinite Ramsey Theorem. Therefore, our initial supposition—that $R(m)$ does not exist—is false.

Note that this proof does not give an upper bounds on $R(m)$.

**Think about:** Is there a proof that gives an upper bound on $R(m)$?

## 2 Proof of Large Ramsey Theorem

In all of the theorems presented in the course so far, the labels on the vertices did *not* matter. In this section, the labels *do* matter.

**Def 2.1** A finite set $F \subseteq \mathbb{N}$ is called *large* if the size of $F$ is at least as large as the smallest element of $F$.

**Example 2.2**

1. The set $\{1, 2, 10\}$ is large: It has 3 elements, the smallest element is 1, and $3 \geq 1$.

2. The set $\{5, 10, 12, 17, 20\}$ is large: It has 5 elements, the smallest element is 5, and $5 \geq 5$.

3. The set $\{20, 30, 40, 50, 60, 70, 80, 90, 100\}$ is not large: It has 9 elements, the smallest element is 20, and $9 < 20$.

4. The set $\{5, 30, 40, 50, 60, 70, 80, 90, 100\}$ is large: It has 9 elements, the smallest element is 5, and $9 \geq 5$.

5. The set $\{101, \ldots , 190\}$ is not large: It has 90 elements, the smallest element is 101, and $90 < 101$.

We will be considering monochromatic $K_m$'s where the underlying set of vertices is a large set. We need a definition to identify the underlying set.

**Def 2.3** Let $COL$ be a 2-coloring of $K_n$. A set $A$ of vertices is *homogeneous* if there exists a color $c$ such that, for all $x, y \in A$ with $x \neq y$, $COL(\{x, y\}) = c$. In other words, all of the edges between elements of $A$ are the same color. One could also say that there is a monochromatic $K_{|A|}$. 

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*End of Proof of Claim*
Let \( \text{COL} \) be a 2-coloring of \( K_n \). Recall that the vertex set of \( K_n \) is \{1, 2, \ldots, n\}. Consider the set \{1, 2\}. It is clearly both homogeneous and large (using our definition of large). Hence the statement

“for every \( n \geq 2 \), every 2-coloring of \( K_n \) has a large homogeneous set”

is true but trivial.

What if we used \( V = \{m, m + 1, \ldots, m + n\} \) as our vertex set? Then a large homogeneous set would have to have size at least \( m \).

**Notation 2.4** \( K_m^n \) is the graph with vertex set \{m, m + 1, \ldots, m + n\} and edge set consisting of all unordered pairs of vertices. The superscript \( (m) \) indicates that we are labeling our vertices starting with \( m \), and the subscript \( (n) \) is one less than the number of vertices.

**Note 2.5** The vertex set of \( K_m^n \) (namely, \{m, m + 1, \ldots, m + n\}) has \( n + 1 \) elements. Hence if \( K_m^n \) has a large homogeneous set, then \( n + 1 \geq m \) (equivalently, \( n \geq m - 1 \)). We could have chosen to use \( K_m^n \) to denote the graph with vertex set \{m + 1, \ldots, m + n\}, so that the smallest vertex is \( m + 1 \) and the number of vertices is \( n \), but the set we have designated as \( K_m^n \) will better serve our purposes.

**Notation 2.6** \( LR(m) \) is the least \( n \), if it exists, such that every 2-coloring of \( K_m^n \) has a large homogeneous set.

We first prove a theorem about infinite graphs and large homogeneous sets.

**Theorem 2.7** If \( \text{COL} \) is any 2-coloring of \( K_N \), then, for every \( m \geq 2 \), there is a large homogeneous set whose smallest element is at least as large as \( m \).

**Proof:** Let \( \text{COL} \) be any 2-coloring of \( K_N \). By The Infinite Ramsey Theorem there exist an infinite set of vertices,

\[ v_1 < v_2 < v_3 < \cdots, \]

and a color \( c \) such that, for all \( i, j \), \( \text{COL}\{v_i, v_j\} = c \). (This could be called an infinite homogeneous set.) Let \( i \) be such that \( v_i \geq m \). The set

\[ \{v_i, \ldots, v_{i+v_i-1}\} \]

is a homogeneous set that contains \( v_i \) elements and whose smallest element is \( v_i \). Since \( v_i \geq v_i \), it is a large set; hence it is a large homogeneous set. \( \square \)
Theorem 2.8  For every $m \geq 2$, $LR(m)$ exists.

Proof:  This proof is similar to our proof of the finite Ramsey Theorem from the infinite Ramsey Theorem.

Suppose, by way of contradiction, that there is some $m \geq 2$ such that $LR(m)$ does not exist. Then, for every $n \geq m - 1$, there is some way to color $K_n^m$ so that there is no large homogeneous set. Hence there exist the following:

1. $COL_1$, a 2-coloring of $K_{m-1}^m$ that has no large homogeneous set
2. $COL_2$, a 2-coloring of $K_m^m$ that has no large homogeneous set
3. $COL_3$, a 2-coloring of $K_{m+1}^m$ that has no large homogeneous set
   ...
   j. $COL_j$, a 2-coloring of $K_{m+j-2}^m$ that has no large homogeneous set
   ...

We will use these 2-colorings to form a 2-coloring $COL$ of $K_N$ that has no large homogeneous set whose smallest element is at least as large as $m$.

Let $e_1, e_2, e_3, \ldots$ be a list of all unordered pairs of elements of $N$ such that every unordered pair appears exactly once. We will color $e_1$, then $e_2$, etc.

How should we color $e_1$? We will color it the way an infinite number of the $COL_i$’s color it. Call that color $c_1$. Then how to color $e_2$? Well, first consider ONLY the colorings that colored $e_1$ with color $c_1$. Color $e_2$ the way an infinite number of those colorings color it. And so forth.

We now proceed formally:

$J_0 = N$

$$COL(e_1) = \begin{cases} 
\text{RED} & \text{if } |\{j \in J_0 \mid COL_j(e_1) = \text{RED}\}| \text{ is infinite} \\
\text{BLUE} & \text{otherwise}
\end{cases} \quad \text{(3)}$$

$J_1 = \{j \in J_0 \mid COL(e_1) = COL_j(e_1)\}$

Let $i \geq 2$, and assume that $e_1, \ldots, e_{i-1}$ have been colored. Assume, furthermore, that $J_{i-1}$ is infinite and, for every $j \in J_{i-1}$,
\[ \text{We now color } e_i: \]

\[ \text{COL}(e_i) = \begin{cases} 
\text{RED} & \text{if } |\{ j \in J_{i-1} \mid \text{COL}_j(e_i) = \text{RED} \}| \text{ is infinite} \\
\text{BLUE} & \text{otherwise} 
\end{cases} \quad (4) \]

\[ J_i = \{ j \in J_{i-1} \mid \text{COL}(e_i) = \text{COL}_j(e_i) \} \]

One can show by induction that, for every \( i \), \( J_i \) is infinite. Hence this process never stops.

Claim: If \( K_N \) is 2-colored with \( \text{COL} \), then there is no large homogeneous set whose smallest element is at least as large as \( m \).

Proof of Claim:

Suppose, by way of contradiction, that there is a large homogeneous set whose smallest element is at least as large as \( m \). Without loss of generality, we can assume that the size of the large homogeneous set is equal to its smallest element. Let the vertices of that large homogeneous set be \( v_1, v_2, \ldots, v_{v_1} \), where \( m \leq v_1 < v_2 < \cdots < v_{v_1} \), and let the edges between those vertices be \( e_{i_1}, \ldots, e_{i_M} \), where \( i_1 < i_2 < \cdots < i_M \) and \( M = \binom{v_1}{2} \). For every \( j \in J_{i_M} \), \( \text{COL}_j \) and \( \text{COL} \) agree on the colors of those edges. Choose \( j \in J_{i_M} \) so that all the vertices of the large homogeneous set are elements of the vertex set of \( K_{m+j-2}^m \). Then \( \text{COL}_j \) is a 2-coloring of the edges of \( K_{m+j-2}^m \) that has a large homogeneous set, in contradiction to the definition of \( \text{COL}_j \).

End of Proof of Claim

Hence we have produced a 2-coloring of \( K_N \) that has no large homogeneous set whose smallest element is at least as large as \( m \). This contradicts The Infinite Ramsey Theorem. Therefore, our initial supposition—that \( LR(m) \) does not exist—is false.

Note that this proof does not give an upper bounds on \( LR(m) \).

Think about: Is there a proof that gives an upper bound on \( LR(m) \)?
References

